Some asymptotic results for integrated empirical processes with applications to statistical tests

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1. Settings

1. Settings — Sampling

Definitions –

1. Let $F = \{F(t),\ t \in \mathbb{R}\}$ be a *continuous* distribution function [d.f.] and denote by $Q = \{Q(u), u \in [0,1]\}$ the usual quantile function (generalized inverse) pertaining to F defined as

$$Q(u) := \inf\{t \in \mathbb{R} : F(t) \geq u\} \qquad u \in (0,1)$$

$$Q(0) := \lim_{u \downarrow 0} Q(u) \quad \text{and} \quad Q(1) := \lim_{u \uparrow 1} Q(u)$$

2. Let $\{U_i,\ i \in \mathbb{N}^*\}$ be a sequence of i.i.d. random variables [r.v.] uniformly distributed on $[0,1]$.
   - Set for any $i \in \mathbb{N}^*$: $\quad X_i := Q(U_i)$

     The *sample* $\{X_i, i \in \mathbb{N}^*\}$ consists of i.i.d. r.v.'s with d.f. F: $F(t) = \mathbb{P}\{X_1 \leq t\}$ for $t \in \mathbb{R}$.
   - Conversely: $\quad U_i = F(X_i)$

1. Settings — Empirical d.f./processes

Definitions –

3. ***Empirical d.f.'s*** **based upon the samples $X_1, \ldots, X_n$ and $U_1, \ldots, U_n$: for each $n \in \mathbb{N}^*$,**

$$\mathbb{F}_n(t) := \frac{1}{n} \# \Big\{ i \in \{1, \ldots, n\} : X_i \leq t \Big\} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}} \qquad t \in \mathbb{R}$$

$$\mathbb{U}_n(u) := \frac{1}{n} \# \Big\{ i \in \{1, \ldots, n\} : U_i \leq u \Big\} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{U_i \leq u\}} \qquad u \in [0, 1]$$

4. **Related *empirical processes:***

$$\alpha_n(t) := \sqrt{n} \Big(\mathbb{F}_n(t) - F(t) \Big) \qquad t \in \mathbb{R}$$

$$\beta_n(u) := \sqrt{n} \Big(\mathbb{U}_n(u) - u \Big) \qquad u \in [0, 1]$$

Proposition –

$$\alpha_n(t) = \beta_n(F(t)) \qquad t \in \mathbb{R}$$

$\longrightarrow$ *It is enough to work with the uniform law (i.e. the U_i's, $\mathbb{U}_n, \beta_n$)*

1. Settings — Empirical d.f./processes

Definitions –

5. **Families of *integrated d.f.'s* and *integrated empirical d.f.'s* associated with the uniform d.f.,** for any $n \in \mathbb{N}^*$, $u \in [0,1]$:

$$U^{(0)}(u) := u \qquad \mathbb{U}_n^{(0)}(u) := \mathbb{U}_n(u)$$

$$U^{(1)}(u) := \int_0^u v\,dv \qquad \mathbb{U}_n^{(1)}(u) := \int_0^u \mathbb{U}_n(v)\,d\mathbb{U}_n(v)$$

$\longrightarrow$ ***Introduced by Henze & Nikitin – 2003***

Definitions –

5. **Families of *integrated d.f.'s* and *integrated empirical d.f.'s* associated with the uniform d.f., for any $n \in \mathbb{N}^*$, $u \in [0,1]$:**

$$U^{(0)}(u) := u \qquad \mathbb{U}_n^{(0)}(u) := \mathbb{U}_n(u)$$
$$U^{(1)}(u) := \int_0^u v\,dv \qquad \mathbb{U}_n^{(1)}(u) := \int_0^u \mathbb{U}_n(v)\,d\mathbb{U}_n(v)$$

and for any $p \in \mathbb{N}^*$:

$$U^{(p)}(u) := \int_0^u dv_1 \int_0^{v_1} dv_2 \ldots \int_0^{v_{p-1}} v_p\,dv_p$$
$$\mathbb{U}_n^{(p)}(u) := \int_0^u d\mathbb{U}_n(v_1) \int_0^{v_1} d\mathbb{U}_n(v_2) \ldots \int_0^{v_{p-1}} \mathbb{U}_n(v_p)\,d\mathbb{U}_n(v_p)$$

6. **Family of *integrated empirical processes:***

$$\beta_n^{(p)}(u) := \sqrt{n}\left(\mathbb{U}_n^{(p)}(u) - U^{(p)}(u)\right)$$

1. Settings — Empirical d.f./processes

Proposition – **For each $p \in \mathbb{N}$:**

- ***Integrated d.f.'s:***

$$U^{(p)}(u) = \frac{u^{p+1}}{(p+1)!}$$

- ***Integrated empirical d.f.'s:***

$$\mathbb{U}_n^{(p)}(u) = \frac{1}{n^{p+1}} \binom{n\mathbb{U}_n(u)+p}{p+1} \quad \text{a.s.}$$

$$= \frac{\mathbb{U}_n(u)^{p+1}}{(p+1)!} + \sum_{k=1}^{p} \left[{p+1 \atop k+1} \right] \frac{\mathbb{U}_n(u)^k}{n^{p-k+1}} \qquad n \in \mathbb{N}^* \quad u \in [0,1]$$

where $\left[{p+1 \atop k+1} \right]$, $1 \leq k \leq p$, are the unsigned Stirling numbers:

$$\prod_{i=1}^{p} (x+i) = \sum_{k=0}^{p} \left[{p+1 \atop k+1} \right] x^k$$

1. Settings — Some Gaussian processes

Definitions –

1. ***Wiener process*** $\mathbb{W} = \{\mathbb{W}(s),\ s \geq 0\}$:

$$\mathbb{E}\Big(\mathbb{W}(s)\mathbb{W}(t)\Big) = s \wedge t \qquad s,t \geq 0$$

2. ***Brownian bridge*** $\mathbb{B} = \{\mathbb{B}(u),\ u \in [0,1]\}$:

$$\mathbb{E}\Big(\mathbb{B}(u)\mathbb{B}(v)\Big) = u \wedge v - uv \qquad u,v \in [0,1]$$

3. ***Kiefer process*** $\mathbb{K} = \{\mathbb{K}(s,u),\ s \geq 0,\ u \in [0,1]\}$:

$$\mathbb{E}\Big(\mathbb{K}(s,u)\mathbb{K}(t,v)\Big) = (s \wedge t)\,(u \wedge v - uv) \quad s,t \geq 0 \quad u,v \in [0,1]$$

Recall – **'Extracted' Kiefer process** $\{\mathbb{K}(n,u),\ n \in \mathbb{N}^*,\ u \in [0,1]\}$:

$$\mathbb{K}(n,u) = \sum_{i=1}^{n} \mathbb{B}_i(u) \qquad n \in \mathbb{N}^* \quad u \in [0,1]$$

where $\{\mathbb{B}_i,\ i \in \mathbb{N}^*\}$ **is a sequence of independent Brownian bridges**

1. Settings — Aims/Motivations

Aims –

1. **To derive *strong approximations* for *integrated empirical processes* by a family of Gaussian processes involving Brownian bridge and Kiefer process similar to the famous *'Hungarian construction'***
2. **To provide *new statistical tests:***
   - **Kolmogorov-Smirnov-like tests**
   - **Cramér-von Mises-like tests**
   - **Change-point detection, etc.**

Motivations –

1. **Study of *integrated Brownian motions* and *bridges***
2. **Computation of *Bahadur-efficiency* for various tests**
   - **Leading to *more performant procedures...***

2. Asymptotics

2. Asymptotics — 'Hungarian construction'

***Theorem** – Gaussian approximations (Komlós–Major–Tusnády 1975)*

1. **On a suitable probability space, we may define the uniform empirical process $\{\beta_n,\ n\in\mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n,\ n\in\mathbb{N}^*\}$, such that**

$$\sup_{u\in[0,1]} \left|\beta_n(u) - \mathbb{B}_n(u)\right| \underset{n\to\infty}{=} \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{a.s.}$$

2. **On a suitable probability space, we may define the uniform empirical process $\{\beta_n,\ n\in\mathbb{N}^*\}$, in combination with a Kiefer process $\{\mathbb{K}(s,u),\ s\geq 0,\ u\in[0,1]\}$, such that**

$$\max_{1\leq k\leq n} \sup_{u\in[0,1]} \left|\sqrt{k}\,\beta_k(u) - \mathbb{K}(k,u)\right| \underset{n\to\infty}{=} \mathcal{O}\left((\log n)^2\right) \quad \text{a.s.}$$

***Theorem 1 – Gaussian approximation 1** (AA–B–L 2017)*

On a suitable probability space, we may define the p-fold integrated uniform empirical process $\{\beta_n^{(p)},\ n\in\mathbb{N}^*\}$, in combination with a sequence of Brownian bridges $\{\mathbb{B}_n,\ n\in\mathbb{N}^*\}$, such that

$$\sup_{u\in[0,1]} \left|\beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u)\right| \underset{n\to\infty}{=} \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{a.s.}$$

where $\mathbb{B}_n^{(p)}$ is the process defined by

$$\mathbb{B}_n^{(p)}(u) := \frac{1}{p!}\, u^p\, \mathbb{B}_n(u) \qquad u\in[0,1]$$

2. Asymptotics — Gaussian approximations

Theorem 2 – Gaussian approximation 2 *(AA–B–L 2017)*

On a suitable probability space, we may define the p-fold integrated uniform empirical process $\{\beta_n^{(p)},\ n\in\mathbb{N}^*\}$, in combination with a Kiefer process $\{\mathbb{K}(s,u),\ s\geq 0,\ u\in[0,1]\}$, such that

$$\max_{1\leq k\leq n}\sup_{u\in[0,1]}\left|\sqrt{k}\,\beta_k^{(p)}(u)-\mathbb{K}^{(p)}(k,u)\right| \underset{n\to\infty}{=} \mathcal{O}\Big((\log n)^2\Big) \quad \text{a.s.}$$

where $\mathbb{K}^{(p)}$ is the process defined by

$$\mathbb{K}^{(p)}(s,u) := \frac{1}{p!}\,u^p\,\mathbb{K}(s,u) \qquad s\geq 0 \quad u\in[0,1]$$

2. Asymptotics — Gaussian approximations

Sketch of proof –

Lemma 1 – Representation of $\beta_n^{(p)}$ by means of β_n and $\mathbb{U}_n$

$$\beta_n^{(p)}(u) = \frac{1}{p!}\, u^p\, \beta_n(u) + \sum_{k=2}^{p+1} \frac{\binom{p+1}{k}}{(p+1)!}\, \frac{u^{p+1-k}}{n^{(k-1)/2}}\, \beta_n(u)^k + \sum_{k=1}^{p} \begin{bmatrix} p+1 \\ k+1 \end{bmatrix} \frac{\mathbb{U}_n(u)^k}{n^{p-k+1/2}} \qquad n \in \mathbb{N}^* \quad u \in [0,1]$$

Lemma 2 – Chung's law of the iterated logarithm for β_n

$$\sup_{u\in[0,1]} \left|\beta_n(u)\right| \underset{n\to\infty}{=} \mathcal{O}\left(\sqrt{\log\log n}\right) \quad \text{a.s.}$$

2. Asymptotics — Gaussian approximations

Corollary – Law of the iterated logarithm for $\beta_n^{(p)}$

$$\limsup_{n\to\infty} \frac{\sup_{u\in[0,1]} \left|\beta_n^{(p)}(u)\right|}{\sqrt{\log\log n}} = \frac{(p+1/2)^{p+1/2}}{p!\,(p+1)^{p+1}} \quad \text{a.s.}$$

Examples –

$$\limsup_{n\to\infty} \frac{\sup_{u\in[0,1]} \left|\beta_n^{(0)}(u)\right|}{\sqrt{\log\log n}} = \frac{1}{\sqrt{2}} \quad \text{a.s.}$$ *Chung's LIL*

$$\limsup_{n\to\infty} \frac{\sup_{u\in[0,1]} \left|\beta_n^{(1)}(u)\right|}{\sqrt{\log\log n}} = \frac{3}{8}\sqrt{\frac{3}{2}} \quad \text{a.s.}$$

2. Asymptotics — A key inequality

Complement* – *An exponential bound *(AA–B–L 2017)*

For any $n \in \mathbb{N}^*$ and large enough x:

$$\mathbb{P}\left\{\sup_{u\in[0,1]}\left|\beta_n^{(p)}(u)-\mathbb{B}_n^{(p)}(u)\right| \geq \frac{1}{\sqrt{n}}\left(x+12\log(n)\right)\right\} \leq A_p \sum_{k=2}^{p+1} e^{-B_p\, x^{2/k} n^{1-2/k}}$$

where A_p and B_p are positive constants depending on p

$\longrightarrow$ Supplies Theorem 1 and other consequences...

3. Applications

3. Applications — Goodness-of-fit

Problem 1* – Testing the null hypothesis *(fitting a distribution)

$$\mathcal{H}_0 : F = F_0$$

Tools: some statistics

Definitions –

1. ***Kolmogorov-Smirnov-like statistic***

$$\mathbf{S}_n^{(p)} := \sqrt{n} \sup_{t \in \mathbb{R}} \left| \left(\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t) \right) \right|$$

2. ***Cramér-von Mises-like statistic***

$$\mathbf{T}_n^{(p)} := n \int_{\mathbb{R}} \left(\mathbb{F}_n^{(p)}(t) - F_0^{(p)}(t) \right)^2 dF_0(t)$$

Problem 1 – **Testing the null hypothesis** ***(fitting a distribution)***

$$\boxed{\mathcal{H}_0 : F = F_0}$$

Theorem 3 – ***(AA–B–L 2017)***

Under $\mathcal{H}_0$:

$$\left| \mathsf{S}_n^{(p)} - \sup_{u\in[0,1]} \left|\mathbb{B}_n^{(p)}(u)\right| \right| \underset{n\to\infty}{=} \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{a.s.}$$

$$\left| \mathsf{T}_n^{(p)} - \int_0^1 \left[\mathbb{B}_n^{(p)}(u)\right]^2 du \right| \underset{n\to\infty}{=} \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}\,\log n\right) \quad \text{a.s.}$$

3. Applications — Goodness-of-fit

Problem 1 – Hypothesis testing *(fitting a distribution)*

$$\mathcal{H}_0 : F = F_0 \quad \text{versus} \quad \mathcal{H}_1 : F = F_1$$

Simulations – Tests for uniformity: $F_0(u) = u$

- Random samples of size $n = 100$ at level $\alpha = 0.05$ based on 10000 replications for each $p \in \{0, 1, 2, 3\}$
- Computation of powers

Various alternatives

$A_k : F_1(u) = 1 - (1-u)^k \quad u \in [0,1]$

$$B_k : F_1(u) = \begin{cases} 2^{k-1}u^k & u \in [0, 0.5] \\ 1 - 2^{k-1}(1-u)^k & u \in [0.5, 1] \end{cases}$$

$$C_k : F_1(u) = \begin{cases} 0.5 - 2^{k-1}(0.5-u)^k & u \in [0, 0.5] \\ 0.5 + 2^{k-1}(u-0.5)^k & u \in [0.5, 1] \end{cases}$$

F_1	$S_n^{(0)}$	$S_n^{(1)}$	$S_n^{(2)}$	$S_n^{(3)}$
$A_{1.5}$	0.91	0.96	0.94	0.58
A_2	1	1	1	0.99
$B_{1.5}$	0.34	0.54	0.61	0.17
B_2	0.94	0.97	0.98	0.76
B_3	1	1	1	0.99
$C_{1.5}$	0.48	0.34	0.31	0.48
C_2	0.96	0.81	0.75	0.86
C_3	1	0.99	0.99	0.99

3. Applications — Two-sample problem

Problem 2 – **Testing the null hypothesis** *(comparison of two samples)*

$$\mathcal{H}_0' : F = G$$

Settings –

- $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$: independent random *samples* from *continuous* d.f.'s F and G
- $\mathbb{F}_m^{(p)}$ and $\mathbb{G}_n^{(p)}$: their *p-fold integrated empirical d.f.'s*

Definitions –

1. *Kolmogorov-Smirnov-like statistic*

$$\mathbf{S}_{m,n}^{(p)} := \sqrt{\frac{mn}{m+n}} \sup_{t\in\mathbb{R}} \left|\left(\mathbb{F}_m^{(p)}(t) - \mathbb{G}_n^{(p)}(t)\right)\right|$$

2. *Cramér-von Mises-like statistic*

$$\mathbf{T}_{m,n}^{(p)} := \frac{mn}{m+n} \int_{\mathbb{R}} \left(\mathbb{F}_m^{(p)}(t) - \mathbb{G}_n^{(p)}(t)\right)^2 dF(t)$$

3. Applications — Two-sample problem

Problem 2 – **Testing the null hypothesis** ***(comparison of two samples)***

$$\mathcal{H}_0' : F = G$$

Theorem 4 – *(AA–B–L 2017)*

Under $\mathcal{H}_0'$:

$$\left| \mathbf{S}_{m,n}^{(p)} - \sup_{u\in[0,1]} \left|\mathbb{B}_{m,n}^{(p)}(u)\right| \right| \underset{m,n\to\infty}{=} \mathcal{O}(\varphi(m,n)) \quad \text{a.s.}$$

$$\left| \mathbf{T}_{m,n}^{(p)} - \int_0^1 \left[\mathbb{B}_{m,n}^{(p)}(u)\right]^2 du \right| \underset{m,n\to\infty}{=} \mathcal{O}(\phi(m,n)) \quad \text{a.s.}$$

where

$$\varphi(m,n) := \max\left(\frac{\log m}{\sqrt{m}}, \frac{\log n}{\sqrt{n}}\right)$$

$$\phi(m,n) := \max\left(\sqrt{\frac{\log\log m}{m}}\,\log m, \sqrt{\frac{\log\log n}{n}}\,\log n\right)$$

3. Applications — Two-sample problem

Problem 2 – **Testing the null hypothesis** ***(comparison of two samples)***

$$\mathcal{H}'_0 : F = G$$

Theorem 4 – ***(AA–B–L 2017)***

Under $\mathcal{H}'_0$:

$$\left| \mathbf{S}^{(p)}_{m,n} - \sup_{u\in[0,1]} \left|\mathbb{B}^{(p)}_{m,n}(u)\right| \right| \underset{m,n\to\infty}{=} \mathcal{O}(\varphi(m,n)) \quad \text{a.s.}$$

$$\left| \mathbf{T}^{(p)}_{m,n} - \int_0^1 \left[\mathbb{B}^{(p)}_{m,n}(u)\right]^2 du \right| \underset{m,n\to\infty}{=} \mathcal{O}(\phi(m,n)) \quad \text{a.s.}$$

where

$$\mathbb{B}^{(p)}_{m,n}(u) := \frac{1}{p!}\, u^p \left(\sqrt{\frac{n}{m+n}}\, \mathbb{B}^1_m(u) - \sqrt{\frac{m}{m+n}}\, \mathbb{B}^2_n(u) \right) \qquad u \in [0,1]$$

$\{\mathbb{B}^1_n,\ n\in\mathbb{N}^*\}$ and $\{\mathbb{B}^2_n,\ n\in\mathbb{N}^*\}$ being two independent sequences of Brownian bridges

3. Applications — Change-point problem

Problem 3 – **Testing the null hypothesis** ***(homogeneity of the sample)***

$$\mathcal{H}_0'' : X_1, \ldots, X_n \text{ have d.f. } F_0$$

Against the alternative hypothesis:

$$\mathcal{H}_1'' : \exists\, k^* \in \{1, \ldots, n-1\}, \begin{cases} X_1, \ldots, X_{k^*} \text{ have d.f. } F_0 \\ X_{k^*+1}, \ldots, X_n \text{ have d.f. } F_1 \end{cases}$$

Settings –

- $\mathbb{F}_k^{(p)-}$: **p-fold integrated empirical d.f. based upon the k first observations**
- $\mathbb{F}_{n-k}^{(p)+}$: **p-fold integrated empirical d.f. based upon the $(n-k)$ last observations**

3. Applications — Change-point problem

Problem 3 – **Testing the null hypothesis** ***(homogeneity of the sample)***

$$\mathcal{H}_0'' : X_1, \ldots, X_n \text{ have d.f. } F_0$$

Definitions –

- ***Ad hoc empirical process***

$$\widetilde{\alpha}_n^{(p)}(s,t) := \frac{\lfloor ns \rfloor (n - \lfloor ns \rfloor)}{n^{3/2}} \left(\mathbb{F}_{\lfloor ns \rfloor}^{(p)-}(t) - \mathbb{F}_{n-\lfloor ns \rfloor}^{(p)+}(t) \right) \quad s \in (0,1) \quad t \in \mathbb{R}$$

- ***'Hybrid' Kiefer process***

For $\{\mathbb{K}_1(s,u),\ s \in \mathbb{R},\ u \in [0,1]\}$ and $\{\mathbb{K}_2(s,u),\ s \in \mathbb{R},\ u \in [0,1]\}$ two independent Kiefer processes, set

$$\mathring{\mathbb{K}}_n(s,u) := \begin{cases} \frac{1}{\sqrt{n}} \Big[\mathbb{K}_2(\lfloor ns \rfloor, u) - s\Big(\mathbb{K}_1(\lfloor n/2 \rfloor, u) \\ \quad + \mathbb{K}_2(\lfloor n/2 \rfloor, u)\Big)\Big] \qquad s \in \left[0, \frac{1}{2}\right] \quad u \in [0,1] \\ \frac{1}{\sqrt{n}} \Big[-\mathbb{K}_1(\lfloor n(1-s) \rfloor, u) + (1-s)\Big(\mathbb{K}_1(\lfloor n/2 \rfloor, u) \\ \quad + \mathbb{K}_2(\lfloor n/2 \rfloor, u)\Big)\Big] \qquad s \in \left[\frac{1}{2}, 1\right] \quad u \in [0,1] \end{cases}$$

Problem 3* – Testing the null hypothesis *(homogeneity of the sample)

$$\mathcal{H}_0'' : X_1, \ldots, X_n \text{ have d.f. } F_0$$

Theorem 5* – *Strong approximation* *(AA–B–L 2017)

On a suitable probability space, it is possible to define $\{\widetilde{\alpha}_n^{(p)},\ n \in \mathbb{N}^*\}$, jointly with two independent Kiefer processes $\{\mathbb{K}_1(s,u),\ s \in \mathbb{R},\ u \in [0,1]\}$ and $\{\mathbb{K}_2(s,u),\ s \in \mathbb{R},\ u \in [0,1]\}$, such that, under $\mathcal{H}_0''$:

$$\sup_{\substack{s \in (0,1) \\ t \in \mathbb{R}}} \left| \widetilde{\alpha}_n^{(p)}(s,t) - \mathring{\mathbb{K}}_n^{(p)}(s, F_0(t)) \right| \underset{n \to \infty}{=} \mathcal{O}\left(\frac{(\log n)^2}{\sqrt{n}} \right) \quad \text{a.s.}$$

where

$$\mathring{\mathbb{K}}_n^{(p)}(s,u) := \frac{1}{p!}\, u^p\, \mathring{\mathbb{K}}_n(s,u) \qquad n \in \mathbb{N}^* \quad s, u \in [0,1]$$

4. Further investigations

4. Further investigations

1. ***Weighted approximations***

   $\longrightarrow$ ***problem:*** $$\sup_{u\in(0,1)} \frac{\left|\beta_n^{(p)}(u) - \mathbb{B}_n^{(p)}(u)\right|}{u^\gamma} \underset{n\to\infty}{=} \mathcal{O}(?)$$

2. ***Poisson approximation***

   Let $\{\nu_n,\ n\in\mathbb{N}^*\}$ be a sequence of Poisson r.v.'s independent of the sequence $\{X_i,\ i\in\mathbb{N}^*\}$ such that $\mathbb{E}(\nu_n) = n$. Introduce the Poisson bridges $\{\mathbb{P}_n,\ n\in\mathbb{N}^*\}$ defined, for each $n\in\mathbb{N}^*$, by

   $$\mathbb{P}_n(u) := \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{\nu_n} \mathbb{1}_{\{U_i\leq u\}} - \nu_n u\right) \qquad u\in[0,1]$$

   $\longrightarrow$ ***problem:*** $$\sup_{u\in[0,1]} \left|\beta_n^{(p)}(u) - \mathbb{P}_n^{(p)}(u)\right| \underset{n\to\infty}{=} \mathcal{O}(?)$$

3. ***Higher dimension, multivariate case...***

Thank you for your attention!

References:

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Slides available at http://math.univ-lyon1.fr/~alachal/exposes/slides_london_2017.pdf