Automorphism groups of metric structures

1. Polish groups; the space of actions.

J. Melleray

Institut Camille Jordan (Lyon)

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Polish groups: some examples

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- Topological group: $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous.
- Completeness implies that one can use the Baire category theorem - a countable intersection of dense open sets is dense.

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- and so on.

Our first permutation group: S_∞

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- Is the distance d complete?

Complete and noncomplete distances

Recall that $d(\sigma, \tau) = \inf\{2^{-n} \colon \forall i < n \; \sigma(i) = \tau(i)\}$. Define

$$\sigma_i(n) = \begin{cases} n+1 & \text{if } n \leq i \\ 0 & \text{if } n = i \\ n & \text{otherwise} \end{cases}$$

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Theorem

Let G be a Polish group, and d be a left-invariant metric on G. Then d' defined by $d'(g, h) = d(g, h) + d(g^{-1}, h^{-1})$ is complete.

The unitary group

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Above F is finite and $\varepsilon > 0$ (these are not subgroups!).

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It is tempting to use the operator norm: it is complete, bi-invariant... and nonseparable: permutations of a Hilbert basis provide a continuum of group elements pairwise at distance $\sqrt{2}$.

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The first topology one may think of is given by the measure of the support:

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The Polish topology commonly used in ergodic theory is the corresponding pointwise convergence topology.

Polish groups are isometry groups

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Theorem (Gao–Kechris)

Any Polish group is (isomorphic to) the isometry group of a Polish metric space.

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Theorem (Uspenskij)

Any Polish group is isomorphic to a closed subgroup of $\mathrm{Iso}(\mathbb{U})$.

Baire category methods

Why Baire category?

Locally compact groups are a classical object of study, which was investigated in depth already during the first part of the 20th century. It is well-known and important that locally compact groups admit a (essentially unique) Haar measure, which is preserved by (say) left translation.

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Baire category notions provide a well-behaved notion of largeness that can be used to replace some measure-theoretic arguments in the context of non locally compact groups.

Leaning to love G_δ subsets

Theorem (Alexandrov)

Let X be a Polish topological space, and Y be a metric space containing X. Then X is a G_δ subset of Y: that is, X is a countable intersection of open subsets of Y.
Conversely, if X is Polish and $A \subseteq X$ is G_δ, then A is Polish.

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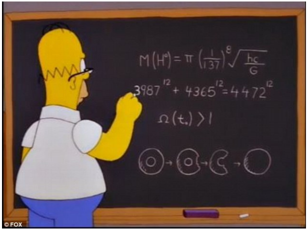

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A meagre set can be dense! A countable intersection of comeagre sets is again comeagre; they should be thought of as analogues of sets of full measure.

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Just like measurable sets, Baire-measurable sets form a σ-algebra which contains all Borel sets. There are more...

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Theorem (Lusin–Sierpinski)

Analytic subsets of Polish spaces are Baire-measurable.

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This is the category analogue of Fubini's theorem.

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These maps come up in a variety of settings and under different guises.

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When f is a projection map, this is the Kuratowski–Ulam theorem; this is in some sense analoguous to the measure disintegration theorem (here category is split along the fibers of f).

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Theorem

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Thus we have a canonical witness of the fact that A is Baire measurable.

The Pettis lemma

Lemma (Pettis)

Let G be a Polish group, and $A, B \subseteq G$. Then

$$U(A) \cdot U(B) \subseteq A \cdot B \ .$$

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Theorem (Banach)

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In particular, if τ, τ' are two Polish topologies on the same group G, and one is contained in the other (or even: all τ-open sets are τ'-Borel), then they are equal.

Topologically transitive actions on Polish spaces.

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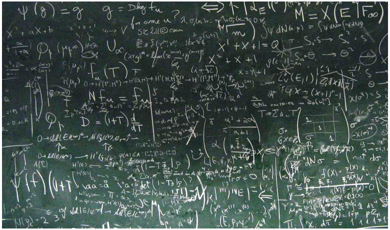
Proof.



$\square$

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Assume that G is a group acting by homeomorphisms on a Polish space X, that the action is topologically transitive, and that A is a Baire-measurable subset of X which is G-invariant. Then A is either meagre or comeagre.

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- This applies for instance for the groups S_∞, $U(\ell_2)$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbb{U})$.