Some distributions on pseudo-Brownian motion and pseudo-random walk

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Recent developments in probability theory and stochastic processes
A conference in honour of Enzo Orsingher
On the occasion of his 70th birthday
Rome — September 23, 2016

Acknowledgments

It is a great honor and a huge pleasure for me to open this conference dedicated to Professor Enzo Orsingher.

I warmly thank the organizers for having invited me at this exceptional event.

Acknowledgments

I met Enzo the first time in 2004,
it was the beginning
of several collaborations
and many correspondences.

Acknowledgments

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ScienceDirect
ELSEVIER
Ann. I. H. Poincaré – PR 42 (2006) 753–772
ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES
www.elsevier.com/locate/anihpb

Minimal cyclic random motion in $\mathbb{R}^n$ and hyper-Bessel functions

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IHP 2006

Statistics and Probability Letters 79 (2009) 243–254
Contents lists available at ScienceDirect
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Some Darling–Siegert relationships connected with random flights

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SPL 2008

Electronic Journal of Probability
Vol. 15 (2010), Paper no. 28, pages 895–931.
Journal URL
http://www.math.washington.edu/~ejpecp/

Joint distribution of the process and its sojourn time on the positive half-line for pseudo-processes governed by high-order heat equation

Valentina Cammarota* and Aimé Lachal†

EJP 2010

Available online at www.sciencedirect.com
SciVerse ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 122 (2012) 217–249
stochastic processes and their applications
www.elsevier.com/locate/spa

Joint distribution of the process and its sojourn time in a half-line $[a, +\infty)$ for pseudo-processes driven by a high-order heat-type equation

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SPA 2012

Markov Processes Relat. Fields 21, 887–938 (2015)
Markov Processes and Related Fields © Polymat, Moscow 2015

Entrance and Sojourn Times for Markov Chains. Application to (L, R)-random walks

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[2] Pôle de Mathématiques, Institut Camille Jordan, CNRS UMR5208, Bât. L. de Vinci, Institut National des Sciences Appliquées de Lyon, 20 av. A. Einstein, 69621 Villeurbanne Cedex, France. E-mail: aime.lachal@insa-lyon.fr, Web page: http://math.univ-lyon1.fr/~alachal

MPRF 2015

Nice collaborations...

(with Enzo Orsingher, Samantha Leorato and Valentina Cammarota)

Acknowledgments

A common point:

"High-order" (i.e., order > 2) partial differential equations...

Acknowledgments

Not simply a probabilist colleague,
not simply a collaborator of mine,
Enzo is a genuine friend.

I would like to thank him
especially for his immense generosity:
Each time I came in Roma,
I have always been very well received.

Acknowledgments

Assisi 2004

Ravello 2006

Gubbio 2008

Nice memories...

Prologue

Some fascinating formulae...

A new topic in my life of "stochastician"...

Some fascinating formulae...

$$\begin{cases} \mathbb{P}\left\{T_t^+ \in ds\right\}/ds = \dfrac{\sqrt{3}}{2\pi} \dfrac{1}{\sqrt[3]{s(t-s)^2}} \\ \mathbb{P}\left\{T_t^- \in ds\right\}/ds = \dfrac{\sqrt{3}}{2\pi} \dfrac{1}{\sqrt[3]{s^2(t-s)}} \end{cases} \qquad s \in (0,t)$$

E. Orsingher — 1991

Processes governed by signed measures connected with third-order "heat-type" equations

Litovskii Matematicheskii Sbornik (Lithuanian Mathematical Journal)

A kind of Arcsine Law?!

(i.e. Paul Lévy's arcsine law for Brownian motion)

Recalling a well-known connection

Heat equation (of order 2)

(and parabolic/elliptic equations of order 2)

$$\Updownarrow$$

Brownian motion / Random walk

(and diffusion processes)

What about the higher-order?

High-order heat-type equation

—

Continuous space/time

- ***Equation:***

$$\partial_t u(t,x) = \mathcal{D}_x u(t,x) \qquad t>0,\, x\in\mathbb{R}$$

where

$$\partial_t = \frac{\partial}{\partial t} \text{ and } \mathcal{D}_x = \begin{cases} (-1)^{N-1}\dfrac{\partial^{2N}}{\partial x^{2N}} = (-1)^{N-1}\Delta_x^N & (N \text{ integer} \geq 2\text{, even order}) \\ \pm\dfrac{\partial^{2N+1}}{\partial x^{2N+1}} & (N \text{ integer} \geq 1\text{, odd order}) \end{cases}$$

- ***Equation:***

$$\partial_t u(t,x) = \mathcal{D}_x u(t,x) \qquad t>0,\, x\in\mathbb{R}$$

where

$$\partial_t = \frac{\partial}{\partial t} \text{ and } \mathcal{D}_x = \begin{cases} (-1)^{N-1}\dfrac{\partial^{2N}}{\partial x^{2N}} = (-1)^{N-1}\Delta_x^N & (N \text{ integer} \geq 2, \text{ even order}) \\ \pm\dfrac{\partial^{2N+1}}{\partial x^{2N+1}} & (N \text{ integer} \geq 1, \text{ odd order}) \end{cases}$$

$\longrightarrow$ ***Elementary solution:***

$$p(t,x) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} e^{ixu - tu^{2N}}\,du & \text{(even order)} \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} e^{ixu \pm iu^{2N+1}}\,du & \text{(odd order)} \end{cases}$$

High-order heat-type equation – Continuous space/time

- ***Equation:***

$$\partial_t u(t,x) = \mathcal{D}_x u(t,x) \qquad t>0,\, x\in\mathbb{R}$$

where

$$\partial_t = \frac{\partial}{\partial t} \text{ and } \mathcal{D}_x = \begin{cases} (-1)^{N-1}\dfrac{\partial^{2N}}{\partial x^{2N}} = (-1)^{N-1}\Delta_x^N & (N \text{ integer} \geq 2\text{, even order}) \\ \pm\dfrac{\partial^{2N+1}}{\partial x^{2N+1}} & (N \text{ integer} \geq 1\text{, odd order}) \end{cases}$$

⟶ Pseudo-Markov process (pseudo-Brownian motion) $(B_t)_{t\geq 0}$ with pseudo-transition densities

$$p(t;x,y) = \mathbb{P}_x\{B_t \in dy\}/dy = p(t,x-y)$$

High-order heat-type equation – Continuous space/time

- ***Equation:***

$$\partial_t u(t,x) = \mathcal{D}_x u(t,x) \qquad t>0,\, x\in\mathbb{R}$$

Warning:

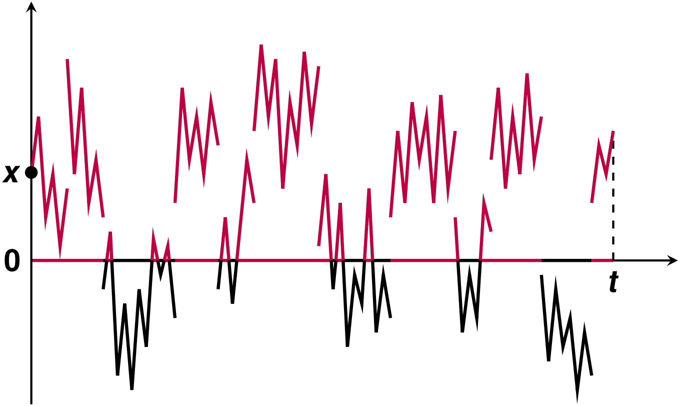
p has a varying sign,
so $\mathbb{P}_x$ is a signed measure
with total mass 1 and infinite total variation!

$\longrightarrow$ ***Problems in defining properly the pseudo-process...***

The results to be announced
are valid at least formally...

- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\, ds$$



- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\, ds$$

⟶ ***Pseudo-distribution of T_t:***

- **Even order (Krylov 1960), arcsine law**

$$\mathbb{P}_0\{T_t \in ds\}/ds = \frac{1}{\pi\sqrt{s(t-s)}} \qquad s \in (0,t)$$

$$\mathbb{P}_0\{T_t \leq s\} = \frac{2}{\pi} \arcsin\sqrt{\frac{s}{t}} \qquad s \in [0,t]$$

High-order heat-type equation – Continuous space/time

- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\,ds$$

⟶ ***Pseudo-distribution of T_t:***

- **Order 3** (Orsingher 1991), **Beta law**

$$\mathbb{P}_0\{T_t \in ds\}/ds = \begin{cases} \dfrac{\sqrt{3}}{2\pi}\,\dfrac{1}{\sqrt[3]{s(t-s)^2}} & \text{for } +\dfrac{\partial^3}{\partial x^3} \\ \dfrac{\sqrt{3}}{2\pi}\,\dfrac{1}{\sqrt[3]{s^2(t-s)}} & \text{for } -\dfrac{\partial^3}{\partial x^3} \end{cases} \quad s \in (0,t)$$

- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\,ds$$

⟶ ***Pseudo-distribution of T_t:***

- **Odd order (AL 2003), Beta law** – e.g. for $+\partial^{2N+1}/\partial x^{2N+1}$

$$\mathbb{P}_0\{T_t \in ds\}/ds = \frac{1}{\pi}\sin\left(\frac{N}{2N+1}\pi\right) \times \begin{cases} s^{-\frac{N}{2N+1}}(t-s)^{-\frac{N+1}{2N+1}} & \text{for even } N \\ s^{-\frac{N+1}{2N+1}}(t-s)^{-\frac{N}{2N+1}} & \text{for odd } N \end{cases}$$

Examples: orders 5 and 7 (Hochberg & Orsingher 1991–1994)

- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\, ds$$

⟶ ***Joint pseudo-distribution of (T_t, B_t):***

- **Intermediate solution (AL 2003), Pseudo-Brownian bridge**

$$\begin{cases} \mathbb{P}_0\{T_t \in ds \mid B_t = 0\}/ds = \dfrac{1}{t} & s \in (0,t) \quad \text{Uniform law} \\ \mathbb{P}_0\{T_t \in ds \mid B_t > \text{ (or } <\text{) } 0\}/ds = \ldots & \quad \text{Beta laws} \end{cases}$$

Examples: orders 3 and 4 (Nikitin & Orsingher 2000)

High-order heat-type equation – Continuous space/time

- ***Sojourn time:***

$$T_t = \text{measure}\{s \in [0,t] : B_s \geq 0\} = \int_0^t \mathbb{1}_{[0,+\infty)}(B_s)\,ds$$

$\longrightarrow$ ***Joint pseudo-distribution of (T_t, B_t):***

- **Intermediate solution (AL 2003), Pseudo-Brownian bridge**

$$\begin{cases} \mathbb{P}_0\{T_t \in ds \mid B_t = 0\}/ds = \dfrac{1}{t} \quad {}_{s\in(0,t)} & \text{Uniform law} \\ \mathbb{P}_0\{T_t \in ds \mid B_t > \text{ (or <) } 0\}/ds = \dots & \text{Beta laws} \end{cases}$$

Examples: orders 3 and 4 (Nikitin & Orsingher 2000)

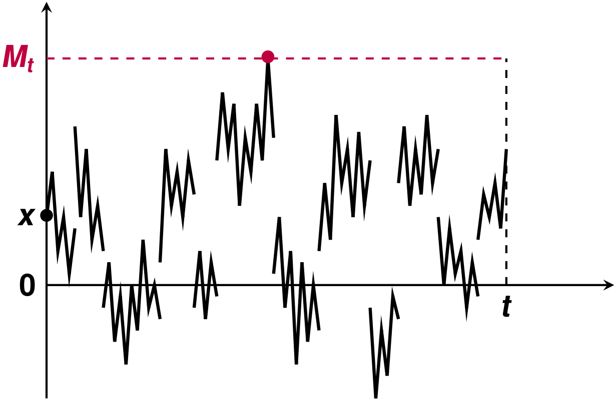
- **General Solution (V. Cammarota & AL 2010–2012)**

$$\mathbb{P}_x\{T_t \in ds, B_t \in dy\}/(ds\,dy) = \dots$$

High-order heat-type equation – Continuous space/time

- ***Running maximum:***

$$M_t = \max_{0 \le s \le t} B_s$$



High-order heat-type equation – Continuous space/time

- ***Running maximum:***

$$M_t = \max_{0\le s\le t} B_s$$

$\longrightarrow$ ***Pseudo-distribution of M_t:***

- Order 3 (Orsingher 1991)

$$\mathbb{P}_0\{M_t \in dy\}/dy = \begin{cases} 3\,p(t,y) & \text{for } +\frac{\partial^3}{\partial x^3} \\ p(t,y) + \frac{1}{\Gamma(\frac{1}{3})}\int_0^t \frac{\frac{\partial p}{\partial y}(s,y)}{(t-s)^{2/3}}\,ds & \text{for } -\frac{\partial^3}{\partial x^3} \end{cases} \quad y>0$$

with

$$p(t,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iyu \mp itu^3}\,du = \frac{1}{\sqrt[3]{3t}}\,Ai\left(\pm\frac{x}{\sqrt[3]{3t}}\right)$$

where Ai is the Airy function

High-order heat-type equation – Continuous space/time

- ***Running maximum:***

$$M_t = \max_{0 \le s \le t} B_s$$

$\longrightarrow$ ***Pseudo-distribution of M_t:***

- **Arbitrary order (AL 2003) – e.g. for $(-1)^{N-1}\partial^{2N}/\partial x^{2N}$**

$$\mathbb{P}_x\{M_t \in dy\}/dy = \sum_{k=1}^{N} \alpha_k \int_0^t \frac{\partial^k p}{\partial x^k}(s; x, y) \frac{ds}{(t-s)^{1-k/N}} \qquad x<a$$

for some constants α_k

- ***Running maximum:***

$$M_t = \max_{0 \le s \le t} B_s$$

⟶ Joint pseudo-distribution of (M_t, B_t):

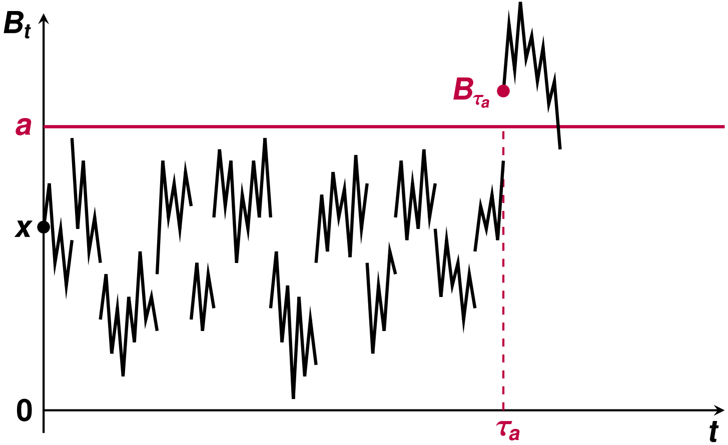
- **Arbitrary order (AL 2007)**

$$\mathbb{P}_x\{M_t \in dy, B_t \in dz\}/dy\,dz = \dots$$

Examples: orders 3 and 4 (Beghin, Hochberg, Orsingher & Ragozina 2000–2001)

- ***First overshooting time:***

$$\tau_a = \inf\left\{t \geq 0 : B_t > a\right\} \qquad x<a$$



High-order heat-type equation – Continuous space/time

- ***First overshooting time:***

$$\tau_a = \inf\left\{t \geq 0 : B_t > a\right\} \qquad x<a$$

$\longrightarrow$ ***Joint pseudo-distribution of (τ_a, B_{τ_a}):***

- **Order 2 – Brownian motion!**

$$B_{\tau_a} = a \implies \mathbb{P}_x\{B_{\tau_a} \in dz\}/dz = \delta_a(z)$$

High-order heat-type equation – Continuous space/time

- ***First overshooting time:***

$$\tau_a = \inf\left\{t \geq 0 : B_t > a\right\} \qquad x<a$$

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- **Order 2 – Brownian motion!**

$$B_{\tau_a} = a \implies \mathbb{P}_x\{B_{\tau_a} \in dz\}/dz = \delta_a(z)$$

- **Order 4 – "Biharmonic" Brownian motion (Nishioka 1996)**

  $\longrightarrow$ ***"Monopoles and dipoles"***

$$\mathbb{P}_x\{B_{\tau_a} \in dz\}/dz = \delta_a(z) - (x-a)\delta_a'(z)$$

in the sense

$$\mathbb{E}_x[\varphi(B_{\tau_a})] = \varphi(a) + (x-a)\varphi'(a) \quad \text{for any test function } \varphi$$

- ***First overshooting time:***

$$\tau_a = \inf\left\{t \geq 0 : B_t > a\right\} \qquad x<a$$

$\longrightarrow$ ***Joint pseudo-distribution of (τ_a, B_{τ_a}):***

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in the sense

$$\mathbb{E}_x[\varphi(B_{\tau_a})] = \varphi(a) + (x-a)\varphi'(a) \quad \text{for any test function } \varphi$$

$\implies$ ***B_{τ_a} seems to be concentrated at a...***

- ***First overshooting time:***

$$\tau_a = \inf\left\{t \geq 0 : B_t > a\right\} \qquad x<a$$

⟶ ***Joint pseudo-distribution of (τ_a, B_{τ_a}):***

- **Arbitrary order** (AL 2007) – e.g. for $(-1)^{N-1}\partial^{2N}/\partial x^{2N}$

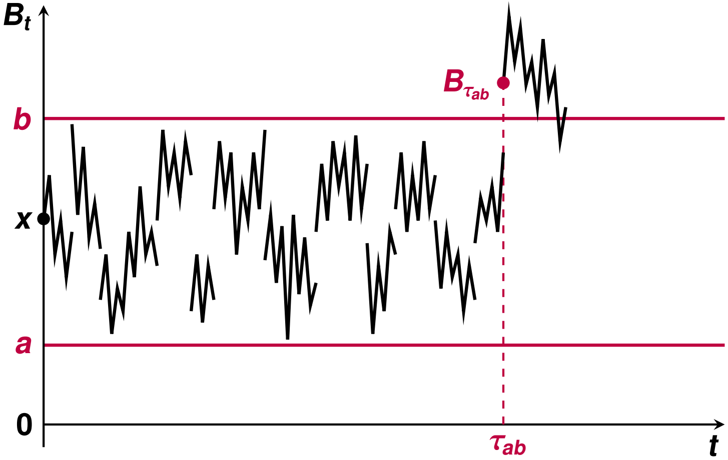
  ⟶ ***"Multipoles"***

$$\mathbb{P}_x\{B_{\tau_a} \in dz\}/dz = \sum_{q=0}^{N-1} (-1)^q \frac{(x-a)^q}{q!}\, \delta_a^{(q)}(z)$$

$$\mathbb{P}_x\{\tau_a \in dt, B_{\tau_a} \in dz\}/dt\, dz = \ldots$$

- ***First exit time:***

$$\tau_{ab} = \inf\left\{t \geq 0 \colon B_t \notin (a,b)\right\} \qquad x \in (a,b)$$



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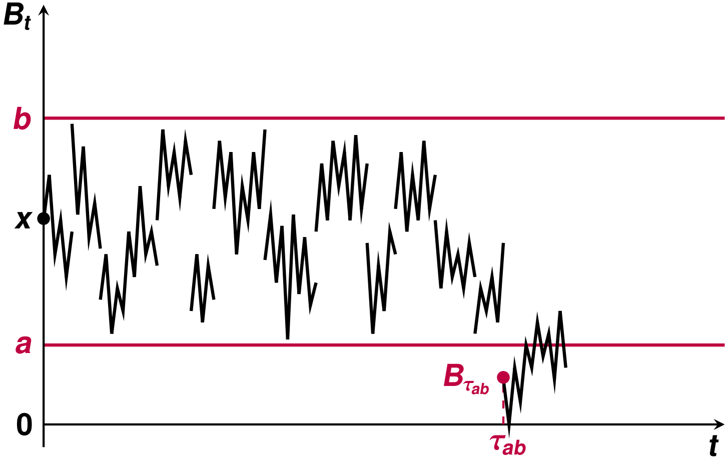
$$\tau_{ab} = \inf\left\{t \geq 0 : B_t \notin (a,b)\right\} \qquad x\in(a,b)$$



High-order heat-type equation – Continuous space/time

- ***First exit time:***

$$\tau_{ab} = \inf\left\{t \geq 0 : B_t \notin (a,b)\right\} \qquad x \in (a,b)$$

$\longrightarrow$ ***Joint pseudo-distribution of $(\tau_{ab}, B_{\tau_{ab}})$:***

- **Order 2 – Brownian motion and gambler's ruin!**

$$\mathbb{P}_x\{B_{\tau_{ab}} \in dz\}/dz = \frac{b-x}{b-a}\,\delta_a(z) + \frac{x-a}{b-a}\,\delta_b(z)$$

In particular, the "ruin probabilities" are given by

$$\mathbb{P}_x\left\{\tau_a^- < \tau_b^+\right\} = \frac{b-x}{b-a} \quad \text{and} \quad \mathbb{P}_x\left\{\tau_b^+ < \tau_a^-\right\} = \frac{x-a}{b-a}$$

where $\begin{cases} \tau_b^+ = \inf\{t \geq 0 : B_t > b\} \\ \tau_a^- = \inf\{t \geq 0 : B_t < a\} \end{cases}$

- ***First exit time:***

$$\tau_{ab} = \inf\left\{t \geq 0 : B_t \notin (a,b)\right\} \qquad x \in (a,b)$$

$\longrightarrow$ ***Joint pseudo-distribution of $(\tau_{ab}, B_{\tau_{ab}})$:***

- **Even order (AL 2014) – for $(-1)^{N-1}\partial^{2N}/\partial x^{2N}$**

$$\mathbb{P}_x\{B_{\tau_{ab}} \in dz\}/dz = \sum_{q=0}^{N-1}\left(H_q^-(x)\,\delta_a^{(q)}(z) + H_q^+(x)\,\delta_b^{(q)}(z)\right)$$

where H_q^-, H_q^+ are the interpolation Hermite polynomials such that

$$\begin{cases} \dfrac{d^p H_q^-}{dx^p}(a) = \delta_{pq} & \dfrac{d^p H_q^-}{dx^p}(b) = 0 \\ \dfrac{d^p H_q^+}{dx^p}(a) = 0 & \dfrac{d^p H_q^+}{dx^p}(b) = \delta_{pq} \end{cases} \qquad \text{for } 0 \leq p \leq N-1$$

$$\mathbb{P}_x\{\tau_{ab} \in dt, B_{\tau_{ab}} \in dz\}/dt\,dz = \ldots$$

High-order heat-type equation – Continuous space/time

- ***First exit time:***

$$\tau_{ab} = \inf\left\{t \geq 0 : B_t \notin (a,b)\right\} \qquad x\in(a,b)$$

→ ***Joint pseudo-distribution of $(\tau_{ab}, B_{\tau_{ab}})$:***

- **Example: order 4 – Biharmonic Brownian motion**

$$\mathbb{P}_x\{B_{\tau_{ab}} \in dz\}/dz = \frac{(x-b)^2(2x-3a+b)}{(b-a)^3}\,\delta_a(z) - \frac{(x-a)(x-b)^2}{(b-a)^2}\,\delta_a'(z) - \frac{(x-a)^2(2x+a-3b)}{(b-a)^3}\,\delta_b(z) - \frac{(x-a)^2(x-b)}{(b-a)^2}\,\delta_b'(z)$$

In particular, the "ruin pseudo-probabilities" are given by

$$\mathbb{P}_x\left\{\tau_a^- < \tau_b^+\right\} = \frac{(x-b)^2(2x-3a+b)}{(b-a)^3} \qquad \mathbb{P}_x\left\{\tau_b^+ < \tau_a^-\right\} = -\frac{(x-a)^2(2x+a-3b)}{(b-a)^3}$$

High-order heat-type equation – Continuous space/time

- ***Tools:***
  - **Generalized Feynman-Kac formula**
  - **Boundary value problems**
  - **Polyharmonic functions**
  - **Spitzer identities (even orders)**
  - **Algebra: Vandermonde systems, special functions...**

Heat-type equation of order 4

—

Discrete space/time

Heat-type equation of order 4 – Discrete space/time

- ***Equation:***

$$\partial_n u(n,x) = -c\Delta_x^2 u(n,x) \qquad n\in\mathbb{N},\, x\in\mathbb{Z}$$

where $c > 0$ and

$$\begin{cases} \partial_n u(n,x) & = u(n+1,x) - u(n,x) \\ \Delta_x^2 u(n,x) & = u(n,x+2) - 4u(n,x+1) + 6u(n,x) \\ & \quad -4u(n,x-1) + u(n,x-2) \end{cases}$$

- ***Equation:***

$$\partial_n u(n,x) = -c\Delta_x^2 u(n,x) \qquad n\in\mathbb{N},\, x\in\mathbb{Z}$$

where $c>0$ and

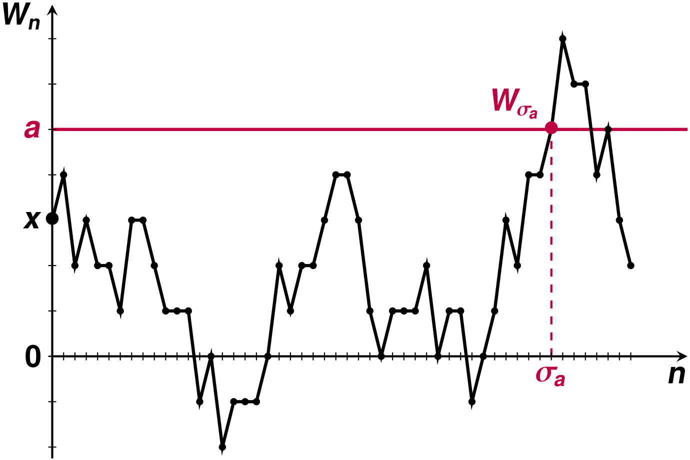
$$\begin{cases} \partial_n u(n,x) &= u(n+1,x) - u(n,x) \\ \Delta_x^2 u(n,x) &= u(n,x+2) - 4u(n,x+1) + 6u(n,x) \\ & \quad -4u(n,x-1) + u(n,x-2) \end{cases}$$

$\longrightarrow$ ***Pseudo-random walk*** $(W_m)_{m\geq 0}$: $W_m = W_0 + \sum_{i=1}^m U_i$ ***where*** $(U_i)_{i\geq 1}$ ***is a sequence of i.i.d. pseudo-r.v. with pseudo-distribution***

$$\begin{cases} \mathbb{P}\{U_1 = 2\} = \mathbb{P}\{U_1 = -2\} = -c \\ \mathbb{P}\{U_1 = 1\} = \mathbb{P}\{U_1 = -1\} = 4c \\ \mathbb{P}\{U_1 = 0\} = 1 - 6c \end{cases}$$

- ***First overshooting time:***

$$\sigma_a = \inf\left\{n \geq 0 : W_n \geq a\right\} \qquad x \leq a$$



- ***First overshooting time:***

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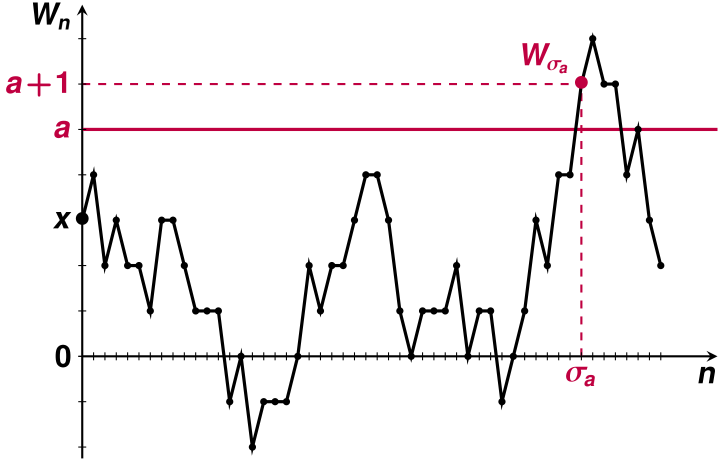


Heat-type equation of order 4 – Discrete space/time

- ***First overshooting time:***

$$\sigma_a = \inf\left\{n \geq 0 : W_n \geq a\right\} \qquad x \leq a$$

⟶ ***Joint pseudo-distribution of (σ_a, W_{σ_a}) (Sato 2002):***

- **In particular: marginal pseudo-distribution of W_{σ_a}**

$$\mathbb{P}_x\{W_{\sigma_a} = a\} = a + 1 - x \qquad \mathbb{P}_x\{W_{\sigma_a} = a + 1\} = x - a$$

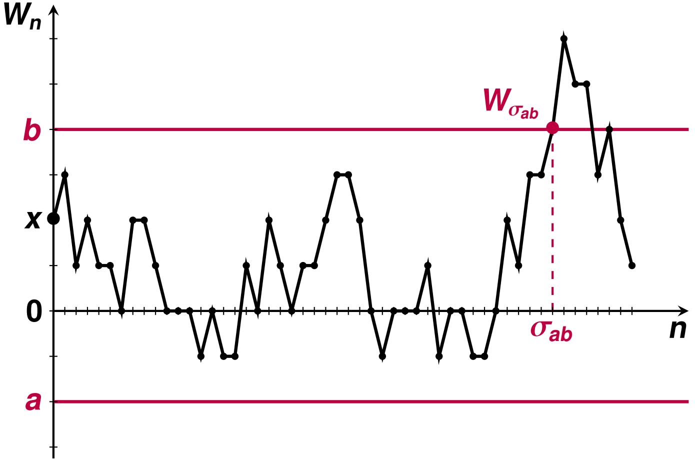
- **"Monopoles and dipoles"**

Pseudo-Brownian motion vs. pseudo-random walk

$$\mathbb{E}_x[\varphi(W_{\sigma_a})] = \varphi(a) + (x - a)[\varphi(a + 1) - \varphi(a)]$$
$$\mathbb{E}_x[\varphi(B_{\tau_a})] = \varphi(a) + (x - a)\varphi'(a)$$

- ***First exit time:***

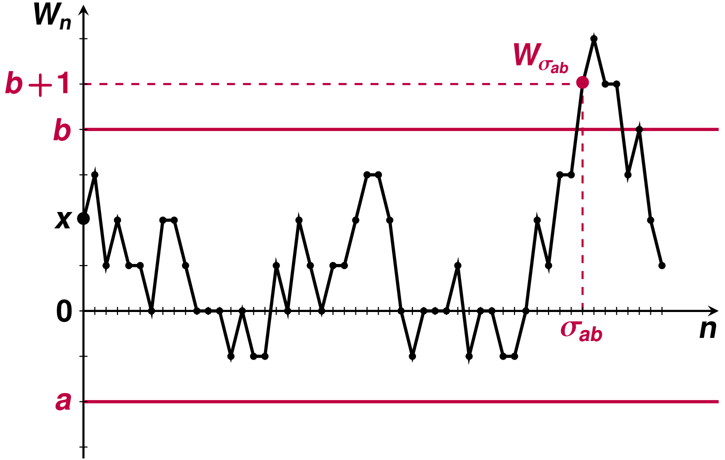
$$\sigma_{ab} = \inf\left\{n \geq 0 : W_n \notin (a,b)\right\} \qquad x \in [a,b]$$



Heat-type equation of order 4 – Discrete space/time

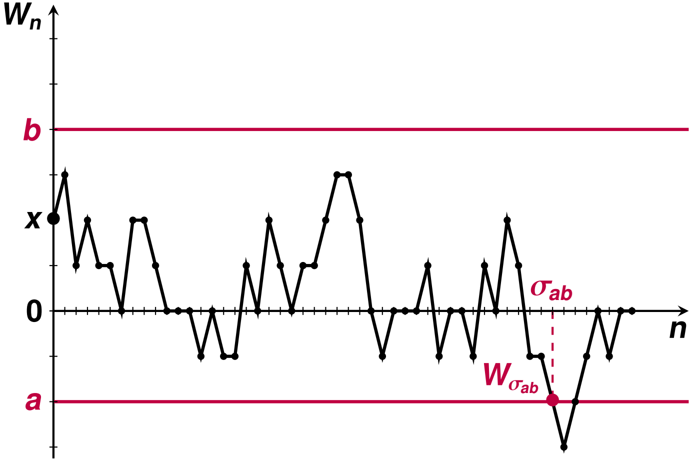
- ***First exit time:***

$$\sigma_{ab} = \inf\left\{n \geq 0 : W_n \notin (a,b)\right\} \qquad x \in [a,b]$$



- ***First exit time:***

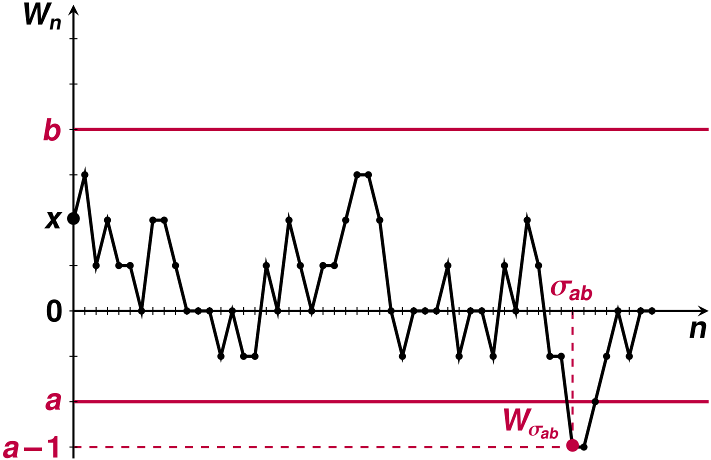
$$\sigma_{ab} = \inf\left\{n \geq 0 : W_n \notin (a,b)\right\} \qquad x \in [a,b]$$



Heat-type equation of order 4 – Discrete space/time

- ***First exit time:***

$$\sigma_{ab} = \inf\left\{n \geq 0 \colon W_n \notin (a,b)\right\} \qquad x\in[a,b]$$



- ***First exit time:***

$$\sigma_{ab} = \inf\left\{n \geq 0 : W_n \notin (a,b)\right\} \qquad x\in[a,b]$$

$\longrightarrow$ ***Joint pseudo-distribution of*** $(\sigma_{ab}, W_{\sigma_{ab}})$ *(Sato 2002)*:

- **In particular: marginal pseudo-distribution of** $W_{\sigma_{ab}}$

$$\begin{cases} \mathbb{P}_x\{W_{\sigma_{ab}} = a-1\} = \dfrac{(x-a)(x-b)(b-x+1)}{(b-a+1)(b-a+2)} \\ \mathbb{P}_x\{W_{\sigma_{ab}} = a\} = -\dfrac{(x-a+1)(x-b)(b-x+1)}{(b-a)(b-a+1)} \\ \mathbb{P}_x\{W_{\sigma_{ab}} = b\} = \dfrac{(x-a)(x-a+1)(b-x+1)}{(b-a)(b-a+1)} \\ \mathbb{P}_x\{W_{\sigma_{ab}} = b+1\} = -\dfrac{(x-a)(x-a+1)(b-x)}{(b-a+1)(b-a+2)} \end{cases}$$

High-order heat-type equation

—

Discrete space/time

- ***Equation:***

$$\partial_n u(n,x) = (-1)^{N-1} c \Delta_x^N u(n,x) \qquad n\in\mathbb{N}, x\in\mathbb{Z}$$

where $c > 0$ and

$$\begin{cases} \partial_n u(n,x) &= u(n+1,x) - u(n,x) \\ \Delta_x^N u(n,x) &= \displaystyle\sum_{k=-N}^{N} (-1)^k \binom{2N}{k+N} u(n,x+k) \end{cases}$$

High-order heat-type equation – Discrete space/time

- ***Equation:***

$$\boxed{\partial_n u(n,x) = (-1)^{N-1} c \Delta_x^N u(n,x) \qquad n\in\mathbb{N}, x\in\mathbb{Z}}$$

where $c > 0$ and

$$\begin{cases} \partial_n u(n,x) &= u(n+1,x) - u(n,x) \\ \Delta_x^N u(n,x) &= \displaystyle\sum_{k=-N}^{N} (-1)^k \binom{2N}{k+N} u(n,x+k) \end{cases}$$

⟶ Symmetric pseudo-random walk $(W_m)_{m\geq 0}$:
$W_m = W_0 + \sum_{i=1}^{m} U_i$ where $(U_i)_{i\geq 1}$ is a sequence of i.i.d. pseudo-r.v. with pseudo-distribution

$$\begin{cases} \mathbb{P}\{U_1 = i\} = (-1)^{i-1} c \binom{2N}{i+N} & \text{if } \begin{cases} -N \leq i \leq N \\ i \neq 0 \end{cases} \\ \mathbb{P}\{U_1 = 0\} = 1 - c\binom{2N}{N} \end{cases}$$

High-order heat-type equation – Discrete space/time

- ***First overshooting time:***

$$\sigma_a = \inf\left\{n \geq 0 : W_n \geq a\right\} \qquad {}_{x<a}$$

$\longrightarrow$ ***Joint pseudo-distribution of*** (σ_a, W_{σ_a}) ***(AL 2014):***

$$\mathbb{P}_x\{W_{\sigma_a} = k\} = (-1)^{k+a} \frac{a}{k} \binom{N-1}{k-a} \binom{a+N-1}{a} \qquad {}_{k \in \{a, a+1, \dots, a+N-1\}}$$

$$\mathbb{E}_x\left(z^{\sigma_a} \mathbb{1}_{\{W_{\sigma_a} = k\}}\right) = \dots$$

High-order heat-type equation – Discrete space/time

- ***First exit time:***

$$\sigma_{ab} = \inf\left\{n \geq 0 : W_n \notin (a,b)\right\} \qquad x\in[a,b]$$

$\longrightarrow$ ***Joint pseudo-distribution of*** $(\sigma_{ab}, W_{\sigma_{ab}})$ ***(AL 2014):***

$$\begin{cases} \mathbb{P}_x\{W_{\sigma_{ab}}=k\} = (-1)^{k+a-1}\,\dfrac{N}{k}\,\dfrac{\binom{N-a-1}{N}\binom{N+b-1}{N}\binom{N-1}{a-k}}{\binom{b-k+N-1}{N}} & k\in\{a,a-1,\dots,a-N+1\} \\ \mathbb{P}_x\{W_{\sigma_{ab}}=k\} = (-1)^{k+b}\,\dfrac{N}{k}\,\dfrac{\binom{N-a-1}{N}\binom{N+b-1}{N}\binom{N-1}{k-b}}{\binom{k-a+N-1}{N}} & k\in\{b,b+1,\dots,b+N-1\} \end{cases}$$

$$\mathbb{E}_x\left(z^{\sigma_{ab}}\,\mathbb{1}_{\{W_{\sigma_{ab}}=k\}}\right) = \dots$$

Thank you
for your attention!

http://math.univ-lyon1.fr/~alachal/exposes/slides_rome_2016.pdf