Bridges of certain Wiener integrals: some properties and application to goodness-of-fit testing

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1 Settings

Let $(W(t))_{t\geqslant 0}$ and $(\beta(t))_{t\geqslant 0}$ be respectively standard Wiener process started at 0 and standard Brownian bridge on $[0,1]$, and denote by

$$W_n(t) = \int_0^t \frac{(t-s)^n}{n!}\,dW(s)$$

and

$$\beta_n(t) = \int_0^t \frac{(t-s)^n}{n!}\,d\beta(s)$$

their respective n-fold primitives for any integer $n \geqslant 0$ ($W_0 = W$, $\beta_0 = \beta$).

Let us imbed W_n into the $(n+1)$-dimensional Gaussian process

$$U_n = (W_0, W_1, \dots, W_n)$$

and introduce the process $(U_n(t)\,|\,U_n(1) = \mathbf{0})_{0\leqslant t\leqslant 1}$ we shall call "bridge" associated with U_n pinned at times 0 and 1 at the origin $\mathbf{0}$ in $\mathbb{R}^{n+1}$.

Since $(d^i/dt^i)W_n = W_{n-i}$ for $0 \leqslant i \leqslant n$, we shall only consider the bridge $(W_n(t)\,|\,U_n(1) = \mathbf{0})_{0\leqslant t\leqslant 1}$.

Referring to the classical representations for standard Brownian bridge $(\beta(t))_{0\leqslant t\leqslant 1}$,

$$\begin{array}{ll} (W(t)\,|\,W(1)=0)_{0\leqslant t\leqslant 1} & \text{(conditioning)} \\ (tW(\frac{1}{t}-1))_{0\leqslant t\leqslant 1} & \text{(time-inversion)} \\ (W(t)-tW(1))_{0\leqslant t\leqslant 1} & \text{(random drift)}\ , \end{array}$$

we have similar representations for B_n:

- **Conditioning W_n**

$$\boxed{(B_n(t))_{0\leqslant t\leqslant 1} \stackrel{\text{law}}{=} (W_n(t)\,|\,U_n(1)=\mathbf{0})_{0\leqslant t\leqslant 1}.}$$

- **Time-inversion**

$$\begin{aligned} (B_n(t))_{0<t<1} &\stackrel{\text{law}}{=} \left(t^{2n+1}W_n\left(\frac{1}{t}-1\right)\right)_{0<t<1} \\ &\stackrel{\text{law}}{=} \left((1-t)^{2n+1}W_n\left(\frac{t}{1-t}\right)\right)_{0<t<1}. \end{aligned}$$

- **Random drift**

$$\boxed{(B_n(t))_{0\leqslant t\leqslant 1} \stackrel{\text{law}}{=} (W_n(t)-\textstyle\sum_{i=0}^{n} p_{n-i\,n}(t)W_i(1))_{0\leqslant t\leqslant 1},}$$

where $p_{in}(t)=\frac{(-1)^i}{i!}t^{n+1}(1-t)^i\sum_{j=0}^{n-i}\binom{n+j}{n}(1-t)^j$.

The $(p_{in})_{0\leqslant i\leqslant n}$ are the classical interpolation polynomials at 0 and 1 with degree $2n+1$ such that

$$\frac{d^j p_{in}}{dt^j}(0)=0 \ \text{ and } \ \frac{d^j p_{in}}{dt^j}(1)=\delta_{ij} \ \text{ for } \ 0\leqslant j\leqslant n.$$

- **Conditioning β_n and random drift**

$$\begin{aligned} (B_n(t))_{0\leqslant t\leqslant 1} &\stackrel{\text{law}}{=} (\beta_n(t)\,|\,\beta_1(1)=\ldots=\beta_n(1)=0)_{0\leqslant t\leqslant 1} \\ &\stackrel{\text{law}}{=} (\beta_n(t)-\textstyle\sum_{i=1}^{n} p_{n-i\,n}(t)\beta_i(1))_{0\leqslant t\leqslant 1}. \end{aligned}$$

2 Prediction property

Set $B_{n\ell}(t) = (W(t) \,|\, U_n(\ell) = 0),\ 0 \leqslant t \leqslant \ell$ (bridge with length ℓ).

Theorem 2.1 *Fix an instant $t_0 \in (0,1)$. We have for $0 \leqslant t \leqslant 1 - t_0$:*

$$\boxed{B_n(t + t_0) = \widetilde{B}_{n\,1-t_0}(t) + \sum_{i=0}^{n} p_{in}(t_0; t + t_0) \frac{d^i B_n}{dt^i}(t_0)}$$

where $\widetilde{B}_{n\,1-t_0}$ is a copy of the process $B_{n\,1-t_0}$ which is independent of $(B_n(t))_{0 \leqslant t \leqslant t_0}$, and

$$p_{in}(t_0; t) = \frac{1}{i!} \left(\frac{1-t}{1-t_0} \right)^{n+1} (t - t_0)^i \sum_{j=0}^{n-i} \binom{n+j}{n} \left(\frac{t - t_0}{1 - t_0} \right)^j$$

are the interpolation polynomials at t_0 and 1 such that

$$\frac{d^j p_{in}}{dt^j}(t_0; t_0) = \delta_{ij} \ \ \textit{and} \ \ \frac{d^j p_{in}}{dt^j}(t_0; 1) = 0 \ \ \textit{for} \ \ 0 \leqslant j \leqslant n.$$

3 Boundary value problems

We consider the following differential equation:

$$\frac{d^{2n+2}f}{dx^{2n+2}}(x) = (-1)^{n+1}g(x),\ x \in (0,1)$$

(where g is any given continuous function defined on $[0,1]$) subject to different types of boundary value conditions:

$$\begin{array}{lrcl}
\text{(I)} & \dfrac{d^i f}{dx^i}(0) & = & 0 \text{ for } 0 \leqslant i \leqslant n; \\
\text{(II)} & \dfrac{d^i f}{dx^i}(1) & = & 0 \text{ for } 0 \leqslant i \leqslant n; \\
\text{(III)} & \dfrac{d^i f}{dx^i}(1) & = & 0 \text{ for } n+1 \leqslant i \leqslant 2n+1; \\
\text{(IV)} & \dfrac{d^i f}{dx^i}(1) & = & 0 \text{ for } i = n \text{ and } n+2 \leqslant i \leqslant 2n+1.
\end{array}$$

Theorem 3.1 *The unique solution of each boundary value problem (I–II), (I–III) and (I–IV) is given by*

$$\boxed{f(x) = \int_0^1 G(x,y)g(y)\,dy,\ x \in [0,1]}$$

where G is respectively one of the following Green functions G_{B_n}, G_{W_n} and G_{β_n}:

(I–II): $G_{B_n}(x,y) = \mathbb{E}[B_n(x)B_n(y)]$;

(I–III): $G_{W_n}(x,y) = \mathbb{E}[W_n(x)W_n(y)]$;

(I–IV): $G_{\beta_n}(x,y) = \mathbb{E}[\beta_n(x)\beta_n(y)]$,

for $(x,y) \in [0,1] \times [0,1]$.

Let i be an integer such that $0 \leqslant i \leqslant n$ and set

$$\mathbf{W}_n(t) = [W_n(t) \,|\, W_i(1) = 0] \stackrel{\text{law}}{=} W_n(t) - \frac{\mathbb{E}[W_n(t)W_i(1)]}{\mathbb{E}[W_i(1)^2]} W_i(1).$$

We have $\beta_n \stackrel{\text{law}}{=} \mathbf{W}_{0n}$.

Let us consider the following boundary value conditions:

(I) $\displaystyle \frac{d^i f}{dx^i}(0) = 0$ for $0 \leqslant i \leqslant n$;

(V) $\displaystyle \frac{d^j f}{dx^i}(1) = 0$ for $j \in \{n-i, n+1, n+2, \ldots, 2n+1\} \backslash \{n+i+1\}$.

Theorem 3.2 *The unique solution of the boundary value problem (I–V) is given by*

$$\boxed{f(x) = \int_0^1 G_{\mathbf{W}_{in}}(x,y) g(y)\, dy, \; x \in [0,1]}$$

where

$$G_{\mathbf{W}_{in}}(x,y) = \mathbb{E}[\mathbf{W}_{in}(x)\mathbf{W}_{in}(y)] \;\; \textit{for} \;\; (x,y) \in [0,1] \times [0,1].$$

4 The distribution of the maximum

Let us introduce the probability measure $\mathbb{Q}^x$ defined by the following Cameron-Martin-Girsanov density with respect to $\mathbb{P}$:

$$\left.\frac{d\mathbb{Q}^{\,x}_{w_n(x)}}{d\mathbb{P}_{w_n(x)}}\right|_{\mathfrak{F}_t} = \exp\Big[-(n+\tfrac{1}{2})\,\frac{(2n)!^{\,2}}{n!^{\,2}}\,x^2[(t+1)^{2n+1}+1] -(2n+1)!\,x\sum_{i=0}^{n}(-1)^i\frac{(t+1)^{n-i}}{(n-i)!}W_i(t)\Big]$$

where $w_n(x) = -(2n+1)!\,x\Big(\frac{1}{(n+1)!},\frac{1}{(n+2)!},\dots,\frac{1}{(2n+1)!}\Big)$ and $(\mathfrak{F}_t)_{t\geqslant 0}$ is the Brownian filtration.

Write $\tau_0 = \min\{t>0 : W_n(t)=0\}$ (with $\min\emptyset = +\infty$) for the first hitting time through 0 for W_n.

Theorem 4.1 *The distribution function of* $\max_{0\leqslant t\leqslant 1} B_n(t)$ *is expressible by means of the law of* τ_0 *as follows:*

$$\begin{aligned}\mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1} B_n(t) < x\Big\} &= \mathbb{Q}^{\,x}_{w_n(x)}\{\tau_0=+\infty\}\\ &= \exp\Big[-(n+\tfrac{1}{2})\,\tfrac{(2n)!^{\,2}}{n!^{\,2}}\,x^2\Big]\\ &\quad\times\mathbb{E}_{w_n(x)}\Big\{\exp\Big[-(n+\tfrac{1}{2})\,\frac{(2n)!^{\,2}}{n!^{\,2}}\,x^2(\tau_0+1)^{2n+1}\\ &\quad -(2n+1)!\,x\sum_{i=0}^{n-1}(-1)^i\frac{(\tau_0+1)^{n-i}}{(n-i)!}W_i(\tau_0)\Big]\Big\}.\end{aligned}$$

In particular,

$$\mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1} B_1(t) < x\Big\} = 1-e^{-6x^2}\mathbb{E}_{(-3x,-x)}\Big\{\exp\big[-6x^2(\tau_0+1)^3-6x(\tau_0+1)W_0(\tau_0)\big]\Big\},$$

$$\mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1} \beta(t) < x\Big\} = 1-e^{-x^2}\mathbb{E}_{-x}\Big\{e^{-\frac{1}{2}x^2\tau_0}\Big\} = 1-e^{-2x^2}.$$

Theorem 4.2 *The distribution function of* $\max_{0\leqslant t\leqslant 1}\beta_n(t)$ *is given by*

$$\mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1}\beta_n(t)<x\Big\}=\int_{\mathbb{R}^n}\mathbb{Q}^x_{\gamma_n(x,\omega)}\{\tau_0=+\infty\}\mathfrak{p}_n(1;d\omega)$$

where the components of point $\gamma_n(x,\omega)$ *are given by*

$$\begin{cases}\gamma_n(x,\omega)_0 = \displaystyle\sum_{j=1}^{n}\alpha_{n\,n-j\,n}\omega_j-\frac{(2n+1)!}{(n+1)!}\,x,\\ \gamma_n(x,\omega)_i = \displaystyle\sum_{j=i}^{n}\alpha_{n-i\,n-j\,n}\omega_j-\frac{(2n+1)!}{(n+1+i)!}\,x \text{ if } 1\leqslant i\leqslant n,\end{cases}$$

with

$$\alpha_{ijn}=(-1)^j\frac{i!}{j!}\binom{2n+1-j}{i-j} \text{ for } 0\leqslant j\leqslant i\leqslant n$$

and

$$\mathfrak{p}_n(1;d\omega)=\mathbb{P}\{(\beta_1(1),\ldots,\beta_n(1))\in d\omega_1\ldots d\omega_n\}.$$

Theorem 4.3

$$\mathfrak{p}_n(1;\omega)=\frac{A_n}{(2\pi)^{(n+1)/2}}\exp\Big[-\tfrac{1}{2}\sum_{1\leqslant i,j\leqslant n}g_{ij}\omega_i\omega_j\Big]$$

where the matrix $(g_{ij})_{0\leqslant i,j\leqslant n}$ *is the inverse of the matrix* $(\gamma_{ij})_{0\leqslant i,j\leqslant n}$ *with*

$$\gamma_{ij}=\frac{1}{i!\,j!\,(i+j+1)} \text{ and } A_n=\Big(\textstyle\prod_{i=n+1}^{2n+1}i!/\prod_{i=1}^{n}i!\Big)^{\frac{1}{2}}.$$

Write

$$\tau_x = \inf\{t > 0 : W_n(t) \notin (-x, x)\},\ x > 0$$

for the first exit time from the interval $(-x, x)$ for W_n, and

$$\begin{aligned} p_n(t; dw) &= \mathbb{P}\{U_n(t) \in dw\}, \quad w = (w_0, w_1, \ldots, w_n), \\ (\text{ resp. } \quad q_{n,x}(t; dw) &= \mathbb{P}\{U_n(t) \in dw, t < \tau_x\}) \end{aligned}$$

for the density of the process $(U_n(t))_{t\geqslant 0}$ (resp. killed process $(U_n(t))_{0\leqslant t<\tau_x}$).

Theorem 4.4 *The distribution functions of* $\max_{0\leqslant t\leqslant 1} |B_n(t)|$ *and* $\max_{0\leqslant t\leqslant 1} |\beta_n(t)|$ *can be written for* $x > 0$ *as follows:*

$$\boxed{\begin{aligned} \mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1} |B_n(t)| < x\Big\} &= \frac{q_{n,x}(1; \mathbf{0})}{p_n(1; \mathbf{0})}, \\ \mathbb{P}\Big\{\max_{0\leqslant t\leqslant 1} |\beta_n(t)| < x\Big\} &= \sqrt{2\pi} \int_{\mathbb{R}^n} q_{n,x}(1; 0, dw_1, \ldots, dw_n). \end{aligned}}$$

5 Goodness-of-fit testing

Let $X_1, X_2, \ldots, X_N, \ldots$ be independent random variables with continuous distribution function F and write

$$F_N(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{X_i < x\}}$$

for the usual empirical distribution function of the sample $(X_1, X_2, \ldots, X_N)$. The random variables $U_i = F(X_i), i \geqslant 1$, are independent and uniformly distributed on $(0, 1)$. Let

$$G_N(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{U_i < x\}}.$$

We have $F_N = G_N \circ F$. For testing the null hypothesis H_0: $F = F_0$ where F_0 is an *a priori* specified continuous distribution function, we introduce

$$\begin{aligned} F_{n,N}(x) &= \int_{-\infty}^{x} \frac{[F_0(x) - F_0(y)]^n}{n!} \, dF_N(y), \\ F_{n,0}(x) &= \int_{-\infty}^{x} \frac{[F_0(x) - F_0(y)]^n}{n!} \, dF_0(y) = \frac{F_0(x)^{n+1}}{(n+1)!}, \\ \mathcal{F}_{n,N}(x) &= \sqrt{N}[F_{n,N}(x) - F_{n,0}(x)], \end{aligned}$$

and

$$\begin{aligned} G_{n,N}(t) &= \int_{0}^{t} \frac{(t-s)^n}{n!} \, dG_N(s) = \frac{1}{N} \sum_{i=1}^{N} \frac{[(t-U_i)^+]^n}{n!}, \\ G_{n,0}(t) &= \int_{0}^{t} \frac{(t-s)^n}{n!} \, ds = \frac{t^{n+1}}{(n+1)!}, \\ \mathcal{G}_{n,N}(t) &= \sqrt{N}[G_{n,N}(t) - G_{n,0}(t)]. \end{aligned}$$

We have under H_0: $\mathcal{F}_{n,N} = \mathcal{G}_{n,N} \circ F_0$ a.s.

Let us define the following statistics:

$$\begin{aligned}\mathcal{T}^1_{n,N} &= \sup_{-\infty<x<+\infty} \mathcal{F}_{n,N}(x),\\ \mathcal{T}^2_{n,N} &= \sup_{-\infty<x<+\infty} |\mathcal{F}_{n,N}(x)|,\\ \mathcal{T}^3_{n,N} &= \int_{-\infty}^{+\infty} \mathcal{F}_{n,N}(x)\,dF_0(x).\end{aligned}$$

We have under H_0:

$$\begin{aligned}\mathcal{T}^1_{n,N} &= \max_{0\leqslant t\leqslant 1} \mathcal{G}_{n,N}(t) \ \text{ a.s.},\\ \mathcal{T}^2_{n,N} &= \max_{0\leqslant t\leqslant 1} |\mathcal{G}_{n,N}(t)| \ \text{ a.s.},\\ \mathcal{T}^3_{n,N} &= \int_0^1 \mathcal{G}_{n,N}(t)\,dt = \mathcal{G}_{n+1,N}(1) \ \text{ a.s.}\\ &\stackrel{\text{law}}{=} \frac{\sqrt{N}}{(n+1)!}\left[\frac{1}{N}\sum_{i=1}^{N} U_i^{n+1} - \frac{1}{n+2}\right].\end{aligned}$$

Theorem 5.1 *The process $\mathcal{G}_{n,N}$ converges weakly towards β_n as $N \longrightarrow +\infty$, and then, under H_0,*

$$\boxed{\begin{aligned}\mathcal{T}^1_{n,N} &\xrightarrow[N\to\infty]{law} \max_{0\leqslant t\leqslant 1} \beta_n(t),\\ \mathcal{T}^2_{n,N} &\xrightarrow[N\to\infty]{law} \max_{0\leqslant t\leqslant 1} |\beta_n(t)|,\\ \mathcal{T}^3_{n,N} &\xrightarrow[N\to\infty]{law} \beta_{n+1}(1).\end{aligned}}$$

Particular case $n = 1$:

- Henze & Nikitin: large deviations results, computation of the Bahadur efficiency under $H_1 : F = F(\cdot + \theta)$, $\theta > 0$.
- Schmid & Trede, Hawkins & Kochar: stochastic dominance.

We define in a similar manner:

$$\begin{aligned}\widetilde{F}_{n,N}(x) &= \int_{-\infty}^{x} \frac{[F_0(x)-F_0(y)]^n}{n!}\, dF_N(y) \\ &\quad - \sum_{j=0}^{n} p_{n-j\,n}(F_0(x)) \int_{-\infty}^{x} \frac{[1-F_0(y)]^j}{j!}\, dF_N(y), \\ \widetilde{F}_{n,0}(x) &= \int_{-\infty}^{x} \frac{[F_0(x)-F_0(y)]^n}{n!}\, dF_0(y) \\ &\quad - \sum_{j=0}^{n} p_{n-j\,n}(F_0(x)) \int_{-\infty}^{x} \frac{[1-F_0(y)]^j}{j!}\, dF_0(y), \\ \widetilde{\mathcal{F}}_{n,N}(x) &= \sqrt{N}[\widetilde{F}_{n,N}(x) - \widetilde{F}_{n,0}(x)],\end{aligned}$$

and

$$\begin{aligned}\widetilde{G}_{n,N}(t) &= \int_0^t \frac{(t-s)^n}{n!}\, dG_N(s) - \sum_{j=0}^{n} p_{n-j\,n}(t) \int_0^1 \frac{(1-s)^j}{j!}\, dG_N(s) \\ &= \frac{1}{N}\sum_{i=1}^{N} \frac{[(t-U_i)^+]^n}{n!} - \sum_{j=0}^{n} p_{n-j\,n}(t) \Big(\frac{1}{N}\sum_{i=1}^{N} \frac{(1-U_i)^j}{j!} \Big), \\ \widetilde{G}_{n,0}(t) &= \int_0^t \frac{(t-s)^n}{n!}\, ds - \sum_{j=0}^{n} p_{n-j\,n}(t) \int_0^1 \frac{(1-s)^j}{j!}\, ds = 0, \\ \widetilde{\mathcal{G}}_{n,N}(t) &= \sqrt{N}[\widetilde{G}_{n,N}(t) - \widetilde{G}_{n,0}(t)].\end{aligned}$$

We have under H_0: $\widetilde{\mathcal{F}}_{n,N} = \widetilde{\mathcal{G}}_{n,N} \circ F_0$ a.s.

Let us define the new following statistics:

$$\widetilde{\mathcal{T}}^1_{n,N} = \sup_{-\infty<x<+\infty} \widetilde{\mathcal{F}}_{n,N}(x),$$

$$\widetilde{\mathcal{T}}^2_{n,N} = \sup_{-\infty<x<+\infty} |\widetilde{\mathcal{F}}_{n,N}(x)|,$$

$$\widetilde{\mathcal{T}}^3_{n,N} = \int_{-\infty}^{+\infty} \widetilde{\mathcal{F}}_{n,N}(x)\, dF_0(x).$$

We have under H_0:

$$\widetilde{\mathcal{T}}^1_{n,N} = \max_{0\leqslant t\leqslant 1} \widetilde{\mathcal{G}}_{n,N}(t) \text{ a.s.},$$

$$\widetilde{\mathcal{T}}^2_{n,N} = \max_{0\leqslant t\leqslant 1} |\widetilde{\mathcal{G}}_{n,N}(t)| \text{ a.s.},$$

$$\widetilde{\mathcal{T}}^3_{n,N} = \int_0^1 \widetilde{\mathcal{G}}_{n,N}(t)\, dt \text{ a.s.}$$

Theorem 5.2 *The process $G_{n,N}$ converges weakly towards B_n as $N \longrightarrow +\infty$, and then, under H_0,*

$$\widetilde{\mathcal{T}}^1_{n,N} \xrightarrow[N\to\infty]{law} \max_{0\leqslant t\leqslant 1} B_n(t),$$

$$\widetilde{\mathcal{T}}^2_{n,N} \xrightarrow[N\to\infty]{law} \max_{0\leqslant t\leqslant 1} |B_n(t)|,$$

$$\widetilde{\mathcal{T}}^3_{n,N} \xrightarrow[N\to\infty]{law} \int_0^1 B_n(s)\, ds.$$