$\boxed{\text{Moments des temps d'attente d'une queue}}$

W.B. Gong & J.-Q. Hu (1992)
The McLaurin series for the GI/G/1 queue

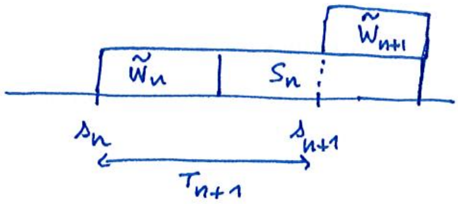
S : temps de service

T : temps inter-arrivées , densité f_T

$\tilde W$: temps d'attente

$W = \tilde W + S$: temps de séjour total

Équation de Lindley : $\boxed{\tilde W_{n+1} = (\tilde W_n + S_n - T_{n+1})^+}$



En régime stationnaire ($n \to +\infty$) :

$$\begin{cases} \tilde W = (\tilde W + S - T)^+ = (W-T)^+ \\ W = S + (W-T)^+ \end{cases}$$

Posons $w_i = E\left(\frac{W^i}{i!}\right)$, $\tilde w_i = E\left(\frac{\tilde W^i}{i!}\right)$, $s_i = E\left(\frac{S^i}{i!}\right)$.

On a $$\tilde w_i = E\left(\frac{\tilde W^i}{i!}\right) = E\left[\frac{(W-T)^{+i}}{i!}\right] = E\left[\int_0^W \frac{(W-t)^i}{i!} \underbrace{f_T(t)}_{\sum_{j=0}^\infty f_T^{(j)}(0)\frac{t^j}{j!}} dt\right] = \sum_{j=0}^\infty f_T^{(j)}(0)\, E\Bigg[\underbrace{\int_0^W \frac{(W-t)^i}{i!}\frac{t^j}{j!}\,dt}_{W^{i+j+1}\underbrace{\frac{B(i+1,j+1)}{i!\,j!}}_{\frac{1}{(i+j+1)!}}}\Bigg]$$

$$= \sum_{j=0}^\infty f_T^{(j)}(0)\, w_{i+j+1}$$

$$= f_T(0)\, w_{i+1} + \sum_{j=1}^\infty f_T^{(j)}(0)\, w_{i+j+1}$$

Or $$w_i = E\left(\frac{W^i}{i!}\right) = E\left[\frac{(S+\tilde W)^i}{i!}\right] = E\left[\sum_{j=0}^i \frac{S^{i-j}}{(i-j)!}\frac{\tilde W^j}{j!}\right] = \sum_{j=0}^i s_{i-j}\,\tilde w_j$$

donc $$\tilde w_i = f_T(0)\sum_{j=0}^{i+1} s_{i+1-j}\,\tilde w_j + \sum_{j=1}^\infty f_T^{(j)}(0)\, w_{i+j+1}$$

$$= f_T(0)\sum_{j=0}^{i-1} s_{i+1-j}\,\tilde w_j + f_T(0)\, s_1 \tilde w_i + f_T(0)\,\tilde w_{i+1} + \sum_{j=1}^\infty f_T^{(j)}(0)\, w_{i+j+1}$$

$$= f_T(0)\sum_{j=0}^{i-1} s_{i+1-j}\,\tilde w_j + f_T(0)\, s_1 \tilde w_i + f_T(0)\underbrace{\left[\sum_{j=0}^\infty f_T^{(j)}(0)\, w_{i+j+2}\right]}_{\sum_{j=1}^\infty f_T^{(j-1)}(0)\, w_{i+j+1}} + \sum_{j=1}^\infty f_T^{(j)}(0)\, w_{i+j+1}$$

$$= f_T(0)\sum_{j=0}^{i-1} s_{i+1-j}\,\tilde w_j + f_T(0)\, s_1 \tilde w_i + \sum_{j=1}^\infty \left[f_T(0) f_T^{(j-1)}(0) + f_T^{(j)}(0)\right] w_{i+j+1}$$

$$\Rightarrow \boxed{E(\tilde W^i) = \frac{1}{1 - f_T(0)E(S)}\left[\frac{f_T(0)}{i+1}\sum_{j=0}^{i-1} C_{i+1}^j\, E(S^{i+1-j})\, E(\tilde W^j) + i\sum (f_T f_T^{(j-1)} + f_T^{(j)})(0)\, E\left(\frac{W^{i+j+1}}{(i+j+1)!}\right)\right]}$$

<u>Exemple</u> : $T : \mathcal{E}(\lambda)$ $f_T(t) = \lambda e^{-\lambda t}$, $f_T^{(j)}(0) = (-1)^j \lambda^{j+1}$ donc $(f_T f_T^{(j-1)} + f_T^{(j)})(0) = 0$ pour tout $j \geq 0$

$$\boxed{E(\tilde W^i) = \frac{\lambda/(i+1)}{1-\lambda E(S)}\sum_{j=0}^{i-1} C_{i+1}^j\, E(S^{i+1-j})\, E(\tilde W^j)}$$

pour $i=1$: $E(\tilde W) = \frac{\lambda E(S^2)}{2(1-\lambda E(S))}$ $\to$ Pollaczek-Khintchine