Actuariat

Thème : modèles boursiers et modèles d'assurance.

Notions théoriques :

- variables aléatoires, espérance mathématique ;
- loi de Bernoulli, loi de Gauss, processus de Poisson ;
- théorème de la limite centrale.

1 Modèle boursier de Louis Bachelier

Un capital initial S_0 est placé en bourse. Le modèle proposé par Louis Bachelier [1] (théorie de la spéculation, 1900) revient à considérer que ce capital rapporte des intérêts variables sur des périodes régulières selon la formule

$$\frac{\Delta S_k}{S_k} = \frac{S_{k+1} - S_k}{S_k} = R_{k+1},$$

S_k désignant le capital à l'issue de la k^{e} période, les R_k étant des v.a. indépendantes prenant deux valeurs a et b, $-1 < a < b$, avec la même probabilité : $\mathbb{P}(R_k = a) = \mathbb{P}(R_k = b) = \frac{1}{2}$ (loi de Bernoulli). Au bout de n périodes (horizon n), le capital devient $S_n = S_0 \prod_{k=1}^{n} (1 + R_k)$. L'espérance r et la variance σ^2 de R_k sont respectivement données par $r = \frac{1}{2}(a + b)$ et $\sigma^2 = \frac{1}{4}(b - a)^2$. On peut donc écrire $R_k = r + \sigma U_k$ où U_k est une v.a. à valeurs dans $\{-1, 1\}$ suivant la loi de Bernoulli $\mathbb{P}(U_k = -1) = \mathbb{P}(U_k = 1) = \frac{1}{2}$. Ainsi

$$\boxed{S_n = S_0 \prod_{k=1}^{n} (1 + r + \sigma U_k).}$$

Dans ce modèle, le temps est discret. Dans la réalité, le temps est une notion plutôt continue et l'on va déduire de l'étude précédente un modèle en temps continu.

Soit l'intervalle de temps $[0, t]$ que l'on décompose en $n = \frac{t}{\varepsilon}$ sous-intervalles $[k\varepsilon, (k+1)\varepsilon[$ de longueur ε petite, sur lesquels le taux d'intérêt et la variance sont proportionnels à la durée de l'intervalle de temps : $r_n = r\frac{t}{n}$, $\sigma_n^2 = \sigma^2 \frac{t}{n}$; r est le taux instantané et σ^2 la volatilité. Notons que $(1 + r_n)^n \sim e^{rt}$ lorsque $n \longrightarrow +\infty$. A l'instant t, le capital final correspondant à ce découpage est $S_t^{(n)} = S_0 \prod_{k=1}^{n} (1 + R_k^{(n)})$. On va étudier la limite de cette v.a. lorsque $n \longrightarrow +\infty$. On a

$$\ln S_t^{(n)} = \ln S_0 + \sum_{k=1}^{n} Y_k^{(n)}$$

avec $Y_k^{(n)} = \ln \left[1 + r\frac{t}{n} + \sigma\sqrt{\frac{t}{n}}\, U_k^{(n)}\right]$.

En écrivant à l'aide d'un développement limité $Y_k^{(n)} = r\frac{t}{n} + \sigma\sqrt{\frac{t}{n}}\, U_k^{(n)} - \frac{\sigma^2 t}{2n}\, {U_k^{(n)}}^2 + o(\frac{1}{n})$, ou encore, puisque ${U_k^{(n)}}^2 = 1$,

$$Y_k^{(n)} = \sigma\sqrt{\frac{t}{n}}\, U_k^{(n)} + (r - \frac{1}{2}\sigma^2)\frac{t}{n} + o(\frac{1}{n}),$$

1. Bachelier, Louis : mathématicien français (Le Havre 1870 – Saint-Servan-sur-Mer 1946)

on trouve

$$\ln S_t^{(n)} = \ln S_0 + (r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\,\Big[\frac{1}{\sqrt{n}}\sum_{k=1}^{n} U_k^{(n)}\Big] + o(1).$$

Le théorème de la limite centrale appliqué à la suite i.i.d. $(U_k^{(n)})_{n\in\mathbb{N}^*, 1\leqslant k\leqslant n}$ assure que la v.a. $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} U_k^{(n)}$ suit asymptotiquement la loi de Gauss que l'on notera N_t. Ainsi, le capital à l'horizon t est donné par la célèbre formule de L. Bachelier (1900) :

$$\boxed{S_t = S_0 \exp\Big[(r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\,N_t\Big].}$$

A partir de cette représentation, on peut effectuer les calculs de prix d'option d'achat (call) ou de vente (put) qui font l'objet des célèbres travaux de Cox-Ross-Rubinstein [2] (temps discret, 1979) et de Black-Scholes-Merton [3] (temps continu, 1973).

2 Assurance-dommage

Une compagnie d'assurance assure des biens sur une période fixée. Les sinistres (accidents, vols, incendies, etc.) étant généralement imprévisibles, on utilise de nouveau des modèles probabilistes pour étudier les profits et pertes de la compagnie.

Le nombre de déclarations de sinistres durant l'intervalle de temps $[0, t]$ est modélisé par une v.a. N_t, le coût du k^{e} sinistre par une v.a. X_k ; on pose $X_0 = 0$. Le coût total de tous les sinistres survenus durant $[0, t]$ est donc $Y_t = \sum_{k=0}^{N_t} X_k$.

En général, on peut supposer la suite $(X_n)_{n\in\mathbb{N}^*}$ indépendante, les sinistres se réalisant indépendamment les uns des autres. Lorsque les biens assurés sont du même type, on peut supposer que la suite $(X_n)_{n\in\mathbb{N}^*}$ est de plus identiquement distribuée. Un modèle couramment usité est celui pour lequel $(N_t)_{t\in\mathbb{R}^+}$ est un processus de Poisson. Dans ce cas, on dit que $(Y_t)_{t\in\mathbb{R}^+}$ est un processus de Poisson composé et l'on a

$$\boxed{\mathbb{E}(Y_t) = \mathbb{E}(N_t)\mathbb{E}(X_1) \quad \text{et} \quad \mathrm{var}\,(Y_t) = \mathbb{E}(N_t)\mathrm{var}\,(X_1) + \mathrm{var}\,(N_t)\mathbb{E}(X_1^2).}$$

Dans le même temps, la compagnie aura perçu les primes régulières des assurés : $\alpha(t) = \alpha([t] + 1)$ où α est un certain coefficient de pondération. La réserve de la compagnie à l'instant t est donc, si elle n'a pas été ruinée entre-temps,

$$Z_t = Z_0 + \alpha(t) - Y_t.$$

Les quantités suivantes sont à la base de la théorie du risque :
- temps de faillite : $T_0 = \inf\{t > 0 : Z_t \leqslant 0\}$;
- probabilité de ruine : ${}_tq_x := \mathbb{P}\{T_0 \leqslant t\} = 1 - {}_tp_x$;
- temps moyen de vie : $\mathring{e}_x := \mathbb{E}(T_0) = \int_0^\infty {}_tp_x\,dt$.

Un des problèmes majeurs de l'assureur est de trouver le montant minimal des primes à imputer $\alpha(t)$ tel que ${}_tp_x \geqslant 1 - \varepsilon$, ou tel que $\mathring{e}_x > t$.

Examinons le cas particulier où $\alpha(t) = \alpha t$ et les X_k suivent la loi exponentielle $\mathcal{E}(\mu)$. Posons $R(z) = \mathbb{P}(\inf_{t\in\mathbb{R}^+} Z_t > 0 \mid Z_0 = z) = \mathbb{P}(T_0 = +\infty \mid Z_0 = z)$; cette quantité est la probabilité pour la compagnie disposant d'un capital initial z de ne pas être ruinée. H. Cramér [4] a montré (1954) que la fonction R satisfait à l'équation intégro-différentielle

$$\alpha R'(z) = \lambda R(z) - \lambda \int_0^z \mu e^{-\mu(z-x)} R(x)\,dx$$

2. Cox, John Carrington : économiste américain (1943 –)
Ross, Stephen Alan : économiste américain (Boston 1944 – 2017)
Rubinstein, Mark Edward : économiste américain (1944 –)
3. Black, Fischer : mathématicien américain (Georgetown 1938 – New York 1995)
Scholes, Myron Samuel : économiste canadien (Timmins 1941 –)
Merton, Robert C. : économiste américain (New York 1944 –)
4. Cramér, Harald : mathématicien et statisticien suédois (Stockholm 1893 – Stockholm 1985)

avec les conditions $\forall z \in \mathbb{R}^+$, $R(z) \in [0,1]$ et $\lim_{z\to+\infty} R(z) = 1$, cette dernière traduisant le fait que si la compagnie dispose d'un capital initial infini, elle ne sera jamais ruinée. On obtient facilement l'équation différentielle

$$R''(z) = (\frac{\lambda}{\alpha} - \mu)R'(z) \quad \text{avec} \quad R'(0) = \frac{\lambda}{\alpha}R(0)$$

de laquelle on tire

$$R(z) = 1 - \frac{\lambda}{\alpha\mu}e^{-(\mu-\frac{\lambda}{\alpha})z} \quad \text{si} \quad \frac{\lambda}{\mu} < \alpha,$$

$$R(z) = 0 \quad \text{si} \quad \frac{\lambda}{\mu} \geqslant \alpha.$$

Pour que la probabilité de ruine soit supérieure à $1-\varepsilon$ ($R(z) \geqslant 1-\varepsilon$) l'assureur a intérêt à choisir α et Z_0 tels que

$$\boxed{Z_0 \geqslant \frac{1}{\mu - \frac{\lambda}{\alpha}} \ln \frac{\lambda}{\alpha\mu\varepsilon}.}$$

3 Assurance-vie

L'assurance-vie est une partie du domaine des sciences actuariales qui fonde des contrats à long terme ou à échéance aléatoire. Une police d'assurance engage deux parties : d'une part, l'assuré, qui verse une prime annuelle (annuité) pendant une période déterminée, éventuellement jusqu'à la date de son décès, et d'autre part, l'assureur, qui remboursera au bénéficiaire désigné par l'assuré un capital fixé dans les termes du contrat en cas de décès de l'assuré avant l'échéance du contrat. Les versements numéraires sont modélisés par la notion de capital-temps.

3.1 Capital-temps

Un capital-temps est la donnée somme s dont on dispose à un instant t ; on le désigne par le couple (s,t). La valeur actualisée d'un capital est basée sur un taux annuel r que l'on supposera, pour simplifier, fixe au cours du temps. Ainsi, une somme s détenue à l'instant 0 deviendra $s(1+r)^n$ au bout de n années. Inversement, pour disposer de la somme s à la date n (en années), il faudra placer une somme sv^n à l'instant 0 au taux annuel r ; $v = \frac{1}{1+r}$ est le coefficient d'actualisation, et sv^t est la valeur actualisée du capital-temps (s,t) (v^t Fr. à la date 0 produisent 1 Fr. à la date t). Lorsque la somme S et/ou la date T sont aléatoires, on appelle prix (moyen) actualisé du capital-temps (S,T) l'espérance $\mathbb{E}(Sv^T)$.

Cas déterministe

- $C^{\circ\circ} = \sum_{k=1}^{n}(c_k, t_k)$: acquisition à chaque instant t_k de la somme c_k ;
- $C^{\circ} = \sum_{k=1}^{n} c_k v^{t_k}$: valeur actualisée du capital-temps $C^{\circ\circ}$.

Cas aléatoire

- $C^{\circ\circ} = \sum_{k=1}^{n}(c_k, T_k)$: acquisition à chaque instant aléatoire T_k de la somme c_k ;
- $C^{\circ} = \sum_{k=1}^{n} c_k v^{T_k}$: valeur actualisée du capital-temps $C^{\circ\circ}$;
- $C = \mathbb{E}(C^{\circ}) = \sum_{k=1}^{n} c_k \mathbb{E}(v^{T_k})$: prix (moyen) actualisé du capital-temps.

Note. — On choisira ici les sommes c_k déterministes et le taux r fixe contrairement aux modèles boursiers.

3.2 Police d'assurance

Une police d'assurance est la donnée d'un couple $(C^{\circ\circ}+D^{\circ\circ}, P^{\circ\circ}+Q^{\circ\circ})$ où

- $P^{\circ\circ}$ est le capital des primes versées par l'assuré jusqu'à une échéance fixe ou sa date de décès ;
- $C^{\circ\circ}$ est le capital remboursé par l'assureur au bénéficiaire en cas de décès du souscripteur,

auxquels se rajoutent éventuellement des frais annexes :

- $D^{\circ\circ}$ est le capital-temps couvrant les frais de l'assureur (frais de gestion, d'administration, etc.) ;
- $Q^{\circ\circ}$ est le capital-temps des charges correspondantes à imputer à l'assuré.

Annuités versées par l'assuré

- $P^{\circ\circ} = \sum_{k=0}^{N}(p,k)$: paiement d'une prime annuelle p pendant $N+1$ années sans différé (paiement au début de chaque année), N étant l'année (aléatoire) du décès de l'assuré à compter de la date de souscription ;
- $P^{\circ} = \sum_{k=0}^{N} p\, v^k = p\,\dfrac{1-v^{N+1}}{1-v}$: valeur actualisée du capital-temps $P^{\circ\circ}$;
- $P = \mathbb{E}\left[\sum_{k=0}^{+\infty} p\, v^k \mathbb{1}_{\{N\geqslant k\}}\right] = \sum_{k=0}^{+\infty} p\, v^k \mathbb{P}(N \geqslant k) = p\,\dfrac{1-\mathbb{E}(v^{N+1})}{1-v}$: prix actualisé du capital-temps $P^{\circ\circ}$.

Responsabilité de la compagnie

- $C^{\circ\circ} = (c,T)$: remboursement du capital c à l'instant T, date (aléatoire) du décès à compter de la date de souscription, $N=[T]$ avec $[x]=n$ si $x\in]n,n+1]$, $n\in\mathbb{N}$;
- $C^{\circ} = cv^T$: valeur actualisée du capital-temps $C^{\circ\circ}$;
- $C = \mathbb{E}(C^{\circ}) = c\mathbb{E}(v^T)$: valeur actualisée du capital-temps $C^{\circ\circ}$.

Le profit de la compagnie est donné par la valeur $(C+D)-(P+Q)$. Le contrat $(C^{\circ\circ}+D^{\circ\circ}, P^{\circ\circ}+Q^{\circ\circ})$ est dit équitable lorsque les prix des capitaux de chaque partie sont identiques : $C+D=P+Q$, soit $c = p\,\dfrac{1-\mathbb{E}(v^{N+1})}{(1-v)\mathbb{E}(v^T)} + Q - D$, où $D=\mathbb{E}(D^{\circ})$ et $Q=\mathbb{E}(Q^{\circ})$. Cette valeur de c est celle du capital maximal que l'assurance doit pouvoir être en mesure de rembourser (en moyenne) pour éviter la faillite.

3.3 Table de mortalité

Notre problème à présent est de fournir une prédiction sur T. On définit une table de mortalité pour une certaine population par une application décroissante

$$l:\ \textit{individu } (x) \textit{ âgé de } x \textit{ années} \longmapsto \textit{durée de vie } l_x$$

telle que $\lim_{x\to+\infty} l_x = 0$, x étant la date de souscription du contrat et $x+T_x$ celle du décès du souscripteur.

3.3.1 Modèle classique

Un modèle classique consiste à poser ${}_tp_x = \mathbb{P}(T_x > t) = \dfrac{l_{x+t}}{l_x}$. On a dans ce cas

$$\mathbb{P}(T_x > s+t) = \frac{l_{x+s+t}}{l_x} = \frac{l_{x+s+t}}{l_{x+s}}\frac{l_{x+s}}{l_x} = \mathbb{P}(T_x > s)\mathbb{P}(T_{x+s} > t),$$

soit :

$$\boxed{{}_{s+t}p_x = {}_sp_x\, {}_tp_{x+s}.}$$

Si $n\in\mathbb{N}$, on a ${}_np_x = p_xp_{x+1}\ldots p_{x+n-1}$ où $p_x = {}_1p_x = \mathbb{P}(T_x > 1)$.

3.3.2 Taux de mortalité

Le taux de mortalité de la population considérée est défini par $\mu_x = \lim_{t \to 0^+} \frac{\mathbb{P}(T_x > t)}{t} = f_{T_x}(0)$ où f_{T_x} est la densité de T_x. On a les relations suivantes entre taux de mortalité et table de mortalité :

$$\mu_x = \lim_{t \to 0^+} \frac{1 - l_{x+t}/l_x}{t} = -\frac{1}{l_x} \lim_{t \to 0^+} \frac{l_{x+t} - l_x}{t},$$

d'où

$$\boxed{\mu_x = -\frac{l'_x}{l_x}, \quad l_x = l_0 \exp\left(-\int_0^x \mu_y \, dy\right), \quad {}_tp_x = \exp\left(-\int_x^{x+t} \mu_y \, dy\right).}$$

La densité de T_x est alors donnée par :

$$f_{T_x}(t) = -\frac{d}{dt}({}_tp_x) = {}_tp_x \, \mu_{x+t}$$

et le temps moyen de vie par :

$$\mathbb{E}(T_x) = \int_0^{+\infty} t \, {}_tp_x \, \mu_{x+t} \, dt = \int_0^{+\infty} {}_tp_x \, dt.$$

3.3.3 Quelques modèles historiques

1. De Moivre (1729) :
   $l_x = (86 - x)^+$, $\mu_x = \mathbb{1}_{]0,86[}(x)$, ${}_tp_x = \frac{(86 - x - t)^+}{(86 - x)^+}$, $\mathbb{E}(T_x) = \frac{1}{2}(86 - x)^+$.
   Cette table est basée sur l'espérance de vie maximale de l'époque : 86 ans, obsolète de nos jours !
2. Gompertz (1824) :
   $\mu_x = \beta e^{-\alpha x}$, $l_x = l_0 e^{-\frac{\beta}{\alpha}(1-e^{-\alpha x})}$, ${}_tp_x = e^{-\frac{\beta}{\alpha} e^{-\alpha x}(1-e^{-\alpha t})}$.
3. Makeham (1860) :
   $\mu_x = \beta e^{-\alpha x} + \delta$, la partie $\beta e^{-\alpha x}$ modélisant les décès de vieillesse, et la partie δ celle des décès accidentels.
4. Weibull (1939) :
   $\mu_x = \beta x^\alpha$, $l_x = l_0 e^{-\frac{\beta}{\alpha+1} x^{\alpha+1}}$, ${}_tp_x = e^{-\frac{\beta}{\alpha+1}[(x+t)^{\alpha+1} - x^{\alpha+1}]}$.

Il est difficile d'établir de telles tables pour des populations connaissant d'importants mouvements d'émigration et d'immigration.

3.4 Durée de vie tronquée

3.4.1 Remboursement différé

Dans certains contrats, la compagnie d'assurance ne rembourse le capital assuré qu'à la date anniversaire du contrat suivant immédiatement la date du décès du souscripteur. On décompose alors $T_x = N_x + S_x$, où N_x est l'année du décès depuis la date de souscription et S_x la fraction de vie restant dans cette dernière année : $S_x = T_x - [T_x] \in]0, 1]$. On a :

$$\mathbb{P}(N_x = n) = \mathbb{P}(n < T_x \leqslant n + 1) = {}_np_x - {}_{n+1}p_x = {}_np_x(1 - p_{x+n}),$$

et

$$\mathbb{P}(S_x \leqslant s \mid N_x = n) = \frac{\mathbb{P}(n < T_x \leqslant n + s)}{\mathbb{P}(N_x = n)} = \frac{1 - {}_sp_{x+n}}{1 - p_{x+n}} \text{ pour } s \in]0, 1].$$

Exemple : considérons le modèle pour lequel la fonction $s \longmapsto \mu_{x+s}$ est constante sur chaque intervalle $]n, n + 1]$ de valeur $\mu_{x+n+1/2}$. On a ${}_tp_x = \exp\left(-\int_x^{x+t} \mu_{x+1/2} \, dy\right) = e^{-t\mu_{x+1/2}}$ pour $t \in]0, 1]$ d'où

$$\mathbb{P}(S_x \leqslant s \mid N_x = n) = \frac{1 - e^{-t\mu_{s+n+1/2}}}{1 - e^{-t\mu_{x+n+1/2}}}$$

et alors $(S_x \mid N_x = n)$ suit la loi exponentielle $\mathcal{E}(\mu_{x+n+1/2})$ tronquée par 1, c'est-à-dire $(S_x \mid N_x = n) = (X \mid X \leqslant 1)$ où X suit la loi exponentielle $\mathcal{E}(\mu_{x+n+1/2})$.

3.4.2 Relation entre les annuités unitaires

Posons :

- $\ddot{a}_x = \mathbb{E}\left[\sum_{k=0}^{N_x} v^k\right]$: versements (moyens) annuels d'une unité par l'assuré ;
- $A_x = \mathbb{E}(v^{N_x+1})$: remboursement (moyen) d'une unité par l'assureur.

On a

$$\ddot{a}_x = \sum_{k=0}^{+\infty} v^k \mathbb{P}(N_x \geqslant k) = \sum_{k=0}^{+\infty} v^k \,{}_k p_x$$

et

$$A_x = \sum_{k=0}^{+\infty} v^{k+1} \mathbb{P}(N_x = k) = \sum_{k=0}^{+\infty} v^k (\,{}_k p_x - \,{}_{k+1} p_x) = v \sum_{k=0}^{+\infty} v^k \,{}_k p_x - \left(\sum_{k=0}^{+\infty} v^k \,{}_k p_x - 1\right)$$

d'où :

$$\boxed{A_x = 1 - (1 - v)\ddot{a}_x.}$$

Annexe 1 : Probabilité conditionnelle, espérance conditionnelle

Si $\mathbb{P}(B) \neq 0$, on définit la probabilité conditionnelle de A sachant B par

$$\boxed{\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}}$$

ainsi que l'espérance conditionnelle d'une v.a. X sachant B par

$$\boxed{\mathbb{E}(X \mid B) = \frac{\mathbb{E}(X \mathbb{1}_B)}{\mathbb{P}(B)}.}$$

On a la formule des probabilités totales : si $(B_1, \ldots, B_n)$ est une partition de B, c'est-à-dire que les B_k sont non vides deux à deux disjoints de réunion totale B, alors

$$\boxed{\mathbb{P}(A \mid B) = \sum_{k=1}^{n} \mathbb{P}(A \mid B_k)\mathbb{P}(B_k) \quad \text{et} \quad \mathbb{E}(X \mid B) = \sum_{k=1}^{n} \mathbb{E}(X \mid B_k)\mathbb{P}(B_k).}$$

Annexe 2 : Théorème de la limite centrale

Soit $(X_n)_{n\in\mathbb{N}^*}$ une suite de v.a. i.i.d. d'espérance $\mathbb{E}(X_1) = m$ et de variance $\mathrm{var}\,(X_1) = \sigma^2$. Alors la v.a. $\dfrac{X_1 + \cdots + X_n - nm}{\sigma\sqrt{n}}$ est asymptotiquement normalement distribuée.

Généralisation (Lindeberg). Soit $(X_k^{(n)})_{n\in\mathbb{N}^*, 1\leqslant k\leqslant n}$ une suite de v.a. i.i.d. d'espérance $\mathbb{E}(X_1^{(1)}) = m$ et de variance $\mathrm{var}\,(X_1^{(1)}) = \sigma^2$. Alors la v.a. $\dfrac{X_1^{(n)} + \cdots + X_n^{(n)} - nm}{\sigma\sqrt{n}}$ est asymptotiquement normalement distribuée.

Louis Bachelier

Born: 11 March 1870 in Le Havre, France
Died: 26 April 1946 in St-Servan-sur-Mer, France

Nul n'est prophète en son pays...

The French mathematician, **Louis Bachelier** is now recognised internationally as the father of financial mathematics, but this fame, which he so justly deserved, was a long time coming. The Bachelier Society, named in his honour, is the world-wide financial mathematics society and mathematical finance is now a scientific discipline of its own. The Society held its first World Congress on 2000 in Paris on the hundredth anniversary of Bachelier's celebrated PhD Thesis *Théorie de la Spéculation*.

Five years before Einstein's famous 1905 paper on Brownian Motion, in which Einstein derived the equation (the partial differential heat/diffusion equation of Fourier) governing Brownian motion and made an estimate for the size of molecules, Bachelier had worked out, for his Thesis, the distribution function for what is now known as the Wiener stochastic process (the stochastic process that underlies Brownian Motion) linking it mathematically with the diffusion equation. The probabilist William Feller had originally called it the Bachelier-Wiener Process. It appears that Einstein in 1905 was ignorant of the work of Bachelier.

Seventy three years before Black and Scholes wrote their famous paper in 1973, Bachelier had derived the price of an option where the share price movement is modelled by a Wiener process and derived the price of what is now called a barrier option (namely the option which depends on whether the share price crosses a barrier). Black and Scholes, following the ideas of Osborne and Samuelson, modelled the share price as a stochastic process known as a Geometric Brownian Motion (with drift).

Louis Bachelier was born in Le Havre in 1870. After education at secondary school in Caen he lost both his parents and had to enter the family business. It was during this period that he seems to have become familiar with the workings of financial markets.

At the age of 22, Bachelier arrived in Paris at the Sorbonne where he followed the lectures of Paul Appell, Joseph Boussinesq and Henri Poincaré (the latter being then aged 38). After some 8 years, in 1900, Bachelier defended his thesis *Théorie de la Spéculation* before these three men, the favourable report being written by no less a figure than Henri Poincaré, one of the most eminent mathematicians in the world at the time.

Quite what his employment was between 1900 and 1914 (when he was drafted into the French Army during the First World War) is not known. It is known, however, that he received occasional scholarships to continue his studies (on the recommendation of Émile Borel (1871–1956) and he gave lectures as a 'free professor' at the Sorbonne between 1909 and 1914. One of his courses was *Probability calculus with applications to financial operations and analogies with certain questions from physics*. In this course he may have drawn out the similarities between the diffusion of probability (the total probability of one being conserved) and the diffusion equation of Fourier (the total heat-energy being conserved). In 1912 he wrote a book *Calcul des Probabilités* and in 1914 a book *Le Jeu, la Chance et le Hasard*. At the end of the War he obtained an academic position (lecturer) at Besançon then moved to Dijon (1922), then to Rennes (1925).

In 1926 he tried to go back to Dijon by applying for the vacant chair but was turned down on account of a critical report from Paul Lévy (1886–1971), then a professor aged 40 at the École Polytechnique.

Bachelier in his Thesis, in progressing from a 'drunkards' random walk with n (discrete) steps in time t, each step being of length d, to a (continuous) distribution for where the drunkard might be at time t, realised that there had to be a relationship between n and d (d equal to $(t/n)^{1/2}$) for the limit process to 'work'.

In a later paper Paul Lévy thought that Bachelier had made a mistake in his paper by making the tangent of the path (up or down) constant and Bachelier failed to be appointed at Dijon. Bachelier was furious and wrote to Lévy, who, apparently, was unrepentant over this calumny.

The algebraic sum of the upwards and downwards steps taken by the drunkard gives the height of the drunkard at time t above the origin while the sum of the squares of the steps is equal to t and the algebraic and absolute sum of the cubes of the upward and downward steps (and higher powers) become closer and closer to zero. It is these properties of continuity, non-differentiability, infinite 1st order variation, finite 2nd order variation and zero 3rd or higher order variation that gives the drunkard's walk and, in the limit, Brownian Motion some of its unique character and leads to Itô's important Lemma.

It seems extraordinary that Lévy was, apparently, unfamiliar with Bachelier's work as Bachelier had by this time (1926) published 3 books and some 13 papers on probability and regarded showing how a continuous distribution could be derived from a discrete distribution as his most important achievement. Lévy once told J.L. Doob that 'reading other writers' mathematics gave him physical pain' so perhaps it was the case that Lévy had never read Bachelier.

Borel, however, must have known Bachelier (he had approved the scholarships to Bachelier). It should be pointed out that Poincaré, who would not have made this mistake over the interpretation of Bachelier's work, had died some 14 years earlier.

It seems that Bachelier, was regarded as being of lesser importance in the eyes of the French mathematical élite (Hadamard, Borel, Lebesgue, Lévy, Baire). His mathematics was not rigorous (it could not be as the mathematical techniques necessary to make it so had not been developed e.g. measure theory and axiomatic probability) although, his results were basically correct.

However, Lévy, a few years later, was apparently surprised to find Kolmogorov referring to Bachelier's work. In 1931, Lévy wrote a letter of apology to Bachelier and they were reconciled.

Bachelier moved back to Besançon (this time as permanent professor) in 1927 and retired aged 67 in 1937. His last publication was in 1941 and he died in 1946 aged 76.

Bachelier's work is remarkable for herein lie the theory of Brownian Motion (one of the most important mathematical discoveries of the 20th century), the connection between random walks and diffusion, diffusion of probability, curves lacking tangents (non-differentiable functions), the distribution of the Wiener process and of the maximum value attained in a given time by a Wiener process, the reflection principle, the pricing of options including barrier options, the Chapman-Kolmogorov equations in the continuous case, (namely $f(x_n|x_s) = \int_{-\infty}^{\infty} f(x_n|x_r)\, f(x_r|x_s)\, dx_r$ for $n > r > s$ where f are the transition densities of a Markov sequence of random variables) and the seeds of Markov Processes, weak convergence of random variables (i.e. convergence in distribution), martingales and Itô stochastic calculus.

Bachelier's treatment and understanding of the theory of Brownian Motion (originally called Brownian Movement) is more elegant and mathematical than in Einstein's 1905 paper. While Einstein had an unsurpassed 'nose' for physics his nose for mathematics was, by his own admission, not so highly developed.

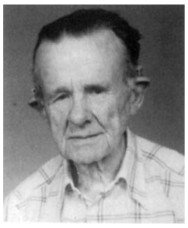
The work of Bachelier leads on to the work of Wiener (1923), Kolmogorov (1931), Itô (1950), and Black, Scholes and Merton (1973).

Bachelier was ahead of his time and his work was not appreciated in his lifetime. In the light of the enormous importance of international derivative exchanges (where the pricing is determined by financial mathematics) the remarkable pioneering work of Bachelier can now be appreciated in its proper context and Bachelier can now be given his proper place.

Article by: *David O. Forfar*

Carl Harald Cramér

Born: 25 September 1893 in Stockholm, Sweden
Died: 5 October 1985 in Stockholm, Sweden



Harald Cramér entered the University of Stockholm in 1912. He embarked on a course of study which involved both chemistry and mathematics and at first the chemistry seemed to be at least as important to him as the mathematics. In fact he worked as a research assistant on a biochemistry project before becoming firmly settled on research in mathematics. Cramér's first five publications are written jointly with the chemist H. von Euler during 1913–14. After this he worked on his doctoral studies in mathematics which were supervised by Marcel Riesz. Also influenced by G.H. Hardy, Cramér's research resulted in the award of a Ph.D. in 1917 for his thesis *On a class of Dirichlet series.*

In 1919 Cramér was appointed assistant professor at the University of Stockholm. He began to produce a series of papers on analytic number theory, and he addressed the Scandinavian Congress of Mathematicians in 1922 on *Contributions to the analytic theory of numbers* detailing his work on the topic up to that time. One interesting paper by Cramér over this period which we should note is one he published in 1920 discussing prime number solutions x, y to the equation $ax + by = c$, where a, b, c are fixed integers. Note that if $a = b = 1$ then the question of whether this equation has a solution for all c is Goldbach's conjecture, while if $a = 1$, $b = -1$, $c = 2$, then the question about prime solutions to $x = y + 2$ is the twin prime conjecture. Cramér's work in prime numbers is put into the context of the history of prime number theory from Eratosthenes to the mid 1990s.

It was not only through his work on number theory that Cramér was led towards probability theory. He also had a second job, namely as an actuary with the Svenska Life Assurance Company. This led him to study probability and statistics which then became the main area of his research. In 1927 he published an elementary text in Swedish *Probability theory and some of its applications.* In 1929 he was appointed to a newly created chair in Stockholm, becoming the first Swedish professor of Actuarial Mathematics and Mathematical Statistics.

Cramér became interested in the rigorous mathematical formulation of probability in work of the French and Russian mathematicians such as Paul Lévy, Sergi Bernstein, and Aleksandr Khinchin in the early 1930's, but in particular the axiomatic approach of Kolmogorov. The results of his studies were written up in his Cambridge publication *Random variables and probability distributions* which appeared in 1937. This was to lead to later work on stationary stochastic processes. By the mid 1930's Cramér's attention had turned to look at the approach of the English and American statisticians such as Fisher, Neyman and Egon Pearson (Karl Pearson's son). These he described as admirable but :—

> *...not quite satisfactory from the point of view of mathematical rigour.*

Masani describes the beginnings of Cramér's work on stochastic processes as follows :—

> *The first phase, beginning at the start of World War II, is devoted to extending the* 1934 *results of* Khinchin *on univariate stationary stochastic processes to multivariate stationary stochastic processes, and to studying the connections between* Khinchin*'s work and the earlier cognate work on generalised harmonic analysis by* Norbert Wiener in 1930.

During World War II Cramér was to some extent cut off from the rest of the academic world. However he gave shelter to W. Feller who was forced out of Germany by Hitler's anti-Jewish policies in 1934. By the end of World War II Cramér had written his masterpiece *Mathematical Methods of Statistics*. The book was first published in 1945, and republished as recently as 1999. The book combines the two approaches to statistics described above and the latest reprinting is described as follows :—

> *In this classic of statistical mathematical theory, Harald Cramér joins the two major lines of development in the field : while British and American statisticians were developing the science of statistical inference, French and Russian probabilists transformed the classical calculus of probability into a rigorous and pure mathematical theory. The result of Cramér's work is a masterly exposition of the mathematical methods of modern statistics that set the standard that others have since sought to follow.*

In 1950 Cramér became the President of Stockholm University. Despite holding this post until he retired in 1961, Cramér still found time to undertake research despite the large administrative burden placed on him. The second phase of Cramér's work on stochastic processes :—

> *...began around 1950 and lasted until the early 1980s... It is devoted to the analysis of non-stationary processes, specifically to determining the extent to which the representations available for stationary process survive for non-stationary ones.*

Cramér's *Collected Works* were published in 1994. Paul Embrechts, in his review of the two volumes, writes :—

> *One finds treated such fields as number theory, function theory, mathematical statistics, probability and stochastic processes, demography, insurance risk theory, functional analysis and the history of mathematics. Such highlights as the probabilistic method in the study of asymptotic properties of prime numbers, the spectral analysis of stationary processes, the mathematical foundation of inference and the fundamental work on risk theory all add up to a brilliant career as a scientist.*

Another reviewer writes :—

> *This book is a classic, not least for its combination of lucidity and rigour. ... It belongs on the shelf of anyone interested in statistical methods.*

We should give two specific results which we have not mentioned previously which will be remembered as major contributions, namely his work on the central limit theorem and his beautiful theorem that if the sum of two independent random variables is normal then all are normal.

There have been many tributes to Cramér. Edward Phragmen (1863–1937) wrote :—

> *Harald Cramér belonged to a generation of mathematicians for which it was self-evident that mathematics constitutes one of the highest forms of human thought, perhaps even the highest. For these mathematicians numbers were a necessary form of human thought, and the science of numbers was a central humanistic discipline with a cultural value of its own, completely independent of its role as auxiliary science in technical or other areas. This does not however mean that they underestimated the importance of 'using theoretical knowledge to obtain practical know-how'.*

Blom sums up Cramér's contribution with simple but effective words :—

> *He was a great scientist and a good man.*

Article by: *J.J. O'Connor* and *E.F. Robertson*

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