Some stochastic processes related to the 1D-polyharmonic differential operator

or

“My favorite processes”

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Introduction

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$$\frac{d^{2n}}{d^{2n}x_1} \searrow \textbf{spatial operator } \frac{d^{2n}}{d^{2n}x} \longrightarrow \textbf{“pseudo”-diffusion processes}$$

1 The Laplacian Δ

1.1 Temporal operator d^2/dt^2

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are Green functions of the equation (φ is given)

$$\boxed{\frac{d^2 f}{dt^2}(t) = -\varphi(t)}$$

with various boundary values at $t = 0$ and $t = 1$:

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$\longrightarrow$ **Solution:**

$$\boxed{f(t) = \int_0^1 c(s,t)\varphi(s)\, ds}$$

- ***Example of use:*** **prediction, construction of bridges ...**

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$$\boxed{\frac{\partial p}{\partial t}(t; x, y) = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(t; x, y) = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}(t; x, y)}$$

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• ***Example of use:*** **computation of the expectation of various functionals of the processes when it starts at** x **(first hitting times, sojourn times . . .)**

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2.1 Temporal operator d^{2n}/dt^{2n}

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- ***Related processes:*** ***iterated integrals of Brownian motion or Brownian bridge, bridges of iterated integrals of Brownian motion . . .***

$$X(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, dW(s) \quad \text{or} \quad \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, dB(s) \ldots$$

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- ***Connection:*** the covariance functions $c(s,t) = \mathbb{E}[X(s)X(t)]$ (on $[0,1]$) are Green functions of the equation

$$\frac{d^{2n}f}{dt^{2n}}(t) = (-1)^n \varphi(t)$$

with various boundary values at $t=0$ and $t=1$: for certain i's,

$$\frac{d^i f}{dt^i}(0) = 0 \quad \text{or/and} \quad \frac{d^i f}{dt^i}(1) = 0$$

2.2 Temporal operator d^4/dt^4

- ***Related processes:*** a) *integrated Brownian motion* (the so-called *Langevin process*)

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Historical context: Langevin equation (1908)

$\longrightarrow$ **Modelling the displacements of a harmonic oscillator excited by a white noise $\dot{W}$**

$$\boxed{m\,\frac{d^2X}{dt^2}(t) - f\,\frac{dX}{dt}(t) + m\omega^2 X(t) = (kT\beta)\dot{W}(t)}$$

- ***Related processes:*** a) *integrated Brownian motion* (the so-called *Langevin process*)

$$X(t) = \int_0^t W(s)\,ds$$

- ***Connection:*** the covariance function (on $[0, 1]$)

$$c_X(s, t) = \mathbb{E}[X(s)X(t)] = \frac{1}{6}[s \wedge t]^2\,[3(s \vee t) - s \wedge t]$$

is the Green function of the equation

$$\frac{d^4 f}{dt^4}(t) = \varphi(t)$$

with boundary values

$$f(0) = \frac{df}{dt}(0) = 0 \quad \text{and} \quad \frac{d^2 f}{dt^2}(1) = \frac{d^3 f}{dt^3}(1) = 0$$

- ***Related processes:*** **b)** ***integrated Brownian bridge***

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- ***Connection:*** **the covariance function (on $[0,1]$)**

$$c_Y(s,t) = \frac{1}{6}[s \wedge t]^2\,[3(s \vee t) - s \wedge t] - \frac{1}{4}\,s^2 t^2$$

is the Green function of the equation

$$\frac{d^4 f}{dt^4}(t) = \varphi(t)$$

with boundary values

$$f(0) = \frac{df}{dt}(0) = 0 \quad \text{and} \quad \frac{df}{dt}(1) = \frac{d^3 f}{dt^3}(1) = 0$$

- ***Related processes:*** c) *bridge of integrated Brownian motion*

$$\boxed{U(t) = \left(\int_0^t W(s)\,ds \,\middle|\, \int_0^1 W(s)\,ds = 0 \right) = (X(t) \mid X(1) = 0)}$$

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- ***Connection:*** the covariance function (on $[0,1]$)

$$c_U(s,t) = \frac{1}{6}[s \wedge t]^2\,[3(s \vee t) - s \wedge t] - \frac{1}{12}\,s^2 t^2 (3-s)(3-t)$$

is the Green function of the equation

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- ***Related processes:*** **d)** ***another bridge of integrated Brownian motion***

$$Z(t) = \left(\int_0^t W(s)\,ds \,\middle|\, \int_0^1 W(s)\,ds = W(1) = 0 \right) = \left(X(t) \,|\, X(1) = W(1) = 0 \right)$$

- ***Related processes:*** d) *another bridge of integrated Brownian motion*

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- ***Connection:*** the covariance function (on $[0, 1]$)

$$c_Z(s,t) = \mathbb{E}[Z(s)Z(t)] = \frac{1}{6}[s \wedge t]^2 \, [1 - s \vee t]^2 \, [3(s \vee t) - s \wedge t - 2st]$$

is the Green function of the equation

$$\frac{d^4 f}{dt^4}(t) = \varphi(t)$$

with boundary values

$$f(0) = f(1) = 0 \quad \text{and} \quad \frac{df}{dt}(0) = \frac{df}{dt}(1) = 0$$

Motivation

A problem in elasticity : deformations of embedded beams, plates ...

$\longrightarrow$ ***Lauricella problem***

$$\begin{aligned} \Delta^2 f &= \varphi \quad \text{on } \mathcal{D} \\ f &= \psi \quad \text{on } \partial\mathcal{D} \\ \frac{\partial f}{\partial n} &= \chi \quad \text{on } \partial\mathcal{D} \end{aligned}$$

$\longrightarrow$ ***Aim: find a probabilistic representation for the solution*** f

Bridges and polynomial drift

$$Z(t)=\left(\int_0^t W(s)ds \,\Big|\, \int_0^1 W(s)ds=W(1)=0\right)$$
$$=(X(t)|X(1)=W(1)=0)$$

For $0 \leq t \leq 1$:

$$\boxed{Z(t) = X(t) - H(t)X(1) - K(t)W(1)}$$

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For $0 \le t \le 1$:

$$\boxed{Z(t) = X(t) - H(t)X(1) - K(t)W(1)}$$

where the functions H and K are the interpolation Hermite polynomials which are solutions of

$$\boxed{\frac{d^4H}{dt^4}(t) = \frac{d^4K}{dt^4}(t) = 0}$$

with boundary values

$$\begin{cases} H(0) = \dfrac{dH}{dt}(0) = 0 \\ H(1) = 1, \quad \dfrac{dH}{dt}(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} K(0) = \dfrac{dK}{dt}(0) = 0 \\ K(1) = 0, \quad \dfrac{dK}{dt}(1) = 1 \end{cases}$$

Prediction

$$Z(t) = \left(\int_0^t W(s)ds \,\middle|\, \int_0^1 W(s)ds = W(1) = 0 \right)$$
$$= (X(t) | X(1) = W(1) = 0)$$

For $t_0 \leq t \leq 1$:

$$\boxed{Z(t) = \tilde{Z}_{t_0}(t - t_0) + H_{t_0}(t) Z(t_0) + K_{t_0}(t) \frac{dZ}{dt}(t_0)}$$

Prediction

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where $\tilde{Z}_{t_0}$ is a bridge of length $1 - t_0$ and the functions H_{t_0} and K_{t_0} are the interpolation Hermite polynomials which are solutions of

$$\boxed{\frac{d^4 H_{t_0}}{dt^4}(t) = \frac{d^4 K_{t_0}}{dt^4}(t) = 0}$$

with boundary values

$$\begin{cases} H_{t_0}(t_0) = 1, \ \dfrac{dH_{t_0}}{dt}(t_0) = 0 \\ H_{t_0}(1) = \dfrac{dH_{t_0}}{dt}(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} K_{t_0}(t_0) = 0, \ \dfrac{dK_{t_0}}{dt}(t_0) = 1 \\ K_{t_0}(1) = \dfrac{dK_{t_0}}{dt}(1) = 0 \end{cases}$$

2 The iterated Laplacian Δ^n 2.3 Spatial operator d^{2n}/dx^{2n}

- ***Connection:*** let us introduce the *pseudo-Markov kernel* $p(t;x)$ which is a solution of the heat-type equation of high order $2n > 2$

$$\boxed{\frac{\partial p}{\partial t}(t;x) = (-1)^{n+1}\frac{\partial^{2n} p}{\partial x^{2n}}(t;x)} \quad \text{and } p(0;x) = \delta(x)$$

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- ***Related processes:*** *pseudo-Brownian motions* driven by a signed measure (which is NOT a probability measure)

- ***Other related processes:*** *iterated Brownian motions*

$$X(t) = B^1_{B^2_{B^3_{\ddots_{B^n(t)}}}}$$

where $B^1, \ldots, B^n$ are independent reflected BMs

The probability density $p(t;x) = \mathbb{P}\{X(t) \in dx\}/dx$ is a solution of

$$\boxed{\frac{\partial p}{\partial t}(t;x) = \frac{1}{2^{2^n-1}}\frac{\partial^{2^n} p}{\partial x^{2^n}}(t;x)}$$

3 The pseudo-Brownian motion

3.1 Construction

Properties of the heat-type kernel $p(t;x)$

- **It is characterized by** $\displaystyle\int_{-\infty}^{+\infty} e^{iux} p(t;x)\,dx = e^{-tu^{2n}}$
- **It satisfies** $\displaystyle\int_{-\infty}^{+\infty} p(t;x)\,dx = 1$ **and** $\displaystyle\int_{-\infty}^{+\infty} x^2 p(t;x)\,dx = 0$

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- **It satisfies** $\displaystyle\int_{-\infty}^{+\infty} p(t;x)\,dx = 1$ **and** $\displaystyle\int_{-\infty}^{+\infty} x^2 p(t;x)\,dx = 0$
- **It defines a** ***pseudo-Markov process*** $(X(t))_{t\geq 0}$ **by**

$$\boxed{\mathbb{P}_x\{X(t)\in dy\} \stackrel{\text{def}}{=} p(t;x-y)\,dy}$$

and for $0=t_0<t_1<\cdots<t_m$ **and** $x_0=x$**:**

$$\boxed{\mathbb{P}_x\{X(t_1)\in dx_1,\dots,X(t_m)\in dx_m\} \stackrel{\text{def}}{=} \prod_{i=1}^{m} p(t_i-t_{i-1};x_{i-1}-x_i)dx_i}$$

- ***Sojourn time in an interval***

$$T_a(t) = \textbf{measure}\{s \in [0,t] : X(s) \gtrless a\} = \int_0^t \mathbb{1}_{\{X(s) \gtrless a\}}\, ds$$

$$T_{ab}(t) = \textbf{measure}\{s \in [0,t] : X(s) \in [a,b]\} = \int_0^t \mathbb{1}_{\{X(s) \in [a,b]\}}\, ds$$

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- ***Maximum/minimum functionals***

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$$M(t) = \max_{0 \le s \le t} X(s), \quad m(t) = \min_{0 \le s \le t} X(s)$$

- ***First hitting/exit time of an interval***

$$\tau_a = \inf\{t \ge 0 : X(t) \gtrless a\}$$

$$\tau_{ab} = \inf\{t \ge 0 : X(t) \notin [a,b]\}$$

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$\longrightarrow$ ***Problems:*** **determine the pseudo-distributions of** $T_a(t)$, $T_{ab}(t)$, $M(t)$, $m(t)$, τ_a, τ_{ab} ...

a) Pseudo-distribution of the sojourn time $T_a(t)$

$T_a(t)=\int_0^t \mathbb{1}_{\{X(s)>a\}}\,ds$

Set $\varphi(t;x) = \mathbb{E}_x\left[e^{-\mu T_a(t)} f(X(t))\right]$ ***(Feynman-Kac functional)***

$$\overset{\text{def}}{=} \lim_{m\to\infty} \mathbb{E}_x\left[e^{-\frac{t}{m}\sum_{k=0}^{m} \mathbb{1}_{\{X(kt/m)>a\}}} f(X(t))\right]$$

The function φ is a solution of the PDE

$$\boxed{\frac{\partial\varphi}{\partial t} = (-1)^{n+1}\frac{\partial^{2n}\varphi}{\partial x^{2n}} - f\varphi}$$

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Set $\Phi(x) = \int_0^{+\infty} e^{-\lambda t}\,\mathbb{E}_x\left[e^{-\mu T_a(t)} f(X(t))\right] dt$

The function Φ is a solution of the differential equation

$$\boxed{(-1)^{n+1}\frac{d^{2n}\Phi}{dx^{2n}} = \begin{cases} (\lambda+\mu)\,\Phi - f & \text{on } (a,+\infty) \\ \lambda\,\Phi - f & \text{on } (-\infty,a) \end{cases}}$$

$\longrightarrow$ ***Solution*** $T_a(t)=\int_0^t \mathbb{1}_{\{X(s)>a\}}\,ds$

- **The equation can be explicitly solved**
  (involving Vandemonde algebra)

- **The iterated Laplace transform can be inverted**
  (involving the Mittag-Leffler function)

  $\longrightarrow$ **Pseudo-distribution of** $(T_a(t), X(t))$ **under** $\mathbb{P}_x$
  (V. Cammarota & A. L., EJP 2010 and SPA 2011)

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- ***A historical result*** **(Krylov, 1960)**

The distribution of $T_0(t)$ **is the Paul Lévy's arcsine law:**

$$\boxed{\mathbb{P}_0\{T_0(t) \in ds\}/ds = \frac{\mathbb{1}_{(0,t)}(s)}{\pi\sqrt{s(t-s)}}}$$

- ***An unsolved problem:*** **compute the distribution of** $T_{ab}(t)$ **...**

b) Pseudo-distribution of the first overshooting time τ_a

- ***Example*** $n = 1$ $\boxed{\mathbb{P}_x\{W(\tau_a) \in dz\}/dz = \delta_a(z)}$

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- ***Example*** $n = 2$ **(Nishioka 1997)**

$\boxed{\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \delta_a(z) - (x-a)\delta'_a(z)}$ **with** $< \delta'_a, \varphi >= -\varphi'(a)$

$$\boxed{\mathbb{E}_x[f(X(\tau_a))] = f(a) + (x-a)f'(a)}$$

b) Pseudo-distribution of the first overshooting time τ_a

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- ***Example $n = 2$*** **(Nishioka 1997)**

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \delta_a(z) - (x-a)\delta'_a(z)$$ **with** $< \delta'_a, \varphi >= -\varphi'(a)$

$$\mathbb{E}_x[f(X(\tau_a))] = f(a) + (x-a)f'(a)$$

- ***General result*** **(A. L., EJP 2007)**

$$\mathbb{P}_x\{X(\tau_a) \in dz\}/dz = \sum_{p=0}^{n-1} \frac{(a-x)^p}{p!}\,\delta_a^{(p)}(z)$$

with $< \delta_a^{(p)}, \varphi >= (-1)^p \varphi^{(p)}(a)$

$x \mapsto \mathbb{P}_x\{X(\tau_a) \in dz\}/dz$ **is a n-harmonic function**

c) Distribution of the first exit time τ_{ab} **(A. L., work in progress)**

$$\mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz = \sum_{p=0}^{n-1} H_p^-(x)\,\delta_a^{(p)}(z) + \sum_{p=0}^{n-1} H_p^+(x)\,\delta_b^{(p)}(z)$$

where the functions H_p^- and H_p^+, $0 \le p \le n-1$, are the interpolation Hermite polynomials such that $\frac{d^q H_p^-}{dx^q}(a) = \delta_{pq}$, $\frac{d^q H_p^-}{dx^q}(b) = 0$ and $\frac{d^q H_p^+}{dx^q}(a) = 0$, $\frac{d^q H_p^+}{dx^q}(b) = \delta_{pq}$ for $0 \le q \le n-1$.

$x \mapsto \mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz$ is a n-harmonic function

c) Distribution of the first exit time τ_{ab} (A. L., work in progress)

$$\mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz = \sum_{p=0}^{n-1} H_p^-(x)\,\delta_a^{(p)}(z) + \sum_{p=0}^{n-1} H_p^+(x)\,\delta_b^{(p)}(z)$$

where the functions H_p^- and H_p^+, $0 \le p \le n-1$, are the interpolation Hermite polynomials such that $\frac{d^q H_p^-}{dx^q}(a) = \delta_{pq}$, $\frac{d^q H_p^-}{dx^q}(b) = 0$ and $\frac{d^q H_p^+}{dx^q}(a) = 0$, $\frac{d^q H_p^+}{dx^q}(b) = \delta_{pq}$ for $0 \le q \le n-1$.

$x \mapsto \mathbb{P}_x\{X(\tau_{ab}) \in dz\}/dz$ is a n-harmonic function

- *"Ruin pseudo-probabilities"*

Set $\begin{cases} \tau_b^+ = \inf\{t \ge 0 : X(t) > b\} \\ \tau_a^- = \inf\{t \ge 0 : X(t) < a\} \end{cases}$

$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^-(x)$ and $\mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^+(x)$

4 The pseudo-random walk

Discrete Laplacian

$$\Delta_{\text{discrete}} f(x) = \frac{1}{2}[f(x+1) - 2f(x) + f(x-1)]$$

$$\Longrightarrow \quad \Delta^n_{\text{discrete}} f(x) = \frac{1}{2^n} \sum_{k=-n}^{n} (-1)^{k-1} \binom{2n}{n+k} f(x+k)$$

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$\longrightarrow$ **viewed as a generator:**

$$\mathcal{G} f(x) \stackrel{\text{def}}{=} \Delta^n_{\text{discrete}} f(x) = \mathbb{E}_x[f(X_1)] - f(x)$$

where X_1 is the *pseudo-random* variable defined by

$$\begin{cases} \mathbb{P}\{X_1 = k\} = \dfrac{(-1)^{k-1}}{2^n} \dbinom{2n}{n+k} \text{ for } 1 \le |k| \le n \\ \mathbb{P}\{X_1 = 0\} = 1 - \dfrac{1}{2^n} \dbinom{2n}{n} \end{cases}$$

4 The pseudo-random walk

Let $(\xi_k)_{k\geq 1}$ be a sequence of independent identically distributed *pseudo-random* variables with the *pseudo-distribution* of X_1 and set for any $k \geq 1$

$$X_k = \xi_1 + \cdots + \xi_k$$

$\longrightarrow (X_k)_{k\geq 1}$ is a *pseudo-random walk*

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$$X_k = \xi_1 + \cdots + \xi_k$$

$\longrightarrow$ $(X_k)_{k\geq 1}$ is a ***pseudo-random walk***

Set

$$B_N(t) = \frac{1}{N^{1/(2n)}} X_{[Nt]}$$

Limiting continuous pseudo-process:

$$B_N(t) \underset{N\to+\infty}{\longrightarrow} B(t)$$

$\longrightarrow$ $(B(t))_{t\geq 0}$ is a ***pseudo-Brownian motion***

THANK YOU

FOR YOUR ATTENTION!

THE END