Representations of finite Morley rank

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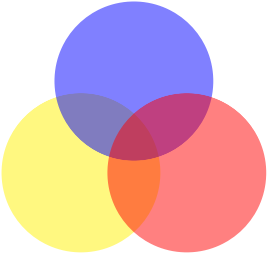
In this talk:

Algebra, geometry, and logic

- Representation theory studies some interactions between algebra and geometry.
  Main idea: "linearize", eg. $\rho : G \to \mathrm{GL}(V)$.
- Model theory analyses structures through their definable sets.
- Model theory has applications to algebra ("model-theoretic algebra"),
- and to geometry ("geometric model theory").



I shall try to say something about group representations in model theory.

Definable sets

The context is that of model theory:

- we consider a structure $\mathcal{M}$ in some language $\mathcal{L}$.
  Eg. a group G, in the language of groups $\{1, \cdot, ^{-1}\}$.
- we look at *definable* subsets $\subseteq M^n$, that is subsets for which one could write a "first-order" definition.
  - the centraliser $C_G(g)$ is defined by $x \cdot g = g \cdot x$.
  - If $H \leq G$ is definable, so is its normaliser $N_G(H)$.
  - If $H \leq G$ is definable, so is the quotient space G/H.
    (In more precise terms, "interpretable".)
  - In $\mathrm{GL}_2(\mathbb{C})$, the subgroup $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$ is definable...
  - but $\mathrm{GL}_2(\mathbb{R})$ is not.

Finite Morley rank

Let $\mathcal{D}$ be the collection of all definable subsets of $\mathcal{M}, \dots, \mathcal{M}^n, \dots$
$\mathcal{M}$ has *finite Morley rank* if there is a function $\operatorname{rk} : \mathcal{D} \to \mathbb{N}$ satisfying "expected" properties of a dimension.

Example

$\mathrm{GL}_2(\mathbb{C})$ has finite Morley rank (in $\mathcal{L} = \{1, \cdot, ^{-1}\}$) and $\operatorname{rk} \mathrm{GL}_2(\mathbb{C}) = 4$.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\}$$

Note for model theorists

In general $\mathcal{M}$ can be nasty. But *if $\mathcal{M}$ is a group*, everything goes well ("ranked grps = grps of finite MR").

Morley, Zariski, Zilber

- If $\mathbb{K}$ is an alg. closed field, the group $(\mathrm{GL}_n(\mathbb{K}), \cdot)$ has fMR.
- An "algebraic group" (matrix group defined by polynomials) has fMR. (Geometers call its rank the *Zariski dimension*.)
- Zilber's original conjecture: if $\mathcal{M}$ has fMR, then it can be analysed and "interesting" bits come from alg. geometry. It's an "atomistic" view of fMR structures.
- Key example: in $\mathrm{GL}_2(\mathbb{C})$ one can recover the field $(\mathbb{C}, +, \cdot)$.
- Original conjecture is not quite true (Hrushovski, Hrushovski-Zilber).

Sum up

- Structures are studied through their definable sets.
- Sometimes definable sets have a dimension (the Morley rank).
- Eg.: a matrix group on an alg. closed field $\mathbb{K}$.
- The dogma: where there's a dimension there's a geometry.
- If G is a fMR group, one can often define an alg. closed field in G (Zilber; see later).
- This is an indication that (despite Hrushovski) Zilber's dogma remains likely for groups.

The Cherlin-Zilber Conjecture

Idea: rk function should make G an object of alg. geometry.

Conjecture (Cherlin-Zilber)

A simple infinite group of finite Morley rank is the $\mathbb{K}$-points of some alg. group ($\mathbb{K} \models \mathrm{ACF}$).

- Large amount of work.
- Conjecture is about "abstract groups" of finite MR.
- This talk does *not* relate to the conjecture.

"Concrete" fMR groups

- In nature groups do not "pop up" as abstract groups, they appear as permutation groups, i.e. groups with an action. Groups fMR are no exception.
- So one should study the general setting:

  G acts on Ω, everything definable in a fMR structure.
- MacPherson-Pillay: one can often reduce to the case where Ω is an abelian group.

fMR Modules

Definition

A *module of fMR* is a pair (G, A) where:

- G is a group
- A is a connected, *abelian* group
- G acts on A
- everything is definable in a fMR structure.
  (Or: the semi-direct product $G \ltimes A$ has fMR and G, A are definable.)

We'll always assume:

- G connected ($G = G^\circ$),
- G almost faithful ($C_G(A)$ is finite),
- A *irreducible*$^\circ$ aka *G-minimal*: A has no proper, non-trivial, G-submodules.

An aside: Abstract Linearity

Question

Which groups of finite MR are linear/definably linear?

(In a sense this amounts to being able to choose A.)

- Building on earlier work by Poizat (see later), Mustafin 2004 has studied definable subgroups of $\mathrm{GL}_n(\mathbb{K})$ for $\mathbb{K}$ a field fMR. But we still do not know everything we could want.
- Frécon has studied sections of abstract fMR G which can be linearized definably.
- His student Tindzogho Ntsiri has studied definable linearity of so-called K-groups.

Pure Linearity

Altınel and Wilson have been working on (*non-definable*) linearity.

Theorem (Altınel-Wilson, 2009)

Every torsion-free nilpotent group G of finite Morley rank has a faithful linear representation over a field of characteristic 0.

Theorem (consequence of Altınel-Wilson, 2011)

Every centerless, solvable group of finite Morley rank with torsion-free Fitting subgroup G has a faithful linear representation over a field of characteristic 0.

Of course the question remains for simple groups.
(Is it any simpler than Cherlin-Zilber?)

The question

Suppose one has a definable module of finite Morley rank $G \ltimes A$. (We assume G connected and A G-minimal.)

Question

- Is G definably linear on A? That is, is there a field structure $\mathbb{K}$ with $A \simeq \mathbb{K}_+^n$ and $G \hookrightarrow \mathrm{GL}(A)$ definably?
- Is the action algebraic? That is, is G Zariski-closed in $\mathrm{GL}(A)$?

We now review some classical results.

Groups on sets: Hrushovski's analysis

One may wish to embed the question into the study of permutation groups of finite MR.
That is, one may drop the assumption that A is an abelian group and do model theory.

Theorem (Hrushovski's analysis, 1989)

Let G be non-solvable and faithful on a strongly minimal set X ($\operatorname{rk} X = \deg X = 1$). Then there is a field $\mathbb{K}$ with $G \simeq \mathrm{PSL}_2(\mathbb{K})$ on $X = \mathbb{K} \cup \{\infty\}$.

The article is actually in a very model-theoretic vein.

Hrushovski's analysis, continued

Corollary

If connected G has a definable subgroup H of corank ($\mathrm{rk}\, G - \mathrm{rk}\, H$) *1 with* $\bigcap_{g \in G} H^g = 1$, *then* $G \simeq \mathrm{PSL}_2(\mathbb{K})$.

The corollary is actually a reconstruction of the Bruhat decomposition of PSL_2. It generalizes Cherlin's 1977 paper.

Remark (on the case where G acts on set X of rank 2)

- Gropp (1992) has studied it from a purely model-theoretic point of view.
- Wiscons (2013) is writing a different and more algebraic analysis.
- This is essentially harder to use than Hrushovski.

Linearization principles 1: Zilber's Field Theorem

Zilber's Field Theorem is a powerful linearization principle.

Theorem (Zilber's Field Theorem, 1977)

Suppose $G = A \rtimes T$ has finite Morley rank, where:

- *A and T are definable, abelian, connected, infinite*
- $C_T(A) = 0$
- *A is T-minimal, that is no non-trivial proper T-invariant definable subgroups*

Then there is a field structure $\mathbb{K}$ with $A \simeq \mathbb{K}_+$, $T \hookrightarrow \mathbb{K}^\times$.

This is *only local* as it requires abelian T! Different field structures obtained at different places may not match!

A digression: bad fields

Zilber's FT led to the question of so-called bad fields.

Definition

A bad field is a structure $(\mathbb{K}, 0, 1, +, \cdot, \ddot{U})$ of finite Morley rank consisting of a field and a proper infinite subgroup $\ddot{U} < \mathbb{K}^{\times}$.

- Have been proved to exist in char. 0 (Baudisch, Hils, Martin-Pizarro, Wagner; 2009).
- Still open in char p.

The possibility of bad fields has considerably complicated the inner analysis of abstract groups of finite Morley rank.

Linearization principles 2: Canada Dry

Theorem (Loveys, Wagner; 1993)

Suppose G is simple, and A is G-minimal and torsion-free. Then there is a field $\mathbb{K}$ such that $A \simeq \mathbb{K}_+^n$ and $G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$.

Remark

The proof is not valid for torsion A, meaning that there is no such thing in characteristic p.

This is *not* the end of the story in char. 0, as one still can ask about algebraicity.

Algebraization principles

Theorem (Poizat, 2001)

Let $\mathbb{K}$ be a field of fMR with char $p \neq 0$. Then every simple, def. subgrp of $\mathrm{GL}_n(\mathbb{K})$ is definably isomorphic to an alg. group over $\mathbb{K}$.

Theorem (Poizat, 2001 says it's mostly Macpherson-Pillay, 1995)

Let $\mathbb{K}$ be a field of fMR with char 0, and G a simple, definable subgroup of $GL_n(\mathbb{K})$ not Zariski closed. Then all elements of G are semi-simple, and solvable subgroups are abelian-by-finite.

Theorem (Borovik-Burdges, 2008 preprint)

Same setting. G has no involutions.

A disaster - could there be a linear bad group?

Summary

Question

Let $A \rtimes G$ have finite Morley rank where A is abelian, irreducible°. Then what?

Toolbox

- Hrushovski's analysis deals with *strongly minimal* set A.
- Zilbers' FT is *purely local*.
- Loveys'-Wagner's Canada Dry linearizes *in char. 0*.
- Poizat's *Quelques modestes remarques* can algebraize linear $G \leq \mathrm{GL}_n(\mathbb{K})$ *in char. p*.

The rest does not quite work for our purposes.

Two directions

The previous slide and the possibility of (linear!) bad groups suggests that one should not be too ambitious.
There are mostly two restrictions one could make:

1. Restrictions on the module, assuming rk A is controlled; or on the action, assuming G is sufficiently transitive.
   Then do *simultaneous* identification: of G, and of the action.
2. Restrictions on G, assuming it is already known, say algebraic.
   Then prove A is a representation of G.

Restrictions on A and simultaneous identification

Theorem (2009)

If $\text{rk } A = 2$ *and* G *not solvable, then there is a field structure* $\mathbb{K}$ *with* $A \simeq \mathbb{K}_+^2$ *and* $G \simeq \text{SL}_2(\mathbb{K})$ *or* $\text{GL}_2(\mathbb{K})$ *natural on* A.

Question

What if $\text{rk } A = 3$?

This is at work with Borovik. Partial results for the moment.

- If G is $2^\perp$, bad groups appear (if they exist). Perhaps even a linear one.
- If $S^\circ \simeq \mathbb{Z}_{2^\infty}$, $G \simeq \text{PSL}_2$ in its "adjoint action" on $A \simeq \mathbb{K}_+^3$.
- in bigger Prüfer rank, we don't know yet (Conjecturally $G \simeq \text{SL}_3(\mathbb{K})$ or $\text{GL}_3(\mathbb{K})$ natural.)

Pseudoreflection groups

Berkman and Borovik assume that the "local" behaviour of G on A is excellent.

Theorem (Berkman-Borovik, 2012)

Let $G \ltimes A$ be a faithful, irreducible module of fMR. Suppose $\text{rk}\, V = n$ and G contains a copy of $\mathbb{Z}_{2^\infty}^n$ (that is, $\mathrm{Pr}_2(\mathrm{G}) \geq \text{rk}\, \mathrm{V}$). Then there is a field $\mathbb{K}$ with $A \simeq \mathbb{K}_+^n$ and $G \simeq \mathrm{GL}_n(\mathbb{K})$ natural.

This is an important step towards highly transitive modules (Berkman-Borovik, at work).

An "algebraic rigidity" conjecture

In order to avoid a discussion of Cherlin-Zilber, we now strongly restrict the structure of G by assuming *it is already algebraic*.

Question

Let $G = \mathbb{G}_{\mathbb{K}}$ be the $\mathbb{K}$-points of a quasi-simple alg. group $\mathbb{G}$, viewed as an abstract group of finite MR. Let A be a definable G-module.

1. Does A bear a $\mathbb{K}$-vector space structure making G linear?
2. Are there a $\mathbb{K}$-rational representation V of G and definable automorphisms φ_i of $\mathbb{K}$ with $A \simeq V^{\varphi_1} \otimes \cdots \otimes V^{\varphi_n}$?

In other words: how much bigger is the category of *fMR* reps of G compared to that of its *algebraic* reps?
In other words: is there a fMR Steinberg tensor theorem?

Evidence for the conjecture

Conj. Let $G = \mathbb{G}_{\mathbb{K}}$ be the $\mathbb{K}$-points of a simple alg. group and A be a definable G-module. Then A is an alg. representation, modulo definable automorphisms.

1. The Borel-Tits Theorem, which indicates that group-theoretic morphisms tend to be algebraic.
2. Corollary to Loveys-Wagner (+BT): conjecture holds in car. 0.
3. The following theorem:

Theorem (Poizat, 2001)

Let $\mathbb{K}$ be a field of fMR of char $p \neq 0$. Then every simple, definable subgroup of $\mathrm{GL}_n(\mathbb{K})$ is definable in the language of fields expanded by a finite number of definable automorphisms.

Main difficulty: no Canada Dry in car. p, so linearizing in the first place could be as expensive as directly proving algebraicity.

fMR representations of algebraic groups

Theorem (Cherlin-D, 2012)

Let $G = (\mathrm{P})\,\mathrm{SL}_2(\mathbb{K})$ and A have rank $\leq 3\,\mathrm{rk}\,\mathbb{K}$. Then ($A$ is a $\mathbb{K}$-vector space and) either $A \simeq \mathbb{K}^2$ natural, or $A \simeq \mathbb{K}^3$ "adjoint".

Theorem (Tindzogho Ntsiri, 2013)

Let $G = (\mathrm{P})\,\mathrm{SL}_2(\mathbb{K})$ and suppose $C_A(T) = 0$ (T the torus). Then A is a direct sum of T-minimal submodules of rank $\mathrm{rk}\,\mathbb{K}$.

A first step towards a weight space decomposition?

Hopeful Corollary; OK for $\mathrm{rk}\,\mathbb{K} = 1$

Let $G = (\mathrm{P})\,\mathrm{SL}_2(\mathbb{K})$ with $\mathrm{rk}\,A \leq 4\,\mathrm{rk}\,\mathbb{K}$. Then ($A$ is a $\mathbb{K}$-vector space and) either $A \simeq \mathbb{K}^4$ rational, or $A \simeq \mathbb{K}^2 \otimes (\mathbb{K}^2)^\varphi$ for some def. automorphism φ of $\mathbb{K}$.

A final request to model-theorists

Problem

What's a nice model-theoretic setting enabling Lie correspondence (group↔Lie algebra)? Is it any helpful?

That's all! My apologies for being so technical!