Automorphism groups of metric structures

4. The talk that was never given

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Extreme amenability

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Since then, numerous examples have been found, including $U(\ell_2)$, $\mathrm{Aut}(\mu)$, and $\mathrm{Iso}(\mathbb{U})$.

Definition

Let G be a group acting continuously and isometrically on a metric space (X, d). The action is finitely oscillation stable if for every 1-Lipschitz $f \colon X \to [0, 1]$, any $\varepsilon > 0$ and any finite $A \subseteq X$ there exists some $g \in G$ such that the oscillation of f on gA is less than ε.

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Theorem (Pestov)

Let G be a Polish group. Then G is extremely amenable if, and only if, there exists a directed collection of bounded left-invariant pseudometrics $(d_i)_{i \in I}$ inducing the topology of G and such that the action of G on each (G, d_i) is finitely oscillation stable.

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Of course, using just one metric above would suffice - but using pseudometrics is sometimes simpler in practice.

Definition

Let G be a Polish group endowed with a left invariant metric d, (K_n) be an increasing sequence of compact subgroups of G and μ_n be the (normalized) Haar measure on K_n.

Concentration of measure

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The sequence (K_n, d, μ_n) is a Lévy family if whenever Borel subsets $A_n \subseteq K_n$ satisfy $\liminf \mu_n(A_n) > 0$ one has

$$\forall \varepsilon > 0 \quad \lim_n \mu_n((A_n)_\varepsilon) = 1$$

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It is immediate that Lévy groups are amenable: a sequence (K_n) as above immediately produces a dense amenable subgroup.

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To produce a dense increasing sequence of compact subgroups, one often produces a dense sequence of finite subgroups, relating this to the extension property; one still has to take care of the concentration of measure!

Extremely amenable nonarchimedean Polish groups

Proposition (Glasner–Weiss)

Let G be an extremely amenable closed subgroup of S_∞. Then there is an ordering on ω which is G-invariant.
In particular, S_∞ is not extremely amenable.

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Proof.

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Corollary

Let $\mathcal{K}$ be a Fraïssé class of finite structures such that the automorphism group of its limit is extremely amenable. Then all elements of $\mathcal{K}$ are rigid, i.e. have no nontrivial automorphisms.

Towards a combinatorial formulation.

When $G = \mathrm{Aut}(\mathcal{M})$, we have an obvious family of pseudometrics to test oscillation stability on: the family (d_A), parameterized by finite subsets of M, with

$$d_A(g,h) = \begin{cases} 0 & \text{if } g_{|A} = h_{|A} \\ 1 & \text{otherwise} \end{cases}$$

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Definition

Given two structures $\mathcal{A}, \mathcal{B}$ in the same language L, we denote by $\binom{\mathcal{B}}{\mathcal{A}}$ the set of all embeddings from $\mathcal{A}$ to $\mathcal{B}$.

The Ramsey property.

Definition

Let $\mathcal{K}$ be a Fraïssé class of finite structures . $\mathcal{K}$ has the Ramsey property if:

For any $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ there exists $\mathcal{C} \in \mathcal{K}$ such that, for any $\gamma \colon \binom{\mathcal{C}}{\mathcal{A}} \to \{0,1\}$ there exists $g \in \binom{\mathcal{C}}{\mathcal{B}}$ such that γ is constant on $g \circ \binom{\mathcal{B}}{\mathcal{A}}$.

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- In the definition above, one could replace (via a compactness argument) $\mathcal{C}$ by $\mathcal{M}$, where $\mathcal{M}$ is the Fraïssé limit of $\mathcal{K}$. Note that each $\binom{\mathcal{B}}{\mathcal{A}}$ above (when $\mathcal{A}, \mathcal{B} \leq K$) is essentially the same thing as a basic open subset of $\mathrm{Aut}(\mathcal{K})$.

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- The definition above forces elements of $\mathcal{K}$ to be rigid, i.e. have a trivial automorphism group.

The criterion

Theorem (Kechris–Pestov–Todorcevic)

Let $\mathcal{K}$ be a Fraïssé class, and G be the automorphism group of its limit. Then G is extremely amenable iff $\mathcal{K}$ has the Ramsey property.

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Example (Pestov)

$\mathrm{Aut}(\mathbb{Q}, \leq)$ is extremely amenable.

Translation in the context of Polish metric structures

Assume now that $\mathcal{K}$ is a metric Fraïssé class with limit $\mathcal{M}$, and $G = \mathrm{Aut}(\mathcal{M})$. Then, as previously, we have a natural directed collection of pseudometrics on G:

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The definition of $\binom{\mathcal{B}}{\mathcal{A}}$ carries over to this context; as is fitting in the continuous setting, it comes equipped with a metric (defined as above).

The approximate Ramsey property

Assume for simplicity that our metric Fraïssé classes have the property that, for all $\mathcal{A} \in \mathcal{K}$, $\mathrm{Aut}(\mathcal{A})$ is compact.

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Assume that $\mathcal{K}$ is a metric Fraïssé class. Then we say that $\mathcal{K}$ has the Ramsey property if: for any $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, and any $\varepsilon > 0$ there exists $\mathcal{C} \in \mathcal{K}$ such that, for any 1-Lipschitz $\gamma \colon \binom{\mathcal{C}}{\mathcal{A}} \to [0,1]$ there exists $g \in \binom{\mathcal{C}}{\mathcal{B}}$ such that the oscillation of γ on $g \circ \binom{\mathcal{B}}{\mathcal{A}}$ is less than ε.

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- Once this is checked, it is rather clear that we just reformulated the finite oscillation stability of the action of G on (G, d_A).

Proposition (M.–Tsankov)

Let $\mathcal{K}$ be a metric Fraïssé class and G be the automorphism group of its limit. Then G is extremely amenable iff $\mathcal{K}$ has the approximate Ramsey property.

A link with the extension property

Definition

A metric Fraïssé class $\mathcal{K}$ has the extension property if for any $\mathcal{A} \in K$ and any finite set P of partial automorphisms of $\mathcal{A}$ there exists $\mathcal{B} \in \mathcal{K}$ in which $\mathcal{A}$ embeds in such a way that all elements of $\mathcal{A}$ extend to automorphisms of $\mathcal{B}$.

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Let $\mathcal{K}$ be a metric Fraïssé class. We say that $\mathcal{K}$ has the ℓ_1-property if for any $\mathcal{A} \in \mathcal{K}$ and all n there is a way to turn (A^n, d_1) into an element of $\mathcal{K}$ so that a product of automorphisms of $\mathcal{A}$ induces an automorphism of that structure.

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Above, $d_1(\bar{a}, \bar{b}) = \dfrac{1}{n} \sum_{i=1}^{n} d(a_i, b_i)$.

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Theorem (M.–Tsankov)

Assume that $\mathcal{K}$ is a metric Fraïssé class with the extension property and the ℓ_1 property. Then $\mathcal{K}$ has the approximate Ramsey property.

Universal minimal flows

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By definition, G is extremely amenable iff its u.m.f is a singleton.

Theorem (Kechris–Pestov–Todorcevic)

The universal minimal flow of a Polish group G is an inverse limit of metrizable G-flows.

Some structure theorems

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This is why a Polish group is extremely amenable iff every continuous action on a compact metric space has a fixed point.

Theorem (Kechris–Pestov–Todorcevic)

The universal minimal flow of an infinite locally compact topological group is not metrizable.

How to compute a universal minimal flow?

Observation

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How true/useful the observation above is depends on the definition of "large": for instance, if G has an extremely amenable subgroup G^* of finite index, then the u.m.f of G is the action of G on the finite set G/G^*.

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Definition

Let G be a Polish group, and H be a closed subgroup. We endow the coset space G/H with the quotient uniformity - If d_r is a right-invariant distance on G, this is induced by

$$d(gH, g'H) = \inf\{d_r(gh, g') \colon h \in H\}$$

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We say that H is coprecompact if the space $(G/H, d_r)$ is precompact (its completion is compact). Equivalently: for every open $V \ni 1$, there exists a finite F such that $VFH = G$.

A general strategy to compute universal minimal flows

Proposition

Assume that G is Polish, and that H is a closed, coprecompact, extremely amenable subgroup. Then any minimal G flow is a quotient of the translation action of G on $\widehat{G/H}$.

So, if the action of G on $\widehat{G/H}$ is minimal, it is the u.m.f of G. In that case the u.m.f of G is metrizable with a G_δ orbit.

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Sketch of proof.

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We then check that this map is uniformly continuous, thus it extends to a continuous map from $\widehat{G/H}$ to X, which is still G-equivariant by continuity of the actions of G on $\widehat{G/H}$ and on X.

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The G_δ orbit is the set G/H: it is Polish, hence G_δ in its completion. $\square$

The universal minimal flow of S_∞.

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During this computation it becomes clear that the completion of $S_\infty/\mathrm{Aut}(\mathbb{Q})$ is isomorphic to LO.
And it is immediate that the action of S_∞ on LO is minimal.

$\square$

Where do metrizable u.m.f come from?

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In that case the u.m.f must have a G_δ orbit (because G/G^* is Polish, hence G_δ in its completion).

Where do metrizable u.m.f come from?

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Question (Angel–Kechris–Lyons)

Let G be a Polish group with a metrizable u.m.f. K. Must there exist a G_δ orbit in K?

The nonarchimedean case

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Theorem (Zucker)

Let G be a nonarchimedean Polish group with metrizable u.m.f. Then there exists a coprecompact extremely amenable closed subgroup G^* such that the u.m.f of G is $\widehat{G/G^*}$.

The case of general Polish groups.

Zucker's construction does not appear to easily translate to all Polish groups. Using a different (and independent/simultaneous) approach, one can show the following.

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Theorem (M.–Nguyen Van Thé–Tsankov)

Let G be a Polish group. Then the u.m.f of G is metrizable and admits a G_δ orbit iff it is of the form $\widehat{G/G^*}$ for an extremely amenable, closed, coprecompact subgroup G^*.

Some structural consequences

Proposition (M.–Nguyen Van Thé–Tsankov)

Let G be a Polish group whose u.m.f is metrizable with a G_δ orbit. Then every minimal G-flow has compact automorphism group and is coalescent: every endomorphism is an automorphism.

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Proposition (M.–Nguyen Van Thé–Tsankov)

Let G be a Polish group whose u.m.f is metrizable with a G_δ orbit. Then the equivalence relation of isomorphism of minimal G-flows is smooth .

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Is the u.m.f of every Roelcke precompact Polish group metrizable?

Proving this for oligomorphic nonarchimedean Polish groups would already be a fantastic result.

Thank you for your attention!