Automorphism groups of countable structures I

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Istanbul, October 2014.

Overview

THEME: Suppose M is a countable first-order structure with a 'rich' automorphism group $\mathrm{Aut}(M)$. Study $\mathrm{Aut}(M)$ as a group and as a topological group.

Involves a mixture of model theory, group theory, combinatorics, basic topology and descriptive set theory.

Rich: homogeneous structures such as the random graph or the rational numbers as an ordered set; ω-categorical structures; the free group of rank ω, ...

Lecture 1

- Background
- The topology of the symmetric group
- Automorphism groups
- Baire category arguments
- Homogeneous structures; amalgamation classes
- Fraïssé's theorem and generalisations.

1.1 Notation and Basics: permutation groups

G is a group acting on a set X and $a \in X$.

- the G-orbit which contains a is $\{ga : g \in G\} \subseteq X$.
- If there is a unique G-orbit on X we say that G is transitive on X.
- $G_a = \{g \in G : ga = a\}$ is the stabilizer of a in G.
- There is a canonical bijection, respecting the G-action, betweeen the set of left cosets of G_a in G and the G-orbit containing a, given by
$$gG_a \mapsto ga.$$
- In particular, the index of G_a in G is the cardinality of the G-orbit which contains a. (*Orbit- Stabilizer Theorem*.)
- Also consider G acting on X^n (for $n \in \mathbb{N}$) or the power set $\mathcal{P}(X)$.
- If $A \subseteq X$ the pointwise stabilizer of A in G is $G_{(A)} = \{g \in G : ga = a \; \forall a \in A\}$.

Exercise: If X is countable and A is a finite subset of X, then $G_{(A)}$ is a subgroup of countable index in G.

Notation and basics: model theory

- L: first-order language (usually countable).
- We do not distinguish between an L-structure M and its domain.
- If M is an L-structure then $\mathrm{Aut}(M)$ is the automorphism group of M.

DEFINITION: Say that a countably infinite L-structure M is ω*-categorical* if it is determined up to isomorphism amongst countable L-structures by its theory $Th(M)$.

RYLL-NARDZEWSKI THEOREM For a countably infinite L-structure M TFAE: (1) M is ω-categorical;
(2) $\mathrm{Aut}(M)$ has finitely many orbits on M^n for all $n \in \mathbb{N}$.

- The orbits are $\emptyset$-definable sets
- Say that $G \leq \mathrm{Sym}(X)$ is *oligomorphic* if it has finitely many orbits on X^n $\forall n$.

1.2 The topology of $\mathrm{Sym}(X)$

Regard the symmetric group $\mathrm{Sym}(X)$ as a topological group: open sets are unions of cosets of pointwise stabilizers of finite sets.

If $G \leq \mathrm{Sym}(X)$ we give this the relative topology. So the basic open sets in $G \leq \mathrm{Sym}(X)$ are of the form $gG_{(A)}$ for $A \subseteq_{fin} X$ and $g \in G$. Note here that

$$G_{(A)} = \{h \in G : ha = a \ \forall a \in A\}$$

so

$$gG_{(A)} = \{h \in G : h|A = g|A\}.$$

- Each basic open set is also closed. So G is *totally disconnected*.
- If X is countable, there are countably many of these basic open sets (each is determined by a map between finite subsets of X): so G is *second countable*.
- In particular, if X is countable then G is *separable*.

Closed subgroups

LEMMA: Suppose $G \leq \mathrm{Sym}(X)$. Then the closure of G in $\mathrm{Sym}(X)$ is

$$\bar{G} = \{g \in \mathrm{Sym}(X) : gY = Y \text{ for all } G\text{-orbits } Y \text{ on } X^n, \ \forall n\}.$$

Proof.

- Show that if $Y \subseteq X^n$ then $\{g \in \mathrm{Sym}(X) : gY = Y\}$ is closed.
- So $\{g \in \mathrm{Sym}(X) : gY = Y \text{ for all } G\text{-orbits } Y \text{ on } X^n, \ \forall n\}$ is closed and clearly it contains G. So it contains $\bar{G}$.
- Suppose $g \in \mathrm{Sym}(X)$ preserves the G-orbits on X^n for all n. An open neighbourhood O of g is specified by $g|\bar{y}$ for some finite tuple $\bar{y}$. As $g\bar{y}$ is in the same G-orbit as $\bar{y}$ there is $h \in G$ with $g\bar{y} = h\bar{y}$. Thus $h \in O$. This shows that $g \in \bar{G}$.

□

Closed subgroups (2)

COROLLARY: A subgroup G of $\mathrm{Sym}(X)$ is closed iff G is the automorphism group of some first-order structure on X.

Proof.

A first-order structure on X is specified by relations and functions on X. So the automorphsim group is the intersection of the setwise stabilisers of certain subsets of M^n for various n. This is a closed subgroup.
Conversely, if $G \leq \mathrm{Sym}(X)$ consider the structure on X which has a relation for each G-orbit on X^n, for each finite n. The automorphism group of this structure is $\bar{G}$. So if G is closed, the automorphism group is G. □

REMARK: The structure on X with relations the G-orbits on X^n is called the *canonical structure* for G on X. If G is oligomorphic this is an ω-categorical structure.
EXERCISE: Suppose $G \leq \mathrm{Sym}(X)$. Then G is compact iff G is closed in $\mathrm{Sym}(X)$ and all G-orbits on X are finite.

Metrizability

If X is countable (say $X = \mathbb{N}$), the topology on $\mathrm{Sym}(X)$ is separable and complete metrizable.

Consider d given by, for $g_1 \neq g_2$,

$$d(g_1, g_2) = 1/n \text{ where } n \text{ is as small as possible with } g_1 n \neq g_2 n.$$

This is a metric for the topology, but it is not complete. To obtain a complete metric, consider

$$d'(g_1, g_2) = d(g_1, g_2) + d(g_1^{-1}, g_2^{-1}).$$

This is a complete metric for the topology. So if X is countable, then any closed $G \leq \mathrm{Sym}(X)$ is a *Polish group* (a topological group which is separable and complete metrizable).

1.3 Using the topology

Let $S_\infty = \mathrm{Sym}(\mathbb{N})$. Note that $|S_\infty| = 2^{\aleph_0}$.

THEOREM: Suppose $G \leq S_\infty$ is closed. Then either $|G| = 2^{\aleph_0}$ or there exists a finite $Y \subseteq \mathbb{N}$ with $G_{(Y)} = 1$.

Proof.

Consider the isolated points in G.
As G is a topological group, either all points are isolated or no points are isolated (i.e. G is *perfect*).
In the first case, the identity element is isolated so there is a basic open set contained in $\{1\}$; the only way this can happen is if $G_{(Y)} = 1$ for some finite Y.
In the second case, G is a non-empty perfect complete space, so contains a copy of the Cantor set. In particular $|G| = 2^{\aleph_0}$. $\square$

Baire Category

DEFINITIONS: Suppose W is a topological space.

- $Z \subseteq W$ is *nowhere dense* if its closure $\bar{Z}$ contains no non-empty open subset of W. Equivalently, $W \setminus \bar{Z}$ is dense in W.
- $Y \subseteq W$ is *meagre* if it is a countable union of nowhere dense sets.
- $X \subseteq W$ is *comeagre* if its complement is meagre. So this means that X contains the intersection of a countable family of dense open sets.

REMARKS: A countable union of meagre sets is meagre and the meagre subsets of W form a σ-ideal in the algebra of subsets of W; so we may think of them as 'small' subsets of W.

THEOREM: (Baire Category Theorem) Suppose W is a complete metrizable space. Then every comeagre subset of W is dense in W. Equivalently, the intersection of any countable family of dense open subsets of W is dense in W.

An application

COROLLARY: Suppose $G \leq S_\infty$ is closed and H is a closed subgroup of G. If $|G : H| \leq \aleph_0$ then H is open in G, that is, $H \geq G_{(A)}$ for some finite set A.

Proof.

Suppose H does not contain $G_{(A)}$ for any finite A.
So the complement of H is dense; therefore it is a dense open set.
The same is true for each coset of H.
If there are only countably many cosets their complements form a countable family of dense open subsets of G with empty intersection.
This contradicts BCT. □

2.1 Homogeneous structures

DEFINITION: An L-structure M is *homogeneous* if isomorphisms beteween finitely generated substructures extend to automorphisms of M.

That is: if $A_1, A_2 \subseteq M$ are f.g. substructures and $f : A_1 \to A_2$ is an isomorphism, then there exists $g \in \mathrm{Aut}(M)$ such that $g|A_1 = f$.

REMARKS:

1. (Warning) Suppose M is any structure. Let M^+ be the canonical structure for $\mathrm{Aut}(M)$ acting on M. Then M^+ is homogeneous and has automorphism group $\mathrm{Aut}(M)$.
2. If L is a finite relational language, then there are only finitely many isomorphism types of L-structure of any finite size. So if M is a homogeneous L-structure, then $\mathrm{Aut}(M)$ is oligomorphic on M.
3. Let L consist of a single 2-ary relation symbol and consider the L-structure $M = (\mathbb{Q}; \leq)$, the rationals with their usual ordering. This is a homogeneous L-structure (use piecewise linear automorphisms).

Amalgamation classes

DEFINITION: A non-empty class $\mathcal{A}$ of finitely generated L-structures is a (Fraïssé) *amalgamation class* if:

1. (IP) $\mathcal{A}$ is closed under isomorphisms;
2. (Hereditary Property, HP) $\mathcal{A}$ is closed under f.g. substructures;
3. (Joint Embedding Property, JEP) if $A_1, A_2 \in \mathcal{A}$ there is $C \in \mathcal{A}$ and embeddings $f_i : A_i \to C$ $(i = 1, 2)$;
4. (Amalgamation Property, AP) if $A_0, A_1, A_2 \in \mathcal{A}$ and $f_i : A_0 \to A_i$ are embeddings, there is $B \in \mathcal{A}$ and embeddings $g_i : A_i \to B$ with $g_1 \circ f_1 = g_2 \circ f_2$.

REMARKS:

1. If $\emptyset \in \mathcal{A}$ then JEP follows from AP.
2. Example: The class $\mathcal{A}$ of all finite graphs is an amalgamation class (where $L = \{R\}$). For AP, regard f_1, f_2 as inclusions and let B be the disjoint union of A_1 and A_2 over A_0 with edges $R^{A_1} \cup R^{A_2}$. Take g_1, g_2 to be the natural inclusions.
   We refer to B as the *free amalgam* of A_1, A_2 over A_0.

Fraïssé's Theorem

DEFINITION Suppose M is an L-structure. The *age* of M, $\mathrm{Age}(M)$ is the class of structures isomorphic to some f.g. substructure of M.

THEOREM: (Fraïssé's Theorem)

1. If M is a homogeneous L-structure, then $\mathrm{Age}(M)$ is an amalgamation class.
2. Conversely, if $\mathcal{A}$ is an amalgamation class of countable L-structures, with countably many isomorphism types, then there is a countable homogeneous L-structure M with $\mathcal{A} = \mathrm{Age}(M)$.
3. Suppose $\mathcal{A}$ is as in (2) and M is a countable homogeneous L-structure with age $\mathcal{A}$. Then M has the property that if $A \subseteq M$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M$ with $g(f(a)) = a$ for all $a \in A$. This property determines M up to isomorphism amongst countable structures with age $\mathcal{A}$.

DEFINITION: The structure M is determined up to isomorphism by $\mathcal{A}$ and is referred to as the *Fraïssé limit*, or *generic structure* of $\mathcal{A}$.

Examples 1

- The class of all finite graphs is an amalgamation class. The Fraïssé limit is the *random graph*.
- If $n \geq 3$, let K_n denote the complete graph on n vertices. The class of all finite graphs which do not embed K_n is an amalgamation class and the Fraïssé limit is sometimes called the generic K_n-free graph.
- (Henson digraphs) We construct continuum many homogeneous directed graphs. A *tournament* is a directed graph with the property that for every two vertices a, b, one of $(a, b), (b, a)$ is a directed edge. There is an infinite set $\mathcal{S}$ of finite tournaments with the property that if A, B are distinct elements of $\mathcal{S}$ then A does not embed in B. If $\mathcal{T}$ is a subset of $\mathcal{S}$, the class $\mathcal{A}(\mathcal{T})$ of finite directed graphs which do not embed any member of $\mathcal{T}$ is an amalgamation class (free amalgamation); the Fraïssé limit $H(\mathcal{T})$ determines $\mathcal{T}$.

Examples 2

- The class of all finite linear orders is an amalgamation class (but we cannot use free amalgamation). The Fraïssé limit is isomorphic to $(\mathbb{Q}; \leq)$.
- The class of all finite partial orders is an amalgamation class.
- The class of all finite groups is an amalgamation class. The generic structure is Philip Hall's universal locally finite group.

Sketch proof of Fraïssé's Theorem (2,3)

GIVEN: countable amalgamation class $\mathcal{A}$.
CONSTRUCTION: Build M inductively as the union of a chain of structures in $\mathcal{A}$:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots .$$

When doing this we ensure that:

- if $C \in \mathcal{A}$, then C embeds into some A_i;
- if A is a f.g. substructure of A_i and $f : A \to B \in \mathcal{A}$, then there is $j > i$ such that there is an embedding $g : B \to A_j$ with $g(f(a)) = a$ for all $a \in A$.

Countably many tasks to perform here.
A task of the first form can be performed using JEP.
For the second, suppose the construction has reached stage $k > i$. At the next stage we can take A_{k+1} which solves the amalgamation problem $A \to A_k$ (inclusion), $f : A \to B$. Specifically, using AP we obtain $h : A_k \to A_{k+1}$ (which can be taken as inclusion), and $g : B \to A_{k+1}$ with $g(f(a)) = h(a) = a$ for all $a \in A$, as required.

Homogeneity of M

- Suppose M_1, M_2 are countable and have the property that if $A \subseteq M_i$ is f.g. and $f : A \to B$ is an embedding with $B \in \mathcal{A}$, then there is an embedding $g : B \to M_i$ with $g(f(a)) = a$ for all $a \in A$.
- Suppse A_i is a f.g. substructure of M_i and $h : A_1 \to A_2$ is an isomorphism.
- Use a back-and-forth argument to show that h extends to an isomorphism $g : M_1 \to M_2$.

2.2 An extension

GIVEN:

- Class $\mathcal{K}$ of f.g L-structures
- A distinguished class of f.g. substructures $A \sqsubseteq B$ ('A is a *nice* substructure of B'

If $B \in \mathcal{K}$, an embedding $f : A \to B$ is a $\sqsubseteq$-embedding if $f(A) \sqsubseteq B$.
ASSUME: $\sqsubseteq$ satisfies:

(N1) If $B \in \mathcal{K}$ then $B \sqsubseteq B$ (so isomorphisms are $\sqsubseteq$-embeddings);

(N2) If $A \sqsubseteq B \sqsubseteq C$ (and $A, B, C \in \mathcal{K}$), then $A \sqsubseteq C$ (so if $f : A \to B$ and $g : B \to C$ are $\sqsubseteq$-embeddings, then $g \circ f : A \to C$ is a $\sqsubseteq$-embedding).

(N3) Suppose $A \sqsubseteq B \in \mathcal{K}$ and $A \subseteq C \subseteq B$ with $C \in \mathcal{K}$. Then $A \sqsubseteq C$.

Nice amalgamation classes

Say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under isomorphisms and has countably many isomorphism types (and countably many embeddings between any pair of elements);
- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures;
- $\mathcal{K}$ has the JEP for $\sqsubseteq$-embeddings;
- $\mathcal{K}$ has AP for $\sqsubseteq$-embeddings: if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_1 : A_0 \to A_1$ and $f_2 : A_0 \to A_2$ are $\sqsubseteq$-embeddings, there is $B \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : A_i \to B$ (for $i = 1, 2$) with $g_1 \circ f_1 = g_2 \circ f_2$.

REMARKS:

1. If $\sqsubseteq$ is just 'substructure' this is as before.
2. The notion $A \sqsubseteq B$ is only defined when B is f.g.
   If M is a countable L-structure and there are f.g. $M_i \subseteq M$ (with $i \in \mathbb{N}$) such that $M = \cup_{i \in \mathbb{N}} M_i$ and $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$. Then for f.g. $A \subseteq M$ we define $A \sqsubseteq M$ to mean that $A \sqsubseteq M_i$ for some $i \in \mathbb{N}$. The condition N3 guarantees that this does not depend on the choice of M_i.

The generalization

THEOREM: Suppose $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class of finitely generated L-structures and $\sqsubseteq$ satisfies (N1, N2, N3). Then there is a countable L-structure M and f.g. substructures $M_i \in \mathcal{K}$ (for $i \in \mathbb{N}$) such that:

1. $M_1 \sqsubseteq M_2 \sqsubseteq M_3 \sqsubseteq \ldots$ and $M = \cup_{i \in \mathbb{N}} M_i$;
2. every $A \in \mathcal{K}$ is isomorphic to a $\sqsubseteq$-substructure of M;
3. (Extension Property) if $A \sqsubseteq M$ is f.g. and $f : A \to B \in \mathcal{K}$ is a $\sqsubseteq$-embedding then there is a $\sqsubseteq$-embedding $g : B \to M$ such that $g(f(a))$ for all $a \in A$.

Moreover, M is determined up to isomorphism by these properties and if $A_1, A_2 \sqsubseteq M$ are f.g. and $h : A_1 \to A_2$ is an isomorphism, then h extends to an automorphsim of M.

The proof is essentially the same as that of Fraïssé's Theorem.

Examples

1. (2-out digraphs) Let $\mathcal{K}$ consist of the set of finite directed graphs where every vertex has at most 2 directed edges coming out of it. For $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ if whenever $a \in A$ and $a \to b$ is a directed edge in B, then $b \in A$. Then $\sqsubseteq$ satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation is just free amalgamation).
2. (Free groups) Let $\mathcal{K}$ be the class of finitely generated free groups. For f.g. $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean that A is a free factor of B. This clearly satisfies N1, N2 and N3 also holds (cf. Magnus, Karrass, Solitar, Ex 2.4.31). Moreover $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class and the generic structure is the free group of rank ω.

The Hrushovski construction

- Suppose A is a finite graph.
- $\delta(A) = 2|A| - |\mathrm{Edges}(A)|$ (Predimension)
- $\mathcal{K} = \{A : \delta(X) \geq 0 \text{ for all } X \subseteq A\}$.
- If $A \subseteq B \in \mathcal{K}$ write $A \sqsubseteq B$ to mean $\delta(A) \leq \delta(B')$ whenever $A \subseteq B' \subseteq B$.

This satisfies N1, N2, N3 and $(\mathcal{K}, \sqsubseteq)$ is an amalgamation class (where the amalgamation can be taken as free amalgamation).