Automorphism groups of countable structures 2,3

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Lectures 2, 3

A quick tour through some major results.

- The small index property (work of Hodges, Hodkinson, Lascar and Shelah and others);
- Extreme amenability and the Ramsey property (work of Kechris, Pestov and Todorcevic);
- Normal subgroup structure of automorphism groups (work of Lascar, Macpherson - Tent, and Tent - Ziegler).

3.1 The small index property

DEFINITION: A countable structure M (or its automorphism group $\mathrm{Aut}(M)$) has the *small index property* (SIP) if whenever H is a subgroup of $G = \mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$, then H is open.
In other words, if $|G : H| < 2^{\aleph_0}$, then there is a finite $A \subseteq M$ with $H \geq G_{(A)}$.
(Note that the first formulation makes sense in an arbitrary topological group.)
REMARKS:

1. If $H \leq G$ is open then $|G : H| \leq \aleph_0$.
2. The SIP implies that we can recover the topology on G from its group-theoretic structure: the open subgroups are precisely the subgroups of small index and the cosets of these form a base for the topology.

Biinterpretability

For a countable ω-categorical structure M the topological group $\mathrm{Aut}(M)$ determines M *up to biinterpretability*.

We cannot expect to recover M completely.
Consider M with automorphism group $G = \mathrm{Sym}(M)$.
This acts on N, the set of subsets of size 2 from M.
Let $G_1 \leq \mathrm{Sym}(N)$ be the set of permutations induced by by this action.
It can be shown that this is closed, so we can regard G_1 as the automorphism group of a structure on N.
The isomorphism $G \to G_1$ (given by the identity map) is a homeomorphism (check this!).
So the structures M and N have isomorphic topological automorphism groups even though they are different structures.

Examples and History

- SIP proved for $\mathrm{Sym}(\mathbb{N})$ by Dixon, Neumann and Thomas (1986).
- SIP for general linear groups and classical groups over countable fields (Evans, 1986, 1991)
- SIP for $\mathrm{Aut}(\mathbb{Q}; \leq)$ (Truss, 1989).
- Different method introduced by Hodges, Hodkinson, Lascar and Shelah (1993) and used to prove SIP for the random graph.
- Approach extended to general Polish groups by Kechris and Rosendal (2007).

In what follows we will follow the presentation of Kechris and Rosendal. There is also work of M Rubin on recovering an ω-categorical structure from its automorphism group.

A counterexample

An ω-categorical structure without SIP (Cherlin and Hrushovski):

Language L: $2n$-ary relation symbol E_n for each $n \in \mathbb{N}$.
$\mathcal{C}$: the class of finite L-structures A in which E_n is an equivalence relation on n-tuples of distinct elements of A with at most 2 classes.

This is an amalgamation class; call the generic structure M.

For each n there are two equivalence classes of distinct n-tuples from M and every permutation of these equivalence classes extends to an automorphism of M.

So $G = \mathrm{Aut}(M)$ has a closed normal subgroup G^0 consisting of automorphisms which fix all equivalence classes and the quotient group is topologically isomorphic to the direct product C_2^ω (where C_2 is the cyclic group with 2 elements).

Assuming the Axiom of Choice, this has non-open subgroups of index 2.

A question

Is the Cherlin - Hrushovski construction essentially the only obstruction to the SIP for an ω-categorical structure?

DEFINITION: Say that an ω-categorical M is *G-finite* if for every open subgroup $H \leq \mathrm{Aut}(M)$, the intersection of the open subgroups of finite index in H is of finite index in H.

Note that in the example, the intersection of the open subgroups of finite index in G is G^0.

Question

If M is a countable ω-categorical structure which is G-finite, does M have the SIP?

Generic automorphisms

Consider the action of a topological group G on the direct product G^n by conjugation:

$$(g_1, \ldots, g_n) \overset{h}{\mapsto} (hg_1h^{-1}, \ldots, hg_nh^{-1}).$$

If we give G^n the product topology, this is a continuous action, which we refer to as the conjugation action.

DEFINITION: Suppose G is a Polish group. We say that G has *ample homogeneous generics* (ahg's) if for each $n > 0$, there is a comeagre orbit of G on G^n (with the conjugation action).

Theorem

(Kechris - Rosendal; Hodges, Hodkinson, Lascar and Shelah) Suppose G is a Polish group with ample homogeneous generics. Then G has the SIP.

REMARKS: Ahg is a strong property. It does not hold for $\mathrm{Aut}(\mathbb{Q}; \leq)$ (fails for $n = 2$).

Finding ahg's

$(\mathcal{K}, \sqsubseteq)$: an amalgamation class of f.g. L-structures (and N1, N2, N3 hold).
M: generic structure of $(\mathcal{K}, \sqsubseteq)$; $G = \mathrm{Aut}(M)$.
$\mathcal{A}(M)$: the set of $A \sqsubseteq M$ with $A \in \mathcal{K}$.
A *partial automorphism* of M is an isomorphism $f : A_1 \to A_2$ where $A_i \in \mathcal{A}(M)$.

Theorem

Suppose the following two conditions hold:
(i) (Amalgamation property for partial automorphisms) Suppose $A \sqsubseteq B_i \in \mathcal{A}(M)$ (for $i = 1, 2$). Then there is $g \in G_{(A)}$ with the following property. If f_1, f_2 are in $\mathrm{Aut}(B_1)$ and $\mathrm{Aut}(gB_2)$ respectively and stabilize A, and $f_1|A = f_2|A$, then $f_1 \cup f_2$ extends to an automorphism of M.
(ii) (Extension property for partial automorphisms) If $f_1, \ldots, f_n$ are partial automorphisms of M then there is $B \in \mathcal{A}(M)$ containing their domains and images and $g_i \in \mathrm{Aut}(B)$ such that $f_i \subseteq g_i$ for all $i \leq n$.
Then $G = \mathrm{Aut}(M)$ has ample homogeneous generics.

Examples 1

Let $\mathcal{K}$ be the class of finite graphs (and $\sqsubseteq$ is just embedding).
So the generic structure M is the random graph.

Amalgamation property for patial automorphisms: If D is the free amalgmation of B_1 and B_2 over A and $f_i \in \mathrm{Aut}(B_i)$ stabilize A and have the same restriction to A, then their union is an automorphism of D.

Extension Property: a theorem of Hrushovski.

The result generalises to other free amalgamation classes.

Examples 2

Let $\mathcal{K}$ be the class of finitely generated free groups and $\sqsubseteq$ dentote being a free factor.

The free product with amalgamation gives the amalgamation property for partial automorphisms.

For the Extension Property we can take B to be any free factor of M which contains the domains and images of the f_i.

So the free group of rank ω has the SIP (Bryant and Evans, 1997).

Sketch proof of Theorem

GIVEN: $(\mathcal{K}, \sqsubseteq)$ with generic structure M such that APPA and EPPA hold.

SHOW: G has a comeagre orbit on G^n in conjugation action.

Say that $(g_1, \ldots, g_n) \in G^n$ is *generic* if

(a) The set of $A \in \mathcal{A}(M)$ such that $g_i(A) = A$ for all $i \leq n$ is cofinal in $\mathcal{A}(M)$; and

(b) Suppose $A \in \mathcal{A}(M)$ and $g_i(A) = A$ for all i; let $A \sqsubseteq B \in \mathcal{A}(M)$ and $h_i \in \mathrm{Aut}(B)$ extend $g_i|A$. Then there is $\alpha \in G_{(A)}$ such that $\alpha g_i \alpha^{-1} \supseteq h_i$ (for $i \leq n$).

Claim that:

(1) the set of generics in G^n is comeagre; and

(2) any two generics in G^n lie in the same G-orbit.

Proof of (2)

Suppose $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_n)$ are generic. We can build an element of G which conjugates one to the other by using a back-and-forth argument and the following obervation:

Claim: If $A \sqsubseteq B \in \mathcal{A}(M)$ is invariant under the g_i and h_i and $g_i|A = h_i|A$ for all i, there is $\beta \in G_{(A)}$ with $\beta g_i \beta^{-1}|B = h_i|B$.

To see the claim, note that using (a) for the h_i, we may assume $h_i(B) = B$ for all i. Now use (b) for the g_i.

Proof of (1)

To show (1), write the set of generics in G^n as a countable intersection of dense open sets.

For $A \in \mathcal{A}(M)$ let

$$X(A) = \{(g_1, \ldots, g_n) \in G^n : \exists B \in \mathcal{A}(M) \text{ with } A \sqsubseteq B \text{ and } g_i B = B \ \forall i \leq n\}.$$

It is easy to see that $X(A)$ is open. It follows from (ii) that $X(A)$ is dense. Note that

$$\bigcap_{A \in \mathcal{A}(M)} X(A)$$

is the set of elements of G^n which satisfy (a).

Proof of (1) cont.

Suppose $A \sqsubseteq B \in \mathcal{A}(M)$ and $h_1, \ldots, h_n \in \mathrm{Aut}(B)$ satisfy $h_i A = A$.
Let $Y(A, B, \bar{h}) = \{(g_1, \ldots, g_n) \in G^n :$

$$\text{if } (\forall i)(g_i|A = h_i|A) \text{ then } (\exists \alpha \in G_{(A)})\ (\forall i)(\alpha g_i \alpha^{-1}|B = h_i)\}.$$

The intersection of these consists of elements of G^n which satisfy (b).
Each $Y(A, B, \bar{h})$ is easily seen to be open.

For denseness, consider a basic open set specified by partial automorphisms $(f_1, \ldots, f_n)$.
By EPPA we can assume these all have the same domain and image C and we can also assume harmlessly that they extend the h_i.

Using APPA, there is $\beta \in G_{(A)}$ and automorphisms g_i such that $g_i \supseteq f_i \cup \beta h_i \beta^{-1}$. Then $(g_1, \ldots, g_n)$ is in the required open set and is in $Y(A, B, \bar{h})$. This shows the denseness and so gives (1).

Extreme amenability and structural Ramsey theory

Discuss some results of Kechris, Pestov and Todorcevic (2005).

DEFINITION: Suppose G is a topological group.
(1) A *G-flow* is a non-empty, compact (Hausdorff) space Y with a continuous G-action $G \times Y \to Y$.
(2) We say that G is *extremely amenable* if whenever Y is a G-flow, there is a G-fixed point in Y.

EXAMPLE: Suppose $G \leq \mathrm{Sym}(X)$. The product space $\{0, 1\}^X$ is a G-flow.

More generally: Suppose $G \leq \mathrm{Sym}(X)$ is closed and $H \leq Y$ is open. Then the left coset space $Z = G/H$ is discrete and for $k \in \mathbb{N}$, the space $Y = \{1, \ldots, k\}^Z$ of functions $f : Z \to \{1, \ldots, k\}$ with the product topology is a G-flow (the action is $(gf)(z) = f(g^{-1}z)$). Note that here, as G is transitive on Z, the only fixed points are the constant functions. We think of Y as the space of colourings of Z with $\leq k$ colours.

Remarks

(1) The group G is *amenable* if every G-flow has an invariant finitely additive probability measure.
(2) An alternative way of expressing extreme amenability is that the *universal minimal G-flow* $M(G)$ is a point.

Invariant orderings

If $G \leq \mathrm{Sym}(X)$ then $\{0,1\}^{X^2}$ is a G-flow, as is every closed G-invariant subset of this. We can think of this as the set of all binary relations on X. Let

$$LO(X) = \{R \in \{0,1\}^{X^2} : \ R \text{ is a linear order on } X\}.$$

This is a closed, G-invariant subset. So we obtain:

Lemma

If $G \leq \mathrm{Sym}(X)$ is extremely amenable, then there is a G-invariant linear order on X.

So, for example, $\mathrm{Sym}(X)$ is not extremely amenable.

A Theorem

Theorem (Kechris, Pestov, Todorcevic, 2005)

Suppose $G \leq \mathrm{Sym}(X)$ is closed. The following are equivalent:

1. *G is extremely amenable;*
2. *Suppose H is an open subgroup of G and $Z = G/H$. If $c : Z \to \{1, \ldots, k\}$ and $A \subseteq Z$ is finite, there is $g \in G$ and $i \leq k$ such that $c(ga) = i$ for all $a \in A$.*
3. *G preserves a linear ordering on X and G has the Ramsey property.*

First discuss the equivalence of (1) and (2) here and then say what the Ramsey property is.

Equivalence of (1) and (2)

(1) $\Rightarrow$ (2): Consider the G-flow $\{1, \ldots, k\}^Z$. Let Y be the closure in this of the G-orbit $\{gc : g \in G\}$. This is a G-flow, so must contain a G-fixed point. So it contains a constant function $f_i(z) = i$ (for some $i \leq k$). In other words, f_i is in the closure of $\{gc : g \in G\}$. This translates into the condition in (2).

(2) $\Rightarrow$ (1): This is a bit harder, but not excessively so. The proof shows that to decide whether G is extremely amenable, it suffices to consider G-flows whch are closed subflows of $\{1, \ldots, k\}^{G/H}$ (for $H \leq G$ open and $k \in \mathbb{N}$). In fact, we can restrict to $k = 2$ here and take H from a base of open neighbourhoods of 1.

An Example

Let $X = \mathbb{Q}$ and $G = \mathrm{Aut}(\mathbb{Q})$. We verify (2).

Take $H = G_{(C)}$ where C is a subset of $\mathbb{Q}$ of size n. Then we can identify $Z = G/H$ with the set of n-tuples $b_1 < b_2 < \cdots < b_n$ from $\mathbb{Q}$, or, indeed, the set $[\mathbb{Q}]^n$ of subsets of $\mathbb{Q}$ of size n.

So we can think of a function $c : Z \to \{1, \ldots, k\}$ as a k-colouring of $[\mathbb{Q}]^n$. By the classical Ramsey Theorem there is an infinite $Y \subseteq \mathbb{Q}$ such that c is constant on $[Y]^n$.

Given a finite $A \subseteq Z$ let S be the elements of $\mathbb{Q}$ appearing in tuples in A. So S is a finite subset of $\mathbb{Q}$ and we can find $g \in G$ with $gS \subseteq Y$. Then c is constant on gA, as required for (2).
It follows:

Corollary (Pestov)

$\mathrm{Aut}(\mathbb{Q}; \leq)$ *is extremely amenable.*

The Ramsey Property

DEFINITION: Suppose $G \leq \mathrm{Sym}(X)$.

(1) A *G-type* σ is a G-orbit on finite subsets of X. If σ, ρ are G-types, write $\rho \leq \sigma$ iff for all $F \in \rho$ there is $F' \in \sigma$ with $F \subseteq F'$.

(2) Suppose $\rho \leq \sigma \leq \tau$ are G-types.

  (i) If $F \in \sigma$ let $\binom{F}{\rho} = \{F' \subseteq F : F' \in \rho\}$.

  (ii) If $k \in \mathbb{N}$ write

$$\tau \to (\sigma)^{\rho}_{k}$$

  to mean that for every $F \in \tau$ and colouring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$ there is $F_0 \in \binom{F}{\sigma}$ which is monochromatic for c (that is, $c|\binom{F_0}{\rho}$ is constant).

(3) We say that G has the *Ramsey property* if for all k and G-types $\rho \leq \sigma$ there is a G-type $\tau \geq \sigma$ such that $\tau \to (\sigma)^{\rho}_{k}$.

EXERCISE: $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ has the Ramsey property - this is the finite Ramsey theorem.

Equivalence of (2) and (3)

(2) $\Rightarrow$ (3):

Suppose (2) holds. We have to show that the Ramsey property holds. Suppose not - so there are $k \in \mathbb{N}$ and G-types $\rho \leq \sigma$ such that for no τ do we have $\tau \to (\sigma)^{\rho}_{k}$.

Let $F_0 \in \sigma$. For every finite $E \supseteq F_0$ the set

$$C_E = \{c : \binom{E}{\rho} \to \{1, \ldots, k\} : \text{ no monochrome } F \in \tbinom{E}{\sigma}\}$$

is non-empty. Restriction gives a directed system $C_{E'} \to C_E$ for $E' \supseteq E$. By König's lemma there is therefore $c : \rho \to \{1, \ldots, k\}$ with no monochrome $F \in \sigma$. This contradicts (2).

The proof of (3) $\Rightarrow$ (2) is also straightforward.

Examples of Ramsey Classes

We can make a similar definition for classes of finite structures - instead of G-types we have structures in the class.
Structural Ramsey Theory investigates classes of (finite) structures with the Ramsey property.
Note that we should expect such structures to carry an ordering. The following shows that there is a strong connection with homogeneous structures:

Theorem

Suppose $\mathcal{C}$ is a class of finite ordered structures for a finite relational language and $\mathcal{C}$ is closed under substructures and has JEP. If $\mathcal{C}$ is a Ramsey class, then $\mathcal{C}$ has the amalgamation property.

Examples of countable homogeneous ordered structures with extremely amenable automorphism group include ordered versions of: the random graph, the universal homogeneous K_n-free graphs, the Henson digraphs.

3.3 Normal subgroup structure

Some classical results:

THEOREM (J. SCHREIER AND S. ULAM, 1933) Suppose X is countably infinite. If $g \in \mathrm{Sym}(X)$ moves infinitely many elements of X, then every element of $\mathrm{Sym}(X)$ is a product of conjugates of g. In particular, $\mathrm{Sym}(X)/\mathrm{FSym}(X)$ is a simple group.

THEOREM (A. ROSENBERG, 1958). Suppose V is a vector space of countably infinite dimension over a field K. If $FGL(V)$ denotes the elements of $GL(V)$ which have a fixed point space of finite codimension, then $GL(V)/(K^\times.FGL(V))$ is a simple group.

THEOREM (G. HIGMAN, 1954). The non-trivial, proper normal subgroups of $G = \mathrm{Aut}(\mathbb{Q}; \leq)$ are the left-bounded automorphisms, $L = \{g \in G : \exists a \; g|(-\infty, a) = id\}$, the right-bounded automorphisms $R = \{g \in G : \exists a \; g|(a, \infty)\}$ and $B = L \cap R$.

THEOREM (J. TRUSS, 1985). Let Γ be the countable random graph. Then $\mathrm{Aut}(\Gamma)$ is simple.

Warning

TEMPTING IDEA: Automorphism groups of 'nice' countable structures should not have any non-obvious normal subgroups.

This is false:

Example (M. Droste, C. Holland, D. Macpherson)

The automorphism group of a countable, homogeneous semilinear order has $2^{2^{\aleph_0}}$ normal subgroups.

A general result

Theorem (D. Lascar, 1992)

Suppose M is a countable saturated structure with a $\emptyset$-definable strongly minimal set D. Suppose that $M = \mathrm{acl}(D)$. Suppose $g \in G = \mathrm{Aut}(M/\mathrm{acl}(\emptyset))$ is unbounded, i.e. for every $n \in \mathbb{N}$ there is some $X \subseteq D$ with $\dim(gX/X) > n$. Then G is generated by the conjugates of g.

- Implies the results for $\mathrm{Sym}(X)$ and $GL(V)$.
- Proof uses Polish group arguments.
- Ideas used by T. Gardener (1995) to prove analogue of Rosenberg's result for classical groups over finite fields.
- Used by Z. Ghadernezhad and K. Tent (2012) to prove simplicity of automorphism groups of certain generalized polygons and so obtain new examples of simple groups with a *BN*-pair.

Recent general results

THEOREM (D. MACPHERSON AND K. TENT, 2011): Suppose M is a countable, transitive homogeneous relational structure whose age has free amalgamation. Suppose $\mathrm{Aut}(M) \neq \mathrm{Sym}(M)$. Then
(a) $\mathrm{Aut}(M)$ is simple;
(b) (Melleray) if $1 \neq g \in \mathrm{Aut}(M)$ then every element of G is a product of 32 conjugates of $g^{\pm 1}$.

NOTE: This implies Truss' result and unpublished results of M. Rubin (1988).

K. Tent and M. Ziegler (2012) generalized this to the case where M has a *stationary independence relation* $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$ and used this to prove:

THEOREM: Suppose U is the Urysohn rational metric space. If $g \in \mathrm{Aut}(U)$ is not bounded, then every automorphism of U is a product of 8 conjugates of g.

Stationary independence relations

NOTATION/ TERMINOLOGY:

- M is a countable first-order structure;
- $G = \mathrm{Aut}(M)$;
- cl is a G invariant, finitary closure operation on subsets of M;
- If $X \subseteq_{fin} M$ and a is fixed by G_X, then $a \in \mathrm{cl}(X)$ (where $G_X = \{g \in G : gx = x \ \forall x \in X\}$).
- $\mathcal{X} = \{\mathrm{cl}(A) : A \subseteq_{fin} M\}$;
- $\mathcal{F}$ consists of all maps $f : X \to Y$ with $X, Y \in \mathcal{X}$ which extend to automorphisms of M. Call these *partial automorphisms*.

EXAMPLE: Take cl to algebraic closure in M. So, for example, if M is the Fraïssé limit of a free amalgamation class, then $\mathrm{acl}(X) = X$ for all $X \subseteq M$.

In what follows, $\mathop{\smile\!\!\!\!\!\!|}$ is a relation between subsets A, B, C of M: written $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and pronounced 'A is independent from C over B.'

DEFINITION:

We say that $\mathop{\smile\!\!\!\!\!\!|}$ is a *stationary independence relation compatible with* cl if for $A, B, C, D \in \mathcal{X}$ and finite tuples a, b:

1. (Compatibility) We have $a \mathop{\smile\!\!\!\!\!\!|}_b C \Leftrightarrow a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}(b)} C$ and
$$a \mathop{\smile\!\!\!\!\!\!|}_B C \Leftrightarrow e \mathop{\smile\!\!\!\!\!\!|}_B C \text{ for all } e \in \mathrm{cl}(a, B) \Leftrightarrow \mathrm{cl}(a, B) \mathop{\smile\!\!\!\!\!\!|}_B C.$$
2. (Invariance) If $g \in G$ and $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $gA \mathop{\smile\!\!\!\!\!\!|}_{gB} gC$.
3. (Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}_B C \cup D$, then $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B \cup C} D$.
4. (Transitivity) If $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{B \cup C} D$, then $A \mathop{\smile\!\!\!\!\!\!|}_B C \cup D$
5. (Symmetry) If $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $C \mathop{\smile\!\!\!\!\!\!|}_B A$.
6. (Existence) There is $g \in G_B$ with $g(A) \mathop{\smile\!\!\!\!\!\!|}_B C$.
7. (Stationarity) Suppose $A_1, A_2, B, C \in \mathcal{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile\!\!\!\!\!\!|}_B C$. Suppose $h : A_1 \to A_2$ is the identity on B and $h \in \mathcal{F}$. Then there is some $k \in \mathcal{F}$ which contains $h \cup \mathrm{id}_C$ (where id_C denotes the identity map on C).

Remarks and examples

1. For all $a \in M$ and finite X we have $a \mathop{\smile\!\!\!\!\!\!|}_X \mathrm{cl}(X)$. Moreover $a \mathop{\smile\!\!\!\!\!\!|}_X a$ iff $a \in \mathrm{cl}(X)$.
2. Tent and Ziegler consider this where $\mathrm{acl}(X) = X$ and $\mathrm{cl}(X) = X\ \forall X$.
3. Suppose M is the Fraïssé limit of a free amalgamation class (of relational structures). Let $\mathrm{cl}(X) = X\ \forall X$. Define $A \mathop{\smile\!\!\!\!\!\!|}_B C$ to mean $A \cap C \subseteq B$ and $A \cup B$, $C \cup B$ are freely amalgamated over B. This is a stationary independence relation on M.
4. Suppose M is a countable-dimensional vector space over a countable field K. So $G = GL(M)$. Let cl be linear closure and take $A \mathop{\smile\!\!\!\!\!\!|}_B C$ to mean that $\mathrm{cl}(A \cup B) \cap \mathrm{cl}(C \cup B) = \mathrm{cl}(B)$. This gives a stationary independence relation.

Moving almost maximally

DEFINITION: Say that $g \in G$ moves almost maximally if for all $B \in \mathcal{X}$ and $a \in M$ there is a' in the G_B-orbit of a such that

$$a' \underset{B}{\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}} ga'.$$

EXAMPLE 1: Suppose $(M; \mathrm{cl}; \mathop{\smile\!\!\!\!\!\mid}\,)$ is the vector space example. If $g \in G$ does not move almost maximally, then for some finite dimensional subspace B, for all $v \in M$ we have $gv \in \langle v, B\rangle$. Thus g acts as a scalar α on M/B. So $(\alpha^{-1}g - 1)v \in B$ for all v and it follows that g is a scalar multiple of a finitary transformation.

EXAMPLE 2: Suppose $(M; \mathrm{cl}; \mathop{\smile\!\!\!\!\!\mid}\,)$ is the free amalgamation example. Suppose also that $G = \mathrm{Aut}(M)$ is transitive on M and $G \neq \mathrm{Sym}(M)$. If $1 \neq g \in G$, then g moves infinitely many points of each G_B-orbit (for each finite $B \subseteq M$) and using a back-and-forth argument, one shows that there is $h \in G$ such that $[g, h] = g^{-1}h^{-1}gh$ moves almost maximally.

Theorem (Evans, Ghadernezhad, Tent (2013))

Suppose M is a countable structure with a stationary independence relation compatible with a closure operation cl. Suppose that $G = \mathrm{Aut}(M)$ fixes every element of $\mathrm{cl}(\emptyset)$. If $g \in G$ moves almost maximally, then every element of G is a product of 16 conjugates of g.

REMARKS:

1. If $\mathrm{cl}(X) = X \ \forall X$, this is proved in the paper of Tent and Ziegler.
2. As observed by Tent and Ziegler, it implies the result of Macpherson and Tent for the free amalgamation example.
3. Proof is essentialy the same as the the Tent - Ziegler result (plus a trick of Lascar in the case where $\mathcal{F}$ is not countable).