Study of some new integrated statistics

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References

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1 Notations

Let $X_1, X_2, \ldots, X_N, \ldots$ be independent random variables with continuous distribution function F_0 and set

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i < x\}}.$$

The random variables $U_i = F_0(X_i), i \geqslant 1$, are independent and uniformly distributed on $(0,1)$. Set also

$$G_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{U_i < t\}}.$$

1.1 p-fold integrated empirical process

Let $p \geqslant 1$ be an fixed integer. Let us introduce the p-fold integrated empirical process as follows:

$$\begin{aligned}
F_{p,n}(x) &\equiv \int_{-\infty}^{x} \frac{[F_0(x) - F_0(y)]^{p-1}}{(p-1)!} F_n(y)\, dF_0(y), \\
F_{p,0}(x) &\equiv \int_{-\infty}^{x} \frac{[F_0(x) - F_0(y)]^{p-1}}{(p-1)!} F_0(y)\, dF_0(y), \\
\mathcal{F}_{p,n}(x) &\equiv \sqrt{n}[F_{p,n}(x) - F_{p,0}(x)].
\end{aligned}$$

We also introduce the similar quantities related to the uniform distribution. For $t \in [0,1]$,

$$\begin{aligned}
G_{p,n}(t) &\equiv \int_{0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} G_n(s)\, ds = \frac{1}{n} \sum_{i=1}^{n} \frac{[(t-U_i)^+]^p}{p\,!}, \\
G_{p,0}(t) &\equiv \int_{0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} s\, ds = \frac{t^{p+1}}{(p+1)!}, \\
\mathcal{G}_{p,n}(t) &\equiv \sqrt{n}[G_{p,n}(t) - G_{p,0}(t)].
\end{aligned}$$

1.2 Some statistics based on the p-fold integrated empirical process

Let us define the statistics $\mathcal{D}_p$, ω_p^1, ω_p^2, $\tilde{U}_p^2$ and $\bar{U}_p^2$ by

$$\begin{aligned}
\mathcal{D}_{p,n} &\equiv \sup_{-\infty<x<+\infty} |\mathcal{F}_{p,n}(x)|, \quad \text{(Kolmogorov-Smirnov)} \\
\omega_{p,n}^1 &= \int_{-\infty}^{+\infty} \mathcal{F}_{p,n}(x)\, dF_0(x), \\
\omega_{p,n}^2 &= \int_{-\infty}^{+\infty} \mathcal{F}_{p,n}(x)^2\, dF_0(x), \quad \text{(Cramér–von Mises)} \\
\tilde{U}_{p,n}^2 &= \int_{-\infty}^{+\infty} \left[\mathcal{F}_{p,n}(x) - \int_{-\infty}^{+\infty} \mathcal{F}_{p,n}(y)\, dF_0(y)\right]^2 dF_0(x), \quad \text{(Watson)} \\
\bar{U}_{p,n}^2 &= \int_{-\infty}^{+\infty} \left[\mathcal{F}_{p,n}(x) - \frac{F_0(x)^p}{p\,!} \int_{-\infty}^{+\infty} \mathcal{F}_{0,n}(y)\, dF_0(y)\right]^2 dF_0(x).
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{D}_{p,n} &= \max_{0\leqslant t\leqslant 1} \mathcal{G}_{p,n}(t), \\
\omega_{p,n}^1 &= \int_0^1 \mathcal{G}_{p,n}(t)\, dt = \mathcal{G}_{p+1,n}(1), \\
\omega_{p,n}^2 &= \int_0^1 \mathcal{G}_{p,n}(t)^2\, dt, \\
\tilde{U}_{p,n}^2 &= \int_0^1 \left[\mathcal{G}_{p,n}(t) - \int_0^1 \mathcal{G}_{p,n}(s)\, ds\right]^2 dt = \omega_{p,n}^2 - (\omega_{p,n}^1)^2, \\
\bar{U}_{p,n}^2 &= \int_0^1 \left[\mathcal{G}_{p,n}(t) - \frac{t^p}{p\,!} \int_0^1 \mathcal{G}_{0,n}(s)\, ds\right]^2 dt.
\end{aligned}$$

2 Limiting processes

Let $\beta \equiv (\beta(t))_{0\leqslant t\leqslant 1}$ be standard Brownian bridge. We introduce the underlying processes $\beta_p = (\beta_p(t))_{0\leqslant t\leqslant 1}$, $\tilde{\gamma}_p = (\tilde{\gamma}_p(t))_{0\leqslant t\leqslant 1}$ and $\bar{\gamma}_p = (\bar{\gamma}_p(t))_{0\leqslant t\leqslant 1}$ associated with our statistics:

$$\begin{aligned}
\beta_p(t) &\equiv \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\,\beta(s)\,ds = \int_0^t \frac{(t-s)^p}{p\,!}\,d\beta(s),\\
\tilde{\gamma}_p(t) &\equiv \beta_p(t) - \int_0^1 \beta_p(s)\,ds = \beta_p(t) - \beta_{p+1}(1),\\
\bar{\gamma}_p(t) &\equiv \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\,[\beta(s) - \int_0^1 \beta(u)\,du]\,ds = \beta_p(t) - \frac{t^p}{p\,!}\,\beta_1(1).
\end{aligned}$$

Theorem 2.1 *The process $\mathcal{G}_{p,n}$ converges weakly towards β_p as $n \longrightarrow +\infty$, and then:*

$$\begin{aligned}
\mathcal{D}_{p,n} &\stackrel{law}{\longrightarrow} \max_{0\leqslant t\leqslant 1} |\beta_p(t)|,\\
\omega^1_{p,n} &\stackrel{law}{\longrightarrow} \beta_{p+1}(1),\\
\omega^2_{p,n} &\stackrel{law}{\longrightarrow} \int_0^1 \beta_p(t)^2\,dt,\\
\tilde{U}^2_{p,n} &\stackrel{law}{\longrightarrow} \int_0^1 \tilde{\gamma}_p(t)^2\,dt = \int_0^1 \beta_p(t)^2\,dt - \beta_{p+1}(1)^2,\\
\bar{U}^2_{p,n} &\stackrel{law}{\longrightarrow} \int_0^1 \bar{\gamma}_p(t)^2\,dt.
\end{aligned}$$

Proof. The foregoing results readily come from the well-known fact: the process $\mathcal{G}_{0,n}$ converges weakly towards Brownian bridge β, together with the continuous mapping theorem.

3 Covariance and Green functions

Proposition 3.1 *The processes β_p, $\tilde{\gamma}_p$ and $\bar{\gamma}_p$ are centered Gaussian processes with respective covariance functions*

$$\begin{aligned} G_{\beta_p}(s,t) &\equiv \mathbb{E}[\beta_p(s)\beta_p(t)] \\ &= \int_0^{s\wedge t} \frac{(s-u)^p}{p\,!}\,\frac{(t-u)^p}{p\,!}\,du - \frac{(st)^{p+1}}{(p+1)!^2}, \end{aligned}$$

$$\begin{aligned} G_{\tilde{\gamma}_p}(s,t) &\equiv \mathbb{E}[\tilde{\gamma}_p(s)\tilde{\gamma}_p(t)] \\ &= \int_0^{s\wedge t} \frac{(s-u)^p}{p\,!}\,\frac{(t-u)^p}{p\,!}\,du \\ &\quad -\int_0^{s} \frac{(s-u)^p}{p\,!}\,\frac{(1-u)^{p+1}}{(p+1)!}\,du \\ &\quad -\int_0^{t} \frac{(t-u)^p}{p\,!}\,\frac{(1-u)^{p+1}}{(p+1)!}\,du \\ &\quad -\frac{(st)^{p+1}}{(p+1)!^2} + \frac{s^{p+1}+t^{p+1}}{(p+1)!(p+2)!} + \frac{(p+1)^2}{(p+2)!^2(2p+3)}, \end{aligned}$$

$$\begin{aligned} G_{\bar{\gamma}_p}(s,t) &\equiv \mathbb{E}[\bar{\gamma}_p(s)\bar{\gamma}_p(t)] \\ &= \int_0^{s\wedge t} \frac{(s-u)^p}{p\,!}\,\frac{(t-u)^p}{p\,!}\,du \\ &\quad +\frac{(st)^p}{p\,!^2}\Big[\frac{1}{12} - \frac{s+t}{2(p+1)} - \frac{st}{(p+1)^2} + \frac{s^2+t^2}{(p+1)(p+2)}\Big]. \end{aligned}$$

PROOF. Use

$$\begin{aligned} \mathbb{E}\left[\int_0^s f(u)\,d\beta(u)\int_0^t g(u)\,d\beta(u)\right] &= \int_0^{s\wedge t} f(u)g(u)\,du \\ &\quad -\int_0^s f(u)\,du\int_0^t g(u)\,du. \end{aligned}$$

4 Boundary value problems

Theorem 4.1 *The function*

$$f(t) = \int_0^1 G_{\beta_p}(s,t)g(s)\,ds,\ t \in [0,1],$$

is the unique solution of the boundary value problem:

$$\begin{cases} \dfrac{d^{2p+2}f}{dt^{2p+2}}(t) & = & (-1)^{p+1}g(t),\ t \in (0,1), \\ \dfrac{d^i f}{dt^i}(0) & = & 0 \ \textit{for}\ 0 \leqslant i \leqslant p, \\ \dfrac{d^i f}{dt^i}(1) & = & 0 \ \textit{for}\ i = p \ \textit{and}\ p+2 \leqslant i \leqslant 2p+1. \end{cases}$$

Theorem 4.2 *The function*

$$f(t) = \int_0^1 G_{\tilde{\gamma}_p}(s,t)g(s)\,ds,\ t \in [0,1],$$

is the unique solution of the boundary value problem:

$$\begin{cases} \dfrac{d^{2p+2}f}{dt^{2p+2}}(t) & = & (-1)^{p+1}\left[g(t) - \displaystyle\int_0^1 g(s)\,ds\right],\ t \in (0,1), \\ \dfrac{d^i f}{dt^i}(0) & = & 0 \ \textit{for}\ 1 \leqslant i \leqslant p, \\ \dfrac{d^i f}{dt^i}(1) & = & 0 \ \textit{for}\ i = p \ \textit{and}\ p+2 \leqslant i \leqslant 2p+1, \\ \displaystyle\int_0^1 f(t)\,dt & = & 0. \end{cases}$$

Theorem 4.3 *For $p \geqslant 2$, the function*

$$f(t) = \int_0^1 G_{\bar{\gamma}_p}(s,t)g(t)\,ds,\ t \in [0,1],$$

is the unique solution of the boundary value problem:

$$\begin{cases} \dfrac{d^{2p+2}f}{dt^{2p+2}}(t) & = & (-1)^{p+1}g(t),\ t \in (0,1), \\ \dfrac{d^i f}{dt^i}(0) & = & 0 \ \text{for}\ 0 \leqslant i \leqslant p-1, \\ \dfrac{d^i f}{dt^i}(1) & = & 0 \ \text{for}\ i = p-1 \ \text{and}\ p+3 \leqslant i \leqslant 2p+1, \\ \dfrac{d^p f}{dt^p}(0) & = & \dfrac{d^p f}{dt^p}(1) \quad \text{and} \quad \dfrac{d^{p+1} f}{dt^{p+1}}(0) = \dfrac{d^{p+1} f}{dt^{p+1}}(1). \end{cases}$$

Some eigenvalue problems : find the largest eigenvalue of the kernels G_{β_p}, $G_{\tilde{\gamma}_p}$ and $G_{\bar{\gamma}_p}$. For instance,

$$\begin{aligned} \lambda_{G_{\beta_p}} &= \sup\left\{\int_0^1\int_0^1 G_{\beta_p}(s,t)f(s)f(t)\,ds\,dt; f \in L^2[0,1], \|f\|_2 = 1\right\} \\ &= \sup\left\{\mathrm{var}\left[\int_0^1 f(s)\beta_p(s)\,ds\right]; f \in L^2[0,1], \|f\|_2 = 1\right\}. \end{aligned}$$

p	$\lambda_{G_{\beta_p}}$	$\lambda_{G_{\tilde{\gamma}_p}}$	$\lambda_{G_{\bar{\gamma}_p}}$
0	$1,01321.10^{-1}$	$2,53302.10^{-2}$	$2,53302.10^{-2}$
1	$3,19639.10^{-2}$	$1,02659.10^{-2}$	$1,99774.10^{-3}$
2	$4,31427.10^{-3}$	$2,10317.10^{-3}$	$5,66559.10^{-4}$
3	$3,01473.10^{-4}$	$1,78626.10^{-4}$	$7,32902.10^{-5}$
4	$1,29354.10^{-5}$	$8,56456.10^{-6}$	$4,81853.10^{-6}$
5	$3,76910.10^{-7}$	$2,68114.10^{-7}$	$1,92942.10^{-7}$

5 Bahadur theory

5.1 Bahadur exact slope

For testing $H_0 : F = F_0$ versus $H_\theta : F = F_0(. + \theta), \theta > 0$, we shall use the following result:

Suppose that the statistic $T = (T_n)_{n \geqslant 1}$ satisfies the following conditions:
1) $\frac{1}{\sqrt{n}} T_n \xrightarrow[n \to \infty]{\mathbb{P}_{H_\theta}} b_T(\theta)$ *under* H_θ *;*
2) $\lim_{n \to +\infty} \frac{1}{n} \ln \mathbb{P}_{H_0}\Big(\frac{1}{\sqrt{n}} T_n \geqslant \varepsilon\Big) = -h_T(\varepsilon)$ *for all* ε *lying within an interval where* h_T *is continous and that contains* $b_T(0, \infty)$*.*
Then the so-called Bahadur exact slope of statistic T *is given by*

$$\boxed{c_T(\theta) = 2h_T(b_T(\theta)).}$$

5.2 B-efficiency

The local B-efficiency of the statistic $T = (T_n)_{n \geqslant 1}$ is defined by

$$\boxed{e_T \equiv \lim_{\theta \to 0^+} \frac{c_T(\theta)}{2K(\theta)} = \frac{l_T}{I(f_0)}}$$

where

$$\begin{aligned} K(\theta) &\equiv \int_{-\infty}^{+\infty} \ln \frac{f_0(x+\theta)}{f_0(x)} f_0(x+\theta)\, dx \\ I(f_0) &\equiv \int_{-\infty}^{+\infty} \frac{f_0'(x)^2}{f_0(x)}\, dx = \int_0^1 \psi_0'(u)^2\, du \end{aligned}$$

are the Kullback-Leibler and Fisher informations, l_T is the local index of T and $\psi_0 = f_0 \circ F_0^{-1}$ is the density-quantile function (in our cases $K(\theta) \underset{\theta \to 0^+}{\sim} \frac{1}{2} I(f_0)\theta^2$ and $c_T(\theta) \underset{\theta \to 0^+}{\sim} l_T \theta^2$).

5.3 The statistic ω_p^1

Proposition 5.1 *1) Under H_θ:*

$$\frac{1}{\sqrt{n}}\,\omega_{p,n}^1 \xrightarrow[n\to\infty]{\mathbb{P}_{H_\theta}} b_{\omega_p^1}(\theta) \underset{\theta\to 0+}{\sim} \theta\int_0^1 \psi_0(u)\frac{(1-u)^p}{p\,!}\,du.$$

2) Under H_0:

$$\lim_{n\to+\infty}\frac{1}{n}\ln\mathbb{P}_{H_0}\left(\frac{1}{\sqrt{n}}\,\omega_{p,n}^1 \geqslant \varepsilon\right) = -h_{\omega_p^1}(\varepsilon) \underset{\varepsilon\to 0+}{\sim} \frac{1}{2}\,p\,!^2(p+2)^2(2p+3)\varepsilon^2.$$

Theorem 5.2 *The local slope of the statistic ω_p^1 is given by*

$$\boxed{c_{\omega_p^1}(\theta) \underset{\theta\to 0+}{\sim} l_{\omega_p^1}\theta^2}$$

where $l_{\omega_p^1}$ is the local index defined by

$$\boxed{l_{\omega_p^1} \equiv (p+2)^2(2p+3)\left[\int_0^1 \psi_0(u)(1-u)^p\,du\right]^2.}$$

Proof. 1) is due to Glivenko-Cantelli theorem.
2) Observe that

$$\omega_{p,n}^1 \overset{\text{law}}{=} \frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i,$$

with $Z_i = \dfrac{U_i^{p+1}}{(p+1)!} - \dfrac{1}{(p+2)!}$. Chernoff's theorem asserts that if $h_{\omega_p^1}(\varepsilon) = \sup_{s\geqslant 0}[\varepsilon s - \ln\mathbb{E}(e^{sZ_i})]$ denotes the Cramér transform of the random variable Z_i, then

$$\lim_{n\to+\infty}\frac{1}{n}\ln\mathbb{P}_{H_0}\left(\frac{1}{\sqrt{n}}\,\omega_{p,n}^1 \geqslant \varepsilon\right) = -h_{\omega_p^1}(\varepsilon) \underset{\varepsilon\to 0+}{\sim} \frac{\varepsilon^2}{2\,\mathrm{var}(Z_1)}.$$

5.4 The statistic $\mathcal{D}_p$

REMARK. We observe that $h_{\omega_p^1}(\varepsilon) \underset{\varepsilon\to 0+}{\sim} h_{\mathcal{D}_{p+1}}(\varepsilon)$ and $l_{\omega_p^1} = l_{\mathcal{D}_{p+1}}$. So, we only need to study one of the statistics $\mathcal{D}_p$ and ω_p^1 to derive similar properties for the other one.

5.5 The statistic ω_p^2

Proposition 5.3 *1) Under H_θ:*

$$\frac{1}{n}\,\omega_{p,n}^2 \xrightarrow[n\to\infty]{\mathbb{P}_{H_\theta}} b_{\omega_p^2}(\theta)^2 \underset{\theta\to 0+}{\sim} \theta^2 \int_0^1 \left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\,\psi_0(t)\,dt\right]^2 du.$$

2) Under H_0:

$$\lim_{n\to+\infty} \frac{1}{n} \ln \mathbb{P}_{H_0}\left(\frac{1}{n}\,\omega_{p,n}^2 \geqslant \varepsilon^2\right) = -h_{\omega_p^2}(\varepsilon) \underset{\varepsilon\to 0+}{\sim} \frac{\varepsilon^2}{2\lambda_{G_{\beta_p}}}$$

where $\lambda_{G_{\beta_p}}$ is the largest eigenvalue of the integral operator with kernel G_{β_p}.

Theorem 5.4 *The local slope of the statistic ω_p^2 is given by*

$$\boxed{c_{\omega_p^2}(\theta) \underset{\theta\to 0+}{\sim} l_{\omega_p^2}\theta^2}$$

where $l_{\omega_p^2}$ is the local index defined by

$$\boxed{l_{\omega_p^2} \equiv \frac{1}{\lambda_{G_{\beta_p}}} \int_0^1 \left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\,\psi_0(t)\,dt\right]^2 du.}$$

PROOF. $\omega_{p,n}^2 = \left|\left|\frac{1}{n}\sum_{i=1}^n Y_i\right|\right|_2^2$ with $Y_i = \frac{[(t-U_i)^+]^p}{p!} - \frac{t^{p+1}}{(p+1)!}$.

$$\lim_{n\to+\infty} \frac{1}{n} \ln \mathbb{P}_{H_0}\left(\frac{1}{n}\,\omega_{p,n}^2 \geqslant \varepsilon^2\right) \underset{\varepsilon\to 0+}{\sim} \frac{\varepsilon^2}{2 \sup\limits_{\{||f||_2=1\}} \operatorname{var}\left[\int_0^1 f(s)\beta_p(s)\,ds\right]}.$$

6 Some examples for f_0

$$
\begin{aligned}
f_1(x) &\equiv \varphi(x) \equiv \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} && \text{(normal density)},\\
f_2(x) &\equiv \frac{e^x}{(1+e^x)^2} && \text{(logistic density)},\\
f_3(x) &\equiv \frac{1}{2} e^{-|x|} && \text{(Laplace density)},\\
f_4(x) &\equiv \frac{1}{\pi \cosh x} && \text{(hyberbolic cosine density)},\\
f_5(x) &\equiv \frac{1}{\pi(1+x^2)} && \text{(Cauchy density)}.
\end{aligned}
$$

	Gauss	Logistic	Laplace	Hyperbolic	Cauchy
D_0	0.637	0.750	1	0.811	0.811
D_1	0.955	1	0.750	0.986	0.608
D_2	0.895	0.938	0.703	0.924	0.570
D_3	0.721	0.840	0.595	0.813	0.453
ω_0^2	0.907	0.987	0.822	1	0.750
ω_1^2	0.912	0.968	0.749	0.964	0.629
ω_2^2	0.851	0.881	0.648	0.863	0.514
ω_3^2	0.787	0.789	0.540	0.756	0.395
$\bar{U}_0^2$	0.486	0.657	0.822	0.758	1
$\bar{U}_1^2$	0.142	0.199	0.261	0.234	0.321
$\bar{U}_2^2$	0.042	0.058	0.071	0.068	0.086
$\bar{U}_3^2$	0.091	0.124	0.146	0.142	0.176
$\bar{U}_4^2$	0.156	0.212	0.247	0.242	0.295
$\bar{U}_5^2$	0.217	0.292	0.333	0.333	0.396

7 Local asymptotic optimality

We are interested in solving the problem of optimality in Bahadur-Raghavachari inequality in regard to our statistics. This inequality can be stated as follows:

for all statistic T,

$$\boxed{c_T(\theta) \leqslant 2K(\theta).}$$

The optimality consists of finding a distribution function F_0 such that the equality holds in the foregoing inequality, that is $c_T(\theta) = 2K(\theta)$ for all $\theta > 0$. This is a difficult problem, and the local asymptotic optimality (for short LAO) is easier to be reached. So, we shall search the distribution function F_0 such that the asymptotics holds:

$$\boxed{c_T(\theta) \underset{\theta \to 0+}{\sim} 2K(\theta).}$$

This problem boils down to finding the probability density function f_0 satisfying

$$\boxed{I(f_0) = l_T.}$$

7.1 LAO for $\mathcal{D}_p$

We have to solve the equation

$$\boxed{\int_0^1 \psi'(u)^2\,du = (2p+1)(p+1)^2\left[\int_0^1 (1-u)^{p-1}\psi(u)\,du\right]^2.}$$

This equation can be formulated as a variational problem: minimize $\int_0^1 \psi'(u)^2\,du$ within the set of the functions $\psi \in L^2[0,1]$ that are absolutely continuous with derivative in $L^2[0,1]$ subject to the conditions $\psi(0)=\psi(1)=0$, $\psi \geqslant 0$ and $\int_0^1 (1-u)^{p-1}\psi(u)\,du = \frac{1}{(p+1)\sqrt{2p+1}}$. The solutions $\psi \in \overset{\circ}{\boldsymbol{W}}_{2,1}\ [0,1]$ of that problem have the form

$$\psi(u) = \mu(1-u)[1-(1-u)^p],$$

where μ is any positive constant.

Now, we seek the corresponding distribution functions F related to ψ by the relation $\psi = F' \circ F^{-1}$. Those functions F satisfies the following differential equation:

$$F'(x) = \mu(1-F(x))[1-(1-F(x))^p]$$

from which we derive the family of distribution functions

$$\boxed{F(x) = 1-(1+e^{ax+b})^{-1/p}, \quad a>0, b\in\mathbb{R},}$$

together with the related probability density functions:

$$\boxed{f(x) = \frac{a}{p}\frac{e^{ax+b}}{(1+e^{ax+b})^{1+1/p}}, \quad a>0, b\in\mathbb{R}.}$$

As a result, we just found a generalization of the logistic distribution introduced by Dubey.

7.2 LAO for ω_p^2

The problem of LAO for ω_p^2 is equivalent to solving the equation

$$\boxed{\int_0^1 \psi'(u)^2\,du = \frac{1}{\lambda_{G_{\beta_p}}}\int_0^1\left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\,\psi(t)\,dt\right]^2 du.}$$

The corresponding variational problem consists of minimizing $\displaystyle\int_0^1 \psi'(u)^2\,du$ within the set of the functions $\psi \in\overset{\circ}{\boldsymbol{W}}_{2,1}[0,1]$ subject to the condition $\displaystyle\int_0^1\left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\,\psi(t)\,dt\right]^2 du = \lambda_{G_{\beta_p}}$.

It may be shown that ψ is the unique solution of the following boundary value problem:

$$\left\{\begin{array}{rcl} \dfrac{d^{2p+2}\psi}{dt^{2p+2}}(t) & = & (-1)^p\lambda\psi(t),\ t\in(0,1),\\[2ex] \dfrac{d^i\psi}{dt^i}(0) & = & 0 \text{ for } i=0 \text{ and } p+2\leqslant i\leqslant 2p+1,\\[2ex] \dfrac{d^i\psi}{dt^i}(1) & = & 0 \text{ for } i=0 \text{ and } 2\leqslant i\leqslant p+1. \end{array}\right.$$

We are not able to solve this problem explicitly.

7.3 LAO for $\tilde{U}_p^2$

The problem of LAO for $\tilde{U}_p^2$ consists of solving the equation

$$\int_0^1 \psi'(u)^2\,du = \frac{1}{\lambda_{G_{\tilde{\gamma}_p}}}\left[\int_0^1 \left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\,\psi_0(t)\,dt\right]^2 du - \left[\int_0^1 \frac{(1-t)^p}{p\,!}\,\psi_0(t)\,dt\right]^2\right].$$

It may be shown that ψ is the unique solution of the following boundary value problem:

$$\begin{cases} \dfrac{d^{2p+2}\psi}{dt^{2p+2}}(t) = (-1)^p\lambda\psi(t),\ t\in(0,1), \\ \dfrac{d^i\psi}{dt^i}(0) = 0 \ \text{ for } \ i=0 \ \text{ and } \ p+3\leqslant i\leqslant 2p+1, \\ \dfrac{d^i\psi}{dt^i}(1) = 0 \ \text{ for } \ i=0 \ \text{ and } \ 2\leqslant i\leqslant p+2. \end{cases}$$

We are not able to solve this problem explicitly.

7.4 LAO for $\bar{U}_p^2$

The problem of LAO for $\bar{U}_p^2$ is equivalent to solving the equation

$$\int_0^1 \psi'(u)^2\,du = \frac{1}{\lambda_{G_{\bar{\gamma}_p}}}\left[\int_0^1\left[\int_0^u \frac{(u-t)^{p-1}}{(p-1)!}\left(\psi(t)-\int_0^1\psi(v)\,dv\right)dt\right]^2 du\right].$$

It may be shown that ψ is the unique solution of the following boundary value problem:

$$\begin{cases} \dfrac{d^{2p+2}\psi}{dt^{2p+2}}(t) = (-1)^p\lambda\left[\psi(t)-\int_0^1\psi(v)\,dv\right],\ t\in(0,1),\\ \dfrac{d^i\psi}{dt^i}(0) = 0 \text{ for } i=0 \text{ and } p+2\leqslant i\leqslant 2p+1,\\ \dfrac{d^i\psi}{dt^i}(1) = 0 \text{ for } i=0,\ 3\leqslant i\leqslant p+2 \text{ and } i=2p+1. \end{cases}$$

This system may be solved by condidering the boundary value problem satisfied by the function $\bar{\psi}$:

$$\begin{cases} \dfrac{d^{2p+2}\bar{\psi}}{dt^{2p+2}}(t) = (-1)^p\lambda\bar{\psi}(t),\ t\in(0,1),\\ \bar{\psi}(0) = \bar{\psi}(1),\\ \dfrac{d^i\bar{\psi}}{dt^i}(0) = 0 \text{ for } p+2\leqslant i\leqslant 2p+1,\\ \dfrac{d^i\bar{\psi}}{dt^i}(1) = 0 \text{ for } 3\leqslant i\leqslant p+2 \text{ and } i=2p+1\\ \int_0^1\bar{\psi}(s)\,ds = 0. \end{cases}$$

We are not able to solve this problem explicitly.