A survey on the pseudo-process driven by the high-order heat-type equation $\partial/\partial t = \pm\partial^n/\partial x^n$ concerning the first hitting times and sojourn times

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1 Construction

- ***Heat-type equation of high order*** $n > 2$: $\boxed{\dfrac{\partial u}{\partial t} = \pm \dfrac{\partial^n u}{\partial x^n}}$
- ***Heat-type kernel:*** $p(t;x)$
  - it is characterized by $\displaystyle\int_{-\infty}^{+\infty} e^{iux} p(t;x)\,dx = \begin{cases} e^{-tu^n} & \text{for even } n \\ e^{\pm itu^n} & \text{for odd } n \end{cases}$
  - it satisfies $\displaystyle\int_{-\infty}^{+\infty} p(t;x)\,dx = 1$ and $\displaystyle 1 < \int_{-\infty}^{+\infty} |p(t;x)|\,dx \begin{cases} < +\infty & \text{for even } n \\ = +\infty & \text{for odd } n \end{cases}$
  - it defines a Markov pseudo-process $(X(t))_{t\geq 0}$ driven by a signed measure (which is NOT a probability measure) by

$$\boxed{\mathbb{P}_x\{X(t) \in dy\} = p(t; x-y)\,dy}$$

and for $0 = t_0 < t_1 < \ldots < t_m$ and $x_0 = x$:

$$\boxed{\mathbb{P}_x\{X(t_1) \in dx_1, \ldots, X(t_m) \in dx_m\} = \prod_{i=1}^{m} p(t_i - t_{i-1}; x_{i-1} - x_i)\,dx_i}$$

2 Study of certain functionals

- ***Sojourn time above a fixed level, or within a finite interval***

$$T_a(t) = \text{measure}\{s \in [0,t] : X(s) > a\} = \int_0^t \mathbb{1}_{\{X(s)>a\}}\, ds$$

$$T_{ab}(t) = \text{measure}\{s \in [0,t] : X(s) \in [a,b]\} = \int_0^t \mathbb{1}_{\{X(s)\in[a,b]\}}\, ds$$

- ***Maximum functional of the pseudo-process***

$$M(t) = \max_{0 \le s \le t} X(s)$$

- ***First overshooting time of a single or double threshold***

$$\tau_a = \inf\{t \ge 0 : X(t) > a\} \text{ for } x < a$$

$$\tau_{ab} = \inf\{t \ge 0 : X(t) \notin [a,b]\} \text{ for } x \in (a,b)$$

$\longrightarrow$ ***Problems:*** What are the pseudo-distributions of $T_a(t)$ and $(X(t), T_a(t))$? $T_{ab}(t)$? $M(t)$ and $(X(t), M(t))$? $(\tau_a, X(\tau_a))$? $(\tau_{ab}, X(\tau_{ab}))$?

Main tools

- Feynman-Kac formula
  ($\longrightarrow$ PDE and Fourier-Laplace transforms, Vandermonde algebra)
- Spitzer's identity
  ($\longrightarrow$ Fourier-Laplace transforms, complex analysis)

***W**arning:* the results are justified for EVEN n, formal for ODD n...

Feynman-Kac formula

The function

$$\varphi(t;x)=\mathbb{E}_x\left[e^{-\int_0^t f(X(s))\,ds}g(X(t))\right]\stackrel{\text{def}}{=}\lim_{m\to\infty}\mathbb{E}_x\left[e^{-\frac{t}{m}\sum_{k=0}^{m}f(X(\frac{kt}{m}))}g(X(t))\right]$$

is a solution of the PDE $\dfrac{\partial\varphi}{\partial t}=\pm\dfrac{\partial^n\varphi}{\partial x^n}-f\varphi$ with the condition $\varphi(0;x)=g(x)$.
The Laplace transform $\Phi(x)=\int_0^{+\infty}e^{-\lambda t}\,\varphi(t;x)\,dt$ is a solution of the ODE

$$\boxed{\pm\frac{d^n\Phi}{dx^n}=(f+\lambda)\Phi-g}$$

3 Selected bibliography

- **Krylov (1960): case even** n
- **Hochberg (1978): case** $n = 4$
- **Orsingher et al. (1991–2001): case** $n = 3$ **and** $n = 4$ **(and also** $n = 5, 7$**)**
- **Nishioka (1996–2001): case** $n = 4$
- **Lachal et al. (2003–2010): général case**

4 Distribution of the sojourn time $T_a(t)$

Set $\Phi(x,y) = \displaystyle\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big(e^{-\mu T_a(t)}, X(t)\in dy\Big)\,dt.$

The function Φ is a solution of the differential equation

$$\pm\frac{\partial^n \Phi}{\partial x^n} = \begin{cases} (\lambda+\mu)\,\Phi - \delta_y & \text{on } (a,+\infty) \\ \lambda\,\Phi - \delta_y & \text{on } (-\infty,a) \end{cases}$$

with regularity conditions

- at a: $\dfrac{\partial^p \Phi}{\partial x^p}(a^+,y) = \dfrac{\partial^p \Phi}{\partial x^p}(a^-,y)$ for $p\in\{0,1,\ldots,n-1\}$
- at y: $\dfrac{\partial^p \Phi}{\partial x^p}(y^+,y) = \dfrac{\partial^p \Phi}{\partial x^p}(y^-,y)$ for $p\in\{0,1,\ldots,n-2\}$

  $\dfrac{\partial^{n-1} \Phi}{\partial x^{n-1}}(y^+,y) - \dfrac{\partial^{n-1} \Phi}{\partial x^{n-1}}(y^-,y) = \pm 1$

$\longrightarrow$ ***Solution:*** A.L. and V. Cammarota (case $x = a$: EJP 15 (2010); case $x \neq a$: submitted)

$\longrightarrow$ ***Particular case: distribution of*** $T_a(t)$ ***when*** $x = a = 0$

Theorem 1 (A.L. (EJP 8, 2003)) *The distribution of* $T_0(t)$ *is a Beta law:*

$$\mathbb{P}_0\{T_0(t) \in ds\}/ds = \frac{\sin(\alpha\pi/n)}{\pi} \frac{\mathbb{1}_{(0,t)}(s)}{s^{\alpha/n}(t-s)^{\beta/n}}$$

where $(\alpha, \beta) = \begin{cases} (n/2, n/2) \text{ if } n \text{ is even} \\ ((n \mp 1)/2, (n \pm 1)/2) \text{ if } n \text{ is odd} \end{cases}$. *We have* $\alpha + \beta = n$.

This result has been obtained by Hochberg & Orsingher for $n = 3, 5, 7$ in 1991-94.

Example (even n, Krylov 1960)

The distribution of $T_0(t)$ is the Paul Lévy's arcsine law:

$$\mathbb{P}_0\{T_0(t) \in ds\}/ds = \frac{\mathbb{1}_{(0,t)}(s)}{\pi\sqrt{s(t-s)}}$$

5 Distribution of the sojourn time $T_{ab}(t)$

Set $\Phi(x) = \int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\big(e^{-\mu T_{ab}(t)}\big)\, dt$. The function Φ solves the ODE

$$\pm \frac{d^n\Phi}{dx^n}(x) = \begin{cases} (\lambda+\mu)\,\Phi(x) - 1 & \text{for } x \in (a,b) \\ \lambda\,\Phi(x) - 1 & \text{for } x \notin (a,b) \end{cases}$$

with regularity conditions

$$\frac{d^p\Phi}{dx^p}(a^+) = \frac{d^p\Phi}{dx^p}(a^-) \text{ and } \frac{d^p\Phi}{dx^p}(b^+) = \frac{d^p\Phi}{dx^p}(b^-) \text{ for } p \in \{0,1,\dots,n-1\}$$

It seems to be difficult to solve explicitly this system...

Example ($x = 0,\ (a,b) = (-\varepsilon,\varepsilon),\ \varepsilon \to 0^+$, Beghin & Orsingher 2005)

Local time for the pseudo-process $(X(t))_{t\geq 0}$: $L(t) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{X(s)\in[-\varepsilon,\varepsilon]\}}\, ds$.

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_0(e^{-\mu L(t)})dt = \frac{1}{\lambda + c\mu \sqrt[n]{\lambda}}$$ where $c = \dfrac{1}{\alpha n \sin\frac{\pi}{\alpha n}}$, $\alpha = \begin{cases} 1 \text{ if } n \text{ is odd,} \\ 2 \text{ if } n \text{ is even.} \end{cases}$

6 Distribution of the maximum $M(t)$

Lemma *Spitzer's identity yields*

$$\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big(e^{i\mu X(t)-\nu M(t)}\Big)dt = \frac{1}{\lambda}\,e^{(i\mu-\nu)x}\,\exp\left[\int_0^{+\infty} e^{-\lambda t}\Big(\mathbb{E}_0\Big(e^{i\mu X(t)-\nu X(t)^+}\Big)-1\Big)\frac{dt}{t}\right]$$

Theorem 2 (A.L. (EJP 12, 2007)) *The distribution of $(X(t), M(t))$ is given by*

$$\mathbb{P}_x\{X(t)<y<z<M(t)\} = \sum_{k,m}\alpha_{km}\int_0^t\int_0^s \frac{\partial^m p}{\partial x^m}(\sigma; x-z)\,\frac{I_k(s-\sigma; z-y)}{(t-s)^{1-(m+1)/n}}\,ds\,d\sigma$$

where α_{km} are some coefficients and $\int_0^{+\infty} e^{-\lambda t} I_k(t;\xi)\,dt = e^{\theta_k \sqrt[n]{\lambda}\,\xi}$ with $(\theta_k)^n = \pm 1$.

The distributions of $(X(t), M(t))$ and $(M(t)|X(t)=0)$ have been obtained by Beghin, Hochberg, Orsingher & Ragozina in the cases $n=3,4$ in 2000-01.

Example ($n=4$, Hochberg 1978)

$$\int_0^{+\infty} e^{-\lambda t}\,[\mathbb{P}_x\{M(t)\in da\}/da]\,dt = -\frac{\sqrt{2}}{\lambda^{3/4}}\,e^{\sqrt[4]{\lambda}(x-a)/\sqrt{2}}\sin\left(\frac{\sqrt[4]{\lambda}(x-a)}{\sqrt{2}}\right)$$

7 Distribution of the first overshooting time τ_a

Suppose that n is even and fix $x < a$.

Lemma *The distributions of* $(\tau_a, X(\tau_a))$ *and* $(X(t), M(t))$ *are related by*

$$\mathbb{E}_x\Big(e^{-\lambda\tau_a + i\mu X(\tau_a)}\Big) = (\lambda \mp (i\mu)^n) \int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\Big(e^{i\mu X(t)} \mathbb{1}_{\{M(t)>a\}}\Big)\, dt$$

Theorem 3 (A.L. (EJP 12, 2007)) *The distribution of* $X(\tau_a)$ *is given by*

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \sum_{p=0}^{n/2-1} \frac{(a-x)^p}{p!}\, \delta_a^{(p)}(z)$$ *with* $< \delta_a^{(p)}, \varphi >= (-1)^p \varphi^{(p)}(a)$

Example ($n = 4$, **Nishioka 1997**)

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \delta_a(z) - (x-a)\delta'_a(z)$$ with $< \delta'_a, \varphi >= -\varphi'(a)$

$$\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt\, dz = \mathcal{J}_0(t; x-a)\, \delta_a(z) + \mathcal{J}_1(t; x-a)\, \delta'_a(z)$$

with
$$\begin{cases} \mathcal{J}_0(t;\xi) = \frac{\xi}{2\pi t} \int_0^{+\infty} \left[e^{\xi\lambda} - \cos(\xi\lambda) + \sin(\xi\lambda)\right] e^{-t\lambda^4}\, d\lambda \\ \mathcal{J}_1(t;\xi) = \frac{2}{\pi} \int_0^{+\infty} \left[\cos(\xi\lambda) + \sin(\xi\lambda) - e^{\xi\lambda}\right] \lambda^2\, e^{-t\lambda^4}\, d\lambda \end{cases}$$

8 Distribution of the first exit time τ_{ab}

Suppose that n is even and fix $x \in (a, b)$.

Theorem 4 (A.L. (in preparation)) *The distribution of $X(\tau_{ab})$ is given by*

$$\mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz = \sum_{p=0}^{n/2-1} H_p^-(x)\,\delta_a^{(p)}(z) + \sum_{p=0}^{n/2-1} H_p^+(x)\,\delta_b^{(p)}(z)$$

where the functions H_p^- and H_p^+, $0 \leq p \leq n/2-1$, are the interpolation Hermite polynomials such that $\frac{d^q H_p^-}{dx^q}(a) = \delta_{pq}$, $\frac{d^q H_p^-}{dx^q}(b) = 0$ and $\frac{d^q H_p^+}{dx^q}(a) = 0$, $\frac{d^q H_p^+}{dx^q}(b) = \delta_{pq}$ for $0 \leq q \leq n/2-1$.

Set $\tau_b^+ = \inf\{t \geq 0 : X(t) > b\}$ for $x < b$ and $\tau_a^- = \inf\{t \geq 0 : X(t) < a\}$ for $x > a$.

Corollary *In particular, the "ruin pseudo-probabilities" are given by*

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^-(x) \text{ and } \mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^+(x).$$