Automorphism groups of metric structures

2. This time the title makes no promises concerning which topics will be covered

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Istanbul - March 27, 2015

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Structures

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- A family of relations $(R_i)_{i \in I}$, where each R_i is a subset of some M^{k_i}. $=$ is always part of our list and will not be mentioned.
- A family of functions $(f_j)_{j \in J}$, where each f_j is a function from some M^{l_j} to M.
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We may go the other way: given a language, consider a structure in that language. In what follows, all our languages will be (at most) countable, and the same will apply to structures.

The permutation group topology

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- $\forall j \in J \ \forall \bar{x} \in M^{l_j} \quad g(f_j(\bar{x})) = f_j(g(\bar{x}))$.
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Observation

Assume the universe of $\mathcal{M}$ is ω. The automorphism group $\mathrm{Aut}(\mathcal{M})$ is a closed subgroup of S_∞.

As such it is a nonarchimedean Polish group: a Polish group in which 1 has a basis of neighborhoods consisting of open subgroups.

Ultrahomogeneous structures

Definition

We say that $\mathcal{M}$ is ultrahomogeneous if it has the following property: whenever A, B are finitely generated substructures of $\mathcal{M}$, and $g\colon A \to B$ is an isomorphism, there is $h \in \mathrm{Aut}(\mathcal{M})$ which extends g.

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Then, once $G \leq S_\infty$, give a name to each orbit for the diagonal action of G on ω^k (for all k); the corresponding structure has G as its automorphism group.

□

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3. Up to isomorphism, there are only countably many elements of $\mathcal{K}_{\mathcal{M}}$.

We say that $\mathcal{K}_{\mathcal{M}}$ satisfies the Hereditarity property(HP), the Joint embedding property (JEP) and is countable.

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Whenever $\mathcal{A}, \mathcal{B}, \mathcal{C}$ belong to $\mathcal{K}_{\mathcal{M}}$, and $i_B \colon \mathcal{A} \to \mathcal{B}$, $i_C \colon \mathcal{A} \to \mathcal{C}$ are embeddings, there exists $\mathcal{D} \in \mathcal{K}_{\mathcal{M}}$ and embeddings $j_B \colon \mathcal{B} \to \mathcal{D}$, $j_C \colon \mathcal{C} \to \mathcal{D}$ such that $j_B \circ i_B = j_C \circ i_C$.

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Theorem (Fraïssé)

Assume $\mathcal{K}$ is a countable class of finitely generated structures which satisfies (HP), (JEP) and (AP) (a Fraïssé class). Then there exists a unique ultrahomogeneous countable structure whose age is $\mathcal{K}$, called the Fraïssé limit of $\mathcal{K}$.

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Interactions between properties of the structure and its automorphism group

Assume that $\mathcal{K}$ is a Fraïssé class in a finite relational language, and that G is the automorphism group of its limit $\mathcal{M}$. Then for all $k < \omega$ there are only finitely many orbits for the natural action of G on M^k.

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Theorem (Tsankov)

Oligomorphic Polish groups are Roelcke precompact; conversely, a nonarchimedean Polish group is Roelcke precompact iff it is an inverse limit of oligomorphic groups (iff it is the automorphism group of an $\aleph_0$-categorical structure with countably many sorts).

The extension property

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Example (Hrushovski)

The class of finite graphs has the extension property.

Topological content of the extension property.

Proposition (Kechris–Rosendal)

Let $\mathcal{K}$ be a Fraïssé class of finite structures, and G be the automorphism group of its limit. Then $\mathcal{K}$ has the extension property iff there exists an increasing sequences (G_n) of compact subgroups of G with dense union.

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Note that, when that happens, G is amenable.

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Theorem (Herwig–Lascar)

Assume that L is a finite relational language, that T is a finite set of L-structures and that $\mathcal{A}$ is a T-free structure. let P be a set of partial isomorphisms of $\mathcal{A}$. If there exists a T-free L-structure $\mathcal{M}$ containing $\mathcal{A}$ as a substructure and such that elements of P extend to automorphisms of $\mathcal{M}$, then there exists a finite such $\mathcal{M}$.

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Corollary

Assume that L is a finite relational language, that T is a finite set of L-structures and that the class of all T-free structures is a Fraïssé class. Then it has the extension property.

Finite metric spaces have the extension property

Theorem (Solecki)

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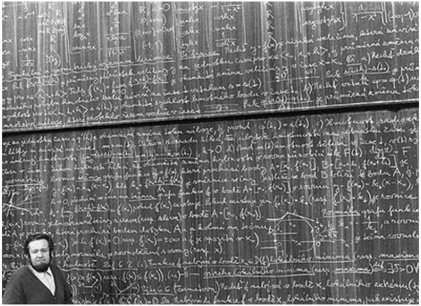
Proof.



$\square$

Metric structures and their automorphism groups

Not all Polish groups are nonarchimedean

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We need to come up with a definition of structure that encompasses these examples.

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Some remarks are in order:

- The distance function now plays the role formerly devoted to $=$.
- Structures/relations are not assumed to be bounded at the moment. They will be whenever we want to do some logic (compactness theorem; categoricity; etc.).
- Assuming relations/functions to be Lipschitz (vs. uniformly continuous) is mostly a matter of convenience; it does not really affect the logic of our structures.

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- Any measure algebra $\mathcal{A}$ is a structure, with the metric $d(A, B) = \mu(A \Delta B)$ and symbols for intersection, union, and complement (with the appropriate Lipschitz constants).

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Note that if $\mathcal{M}$ happens to be discrete we recover the permutation group topology on $\mathrm{Aut}(\mathcal{M})$, so we are talking about the same Polish groups as before in that case.

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Example

ℓ_2, $\mathbb{U}$, $MALG_\mu$ are exactly ultrahomogeneous (one can take $\varepsilon = 0$)

Ultrahomogeneity

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A metric structure $\mathcal{M}$ is ultrahomogeneous if: whenever $A \subset M$ is finite, and g is a partial automorphism of $\mathcal{M}$ with domain $\langle A \rangle$, for any $\varepsilon > 0$ there exists $h \in \mathrm{Aut}(\mathcal{M})$ such that

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Ben Yaacov recently provided the first examples of Polish groups which cannot be realised as the automorphism groups of exactly ultrahomogeneous Polish metric structures.

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The Hereditary property and Joint embedding property are defined in the metric setting just as they are in the classical setting.

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We still have to define an analogue of countability in our context; it is replaced by separability for an appropriate metric.

Separability

Definition

Given a class $\mathcal{K}$ of finitely generated metric L-structures, we let $\mathcal{K}_n$ denote the family of all $(\mathcal{A}, \bar{a})$ where $\mathcal{A} \in \mathcal{K}$, $|\bar{a}| \leq n$ and $\mathcal{A} = \langle \bar{a} \rangle$. We define a pseudo metric d_n on $\mathcal{K}_n$ by setting

$$d((\mathcal{A}, \bar{a}), (\mathcal{B}, \bar{b})) = \inf\{d(i(\bar{a}), j(\bar{b}))\}$$

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It is easy to see that the age of a ultrahomogeneous Polish structure is a Fraïssé class.

A metric version of Fraïssé's theorem

Theorem (Ben Yaacov)

Let $\mathcal{K}$ be a metric Fraïssé class in a language L. Then there exists a unique (up to isomorphism) ultrahomogeneous Polish L-structure whose age is $\mathcal{K}$.

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