Automorphism groups of metric structures

3. The end

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A few more words on metric structures

Let's start all over again

Definition

A metric structure $\mathcal{M}$ is ultrahomogeneous if: whenever $A \subset M$ is finite, and g is a partial automorphism of $\mathcal{M}$ with domain $\langle A \rangle$, for any $\varepsilon > 0$ there exists $h \in \mathrm{Aut}(\mathcal{M})$ such that

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Ben Yaacov recently provided the first examples of Polish groups which cannot be realised as the automorphism groups of exactly ultrahomogeneous Polish metric structures.

Roelcke-precompact Polish groups

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Because the distance is a single object which encodes infinitely many informations at the same time (for instance, the countable basis of neighborhoods of 1 of nonarchimedean Polish groups is completely encoded by a single left-invariant ultrametric).

Metric Fraïssé classes

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We still have to define an analogue of countability in our context; it is replaced by separability for an appropriate metric.

Separability

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Given a class $\mathcal{K}$ of finitely generated metric L-structures, we let $\mathcal{K}_n$ denote the family of all $(\mathcal{A}, \bar{a})$ where $\mathcal{A} \in \mathcal{K}$, $|\bar{a}| \leq n$ and $\mathcal{A} = \langle \bar{a} \rangle$. We define a binary function d_n on $\mathcal{K}_n$ by setting

$$d_n((\mathcal{A}, \bar{a}), (\mathcal{B}, \bar{b})) = \inf\{d(i(\bar{a}), j(\bar{b}))\}$$

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It is easy to see that the age of a ultrahomogeneous Polish structure is a metric Fraïssé class.

Theorem (Ben Yaacov)

Let $\mathcal{K}$ be a metric Fraïssé class in a language L. Then there exists a unique (up to isomorphism) ultrahomogeneous Polish L-structure whose age is $\mathcal{K}$.

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Studying properties of $\mathrm{Hom}(\Gamma, G)$ from the point of view of Baire category can be useful both to extract information on G and on Γ.

Studying generic properties in Hom(Γ, G)

Note that G acts by conjugation on Hom(Γ, G). One can then wonder when there exist dense conjugacy classes and, even better, when there exist comeager conjugacy classes; of course this will depend both on Γ and on G.

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Where do these results come from? In another, better, alternative universe this was explained two days ago.

Topologically transitive actions on Polish spaces.

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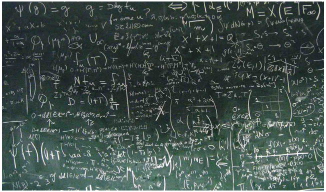

$\square$

The 0–1 topological law

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- Thus when G is a Polish group acting on $\mathrm{Hom}(\Gamma, G)$ by conjugation, and there is a dense conjugay class in $\mathrm{Hom}(\Gamma, G)$, every Baire-measurable conjugacy invariant set (for instance, every conjugacy class) is either meagre or comeagre.

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- This applies for instance for the groups S_∞, $U(\ell_2)$, $\mathrm{Aut}(\mu)$, $\mathrm{Iso}(\mathbb{U})$ (and any countable Γ).

Existence of dense conjugacy classes: towards a combinatorial criterion

Let $\mathcal{K}$ denote a Fraïssé class (in a relational metric language L). Denote by $\mathcal{K}_{aut}^n$ the class of structures $(\mathcal{A}, g_1, \ldots, g_n)$ in the language $L \cup \{f_1, \ldots, f_n\}$ which are such that:

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In other words: if one allows deforming $\mathcal{A}, \mathcal{B}$ a little bit, then they embed in a common element of $\mathcal{K}$.

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In the classical context, one can similarly give a combinatorial criterion for the existence of a comeagre conjugacy class - it involves some more work involving Baire category notions and I will not go into detail here.

Using the space of actions to establish a property of Γ.

Theorem (Glasner–Kitroser/Rosendal)

Assume that Γ is finitely generated. Then a generic element of $\mathrm{Hom}(\Gamma, S_\infty)$ has all of its orbits finite iff Γ is LERF: whenever A is finitely generated subgroups of Γ, A is closed in the profinite topology.

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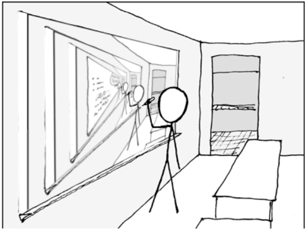

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A countable group Γ has property (RZ_n) if a product of n finitely generated subgroups of Γ is closed in the profinite topology.

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Property (RZ_n) and actions on $\mathbb{U}_n$

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Note that the extension property for metric spaces with distances in $\{0, \ldots, n\}$ amounts to saying that a generic action of a f.g. free group on $\mathbb{U}_n$ has finite orbits for all n.

Complexity computations

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Using the space of actions to establish a property of G.

F_ω denotes the free group on infinitely many generators. For any Polish group G, the set of all $\pi \in \mathrm{Hom}(F_\omega, G) \cong G^\omega$ such that $\pi(F_\omega)$ is dense is dense G_δ in G^ω.

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Thus, to prove that a Polish group is (extremely) amenable, it is enough to show that the set of all $\pi \in \mathrm{Hom}(F_\omega, G)$ which generate an (extremely) amenable subgroup is dense in G.

This leads to an argument that can be used for instance to prove the extreme amenability of $U(\ell_2)$ (Gromov–Millman), $\mathrm{Aut}(\mu)$ (Giordano–Pestov) and $\mathrm{Iso}(\mathbb{U})$ (Pestov).

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Proposition (Glasner–Kitroser–M.)

There exists a comeagre conjugacy class in $\mathrm{Hom}(\Gamma, S_\infty)$ iff isolated subgroups are dense in $Sub(G)$ (the space of subgroups of G, seen as a closed subset of 2^G).

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It appears to be significantly more complicated to give a similar description for other groups than S_∞ - for instance, what happens for groups acting by isometries on $\mathbb{U}_n$?

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Ample generics

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A Polish group G has ample generics if for all k there exists $\bar{g} \in G^k$ such that the set $\{(hg_1h^{-1}, \ldots, hg_kh^{-1}) \colon h \in G\}$ is comeagre in G^k.

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In particular, this implies that G has the small index property: every subgroup of countable index is open in G.

A sufficient condition for ample generics

Definition

Let L be a relational (classical) language, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three $\mathcal{L}$-structures such that $\mathcal{A} \leq \mathcal{B}, \mathcal{C}$. The free amalgam of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ is the L-structure with universe $B \cup C$, such that both $\mathcal{B}$ and $\mathcal{C}$ are substructures and there is no relation between tuples including elements of both $B \setminus A$ and $C \setminus A$.

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This result generalises to contexts where there is a natural amalgam (for instance, where a stationary independence relation in the sense of Tent–Ziegler exists).

Non nonarchimedean Polish groups with ample generics

It is only last week that the first examples of Polish groups with ample generics which are not nonarchimedean appeared on the Arχiv! Actually two different examples appeared within a few days of each other - one by M. Malicki, the other by A. Kaïchouh and F. Le Maître.

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As we mentioned during the previous talk, in many large Polish groups of interest conjugacy classes are meagre - hence ample generics are out of the question. From the point of view of continuous logic, there is a natural, weaker notion, involving topometric Polish groups.

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- (G, τ) is a Polish group.
- ∂ is a bi-invariant distance which refines τ.
- ∂ is τ-lower semicontinuous: each set $\{(g, h) \colon \partial(g, h) \leq r\}$ is τ-closed.

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- If G is the automorphism group of a classical (discrete) structure, the uniform metric, seen from the action on the structure, is certainly discrete; but the coarsest bi-invariant metric refining τ need not be! This metric arises from picking a left-invariant metric d on G setting $\partial(g,h) = \sup_k d(gk, hk)$.

Definition

Assume (G, τ, ∂) is a Polish topometric group, $A \subseteq G$ and $\varepsilon > 0$. Then set $(A)_\varepsilon = \{g \in G\ \exists a \in A\ \partial(g, a) < \varepsilon\}$.

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Theorem

$U(\ell_2)$, $\mathrm{Iso}(\mathbb{U})$ and $\mathrm{Aut}(\mu)$ all have ample generics as Polish topometric groups.

In each case this comes from the existence of a nice countable structure sitting inside the continuous one and whose automorphism group has ample generics (in the usual sense).

What is this good (and not good enough) for?

Theorem

Let (G, τ, ∂) be a Polish topometric group with ample generics. Let $\varphi\colon (G, \partial) \to H$ be a continuous homomorphism from G to a separable topological group H. Then $\varphi\colon (G, \tau) \to H$ is continuous.

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This approach did *not* succeed in proving the following theorem.

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Iso($\mathbb{U}$) has the automatic continuity property.

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Theorem (Sabok)

$\mathrm{Iso}(\mathbb{U})$ has the automatic continuity property.

Sabok's approach also works for $\mathrm{Aut}(\mu)$ and $U(\ell_2)$; it is also based on the existence of nice countable substructures sitting densely in the continuous structure and whose automorphism group has ample generics.

Thank you for your attention!