Joint distribution of the process and its maximum,

joint distribution of the first hitting time and the first hitting place

for a pseudo-process driven by a high-order heat-type equation

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1 High-order heat-type equation

Consider the equation for an **even** integer $N \geqslant 4$:

$$\frac{\partial u}{\partial t} = \kappa_N \frac{\partial^N u}{\partial x^N} \quad \text{with} \quad \kappa_N = (-1)^{N/2+1} = \pm 1.$$

Introduce the heat kernel: $p(t;x)$.

- Characterization:
$$\int_{-\infty}^{+\infty} e^{iux} p(t;x)\, dx = e^{-tu^N};$$

- It defines a Markov pseudo-process $(X(t))_{t \geqslant 0}$ governed by a signed measure with infinite total variation (which is **not** a probability measure) according as
$$\mathbb{P}_x\{X(t) \in dy\} = p(t; x-y)\, dy$$
and for any $n \geqslant 1$ and $0 = t_0 < t_1 < \ldots < t_n$, $x_0 = x$,
$$\mathbb{P}_x\{X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n\} = \prod_{i=1}^{n} p(t_i - t_{i-1}; x_{i-1} - x_i)\, dx_i.$$

2 Problems

2.1 Study of the maximum and the sojourn time (done)

Set

$$M(t) = \max_{0\leqslant s\leqslant t} X(s) \quad \text{and} \quad T_a(t) = \int_0^t \mathbb{1}_{\{X(s)>a\}}\, ds.$$

Relationship between the distributions of $M(t)$ and $T_a(t)$:

$$\boxed{\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\Big[e^{-\mu T_a(t)}\Big]\, dt \underset{\mu\to+\infty}{\longrightarrow} \int_0^{+\infty} e^{-\lambda t}\, \mathbb{P}_x\{M(t)\leqslant a\}\, dt.}$$

2.2 Study of the maximum and first hitting time (to do)

Set $\quad M_t = \max_{0 \leqslant s \leqslant t} X(s) \quad$ and $\quad \tau_a = \inf\{t \geqslant 0 : X(t) > a\}$ for $x < a$.

Relationship between the **joint** *distributions of the couples* $(X(t), M(t))$ *and* $(\tau_a, X(\tau_a))$*:*

$$\boxed{\int_0^{+\infty} e^{-\lambda t}\, \mathbb{E}_x\Big[e^{i\mu X(t) - \nu M(t)}\Big]\, dt \quad ??? \quad \mathbb{E}_x\Big[e^{-\lambda \tau_a + i\mu X(\tau_a)}\Big].}$$

Goal of this talk: computation of the distributions of the couples $(X(t), M(t))$ *and* $(\tau_a, X(\tau_a))$.

Main tools:

- Sampling on dyadic times;
- Spitzer identity;
- Clever algebra related to Vandermonde systems and complex analysis.

3 References

- **Nishioka (1996, 1997, 2001).** — *Case* $\boxed{N = 4\textit{:}}$
  - distributional approach for the first hitting time and place
  - various boundary value problems

- **Nakajima & Sato (1999).** — *Case* $\boxed{N = 4\textit{:}}$
  - pseudo-process with linear drift

- **Beghin, Hochberg, Orsingher & Ragozina (2000, 2001).**
  — *Cases* $\boxed{N = 3 \textit{ and } 4\textit{:}}$
  - joint distribution of $(X(t), M(t))$

- **Lachal (2006).** — *Case* $\boxed{\textit{even } N\textit{:}}$
  - joint distribution of the couple $(X(t), M(t))$
  - joint distribution of the couple $(\tau_a, X(\tau_a))$

  $\longrightarrow$ *"First hitting time and place, monopoles and multipoles for pseudo-processes driven by the equation* $\frac{\partial}{\partial t} = \pm \frac{\partial^N}{\partial x^N}$*"* (submitted)

4 Settings

We assume N **even**. Set

- $\rho = \int_{-\infty}^{+\infty} |p(t;\xi)| \, d\xi > 1$ (for **odd** N, $\rho = +\infty$)

- $(\theta_j)_{0 \leqslant j \leqslant N-1}$: N^{th} roots of $(-1)^{N/2+1}$

- $J = \{j \in \{0, \ldots, N-1\} : \Re\theta_j > 0\}, \; K = \{k \in \{0, \ldots, N-1\} : \Re\theta_k < 0\}$

- $A_j = \prod_{l \in J \setminus \{j\}} \frac{\theta_l}{\theta_l - \theta_j}$ for $j \in J$, $\quad B_k = \prod_{l \in K \setminus \{k\}} \frac{\theta_l}{\theta_l - \theta_k}$ for $k \in K$

- $F(\lambda, \mu, \nu) = \int_0^{+\infty} e^{-\lambda t + i\mu X(t) - \nu M(t)} \, dt$

- $G(\lambda, \mu) = e^{-\lambda \tau_a + i\mu X(\tau_a)}$

5 Step-process

Sampling the pseudo-process $X = (X(t))_{t \geqslant 0}$ on the dyadic times $k/2^n$ $(k, n \in \mathbb{N})$ yields the step-process $X_n = (X_n(t))_{t \geqslant 0}$ defined, for any $n \in \mathbb{N}$, by

$$X_n(t) = \sum_{k=0}^{\infty} X(k/2^n) \mathbb{1}_{[k/2^n,(k+1)/2^n)}(t).$$

Put

- $X_{n,k} = X(k/2^n)$
- $M_n(t) = \max_{0 \leqslant s \leqslant t} X_n(s) = \max_{0 \leqslant j \leqslant \lfloor 2^n t \rfloor} X_{n,j}$
- $M_{n,k} = M_n(k/2^n) = \max(X_{n,0}, X_{n,1}, \ldots, X_{n,k})$

A Distribution of $(X(t), M(t))$

Laplace-Fourier transform of $(X_n(t), M_n(t))$:

$$\begin{aligned} F_n(\lambda,\mu,\nu) &= \int_0^{+\infty} e^{-\lambda t + i\mu X_n(t) - \nu M_n(t)}\,dt \\ &= \sum_{k=0}^{\infty} \left(\int_{k/2^n}^{(k+1)/2^n} e^{-\lambda t}\,dt \right) e^{i\mu X_n - \nu M_{n,k}} \\ &= \frac{1 - e^{-\lambda/2^n}}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda k/2^n + i\mu X_{n,k} - \nu M_{n,k}}. \end{aligned}$$

If $\Re(\lambda) > 2^n \ln \rho$, the series $\left(\sum \mathbb{E}_x\big[e^{-\lambda k/2^n + i\mu X_{n,k} - \nu M_{n,k}}\big]\right)$ is absolutely convergent and then we can write the expectation, with $z = e^{-\lambda/2^n}$:

$$\mathbb{E}_x[F_n(\lambda,\mu,\nu)] = e^{(i\mu-\nu)x}\,\frac{1-z}{\lambda} \sum_{k=0}^{\infty} \mathbb{E}_0\big[e^{i\mu X_{n,k} - \nu M_{n,k}}\big]\, z^k.$$

This generating function can be evaluated thanks to Spitzer identity.

Lemma (Spitzer identity)

Let $(X_n)_{n\geqslant 1}$ *be a sequence of "i.i.d. random variables" and set* $S_0 = 0$, $S_k = \sum_{j=1}^{k} X_j$ *for* $k \geqslant 1$, *and* $M_k = \max(S_0, S_1, \ldots, S_k)$ *for* $k \geqslant 0$. *The following relationship holds for* $|z| < 1$*:*

$$\boxed{\sum_{k=0}^{\infty} \mathbb{E}\left[e^{i\mu S_k - \nu M_k}\right] z^k = \exp\left[\sum_{k=1}^{\infty} \mathbb{E}\left[e^{i\mu S_k - \nu S_k^+}\right] \frac{z^k}{k}\right]}$$

Notice that

- the rhs of the identity depends only on **one** observation of X (at time $k/2^n$, through S_k and S_k^+),
- whereas the lhs depends on **all** observations of X at times $j/2^n$, $1 \leqslant j \leqslant k$ (through S_k and M_k).

This yields in our case, for $|\rho\, e^{-\lambda/2^n}| < 1$ (that is for $\Re(\lambda) > 2^n \ln \rho$):

$$\mathbb{E}_x[F_n(\lambda,\mu,\nu)] = \frac{1}{\lambda}\, e^{(i\mu-\nu)x} \, \exp\left[\sum_{k=1}^{\infty} \frac{z^k}{k}\, \mathbb{E}_0\Big[e^{i\mu X_{n,k} - \nu X_{n,k}^+}\Big] - 1\right]$$

Because of the condition $\Re(\lambda) > 2^n \ln \rho$, we cannot take *a priori* the limit as n tends to infinity. Actually, this equality can be extended to $\Re(\lambda) > 0$ thanks to sharp results on Dirichlet series (Bohr's lemma).

Now, we can pass to the limit when $n \to +\infty$: for $\Re(\lambda) > 0$,

$$\mathbb{E}_x[F(\lambda,\mu,\nu)] = \frac{1}{\lambda}\, e^{(i\mu-\nu)x} \, \exp\left[\int_0^{+\infty} e^{-\lambda t} \left(\mathbb{E}_0\Big(e^{i\mu X(t) - \nu X(t)^+}\Big) - 1\right) \frac{dt}{t}\right].$$

$\longrightarrow$ We need the following integral: for $\Re(\alpha) \leqslant 0$,

$$\int_0^{+\infty} e^{-\lambda t}\, \frac{dt}{t} \int_0^{+\infty} (e^{\alpha\xi} - 1)\mathbb{P}\{X(t) \in d\xi\} = \log\left(\prod_{j\in J} \frac{\sqrt[N]{\lambda}}{\sqrt[N]{\lambda} - \alpha\theta_j}\right).$$

Proposition 1 (Laplace-Fourier transform of $(X(t), M(t))$)

We have for $\Re(\lambda) > 0, \mu \in \mathbb{R}$ *and* $\nu > 0$*:*

$$\mathbb{E}_x\left[\int_0^{+\infty} e^{-\lambda t + i\mu X(t) - \nu M(t)}\, dt\right] = \frac{e^{(i\mu-\nu)x}}{\prod_{j\in J}\left(\sqrt[N]{\lambda} - (i\mu-\nu)\theta_j\right)\prod_{k\in K}\left(\sqrt[N]{\lambda} - i\mu\theta_k\right)}.$$

Successive inversions, distribution of $(X(t), M(t))$

• Inversion with respect to μ and ν:

Proposition 2 (Laplace transform of the joint density of $(X(t), M(t))$)
We have for $z \geqslant x \vee y$ *and* $\Re(\lambda) > 0$*:*

$$\int_0^{+\infty} e^{-\lambda t}\, dt\, \mathbb{P}_x\{X(t) \in dy, M(t) \in dz\}/dy\, dz = \frac{1}{\lambda^{1-2/N}} \sum_{j \in J} \theta_j A_j\, e^{\theta_j \sqrt[N]{\lambda}\,(x-z)} \sum_{k \in K} \theta_k B_k\, e^{\theta_k \sqrt[N]{\lambda}\,(z-y)}.$$

Proposition 3 (Laplace transform of the dist. function of $(X(t), M(t))$)
We have for $z \geqslant x \vee y$ *and* $\Re(\lambda) > 0$*:*

$$\int_0^{+\infty} e^{-\lambda t}\, \mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\}\, dt = \frac{1}{\lambda} \sum_{j \in J, k \in K} \frac{\theta_j A_j B_k}{\theta_j - \theta_k}\, e^{\theta_j \sqrt[N]{\lambda}(x-z) + \theta_k \sqrt[N]{\lambda}(z-y)}.$$

• Inversion with respect to λ:

Theorem A (Distribution function of $(X(t), M(t))$)
We have for $z \geqslant x \vee y$:

$$\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} = \sum_{\substack{k\in K \\ 0\leqslant m\leqslant \# J-1}} a_{km} \int_0^t \int_0^s \frac{\partial^m p}{\partial x^m}(\sigma; x-z) \, \frac{I_{k0}(s-\sigma; z-y)}{(t-s)^{1-(m+1)/N}} \, ds \, d\sigma$$

where the a_{km}'s are some constants expressible by means of the θ_l's and

$$\int_0^{+\infty} e^{-\lambda t} I_{k0}(t;\xi)\, dt = e^{\theta_k \sqrt[N]{\lambda}\, \xi}.$$

Examples

- $\boxed{N=3}$ *(Beghin, Orsingher & Ragozina): although not justified...*

  – *Case* $\kappa_3 = +1$*:*

$$\boxed{\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} = \int_0^t \int_0^s p(\sigma; x-z)\, q(s-\sigma; z-y)\, \frac{ds\, d\sigma}{(t-s)^{2/3}}}$$

with

$$\begin{aligned} p(t;\xi) &= \frac{1}{\pi}\int_0^{+\infty} \cos(\xi\lambda - t\lambda^3)\, d\lambda \quad \text{(Airy function)}, \\ q(t;\xi) &= \frac{\xi}{\pi\Gamma(1/3)t}\int_0^{+\infty} e^{-t\lambda^3 + \frac{1}{2}\,\xi\lambda} \sin\left(\frac{\sqrt{3}}{2}\,\xi\lambda + \frac{\pi}{3}\right) d\lambda. \end{aligned}$$

– *Case* $\kappa_3 = -1$*:*

$$\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} = \int_0^t \int_0^s p(\sigma; z-y)\, q(s-\sigma; x-z)\, \frac{ds\, d\sigma}{(t-s)^{2/3}}$$

with

$$\begin{aligned} p(t;\xi) &= \frac{1}{\pi} \int_0^{+\infty} \cos(\xi\lambda + t\lambda^3)\, d\lambda, \\ q(t;\xi) &= \frac{\xi}{\pi\Gamma(1/3)t} \left[\sqrt{3} \int_0^{+\infty} e^{-t\lambda^3 + \xi\lambda}\, d\lambda \right. \\ &\quad \left. + \int_0^{+\infty} e^{-t\lambda^3 - \frac{1}{2}\xi\lambda} \sin\left(\frac{\sqrt{3}}{2}\xi\lambda + \frac{\pi}{3} \right) d\lambda \right]. \end{aligned}$$

- $\boxed{N = 4}$ *(Beghin, Orsingher & Ragozina):*

$$\boxed{\begin{aligned}\mathbb{P}_x\{X(t) \leqslant y \leqslant z \leqslant M(t)\} \; &= \int_0^t \int_0^s p(\sigma; x - z)\, q_1(s - \sigma; z - y)\, \frac{ds\, d\sigma}{(t - s)^{3/4}} \\ &\quad + \int_0^t \int_0^s \frac{\partial p}{\partial x}(\sigma; x - z)\, q_2(s - \sigma; z - y)\, \frac{ds\, d\sigma}{\sqrt{t - s}}\end{aligned}}$$

with

$$\begin{aligned} p(t; \xi) &= \frac{1}{\pi} \int_0^{+\infty} e^{-t\lambda^4} \cos(\xi\lambda)\, d\lambda, \\ q_1(t; \xi) &= \frac{\xi}{\pi\sqrt{2}\,\Gamma(1/4)\, t} \int_0^{+\infty} e^{-t\lambda^4} \cos(\xi\lambda)\, d\lambda, \\ q_2(t; \xi) &= \frac{\xi}{2\pi^2\, t} \int_0^{+\infty} e^{-t\lambda^4} \left(\cos(\xi\lambda) + \sin(\xi\lambda) - e^{-\xi\lambda}\right) d\lambda. \end{aligned}$$

B Distribution of $(\tau_a, X(\tau_a))$

Recall that $\tau_a = \inf\{t \geqslant 0 : X(t) > a\}$.

Laplace-Fourier transform of $(\tau_a, X(\tau_a))$:

For the step-process $(X_n(t))_{t\geqslant 0}$, the corresponding first hitting time $\tau_{a,n}$ is the instant $k/2^n$ with k such that

$$X(0), X(1/2^n), \ldots, X((k-1)/2^n) \leqslant a < X(k/2^n)$$

or, equivalently:

$$M_{n,k-1} \leqslant a < M_{n,k}$$

where $M_{n,k} = \max_{0\leqslant j\leqslant k} X_{n,j}$ and $X_{n,j} = X(j/2^n)$ for $j \geqslant 0$. We have

$$\begin{aligned}
e^{-\lambda\tau_{a,n}+i\mu X_n(\tau_{a,n})} &= \sum_{k=0}^{\infty} e^{-\lambda k/2^n+i\mu X_{n,k}} 1\!\!1_{\{M_{n,k-1}\leqslant a<M_{n,k}\}} \\
&= \sum_{k=0}^{\infty} \left[e^{-\lambda k/2^n+i\mu X_{n,k}} - e^{-\lambda(k+1)/2^n+i\mu X_{n,k+1}}\right] 1\!\!1_{\{M_{n,k}>a\}}.
\end{aligned}$$

Proposition 4 (Relationship between $(\tau_{a,n}, X(\tau_{a,n}))$ and $(X_{n,k}, M_{n,k})$)
We have for $x \leqslant a$ and $\Re(\lambda) > 2^n \ln \rho$:

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\lambda\tau_{a,n}+i\mu X_n(\tau_{a,n})}\Big] &= \Big(1-e^{-\left(\lambda-\kappa_N(i\mu)^N\right)/2^n}\Big)\\ &\quad\times\sum_{k=0}^{\infty} e^{-\lambda k/2^n}\,\mathbb{E}_x\Big[e^{i\mu X_{n,k}}\,\mathbb{1}_{\{M_{n,k}>a\}}\Big].\end{aligned}$$

Actually, this relation can be extended to the domain $\Re(\lambda) > 0$. Hence, we can take the limit as $n \to +\infty$.

Proposition 5 (Relationship between $(\tau_a, X(\tau_a))$ and $(X(t), M(t))$)
We have for $x \leqslant a$ and $\Re(\lambda) > 0$:

$$\mathbb{E}_x\Big[e^{-\lambda\tau_a+i\mu X(\tau_a)}\Big] = \big(\lambda-\kappa_N(i\mu)^N\big)\int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\Big[e^{i\mu X(t)}\,\mathbb{1}_{\{M(t)>a\}}\Big]\,dt.$$

Proposition 6 (Laplace-Fourier transform of $(\tau_a, X(\tau_a))$)
We have for $x \leqslant a$ and $\Re(\lambda) > 0$ and $\mu \in \mathbb{R}$:

$$\boxed{\mathbb{E}_x\Big[e^{-\lambda\tau_a + i\mu X(\tau_a)}\Big] = \sum_{j\in J} A_j \prod_{l\in J\setminus\{j\}} \left(1 - \frac{i\mu}{\sqrt[N]{\lambda}}\,\bar{\theta}_l\right) e^{\theta_j \sqrt[N]{\lambda}\,(x-a)}\, e^{i\mu a}}$$

where $A_j = \displaystyle\prod_{l\in J\setminus\{j\}} \frac{\theta_l}{\theta_l - \theta_j}$ *for* $j \in J$.

Successive inversions

Expanding the above product yields a sum of $\mu^q\, e^{i\mu a} \times \lambda^{-q/N} e^{\theta_l \sqrt[N]{\lambda}\,(x-a)}$.

Lemma

$$\boxed{(-i\mu)^q\, e^{i\mu a} = \int_{-\infty}^{+\infty} e^{i\mu z}\, \delta_a^{(q)}(z)\, dz \text{ and } \lambda^{-q/N} e^{\theta_l \sqrt[N]{\lambda}\,\xi} = \int_0^{+\infty} e^{-\lambda t} I_{lq}(t;\xi)\, dt.}$$

Distribution of $(\tau_a, X(\tau_a))$

Theorem B (Joint density of the couple $(\tau_a, X(\tau_a))$)

$$\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt\,dz = \sum_{q=0}^{\#J-1} \mathcal{J}_q(t; x-a)\,\delta_a^{(q)}(z)$$

with $\mathcal{J}_q(t;\xi) = \sum_{j\in J} c_{jq}\,I_{jq}(t;\xi)$ *where the* c_{jq}*'s are some constants.*

Corollary (Density of $X(\tau_a)$, "monopoles and multipoles")

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \sum_{q=0}^{\#J-1} (-1)^q\,\frac{(x-a)^q}{q!}\,\delta_a^{(q)}(z)$$

Examples

- $\boxed{N=3}$: *although not justified...*

  – *Case* $\kappa_3 = +1$:

$$\boxed{\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt\,dz = \mathcal{J}_0(t; x-a)\,\delta_a(z)}$$

with $\mathcal{J}_0(t;\xi) = -\dfrac{\xi}{\pi t}\displaystyle\int_0^{+\infty} e^{\frac{1}{2}\xi\lambda - t\lambda^3} \sin\left(\frac{\sqrt{3}}{2}\,\xi\lambda + \frac{\pi}{3}\right) d\lambda.$

  – *Case* $\kappa_3 = -1$:

$$\boxed{\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt\,dz = \mathcal{J}_0(t; x-a)\,\delta_a(z) + \mathcal{J}_1(t; x-a)\,\delta'_a(z)}$$

with

$$\begin{aligned}
\mathcal{J}_0(t;\xi) &= -\frac{\xi}{\pi\sqrt{3}\,t}\int_0^{+\infty}\left[\frac{1}{2}\,e^{\xi\lambda} + e^{-\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda + \frac{\pi}{3}\right)\right] e^{-t\lambda^3}\,d\lambda,\\
\mathcal{J}_1(t;\xi) &= -\frac{\sqrt{3}}{\pi}\int_0^{+\infty}\left[\frac{1}{2}\,e^{\xi\lambda} - e^{-\frac{1}{2}\xi\lambda}\cos\left(\frac{\sqrt{3}}{2}\,\xi\lambda - \frac{\pi}{3}\right)\right] \lambda\,e^{-t\lambda^3}\,d\lambda.
\end{aligned}$$

- $\boxed{N=4}$ *(Nishioka):*

$$\boxed{\mathbb{P}_x\{\tau_a \in dt, X(\tau_a) \in dz\}/dt\,dz = \mathcal{J}_0(t; x-a)\,\delta_a(z) + \mathcal{J}_1(t; x-a)\,\delta'_a(z)}$$

with

$$\begin{aligned}
\mathcal{J}_0(t;\xi) &= \frac{\xi}{2\pi t}\int_0^{+\infty}\left[e^{\xi\lambda} - \cos(\xi\lambda) + \sin(\xi\lambda)\right] e^{-t\lambda^4}\,d\lambda, \\
\mathcal{J}_1(t;\xi) &= \frac{2}{\pi}\int_0^{+\infty}\left[\cos(\xi\lambda) + \sin(\xi\lambda) - e^{\xi\lambda}\right] \lambda^2\, e^{-t\lambda^4}\,d\lambda.
\end{aligned}$$