THE AUTOMORPHISM TOWER OF A CENTERLESS GROUP (MOSTLY) WITHOUT CHOICE

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Abstract. For a centerless group G, we can define its automorphism tower. We define G^{α}: $G^0 = G$, $G^{\alpha+1} = Aut\left(G^{\alpha}\right)$ and for limit ordinals $G^{\delta} = \bigcup_{\alpha<\delta} G^{\alpha}$. Let τ_G be the ordinal when the sequence stabilizes. Thomas' celebrated theorem says $\tau_G < \left(2^{|G|}\right)^{+}$ and more.

If we consider Thomas' proof too set theoretical, we have here a shorter proof with little set theory. However, set theoretically we get a parallel theorem without the axiom of choice.

We attach to every element in G^{α}, the α-th member of the automorphism tower of G, a unique quantifier free type over G (whish is a set of words from $G * \langle x\rangle$) . This situation is generalized by defining "(G, A) is a special pair".

1. Introduction

background. Given any centerless group G, we can embed G into its automorphism group $Aut(G)$. Since $Aut(G)$ is also without center, we can do this again, and again. Thus we can define an increasing continuous series $\langle G^{\alpha} \,|\alpha \in \mathbf{ord}\rangle$ - The automorphism tower. The natural question that rises, is whether this process stops, and when. We define $\tau_G = min\left\{\alpha \,|G^{\alpha+1} = G^{\alpha}\right\}$.

In 1939 (see [3]) Weilant proved that for finite G, τ_G is finite. But there exist examples of centerless infinite groups such that this process does not stop in any finite stage. For example - the infinite dihedral group $D_{\infty} = \langle x, y \,|x^2 = y^2 = 1\rangle$ satisfies $Aut\left(D_{\infty}\right) \cong D_{\infty}$. So the question remained open until 1984, when Simon Thomas' celebrated work (see [4]) proved that $\tau_G \leq \left(2^{|G|}\right)^{+}$. He later (see [2]) improved this to $\tau_G < \left(2^{|G|}\right)^{+}$.

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For a cardinal κ we define τ_κ as the smallest ordinal such that $\tau_\kappa > \tau_G$ for all centerless groups G of cardinality $\leq \kappa$. As an immediate conclusion from Thomas' theorem we have $\tau_\kappa < (2^\kappa)^+$.

We also define the normalizer tower of H - a subgroup of a group G - in G:
$\langle nor_G^\alpha(H)\,|\alpha \in \mathbf{ord}\rangle$ by $nor_G^0(H) = H$, $nor_G^{\alpha+1}(H) = nor(nor_G^\alpha(H))$ and $nor_G^\delta(H) = \bigcup\{nor_G^\alpha(H)\,|\alpha < \delta\}$ for δ limit. Let
$\tau_{G,H} = min\left\{\alpha \,\middle|nor_G^{\alpha+1}(H) = nor_G^\alpha(H)\right\}$. This construction turns out to be very useful, thanks to the following:

For a cardinal κ, let τ_κ^{nlg} be the smallest ordinal such that $\tau_\kappa^{nlg} > \tau_{Aut(\mathfrak{A}),H}$, for every structure of cardinality $\leq \kappa$ and $H \leq Aut(\mathfrak{A})$ of cardinality $\leq \kappa$.

In [1], Just, Shelah and Thomas, found a connection between these ordinals: $\tau_\kappa \geq \tau_\kappa^{nlg}$.

In this paper we deal with an upper bound of τ_κ, but there are conclusion regarding lower bounds as well, and the inequality above is used to prove the existence of such lower bounds by finding structures with long normalizer towers. In [4], Thomas proved that $\tau_\kappa \geq \kappa^+$, and in [1] the authors found that one cannot prove in ZFC a better explicit upper bound for τ_κ then $(2^\kappa)^+$ (using set theoretic forcing). In [5], Shelah proved that if κ is strong limit singular of uncountable cofinality then $\tau_\kappa > 2^\kappa$ (using results from PCF theory).

It remains an open question whether or not there exists a countable centerless group G such that $\tau_G \geq \omega_1$.

In a subsequent paper we plan to prove that $\tau_\kappa^{nlg} \leq \tau_\kappa$ is true even without choice.

Results. Our main theorem: (of course, Thomas' did not need to distinguish G and ${}^{\omega>}G$)

Theorem 1.1. *(ZF) $\tau_{|G|} < \theta_{\mathcal{P}({}^{\omega>}G)}$ for a centerless group G. That is, there is an ordinal α and a function from ${}^{\omega>}G$ onto it such that $\tau_G < \alpha$. Moreover, $\tau_{G'} < \alpha$ for every centerless group G' such that $|G'| \leq |G|$.*

This is an essentially theorem 3.16.

We deal with finding τ_G without choice, and discover that Thomas' theorem still holds. We prove that given a certain algebraic property of G and a subset A ((G, A) is special

- see definition 3.6) we can reduce the bound. Along the way we give a different proof of the theorem without choice in conclusion 3.14 (Thomas used Fodor's lemma in his proof as you can see in section 5, and it is known that its negation is consistent with ZF) . Then we conclude that if $\mathbf{V}'$ is a subclass of $\mathbf{V}$ which is a model of ZF such that $\mathcal{P}(\kappa) \in \mathbf{V}'$, then $\tau_\kappa < \left(\theta_{\mathcal{P}(\kappa)}\right)^{\mathbf{V}'}$ for every κ and so $\tau_{\aleph_0} < \theta_{\mathbb{R}}^{L[\mathbb{R}]}$. (see conclusion 3.19)
Moreover, we give a descriptive set theoretic approach to finding $\tau_{\aleph_0}$ in section 4.
Finally, we return to the axiom of choice, to see that we can improve the bound for certain groups that hold a weaker algebraic property ((G, A) is weakly special - see definition 5.4).

A note about reading this paper. How should you read this paper if you are not interested in the axiom of choice but only on the new and simple proof of Thomas' Theorem? You can read only section 3, and in there, you:
Start with definition 3.6. Continue to claim 3.8, which is very simple. Then conclusion 3.10 is a simple application of that lemma. Claim 3.12 Is a very important step towards 3.13, and then finally conclusion 3.14 wraps it up.

Notation 1.2.

(1) For a group G, its identity element, will be denoted as $e = e_G$.
(2) if $A \subseteq G$ then$\langle A\rangle_G$ is the subgroup generated by A in G. Similarly, if $x \in G$,$\langle A, x\rangle_G$ is the subgroup generated by $A \cup \{x\}$.
(3) The language of a structure is its vocabulary.
(4) $\mathbf{V}$ will denote the universe of sets; $\mathbf{V}'$ will denote a transitive class which is a model of ZF.

2. The Normalizer Tower Without Choice

Definition 2.1.

(1) For a group G and a subgroup $H \leq G$, we define $nor_G^\alpha(H)$ for every ordinal number α by:
  - $nor_G^0(H) = H$.
  - $nor_G^{\alpha+1}(H) = nor_G\left(nor_G^\alpha(H)\right)$.

- $nor_G^{\delta}(H) = \bigcup\{nor_G^{\alpha}(H) \,|\alpha < \delta\}$, for δ limit.

(2) We define $\tau_{G,H}^{nlg} = \tau_{G,H} = min\left\{\alpha \left| nor_G^{\alpha+1}(H) = nor_G^{\alpha}(H)\right.\right\}$.

(3) For a set k, we define $\tau_{|k|}^{nlg}$ as the smallest ordinal α, such that for every structure $\mathfrak{A}$ of power $\|\mathfrak{A}\| \leq |k|$, $\tau_{Aut(\mathfrak{A}),H} < \alpha$ for every subgroup $H \leq Aut(\mathfrak{A}) = G$ of power $|H| \leq |k|$. Note that $\tau_{|k|}^{nlg} = sup\{\tau_{G,H} + 1 \,|$ for such G and $H\}$.

(4) For a cardinal number κ, define τ_{κ}^{nlg} similarly.

Remark 2.2. Note that $\tau_{|k|}^{nlg}$ is well defined (in ZF) since we can restrict ourselves to structures with languages of power $\leq \sum_{n<\omega} |k|^n$ and universe contained in k. See observation 2.3.

Observation 2.3.

(1) (ZF) For any structure $\mathfrak{A}$ whose universe is $|\mathfrak{A}| = A$ there is a structure $\mathfrak{B}$ such that:
   - $\mathfrak{A}, \mathfrak{B}$ have the same universe (i.e. $A = |\mathfrak{B}|$).
   - $\mathfrak{A}, \mathfrak{B}$ have the same automorphism group (i.e. $Aut(\mathfrak{A}) = Aut(\mathfrak{B})$).
   - the language of $\mathfrak{B}$ is of the form $L_{\mathfrak{B}} = \{R_{\bar{a}} \,|\bar{a} \in {}^{\omega>}A\}$ where each $R_{\bar{a}}$ is a $lg(\bar{a})$ place relation.

(2) (ZFC) If $\mathfrak{A}$ is infinite then the language of $\mathfrak{B}$ has cardinality at most $|A|$.

Proof. Define $\mathfrak{B}$ as follows: its universe is $|\mathfrak{A}|$. Its language is $L = \{R_{\bar{a}} \,|\bar{a} \in {}^{n}A, n < \omega\}$ where $R_{\bar{a}}^{\mathfrak{B}} = o(\bar{a})$, which is defined by $o(\bar{a}) = \{f(\bar{a}) \,| f \in Aut(\mathfrak{A})\}$ - the orbit of $\bar{a}$ under $Aut(\mathfrak{A})$. □

Definition 2.4. For a set A, we define $\theta_A = \theta(A)$ to be the first ordinal $\alpha > 0$ such that there is no function from A onto α.

Remark 2.5.

(1) $ZFC \vdash \theta_A = |A|^+$

(2) $ZF \vdash \theta_A$ is a cardinal number, and if A is infinite (i.e. there is an injection from ω into A) then $\theta_A > \aleph_0$.

(3) Usually, we shall consider $\theta_A^{\mathbf{V}'}$ where $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF.

Claim 2.6. (ZF) If G is a group, $H \leq G$ a subgroup then $\tau_{G,H} < \theta_G$.

Proof. If $\tau_{G,H} = 0$ it is clear. if not, define $F : G \to \tau_{G,H}$ by $F(g) = \alpha$ iff $g \in nor_G^{\alpha+1}(H) \setminus nor_G^{\alpha}(H)$, and if there is no such α, $F(g) = 0$. By definition of $\tau_{G,H}$, F is onto. From the definition of θ, $\tau_{G,H} < \theta_G$. □

We can do even more:

Claim 2.7. (ZF) $\tau_{|k|}^{nlg} < \theta_{\mathcal{P}({}^{\omega>}k)}$.

Proof. Let

$$\begin{aligned} \mathcal{B}_k &= \{(\mathfrak{A}, f, x) \,|\mathfrak{A} \text{ is a structure, } L_{\mathfrak{A}} \subseteq {}^{\omega>}k, \, |\mathfrak{A}| \subseteq k, \, f : k \to {}^k k, \\ & \qquad x \in G = Aut(\mathfrak{A}) \text{ and } H \leq G, H = image(f)\} \end{aligned}$$

Let $F : \mathcal{B}_k \to \tau_{|k|}^{nlg}$ be the following map: $F(\mathfrak{A}, f, x) = \alpha$ iff $x \in nor_G^{\alpha+1}(H) \setminus nor_G^{\alpha}(H)$, and if there is no such α, $F(G, H, x) = 0$ (where $G = Aut(\mathfrak{A})$, and $H = image(f)$). Since F is onto $\tau_{|k|}^{nlg}$, its enough to show that there is a one to one function from $\mathcal{B}_k$ to $\mathcal{P}({}^{\omega>}k)$. But $x \in G$, hence $x \subseteq k \times k$ and $f \in {}^k\left({}^k k\right) \subseteq k \times k \times k$, and $\mathfrak{A}$ is a series of subsets of ${}^{\omega>}k$, i.e. a function in ${}^{{}^{\omega>}k}\mathcal{P}({}^{\omega>}k) \subseteq {}^{\omega>}k \times \mathcal{P}({}^{\omega>}k)$, and we can encode such a function as a member of $\mathcal{P}({}^{\omega>}k)$. (How? define an injective function $f_1 : {}^{\omega>}k \times {}^{\omega>}k \to {}^{\omega>}k$, using the definable injective function $cd : \omega \times \omega \to \omega$. Then, define the encoding $f_2 : {}^{\omega>}k \times \mathcal{P}({}^{\omega>}k) \to \mathcal{P}({}^{\omega>}k)$ using f_1). Hence it is clear. □

Claim 2.8. Assume that $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF, $G \in \mathbf{V}'$ a group, $H \in \mathbf{V}'$ a subgroup then $\tau_{G,H}^{\mathbf{V}} = \tau_{G,H}^{\mathbf{V}'} < \theta_G^{\mathbf{V}'}$.

Proof. By claim 2.6, it remains to show that $\tau_{G,H}^{\mathbf{V}} = \tau_{G,H}^{\mathbf{V}'}$. By induction on $\alpha \in V'$, one can see that $\left(nor_G^{\alpha}(H)\right)^{\mathbf{V}} = \left(nor_G^{\alpha}(H)\right)^{\mathbf{V}'}$ (the formula that says that x is in $nor_{G'}(H')$ is bounded in the parameters G' and H'). □

It is also true that $\tau_{|k|}^{nlg}$ is preserved in $\mathbf{V}'$, for every $k \in \mathbf{V}'$, such that $\mathcal{P}\left({}^{\omega>}k\right) \in \mathbf{V}'$:

Claim 2.9. Assume that $\mathbf{V}'$ is a transitive subclass of $\mathbf{V}$ which is a model of ZF.

(1) If $\mathcal{P}\left({}^{\omega>}k\right) \in \mathbf{V}'$ then $\left(\tau_{|k|}^{nlg}\right)^{\mathbf{V}'} = \left(\tau_{|k|}^{nlg}\right)^{\mathbf{V}} < \theta_{\mathcal{P}({}^{\omega>}k)}^{\mathbf{V}'}$.
(2) If $k = \kappa$ a cardinal number and $\mathcal{P}(\kappa) \in \mathbf{V}'$ then $\left(\tau_{\kappa}^{nlg}\right)^{\mathbf{V}'} = \left(\tau_{\kappa}^{nlg}\right)^{\mathbf{V}} < \theta_{\mathcal{P}(\kappa)}^{\mathbf{V}'}$

Proof. (2) follows from (1), as we have an absolute definable bijection $cd : {}^{\omega>}\kappa \to \kappa$.
For a set $k \in \mathbf{V}'$, such that $\mathcal{P}\left({}^{\omega>}k\right) \in \mathbf{V}'$ let

$$\begin{aligned} \mathcal{A}_k &= \{(G,H)\,|\text{There is a structure } \mathfrak{A}, \text{ with } |\mathfrak{A}| \subseteq k, \text{ such that} \\ &\qquad G = Aut(\mathfrak{A}) \text{ and } H \leq G, |H| \leq |k|\} \end{aligned}$$

It is enough to prove that $(\mathcal{A}_k)^{\mathbf{V}} = (\mathcal{A}_k)^{\mathbf{V}'}$, because by definition

$$\begin{aligned} \left(\tau_{|k|}^{nlg}\right)^{\mathbf{V}} &= \bigcup \left\{ \tau_{G,H} + 1 \,\middle|\, (G,H) \in (\mathcal{A}_k)^{\mathbf{V}} \right\} \\ &= \bigcup \left\{ \tau_{G,H} + 1 \,\middle|\, (G,H) \in (\mathcal{A}_k)^{\mathbf{V}'} \right\} \\ &= \left(\tau_{|k|}^{nlg}\right)^{\mathbf{V}'} < \theta_{\mathcal{P}({}^{\omega>}k)}^{\mathbf{V}'} \end{aligned}$$

So let us prove the above equality: $(\mathcal{A}_k)^{\mathbf{V}'} \subseteq (\mathcal{A}_k)^{\mathbf{V}}$, since if $(G,H) \in (\mathcal{A}_k)^{\mathbf{V}'}$ and $\mathfrak{A} \in \mathbf{V}'$ a structure such that $G = Aut(\mathfrak{A})$ then $\mathfrak{A} \in \mathbf{V}$ and $(Aut(\mathfrak{A}))^{\mathbf{V}} = (Aut(\mathfrak{A}))^{\mathbf{V}'}$, because $(Aut(\mathfrak{A}))^{\mathbf{V}} \subseteq {}^{k}k \subseteq \mathcal{P}(k \times k) \in \mathbf{V}'$. So $(G,H) \in (\mathcal{A}_k)^{\mathbf{V}'}$, as witnessed by the same structure.
On the other hand, suppose $(G,H) \in (\mathcal{A}_k)^{\mathbf{V}}$. So let $\mathfrak{A}$ be a structure on k such that $G = Aut(\mathfrak{A})$. By observation 2.3, we may assume that $L_{\mathfrak{A}} = \{R_{\bar{a}}\,|\bar{a} \in {}^{\omega>}k\}$, and each $R_{\bar{a}}$ is a $lg(\bar{a})$ place relation (This is not necessary, it just makes it more convenient). Define $X_{\mathfrak{A}} = \left\{\bar{a}\hat{\ }\bar{b} \,\middle|\, lg(\bar{a}) = lg\left(\bar{b}\right) \wedge \bar{b} \in R_{\bar{a}}^{\mathfrak{A}}\right\}$. Observe that:

- $X_{\mathfrak{A}} \in \mathbf{V}'$, as $X_{\mathfrak{A}} \subseteq {}^{\omega>}k$.
- $\mathfrak{A}$ can be defined using $X_{\mathfrak{A}}$: its universe is k, and for each $\bar{a} \in {}^{\omega>}k$, $R_{\bar{a}} = \left\{\bar{b} \,\middle|\, lg\left(\bar{b}\right) = lg(\bar{a}) \wedge \bar{a}\hat{\ }\bar{b} \in X_{\mathfrak{A}}\right\}$.

So in conclusion, $\mathfrak{A} \in \mathbf{V}'$, and so $G \in \mathbf{V}'$ as before. In addition $H \in \mathbf{V}'$, because H is the image of a function in ${}^{k}\left({}^{k}k\right)$, and ${}^{k}\left({}^{k}k\right) \subseteq \mathcal{P}\left(k \times k \times k\right) \in \mathbf{V}'$. By definition $(G, H) \in (\mathcal{A}_k)^{\mathbf{V}'}$ and we are done. □

3. The Automorphism Tower Without Choice

Definition 3.1. For a centerless group G, we define the series $\langle G^{\alpha} \,|\alpha \in \mathbf{ord}\rangle$:

- $G^0 = G$.
- $G^{\alpha+1} = Aut\left(G^{\alpha}\right)$
- $G^{\delta} = \cup\left\{G^{\alpha} \,|\alpha < \delta\right\}$ for δ limit.

Remark 3.2. Since G is centerless, this makes sense - $G \cong Inn\left(G\right) \leq Aut\left(G\right)$, and $Aut\left(G\right)$ is again without center. So we identify G with $Inn\left(G\right)$, and so $G^{\alpha} \leq G^{\alpha+1}$. This series is therefore monotone and continuous.

Definition 3.3.

(1) Define an ordinal τ_G by $\tau_G = min\left\{\alpha \,|G^{\alpha+1} = G^{\alpha}\right\}$. We shall show below that τ_G is well defined.
(2) For a set k, we define $\tau_{|k|}$ to be the smallest ordinal α such that $\alpha > \tau_G$ for all groups G with power $\leq |k|$.
(3) For a cardinal number κ, define τ_κ similarly.

Definition 3.4. For a group G (not necessarily centerless) and a subset A, we define an equivalence relation $E_{G,A}$ by $xE_{G,A}y$ iff $tp_{qf}\left(x, A, G\right) = tp_{qf}\left(y, A, G\right)$ where $tp_{qf}\left(x, A, G\right) =$

$$\{\sigma\left(z, \bar{a}\right) | \quad \bar{a} \in {}^{n}A, n < \omega, \sigma \text{ a term in the language of groups (i.e. a word)}$$
$$\text{with parameters from } A,$$
$$z \text{ is it's only free variable and } G \models \sigma\left(x, \bar{a}\right) = e\}$$

Remark 3.5.

(1) Note that $xE_{G,A}y$ iff there is an isomorphism between $\langle A,x\rangle_G$ and $\langle A,y\rangle_G$ taking x to y and fixing A.
(2) The relation $E_{G,A}$ is definable and absolute (since $tp_{qf}(x,A,G)$ is absolute - the formula defining it is bounded).

Definition 3.6. We say (G,A) is a special pair if $A\subseteq G$, G is a group and $E_{G,A}=\{(x,x)\,|x\in G\}$ (i.e. the equality).

Example 3.7.

(1) If $G=\langle A\rangle_G$ then (G,A) is special.
(2) If G is centerless then $(Aut(G),G)$ is special (see claim 3.8), so in general, the converse of (1) is not true.
(3) There is a group G with center such that $(Aut(G),Inn(G))$ is special, e.g. $\mathbb{Z}/2\mathbb{Z}$, but
(4) If G is not centerless then (2) is not necessarily true, even if the $|Z(G)|=2$:

It is enough to find a group which satisfies these properties:

(a) $Z(G)=\{a,e\}$ where $a\neq e$.
(b) $H_i\leq G$ for $i=1,2$ are two different subgroups of index 2.
(c) $Z(G)=Z(H_i)$ for $i=1,2$

Let π be the homomorphism $\pi:G\to Aut(G)$ taking g to i_g ($i_g(x)=gxg^{-1}$). Then $inn(G)=image(\pi)$. We wish to find $x_1\neq x_2\in Aut(G)$ with $x_1E_{inn(G),Aut(G)}x_2$. So define $x_i(g)=\begin{cases} ag & g\notin H_i\\ g & g\in H_i\end{cases}$. Since $x_i^2=id$,$x_i\pi(g)x_i^{-1}=\pi(x_i(g))=\pi(g)$ and the fact that $x_i\notin Inn(G)$ (because $Z(G)=Z(H_i)$) it follows that $tp_{qf}(x_1,Inn(G),Aut(G))=tp_{qf}(x_2,Inn(G),Aut(G))$. Now we have to construct such a group. Notice that it is enough to find a centerless group satisfying only the last two properties, since we can take it's product with $\mathbb{Z}/2\mathbb{Z}$. So take $G=D_\infty=\langle a,b\,|a^2=b^2=e\rangle$, and $H_a=ker\varphi_a$ where $\varphi_a:G\to\mathbb{Z}/2\mathbb{Z}$ takes a to 1 and b to 0. In the same way we define H_b, and finish.

The following is the crucial claim:

Claim 3.8. Assume $G_1 \trianglelefteq G_2$, $C_{G_2}(G_1) = \{e\}$ and that (G_1, A) is a special pair. Then (G_2, A) is a special pair.

Proof. First we show that $C_{G_2}(A) = \{e\}$. Suppose that $x \in C_{G_2}(A)$, so $xax^{-1} = a$ for all $a \in A$. Since conjugation by x (i.e. the map $h \mapsto xhx^{-1}$ in G_1) is an automorphism of G_1, (as G_1 is a normal subgroup of G_2), it follows from (G, A) being a special pair (by remark 3.5, clause (1)) that it must be id. Hence, $x \in C_{G_2}(G_1)$, but we assumed $C_{G_2}(G_1) = \{e\}$ hence $x = e$.

Next assume that $xE_{G_2,A}y$ where $x, y \in G_2$ and we shall prove $x = y$. There is an isomorphism $\pi : \langle x, A\rangle_{G_2} \to \langle y, A\rangle_{G_2}$ taking x to y and fixing A. We wish to show that $x = y$, so it is enough to show that $x^{-1}\pi(x) \in C_{G_2}(A)$. This is equivalent to showing $x^{-1}\pi(x)\, a\pi(x^{-1})\, x = a$, i.e. $x^{-1}\pi(xax^{-1})\, x = a$, i.e. $\pi(xax^{-1}) = xax^{-1}$ (remember that $\pi(a) = a$) for every $a \in A$. But xax^{-1} is an element of G_1 (as $G_1 \trianglelefteq G_2$), and $\pi : \langle xax^{-1}, A\rangle_{G_1} \to \langle \pi(xax^{-1}), A\rangle_{G_1}$ must be id because (G_1, A) is a special pair, and we are done. □

Note 3.9. If G is centerless then $G \trianglelefteq Aut(G)$, and $C_{Aut(G)}(G) = \{e\}$.

Conclusion 3.10. Assume G is centerless and (G, A) is a special pair then:

(1) (G^α, A) is a special pair for every $\alpha \in \mathbf{ord}$.
(2) $C_{G^\alpha}(A) = \{e\}$ for every α.

Proof. (2) follows from (1). Prove (1) by induction on α. For limit ordinal, its clear from the definitions, and for successors, the previous claim finishes the job using the above note. □

Conclusion 3.11. Let γ be an ordinal, G a centerless group then:

(1) $C_{G^\gamma}(G) = \{e\}$.
(2) $nor_{G^\gamma}\left(G^\beta\right) = G^{\beta+1}$, for $\beta < \gamma$.
(3) $nor^\beta_{G^\gamma}(G) = G^\beta$ for $\beta \le \gamma$.

Proof.

(1) Follows from conclusion 3.10 and from the fact that (G, G) is a special pair.

(2) The direction $nor_{G^{\gamma}}\left(G^{\beta}\right) \geq G^{\beta+1}$ is clear from the definition of the action of $G^{\beta+1}$ on G^{β}. The direction $nor_{G^{\gamma}}\left(G^{\beta}\right) \leq G^{\beta+1}$ follows from the previous clause: suppose $y \in nor_{G^{\gamma}}\left(G^{\beta}\right)$, so conjugation by y is in $Aut\left(G^{\beta}\right)$. By definition there is $z \in G^{\beta+1}$ such that $yxy^{-1} = zxz^{-1}$ for all $x \in G^{\beta}$, in particular - for all $x \in G$, So $y = z$ (by (1))

(3) By induction on β.

□

Claim 3.12. If G is centerless and (G, A) is a special pair then:

(1) (ZFC) $|G^{\alpha}| \leq 2^{|A|}$ for all ordinals α.

(2) (ZF) There is a one to one absolutely definable (with parameters G^{α} and A) function from G^{α} into $\mathcal{P}\left({}^{\omega>}A\right)$ for each ordinal α.

Proof. (1) follows from (2). The natural way to define the function f is
$f(g) = tp_{qf}(g, A, G^{\alpha})$, which is a set of equations. Luckily it is easy to encrypt equations as elements of ${}^{\omega>}A$: We can assume that there are at least two elements in A - a, b (if not, $G = \{e\}$ because $C_G(A) = \{e\}$). Let $\sigma(z, \bar{a})$ be a word, so it is of the form $\ldots z^{n_i} a^{n_{i+1}} z^{n_{i+2}} \ldots$ where $n_i \in \mathbb{Z}$, and $i = 0, \ldots, m-1$. First we encrypt the exponents series with a natural number, m, using the bijection $cd : {}^{\omega>}\omega \to \omega$, and then we encrypt the series of indices where z appears, call it k. Then we encrypt σ by $a^{k}\hat{}\,b\hat{}\,a^{m}\hat{}\,b$ and after that - the list of elements of A in σ by order of appearance.
Note that our function is definable as promised. □

Claim 3.13. If G is centerless then:

(1) (ZFC) If $|G^{\alpha}| \leq \lambda$ for all ordinals α, then $\tau_G < \lambda^{+}$.

(2) (ZF) If $|G^{\alpha}| \leq |A|$ for all ordinals α and a set A, then $\tau_G < \theta_A$. It is enough to assume that there is a function from A onto G^{α} for each ordinal α.

Proof. (1) follows from (2), but with choice, it is much simpler - $G_{\lambda^+} = \bigcup \{G_\alpha \,|\alpha < \lambda^+\}$. Since $|G_{\lambda^+}| \leq \lambda$ and $\langle G_\alpha \rangle$ is increasing, it follows that there must be some $\alpha < \lambda^+$ such that $G_\alpha = G_{\alpha+1}$.

For the second part, first we show that τ_G is well defined. For this we note that if $G^\alpha \neq G^{\alpha+1}$ then $\tau_{G^{\alpha+1},G} = \alpha + 1$ (see conclusion 3.11). By claim 2.6, $\theta_A \geq \theta_{G^{\alpha+1}} > \alpha + 1$. Since θ_A is well defined, τ_G is well defined as well. Applying the same argument to G^{τ_G}, we see that $\theta_A \geq \theta_{G^{\tau_G}} > \tau_G$. □

So as promised, we proved Thomas' theorem in a different way, without choice:

Conclusion 3.14. (ZFC) Thomas' theorem: if G is a centerless group then $\tau_G < \left(2^{|G|}\right)^+$. Moreover, $\tau_\kappa < (2^\kappa)^+$.

Proof. Taking $A = G$, (G, A) is a special pair applying 3.12 and 3.13 we get the result regarding τ_G. Noting that $(2^\kappa)^+$ is regular and that there are at most 2^κ groups of order κ we are done. □

Now we deal with the case without choice.

Main Theorem 3.15. *(ZF) If (G, A) is a special pair and G is a centerless group, then $\tau_G < \theta_{\mathcal{P}(^{\omega>}A)}$.*

Proof. By claim 3.13, clause (2), we only need to show that $|G^\alpha| \leq |\mathcal{P}\left(^{\omega>}A\right)|$, but this is exactly claim 3.12, clause (2). □

Now we shall improve this by:

Main Theorem 3.16. *(ZF)* $\tau_{|k|} < \theta_{\mathcal{P}(^{\omega>}k)}$.

Proof. Recall that $\tau_{|k|} = \bigcup \{\tau_G + 1 \,|G$ is centerless and $|G| \leq |k|\}$, but we can replace this by $\tau_{|k|} = \bigcup \{\tau_G + 1 \,|G \in \mathcal{G}\}$ where
$\mathcal{G} = \{G \,|G$ is centerless and $G \subseteq k\}$. By the previous theorem (3.15) we know that $\tau_{|k|} \leq \theta^{\mathbf{V}'}_{\mathcal{P}(^{\omega>}k)}$, (for all $G \in \mathcal{G}$, (G, G) is a special pair, so $\tau_G < \theta_{\mathcal{P}(^{\omega>}G)} \leq \theta_{\mathcal{P}(^{\omega>}k)}$) but we want more.

We may assume WLOG that $\tau_{|k|} > \omega$, since $\theta_{\mathcal{P}(^{\omega>}k)} > \omega$ (see remark 2.5). Let $\mathcal{G}' = \{G \in \mathcal{G} \,|\tau_G \text{ is infinite}\}$, so $\tau_\kappa = \bigcup \{\tau_G + 1 \,|G \in \mathcal{G}'\}$.

For each $G \in \mathcal{G}'$ we define a function $R_G : \mathcal{P}\left({}^{\omega>}k\right) \to \tau_G + 1$ which is onto: first we define a function from $\mathcal{P}\left({}^{\omega>}k\right)$ onto G^{τ_G} (using claim 3.12) , then from G^{τ_G} onto τ_G (using claim 2.6, and claim 3.11), and then from τ_G onto $\tau_G + 1$ (remember that $\tau_G \geq \omega$).

Let $\mathcal{B} = \{(x, G) \,|G \in \mathcal{G}', x \in \mathcal{P}\left({}^{\omega>}k\right)\}$. Define a function $R_1 : \mathcal{B} \to \tau_\kappa$ by $R\left((x, G)\right) = R_G\left(x\right)$ (Note - since R_G is definable, there is no use of AC). By definition, R_1 is onto. Now it is enough to find an onto function $R_2 : \mathcal{P}\left({}^{\omega>}k\right) \to \mathcal{B}$. But there is an injective function from $\mathcal{B}$ to $\mathcal{P}\left({}^{\omega>}k\right)$: G is a triple of nonempty subsets of k, so it is enough to know how to encode pairs (a, b) where $\emptyset \neq a \subseteq k$ and $b \subseteq {}^{\omega>}k$ as a subset $c \subseteq {}^{\omega>}k$. For instance let $c = \{x\hat{\ }\bar{y} \,|x \in a, \bar{y} \in b\}$. ∎

Using the following absoluteness lemma:

Lemma 3.17. *.Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive subclass, which is a model of ZF. Let (G, A) be a special pair, and suppose $G, \mathcal{P}\left({}^{\omega>}A\right) \in \mathbf{V}'$. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\left\langle G^\beta \,|\beta < \delta\right\rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e.*
$\mathbf{V} \models$ "$\left\langle G^\beta \,|\beta < \delta\right\rangle$ is the automorphism tower up to δ").

Which we shall prove in the appendix, we can finally deduce:

Theorem 3.18.

(1) Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive subclass, which is a model of ZF. if $\mathcal{P}\left({}^{\omega>}k\right) \in \mathbf{V}'$, then $\left(\tau_{|k|}\right)^{\mathbf{V}} = \left(\tau_{|k|}\right)^{\mathbf{V}'} < \theta^{\mathbf{V}'}_{\mathcal{P}(^{\omega>}k)}$.
(2) If κ is a cardinal number in $\mathbf{V}'$ such that $\mathcal{P}\left(\kappa\right) \in \mathbf{V}'$, then $\left(\tau_\kappa\right)^{\mathbf{V}} = \left(\tau_\kappa\right)^{\mathbf{V}'} < \theta^{\mathbf{V}'}_{\mathcal{P}(\kappa)}$.
(3) In particular, $\tau_{\aleph_0} < \theta^{L[\mathbb{R}]}_{\mathbb{R}}$.

Proof. Obviously, we need only to see (1). Let
$\mathcal{G} = \{G \,|G \text{ is a centerless group and } G \subseteq k\}$. By the assumption on k, it is easy to see that $\mathcal{G}^{\mathbf{V}} = \mathcal{G}^{\mathbf{V}'}$. Hence
$\tau^{\mathbf{V}}_{|k|} = \bigcup \left\{\tau_G + 1 \,\middle|G \in \mathcal{G}^{\mathbf{V}}\right\} = \bigcup \left\{\tau_G + 1 \,\middle|G \in \mathcal{G}^{\mathbf{V}'}\right\} = \tau^{\mathbf{V}'}_{|k|}$ (the second equality is lemma 3.17). By theorem 3.16, we have $\tau^{\mathbf{V}'}_{|k|} < \theta^{\mathbf{V}'}_{\mathcal{P}(^{\omega>}k)}$. ∎

If we apply lemma 1.8 from [1], which says that $\tau_{\kappa}^{nlg} \leq \tau_{\kappa}$ and get:

Main Conclusion 3.19. Let $\mathbf{V}' \subseteq \mathbf{V}$ be as before (but now assume $\mathbf{V} \models ZFC$) .If $\mathcal{P}\left({}^{\omega>}k\right) \in V'$, then $\tau_{|k|}^{nlg} \leq \tau_{|k|} < \theta_{\mathcal{P}({}^{\omega>}k)}^{V'}$.

Note 3.20. We actually don't need to assume that $\mathbf{V}$ is a model of ZFC. $\tau_{\kappa}^{nlg} \leq \tau_{\kappa}$ is true even without choice, and this subject will be addressed in a later work.

4. The descriptive set theoretic Result

In this short section we give a descriptive set theoretic approach into finding a bound on $\tau_{\aleph_0}$. We start with the definition.

Definition 4.1. Let M be structure.

(1) For a formula $\varphi(x, X)$ - a first order formula in the language of M, where x is a single variable and X is a mondaic variable (i.e. serve as a unary predicate - vary on subset of the structure, so not quantified inside the formula) - we define a sequence $\langle X_{\alpha}^{\varphi} \subseteq M \,|\alpha \in \mathbf{Ord} \rangle$ by:
   - $X_0^{\varphi} = \emptyset$.
   - $X_{\alpha+1}^{\varphi} = X_{\alpha} \cup \{x \in M \,|\varphi(x, X_{\alpha}^{\varphi}) \text{ is satisfied in } M\}$.
   - $X_{\delta}^{\varphi} = \bigcup \left\{X_{\beta}^{\varphi} \,|\beta < \delta\right\}$ for δ limit.

(2) For such a formula φ, let $\delta_{\varphi} = min\left\{\alpha \,\middle|X_{\alpha}^{\varphi} = X_{\alpha+1}^{\varphi}\right\}$.

(3) Let $\delta = \delta(M)$ - the inductive ordinal of the structure - be the first ordinal such that for any such formula (allowing members of M as parameters) φ, $\delta_{\varphi} < \delta$.

Theorem 4.2. *For a centerless group G with set of elements $\subseteq \omega$ the height of its automorphism tower is smaller then the inductive ordinal of the structure $\mathfrak{A}$ with universe $\omega \cup \mathcal{P}(\omega)$ the operations of $\mathbb{N}$, membership, and G (i.e. its product) .*

Note 4.3. In this version of the theorem we do not need to use parameters in definition 4.1. However the theorem holds even without assuming that the structure contains G, but then we need parameters (G can be encoded as a subset of ω). In that case this is second order number theory.

Proof. (sketch) By the definition it is enough to find a formula Δ such that X_{α}^{Δ} encodes G^{α} (including it's multiplication and inverse). By (G, G) being special, we know that we can identify members of G^{α} (G^{α} is in the automorphism tower) as sets of finite sequences of ω (see the proof of claim 3.12). It is well known that the operations of $\mathbb{N}$ allow us to encode finite sequences. Hence, much like the proof of lemma 3.17, we can find a formula $\Delta'\left(x, X_{\alpha}^{\Delta}\right)$, as in definition 4.1, such that x satisfies it in $\mathfrak{A}$ iff x encodes a quantifier free type of an element in $G^{\alpha+1}$. Using a similar technique we can find a formula $\Delta''\left(x, y, X_{\alpha}^{\Delta}\right)$ such that x, y satisfy it iff $x, y \in G^{\alpha+1}$ and $x \circ y = id$ (i.e. the automorphism they encode). Likewise, let $\Delta'''\left(x, y, z, X_{\alpha}^{\Delta}\right)$ say that $x \circ y = z$. Now we can define $\Delta\left(x, X_{\alpha}^{\Delta}\right)$ to say that x encodes a triple (a, b, c) where $a \in G^{\alpha+1}$, b encodes a pair (d, d^{-1}) where $d \in G^{\alpha+1}$ and c encodes a triple $(e, f, e \circ f)$ where e and f are from $G^{\alpha+1}$. Now we have successfully encoded $G^{\alpha+1}$ as required. □

5. Back To Choice

Applying the proof of Thomas (which used Fodor's lemma), from [2], we can reduce the bound on τ_G for some groups. The main theorem we shall prove is:

Theorem 5.1. *(ZFC) Let G be a centerless group and $A \subseteq G$. If for all ordinals α, $C_{G^{\alpha}}(A) = \{e\}$ then $|G^{\alpha}| \leq \left(|G|^{|A|} + \aleph_0\right)$.*

Using it and claim 3.13, clause (1), we have

Conclusion 5.2. If (G, A) are as in the theorem, $\tau_G < \left(|G|^{|A|} + \aleph_0\right)^{+}$.

We know that if (G, A) is special A and G satisfy the conditions of the theorem (see 3.10). Hence in particular we have:

Conclusion 5.3. If G is finitely generated, then $\tau_G < \aleph_1$.

However, we can weaken the definition of a special pair so that more pairs (G, A) will be weakly special. So we shall start with:

Definition 5.4.

(1) For a centerless group G, and subgroups H_1, H_2, we say that a homomorphism (really a monomorphism) $\varphi : H_1 \to H_2$ is good if there is an automorphism $\psi : G^{\tau_G} \to G^{\tau_G}$ (so actually an inner automorphism) such that $\varphi = \psi \restriction H_1$.
(2) If $A \subseteq G$, let $E^k_{G,A}$ be an equivalence relation on G defined by: $xE^k_{G,A}y$ iff there is a good homomorphism taking x to y and fixing A.
(3) We say that the pair (G, A) is weakly special if $E^k_{G,A}$ is $\{(x,x)\,|x \in G\}$.

Remark 5.5. If $xE^k_{G,A}y$ then also $xE_{G,A}y$ but not necessarily the other direction, and so if (G, A) is special, it is also weakly special (so the name is justified)

Claim 5.6. If G_1 is centerless, $G_2 = Aut\,(G_1)$, and (G_1, A) is weakly special, then so is (G_2, A).

Proof. The proof is identical to the proof of 3.8, since conjugation is a good homomorphism, $G_1 \trianglelefteq G_2$ and $G_1^{\tau_{G_1}} = G_2^{\tau_{G_2}}$. □

And much like conclusion 3.10 we have:

Conclusion 5.7. If (G, A) is (weakly) special then so is (G^α, A) for every ordinal α, and $C_{G^\alpha}(A) = \{e\}$.

After giving the definition, let us prove theorem 5.1:

Proof. Similar to the proof in [2].
Denote $\lambda = \left(|G|^{|A|} + \aleph_0\right)$. The proof is by induction on α. For $\alpha = 0$ its clear.
Assume $\alpha = \beta + 1$. $G^\alpha = Aut\left(G^\beta\right)$ but every $\varphi \in G^\alpha$ is determined by $\varphi \restriction A$ (because inside G^α applying φ is the same as conjugating by it, and because of the hypothesis). This means that $|G^\alpha| \leq \left|{}^{A}\left(G^\beta\right)\right| \leq \left(|G|^{|A|} + \aleph_0\right)^{|A|} = \lambda$.
Assume that α is a limit ordinal. If $\alpha < \lambda^+$ it is clear, so suppose that $\alpha \geq \lambda^+$. Assume that $|G^\alpha| > \lambda$. Denote $S = \{\beta < \lambda^+\,|cf\,(\beta) = cf\,(\lambda)\}$. S is a stationary subset of λ^+. Choose $B = \langle h_\beta\,|\beta \in S\rangle$ without repetitions such that $h_\beta \in G^{\beta+1} \backslash G^\beta$ (possible because each G^β before α is small by the induction hypothesis). Denote the conjugation of A by h_β for $\beta \in S$ by A_β. We know that $A_\beta \subseteq G^\beta = \bigcup\{G^\gamma\,|\gamma < \beta\}$ (β is a limit ordinal), so by the

definition of S, and the fact that $cf(\lambda) > |A| = |A_\beta|$, there is $\gamma < \beta$ such that $A_\beta \subseteq G^\gamma$. This defines a regressive function $f : S \to \lambda^+$ by $f(\beta) = \gamma$. By Fodor's lemma, there is a subset $S' \subseteq S$ which is stationary (hence $|S'| = \lambda^+$) such that $f \upharpoonright S'$ is constant. So there is a γ such that for every $\beta \in S'$, $A_\beta \subseteq G^\gamma$. But for every $\beta \in S'$, h_β is determined by $h_\beta \upharpoonright A \in {}^A(G^\gamma)$ so $\lambda^+ = |S'| \leq \left|{}^A(G^\gamma)\right| = \lambda$ - a contradiction. $\square$

6. Appendix

Here we shall prove the absoluteness lemma (lemma 3.17).

Lemma 6.1. *.Let $\mathbf{V}' \subseteq \mathbf{V}$ a transitive subclass, which is a model of ZF. Let (G, A) be a special pair, and suppose $G, \mathcal{P}({}^{\omega>}A) \in \mathbf{V}'$. Then, for every ordinal $\delta \in \mathbf{V}'$, the automorphism tower $\left\langle G^\beta \,|\beta < \delta \right\rangle$ in $\mathbf{V}'$ is the same in $\mathbf{V}$ (i.e.*
$\mathbf{V} \models$ "$\left\langle G^\beta \,|\beta < \delta \right\rangle$ is the automorphism tower up to δ").

Proof. Let $\mathfrak{T} = \left\langle G^\beta \,\middle|\, \beta \in \mathbf{ord}^{\mathbf{V}'} \right\rangle$. We shall prove by induction on $\alpha < \delta$ that $\mathfrak{T} \upharpoonright \alpha + 1$ is the automorphism tower in $\mathbf{V}$ up to $\alpha + 1$.

For $\alpha = 0$ this is clear since $G \in \mathbf{V}'$.

For α limit this follows from the definitions.

Suppose $\alpha = \beta + 1$. By the induction hypothesis $\mathfrak{T} \upharpoonright \alpha$ is the automorphism tower in $\mathbf{V}$, so $\left(G^\beta\right)^{\mathbf{V}} = \left(G^\beta\right)^{\mathbf{V}'} \in \mathbf{V}'$. For every $\rho \in Aut\left(G^\beta\right) = G^{\alpha+1}$ in $\mathbf{V}$, we need to show that $\rho \in (G^\alpha)^{\mathbf{V}'}$.

WLOG A is a subgroup of G - if not, replace it with $\langle A \rangle_G$ (we can define a function from $A^{<\omega}$ onto $\langle A \rangle_G^{<\omega}$ as in claim 3.12). Let $\mathbf{A} = A * \langle x \rangle$ i.e. the free product of A and the infinite cyclic group. As in 3.12 there is an absolute definable function from $A^{<\omega}$ onto $\mathbf{A}$, so $\mathcal{P}(\mathbf{A}) \in \mathbf{V}'$. Let $\mathbf{B} = A * \langle x, y \rangle$, and by the same reasoning $\mathcal{P}(\mathbf{B}) \in \mathbf{V}'$.

For every $g \in G^\alpha$, there is an homomorphism φ_g from $\mathbf{A}$ onto $\langle A \cup \{g\} \rangle_{G^\alpha}$ defined by $x \mapsto g$, and fixing A. By 3.10 ((G^α, A) is special), $g \mapsto ker(\varphi_g)$ is injective, and absolutely definable ($ker(\varphi_g)$ is basically just $tp_{qf}(g, A, G^\alpha)$). Note that by the induction hypothesis, $\varphi_g^{\mathbf{V}} = \varphi_g^{\mathbf{V}'}$ for $g \in G^\beta$. Similarly, for $g, h \in G^\alpha$, there is an homomorphism $\varphi_{g,h}$ from $\mathbf{B}$ onto $\langle A \cup \{g, h\} \rangle_{G^\alpha}$ fixing A and taking x to g and y to h, and $(g, h) \mapsto ker(\varphi_{g,h})$ is injective.

The following definition allows to interpret the type of g in the type of some (h_1, h_2) (see example below):

Definition 6.2. Let $B \subseteq \mathbf{B}$

(1) For every $\sigma \in \mathbf{B}$, Let $\psi_\sigma : \mathbf{A} \to \mathbf{B}$ be the homomorphism defined by $x \mapsto \sigma$, $\psi_\sigma \restriction A = id$.

(2) For $g \in G^\beta$ we say that g is affiliated with B (denoted $g \propto B$) if there is a word $\sigma_g = \sigma(x, y, \overline{a}) \in \mathbf{B}$ ($\overline{a}$ are parameters from A) such that $ker(\varphi_g) = \psi_{\sigma_g}^{-1}(B)$.

Example 6.3. Let $\rho \in G^\alpha, h \in G^\beta$. If $B = ker(\varphi_{\rho,h})$ then for every $g \in G^\beta$, $g \propto B$ iff there exists σ_g such that $\varphi_{\rho,h}(\sigma_g) = g$ (i.e. $g \in \langle A \cup \{h, \rho\}\rangle_{G^\alpha} \cap G^\beta$). It could easily be verified that this is indeed true, using the equality $\varphi_{\varphi_{\rho,h}(\sigma)} = \varphi_{\rho,h} \circ \psi_\sigma$ for every $\sigma \in \mathbf{B}$, and 3.10.

We shall find an absolute first order formula $\Delta\left(H, \mathcal{P}\left({}^{\omega>}A\right), G^\beta\right)$ that will say "H is a normal subgroup of $\mathbf{A} = A * \langle x\rangle$ and there exists an automorphism $\rho \in Aut\left(G^\beta\right) = G^\alpha$ such that $H = ker(\varphi_\rho)$".

If we succeed then if $\rho \in (G^\alpha)^{\mathbf{V}}$ then $\Delta\left(ker(\varphi_\rho), \mathcal{P}\left({}^{\omega>}A\right), G^\beta\right)$ will hold. Since $ker(\varphi_\rho) \in \mathbf{V}'$, and α was absolute, there is some $\rho' \in (G^\alpha)^{\mathbf{V}'}$such that $ker(\varphi_\rho) = ker(\varphi_{\rho'})$ so $\rho = \rho'$ and we are done.

Let us describe Δ. It will say that H is a normal subgroup of $\mathbf{A}$ and that for each $h \in G^\beta$ there exists a subgroup $B = B_h \leq \mathbf{B}$ with the following properties:

(1) B is a normal subgroup of $\mathbf{B}$.
(2) $H \subseteq B$, and $B \cap \mathbf{A} = H$.
(3) For every $a \in A$, $a \propto B$ and $\sigma_a = a$ (it follows that $B \cap A = \{e\}$)
(4) $h \propto B$ and $\sigma_h = y$.
(5) If $g \propto B$ and σ_1 and σ_2 witness that, then $\sigma_1 \sigma_2^{-1} \in B$.
(6) If $g_1, g_2 \propto B$ then so is $g_1 g_2$ and $\sigma_{g_1 g_2} = \sigma_{g_1}\sigma_{g_2}$.
(7) If $g \propto B$ then there exists $g' \in G^\beta$ such that $g' \propto B$ and $x\sigma_g x^{-1} = \sigma_{g'}$.

$B = B_h$ induces a monomorphism ρ_B whose domain is $H_B = \left\{g \in G^\beta \left| g \propto B_h\right.\right\}$. It is a subgroup of G^β containing A and h (why? because of the conditions on B_h). Also, for every $g \in H_B$ define $\rho_B(g)$ to be the element $g' \in G^\beta$ as promised from property (7). In order to show that ρ_B is a well defined monomorphism, we note that for every $g_1, g_2 \in H$, if $\sigma_{g_1}\sigma_{g_2}^{-1} \in B$ then $g_1 = g_2$.

Why? Since B is normal, ψ_σ induces $\psi'_\sigma : \mathbf{A} \to \mathbf{B}/B$, and so the condition $g \propto B$ becomes $ker(\varphi_g) = ker\left(\psi'_{\sigma_g}\right)$. Now, if $\sigma_0 \in B$, then $\psi'_{\sigma_0\sigma} = \psi'_\sigma$ so $\psi'_{\sigma_{g_1}} = \psi'_{\sigma_{g_2}}$ hence $ker(\varphi_{g_1}) = ker(\varphi_{g_2})$.

Now it an easy exercise to see that ρ_B is as promised.

After defining ρ_B we demand that for every $h_1, h_2 \in G^\beta$ and suitable B_1 and B_2, ρ_{B_1} and ρ_{B_2} agree on their common domain. Thus we can define $\rho_H = \bigcup\left\{\rho_{B_h}\left|h \in G^\beta\right.\right\}$, and demand that ρ_H will be an automorphism (i.e. onto). Now all that is left is to say that $H = ker(\varphi_{\rho_H})$, and Δ is written.

(There is no problem with writing this in first order, since we can talk about finite sequences from G^β using $\mathcal{P}\left({}^{\omega>}A\right)$ so we can talk about $\mathbf{B}$,$\mathbf{A}$, etc).

Why is Δ correct? because if $\Delta(H, \ldots)$ is true, then $H = ker(\varphi_{\rho_H})$ by definition. On the other hand, if $H = ker(\varphi_\rho)$ for some ρ, then:

- For each h, $ker(\varphi_{\rho,h})$ will be a suitable B_h (by the example above).
- If $B_h$satisfy the condtions above, then $\rho_B \restriction A = \rho \restriction A$ because by condition (2) $ker\left(\varphi_{\rho_B(a)}\right) = \psi^{-1}_{xax^{-1}}(H) = ker\left(\varphi_{\rho(a)}\right)$. Hence, $\rho^{-1} \circ \rho_B \restriction A = id$ and by 3.10, $\rho_B \restriction H_B = \rho \restriction H_B$.

In conclusion, the demands on H are satisfied, and we are done. □

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