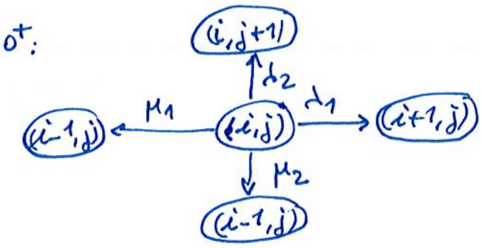
Processus de naissance-mort bidimensionnel

Déf : $(Z_t)_{t\geq 0}$ processus de naissance-mort bidimensionnel : $Z_t = (X_t, Y_t)$

$$\Delta_\varepsilon Z_t = (\Delta_\varepsilon X_t, \Delta_\varepsilon Y_t) = (X_{t+\varepsilon} - X_t, Y_{t+\varepsilon} - Y_t)$$

$(Z_t)_{t\geq 0}$ chaîne de Markov telle que lorsque $\varepsilon \to 0^+$:



$$\begin{cases} \mathbb{P}(\Delta_\varepsilon Z_t = (1,0) \mid Z_t = (i,j)) = \lambda_1(i,j)\varepsilon + o(\varepsilon) \\ \mathbb{P}(\Delta_\varepsilon Z_t = (-1,0) \mid Z_t = (i,j)) = \mu_1(i,j)\varepsilon + o(\varepsilon) \\ \mathbb{P}(\Delta_\varepsilon Z_t = (0,1) \mid Z_t = (i,j)) = \lambda_2(i,j)\varepsilon + o(\varepsilon) \\ \mathbb{P}(\Delta_\varepsilon Z_t = (0,-1) \mid Z_t = (i,j)) = \mu_2(i,j)\varepsilon + o(\varepsilon) \\ \mathbb{P}(|\Delta_\varepsilon X_t| + |\Delta_\varepsilon Y_t| \geq 2 \mid Z_t = (i,j)) = o(\varepsilon) \\ \text{et donc } \mathbb{P}(\Delta_\varepsilon Z_t = (0,0) \mid Z_t = (i,j)) = 1 - [\lambda_1(i,j) + \lambda_2(i,j) + \mu_1(i,j) + \mu_2(i,j)]\varepsilon + o(\varepsilon) \end{cases}$$

Ex : $(X_t)_{t\geq 0}, (Y_t)_{t\geq 0}$: processus de naissance-mort linéaires indépendants.

$$\mathbb{P}(\Delta_\varepsilon Z_t = (k,l) \mid Z_t = (i,j)) = \mathbb{P}(\Delta_\varepsilon X_t = k \mid X_t = i)\, \mathbb{P}(\Delta_\varepsilon Y_t = l \mid Y_t = j)$$

$$= \begin{cases} o(\varepsilon) \text{ si } |k| + |l| \geq 2 \quad (\text{de l'ordre de } \varepsilon^2 \text{ pour } |k| = |l| = 1) \\ [\lambda_1(i)\varepsilon + o(\varepsilon)][1 - (\lambda_2(j) + \mu_2(j))\varepsilon + o(\varepsilon)] = \lambda_1(i)\varepsilon + o(\varepsilon) \text{ si } (k,l) = (1,0) \\ \lambda_2(j)\varepsilon + o(\varepsilon) \quad \text{si } (k,l) = (0,1) \\ \mu_1(i)\varepsilon + o(\varepsilon) \quad \text{si } (k,l) = (-1,0) \\ \mu_2(j)\varepsilon + o(\varepsilon) \quad \text{si } (k,l) = (0,-1) \\ 1 - [\lambda_1(i) + \lambda_2(j) + \mu_1(i) + \mu_2(j)]\varepsilon + o(\varepsilon) \text{ si } (k,l) = (0,0) \end{cases}$$

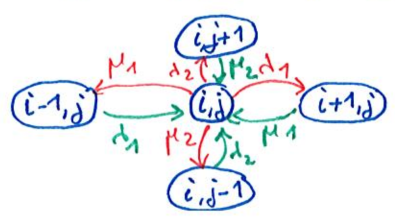
Générateur infinitésimal : $A = (A_{(i,j),(k,l)})_{((i,j),(k,l)) \in \mathbb{N}^2 \times \mathbb{N}^2}$

$$\begin{cases} A_{(i,j),(k,l)} = 0 \quad \text{si } |k-i| + |l-j| \geq 2 \\ A_{(i,j),(i+1,j)} = \lambda_1(i,j), \quad A_{(i,j),(i-1,j)} = \mu_1(i,j) \\ A_{(i,j),(i,j+1)} = \lambda_2(i,j), \quad A_{(i,j),(i,j-1)} = \mu_2(i,j) \\ A_{(i,j)(i,j)} = -[\lambda_1(i,j) + \lambda_2(i,j) + \mu_1(i,j) + \mu_2(i,j)] \end{cases}$$

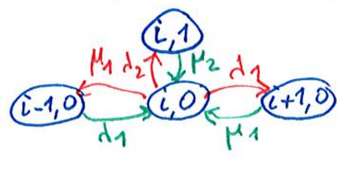
En régime stationnaire : $\pi A = \left(\sum_{(k,l)\in\mathbb{N}^2} \pi_{(k,l)} A_{(k,l),(i,j)} \right)_{(i,j)\in\mathbb{N}^2} = 0$

→ équations de balance globale :

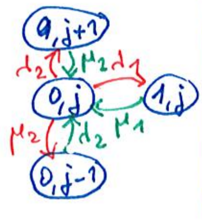
$$\begin{cases} \text{si } i\geq 1, j\geq 1 : [\lambda_1(i,j) + \lambda_2(i,j) + \mu_1(i,j) + \mu_2(i,j)]\,\pi(i,j) = \mu_1(i+1,j)\,\pi(i+1,j) + \lambda_1(i-1,j)\,\pi(i-1,j) + \mu_2(i,j+1)\,\pi(i,j+1) \\ \qquad\qquad + \lambda_2(i,j-1)\,\pi(i,j-1) \\ \text{si } i\geq 1, j=0 : [\lambda_1(i,0) + \lambda_2(i,0) + \mu_1(i,0)]\,\pi(i,0) = \mu_1(i+1,0)\,\pi(i+1,0) + \lambda_1(i-1,0)\,\pi(i-1,0) + \mu_2(i,1)\,\pi(i,1) \\ \text{si } i=0, j\geq 1 : [\lambda_1(0,j) + \lambda_2(0,j) + \mu_2(0,j)]\,\pi(0,j) = \mu_1(1,j)\,\pi(1,j) + \mu_2(0,j+1)\,\pi(0,j+1) + \lambda_2(0,j-1)\,\pi(0,j-1) \\ \text{si } i=0, j=0 : [\lambda_1(0,0) + \lambda_2(0,0)]\,\pi(0,0) = \mu_1(1,0)\,\pi(1,0) + \mu_2(0,1)\,\pi(0,1) \end{cases}$$



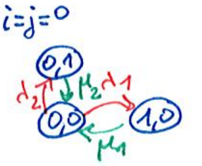
j=0 :



i=0

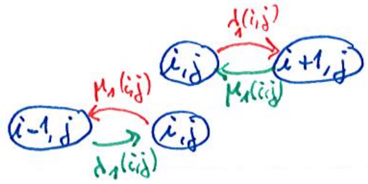


i=j=0

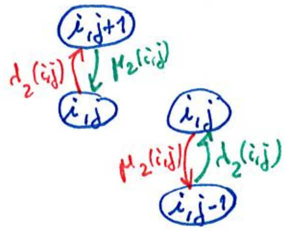


Décomposition en balance locale (si possible)

[si on a une solution pour la balance locale, c'est une solution pour la balance globale]



$$\left.\begin{aligned} \lambda_1(i,j)\pi(i,j) &= \mu_1(i+1,j)\pi(i+1,j) \\ \mu_1(i,j)\pi(i,j) &= \lambda_1(i-1,j)\pi(i-1,j) \end{aligned}\right\}$$ identiques $\Rightarrow \pi(i,j) = \dfrac{\lambda_1(i-1,j)\,\lambda_1(i-2,j)\cdots\lambda_1(0,j)}{\mu_1(i,j)\,\mu_1(i-1,j)\cdots\mu_1(1,j)}\,\pi(0,j)$ ①



$$\left.\begin{aligned} \lambda_2(i,j)\pi(i,j) &= \mu_2(i,j+1)\pi(i,j+1) \\ \mu_2(i,j)\pi(i,j) &= \lambda_2(i,j-1)\pi(i,j-1) \end{aligned}\right\}$$ identiques $\Rightarrow \pi(i,j) = \dfrac{\lambda_2(i,j-1)\,\lambda_2(i,j-2)\cdots\lambda_2(i,0)}{\mu_2(i,j)\,\mu_2(i,j-1)\cdots\mu_2(i,1)}\,\pi(i,0)$ ②

En utilisant d'abord ① puis ② : $\pi(i,j) = \dfrac{\lambda_1(i-1,j)\,\lambda_1(i-2,j)\cdots\lambda_1(0,j)\,\lambda_2(0,j-1)\,\lambda_2(0,j-2)\cdots\lambda_2(0,0)}{\mu_1(i,j)\,\mu_1(i-1,j)\cdots\mu_1(1,j)\,\mu_2(0,j)\,\mu_2(0,j-1)\cdots\mu_2(0,1)}\,\pi(0,0)$

d'abord ② puis ① : $\pi(i,j) = \dfrac{\lambda_2(i,j-1)\,\lambda_2(i,j-2)\cdots\lambda_2(i,0)\,\lambda_1(i-1,0)\,\lambda_1(i-2,0)\cdots\lambda_1(0,0)}{\mu_2(i,j)\,\mu_2(i,j-1)\cdots\mu_2(i,1)\,\mu_1(i,0)\,\mu_1(i-1,0)\cdots\mu_1(1,0)}\,\pi(0,0)$

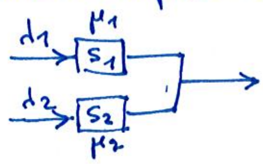
→ problème de cohérence ! (réversibilité : si les équations de balance locale sont vérifiées, la chaîne est reversible : $\pi(i,j)A_{(i,j),(k,l)} = \pi(k,l)A_{(k,l),(i,j)}$

Cas des taux constants :

$$\begin{cases} \lambda_1(i,j) = \lambda_1 \\ \lambda_2(i,j) = \lambda_2 \\ \mu_1(i,j) = \mu_1 \\ \mu_2(i,j) = \mu_2 \end{cases}$$

Posons $\rho_1 = \dfrac{\lambda_1}{\mu_1}$, $\rho_2 = \dfrac{\lambda_2}{\mu_2}$ → chaîne réversible

files d'attente en parallèle



On trouve $\pi(i,j) = \dfrac{\lambda_1^i\,\lambda_2^j}{\mu_1^i\,\mu_2^j}\,\pi(0,0) = \rho_1^i\,\rho_2^j\,\pi(0,0)$

$$\sum_{(i,j)\in\mathbb{N}^2} \pi(i,j) = 1 \Rightarrow \pi(0,0) = \frac{1}{\sum_{i=0}^{\infty}\rho_1^i \sum_{j=0}^{\infty}\rho_2^j} = (1-\rho_1)(1-\rho_2)$$

$$\boxed{\pi(i,j) = (1-\rho_1)\rho_1^i \times (1-\rho_2)\rho_2^j}$$

Ainsi X_∞ et Y_∞ sont indépendantes $\mathcal{G}(1-\rho_1)$ et $\mathcal{G}(1-\rho_2)$.