Metric groups in continuous logic

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Continuous structures

A countable continuous signature:

$$L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\}.$$

Definition

A **metric L-structure** is a complete metric space (M, d) with d bounded by d_0 (usually $d_0 = 1$), along with a family of uniformly continuous operations F_k on M and a family of predicates R_l, i.e. uniformly continuous maps from appropriate M^{k_l} to $[0, 1]$.

It is usually assumed that to a predicate symbol R_i a continuity modulus γ_i is assigned so that when $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of M satisfies

$$|R_i(x_1, ..., x_j, ..., x_{k_i}) - R_i(x_1, ..., x'_j, ..., x_{k_i})| < \varepsilon.$$

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Metric groups - continuous structures

Metric groups will be considered as one-sorted continuous structures in the continuous signature

$$L = \{d, \cdot, ^{-1}\},$$

where d denotes the metric (bounded by 1)
To the symbols $^{-1}$ and $\cdot$ continuity moduli γ_1 and γ_2 are assigned so that

$$d(x_1, x_2) < \gamma_1(\varepsilon) \text{ implies } d(x_1^{-1}, x_2^{-1}) < \varepsilon \text{ and}$$

$$d(x_1, x_2) < \gamma_2(\varepsilon) \text{ implies } max(d(x_1 \cdot y, x_2 \cdot y), d(y \cdot x_1, y \cdot x_2)) < \varepsilon$$

$$\text{for all } y \in M.$$

Formulas

A **continuous formula** is an expression built from 0,1 (assuming $d_0 = 1$) and atomic formulas by applications of the following functions:

$$x/2\ ,\ x\dot{-}y = max(x-y,0)\ ,\ min(x,y)\ ,\ max(x,y)\ ,\ |x-y|\ ,$$

$$\neg(x) = 1-x\ ,\ x\dot{+}y = min(x+y,1)\ ,\ x\cdot y = min(x\cdot y,1)\ ,\ sup_x \text{ and } inf_x.$$

Any **atomic formula** has one of the following forms
- $R(t_1,...t_k)$ for a relational symbol R or
- $d(t_1,t_2)$, where t_i are terms.

Any continuous sentence $\phi(\bar{c})$ can be naturally interpreted in a continuous structure M by the **value** $\phi^M(\bar{c})$.

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An example of a formula of group theory

$$max(sup_y d(x_1 \cdot y, x_2 \cdot y), sup_y d(y \cdot x_1, y \cdot x_2))$$

Continuous theories

The **continuous theory** of a class K of structures of the signature L is the set of all **statements**

$$\phi \leq 0$$

where ϕ is a continuous L-sentence so that the **value** ϕ^M is 0 for all $M \in K$.

Metric groups which are continuous structures

Not all metric groups can be viewed as metric structures!

Lemma

Let a group (G, d) be a metric L-structure with respect to continuity moduli γ_1 and γ_2 as above. Then G admits a two-sided-invariant metric d^* which defines the same topology with d.

Proof. We assume that (G, d) is not discrete . Let $d^*(x,y) = sup_{u,v} d(u \cdot x \cdot v, u \cdot y \cdot v),$. Then $d^*(x,y)$ is a two-sided-invariant metric with $d(x,y) \leq d^*(x,y)$. Since for every ε we have

$$d(x,y) < \gamma_2(\gamma_2(\varepsilon)) \Rightarrow d^*(x,y) < \varepsilon,$$

each open d-ball contains an open d^*-ball and vice versa. $\square$

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SIN

SIN-groups are topological groups having arbitrary small neighbourhoods of identity which are invariant under all inner automorphisms .

Lemma

Let (G, d) be a **SIN***-group with a left-invariant metric $d \leq 1$. Then $(G, \cdot, ^{-1}, d)$ is a metric structure with respect to some continuity moduli γ_1 and γ_2 as above.*

The class of topological groups admitting bi-invariant complete metrics coincides with the class of completely metrizable SIN-groups (V.Klee).

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1. Separably categorical l.c.groups

2. Axiomatisable classes of metric groups

3. Axiomatisability of some geometric properties (connected with actions on metric spaces).

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Separably cateogorical l.c. groups

SEPARABLY CATEGORICAL STRUCTURES

Separable categoricity

In the discrete case a theory T is **countably categorical** if any two countable models of T are isomorphic.

A continuous theory T is **separably categorical** if any two separable models of T are isomorphic.

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Oligomorfic actions

Ryll-Nardzewski: A countable discrete structure M is ω-categorical if and only if $Aut(M)$ is **oligomorphic**, i.e. for every n, $Aut(M)$ has finitely many orbits on M^n.

Definition. An isometric action of Γ on a metric space $(\mathbf{X}, d)$ is said to be **approximately oligomorphic** if for every $n \geq 1$ and $\varepsilon > 0$ there is a finite set $F \subset \mathbf{X}^n$ such that

$$\Gamma \cdot F = \{g\bar{x} : g \in \Gamma \text{ and } \bar{x} \in F\}$$

is ε-dense in $(\mathbf{X}^n, d)$.

C. Ward Henson: *Assuming that Γ is the automorphism group of a non-compact separable continuous metric structure M, Γ is approximately oligomorphic if and only if the structure M is separably categorical.*

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S.c.l.c. groups

Theorem

Let (G, d) be a continuous structure which is a locally compact group with respect to a left invariant metric d.

Then (G, d) is **separably categorical** *if and only if there is a compact clopen subgroup $H < G$ which is invariant with respect to all metric automorphisms of G, and the induced action of $Aut(G, d)$ on the coset space G/H is oligomorphic.*

In this case if the connected component of the unity G^0 is not trivial, H can be taken to be G^0.
When this happens or when d is two-sided-invariant, the subgroup H is normal and G/H is an ω-categorical discrete group.

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Proof. Sketch.

Necessity. We may assume that G is not discrete.
There is a rational number $\rho < 1$ such that the ρ-ball of the unity $B_\rho(e) = \{x \in G : d(x,e) \le \rho\}$ is compact.
Let G_ρ be the subgroup generated by $B_\rho(e)$. Note that G_ρ is an open (in fact clopen) subgroup.

Lemma

The predicate $P(x) = d(x, G_\rho)$ is definable in G,
i.e. there is a sequence of formulas $\phi_k(x)$, $k \ge 1$, so that the maps interpreting $\phi_k(x)$ in G converge to $P(x)$ uniformly.

Using this one can deduce that there is a natural number n such that $G_\rho = B^n_\rho(e)$.

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Locally compact groups - s.c. metric structures

Corollary

A locally compact group G can be presented as a **separably categorical metric structure** *with respect to some metric if and only if*

there is a normal compact clopen subgroup $H < G$ with a bi-invariant metric d

so that d is conjugacy invariant and

the group of automorphisms of G/H which are induced by automorphism of G preserving (H, d) is oligomorphic

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Construction of appropriate metrics

Assume that $H < G$ is a compact, clopen normal subgroup and d be a bi-invariant metric on H as in the formulation of diameter $\leq \frac{1}{2}$.
Consider G/H with respect to the $\{0,1\}$-metric, say d_0.
Choose a set X representing the H-cosets of G.
Wreath product of metrics $d^* = d_0$ wr d:
Assuming that $x_i y_i$ represents the coset $x_i H$ with $x_i \in X$, $i \in \{1,2\}$,

$$\text{if } x_1 \neq x_2 \text{ then } d^*(x_1 y_1, x_2 y_2) = 1,$$

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Stability

A continuous theory T **has the order property** if there is a formula $\psi(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are of the same length and sorts, and there is a model M of T with $(\bar{a}_i : i \in \omega) \subseteq M$, so that

$$\psi(\bar{a}_i, \bar{a}_j) = 0 \Leftrightarrow i < j \text{ and } \psi(\bar{a}_i, \bar{a}_j) = 1 \Leftrightarrow i \geq j.$$

By compactness this condition is equivalent to the property that for all n and $\delta \in [0, 1]$ there are $\bar{a}_1, ..., \bar{a}_n$ such that

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Stable l.c. s.c. groups

Theorem

Let G be a locally compact group which can be presented as a separably categorical metric structure and let H and d^* be as in the construction of the corollary.

Then the continuous theory $Th(G, d^*)$ is stable if and only if the elementary theory of G/H is stable.

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Axiomatisability

AXIOMATISABILITY

$N \models Th(\mathcal{K})$

Proposition. Let $\mathcal{K}$ be a class of continuous structures. A metric structure N is a model of the continuous theory of $\mathcal{K}$ if and only if it is elementarily embeddable into a metric ultraproduct of structures from $\mathcal{K}$.

N is an **elementary substructure** of M,
if $N \subset M$ and for every L-formula $\phi(x_1, ..., x_n)$ and $a_1, ..., a_n \in N$ the values of $\phi(a_1, ..., a_n)$ in N and in M are the same.

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Ultraproducts

The **metric** in the ultraproduct $\prod_I (G_i, d_i)/D$ is defined by the rule that
the distance between $(g_i)_I$ and $(g'_i)_I$ is in the interval $(\varepsilon_1, \varepsilon_2)$ if and only if
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SUP

The continuous *sup*-**theory** of $\mathcal{K}$, $Th^c_{\sup}(\mathcal{K})$, consists of all closed L-conditions

$$sup_{x_1} sup_{x_2} ... sup_{x_n} \varphi = 0 \text{ (} \varphi \text{ does not contain } inf_{x_i}, \; sup_{x_i} \text{),}$$

which hold in all structures of $\mathcal{K}$.

Proposition. N is a model of the continuous *sup*-theory of $\mathcal{K}$ if and only if it is embeddable into a metric ultraproduct of structures from $\mathcal{K}$ as a closed subspace.

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Axiomatisability

$\mathcal{C}$ is **axiomatizable** if any model of $Th(\mathcal{C})$ belongs to $\mathcal{C}$.

Corollary

(1) The class $\mathcal{C}$ is **axiomatizable** *in continuous logic if an only if it is closed under metric isomorphisms, ultraproducts and taking elementary submodels.*
(2) The class $\mathcal{C}$ is **axiomatizable** *in continuous logic* **by** $Th^c_{sup}(\mathcal{C})$ *if an only if it is closed under metric isomorphisms, ultraproducts and taking substructures.*

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Soficity

An abstract group is **sofic** if it is embeddable into a metric ultraproduct of finite symmetric groups with Hamming metrics.

Metric sofic groups. Let $\mathcal{S}$ be the class of complete *id*-continuous metric groups of diameter 1, which are embeddable as closed subgroups into a metric ultraproduct of finite symmetric groups with Hamming metrics.

Corollary

The class of metric sofic groups is sup-axiomatizable (i.e. by its theory Th^c_{sup}).

The set of all abstract sofic groups consists of all discrete structures of the class $\mathcal{S}$.

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FH: any strongly continuous isometric affine action on a real Hilbert space has a fixed point.
F$\mathbb{R}$: any isometric action on a real tree has a fixed point.
OB: any isometric strongly continuous action on a metric space has bounded orbits. (implies **FH** and **F**$\mathbb{R}$)

There are no natural versions of these properties which can be axiomatized.

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Bountiful classes

BOUNTIFUL CLASSES OF GROUPS

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A class of groups $\mathcal{K}$ is called **bountiful** if for any pair of infinite groups $G \leq H$ with $H \in \mathcal{K}$ there is $K \in \mathcal{K}$ such that $G \leq K \leq H$ and $|G| = |K|$ (Ph.Hall).

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Löwenheim-Skolem

Proof. non-**OB**. Let $G \leq H$ with $H \in \mathcal{K}$. Take an action of H on a metric space with an unbounded orbit. Extend G by countably many elements witnessing this unboundedness.

Löwenheim-Skolem Theorem. *Let κ be an infinite cardinal number and assume $|L| \leq \kappa$.*
Let M be an L-structure and suppose $A \subset M$ has density $\leq \kappa$. Then there exists a substructure $N \subseteq M$ containing A such that density$(N) \leq \kappa$ and N is an elementary substructure of M.

Corollary. non-**OB**. By L-S theorem there is an elementary substructure K of H such that $G \leq K$ and the density character of G and K is the same.

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Kopperman and Mathias: By the Löwenheim-Skolem theorem if $\mathcal{K}$ is a reduct of a class axiomatizable in a countable fragment of $L_{\omega_1\omega}$ then $\mathcal{K}$ is bountiful.

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Kazhdan

Let a topological group G have a strongly continuous unitary representation on a Hilbert space $\mathbf{H}$. A closed subset $Q \subset G$ has an *almost ε-invariant vector* in $\mathbf{H}$ if

there exists $v \in \mathbf{H}$ such that $sup_{x \in Q} \parallel x \circ v - v \parallel < \varepsilon \cdot \parallel v \parallel$.

We call a closed subset Q of the group G a *Kazhdan set* if there is ε s.t. every unitary representation of G on a Hilbert space with almost (Q, ε)-invariant vectors also has a non-zero invariant vector. If the group G has a compact Kazhdan subset then it is said that G has *property* $(\mathbf{T})$ *of Kazhdan*.

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Hilbert spaces over $\mathbb{R}$

We identify a **Hilbert space** over $\mathbb{R}$ with a many-sorted metric structure

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r\in\mathbb{R}}, +, -, \langle\rangle),$$

where B_n is the ball of elements of norm $\leq n$ (i.e. we allow diameters of sorts greater than 1),
$I_{mn} : B_m \to B_n$ is the inclusion map,
$\lambda_r : B_m \to B_{km}$ is scalar multiplication by r, with k the unique integer satisfying $k \geq 1$ and $k-1 \leq |r| < k$;
futhermore, $+, - : B_n \times B_n \to B_{2n}$ are vector addition and subtraction and
$\langle\rangle : B_n \times B_n \to [-n^2, n^2]$ is the inner product.
The metric on each sort is given by $d(x,y) = \sqrt{\langle x-y, x-y\rangle}$.
For every operation the continuous modulus is standard.

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This approach can be extended to complex Hilbert spaces.

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}),$$

We only extend the family $\lambda_r : B_m \to B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \to B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k-1 \leq |c| < k$.
We also introduce *Re*- and *Im*-parts of the inner product.

We axiomatise infinite dimensional Hilbert spaces:

$$inf_{x_1,\ldots,x_n} max_{1\leq i<j\leq n}(|\langle x_i, x_j\rangle - \delta_{i,j}|) = 0,$$

$$\delta_{i,j} \in \{0,1\} \text{ with } \delta_{i,j} = 1 \leftrightarrow i = j,$$

It is known that this class is κ-**categorical** for all infinite κ, and the corresponding continuous theory admits elimination of quantifiers.

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non-(T) is bountiful

Definition. Let (G, d) be a metric group. Let $\mathcal{F} = \{F_1, F_2, ...\}$ be a family of continuity moduli for continuous functions $G \times B_i \to B_i$.
We call a closed subset Q of the group G an $\mathcal{F}$-**Kazhdan set** if there is ε s.t. every $\mathcal{F}$-continuous unitary representation of G on a Hilbert space with almost (Q, ε)-invariant vectors also has a non-zero invariant vector.

Theorem

Let (G, d) be a locally compact metric group which does not have property $(\mathbf{T})$. Then for any infinite subset $C \subset G$ there is a closed elementary (in continuous logic) subgroup G_0 of G containing C, with the same density character as C so that the G_0 does not have property $(\mathbf{T})$.

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Proof

Let us consider a class of continuous metric structures which are unions of groups $(\Gamma, \cdot, ^{-1}, d)$ together with metric structures of complex Hilbert spaces

$$(\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c\in\mathbb{C}}, +, -, \langle\rangle_{Re}, \langle\rangle_{Im}).$$

Such a structure (say $A(\Gamma, \mathbf{H})$) also contains a binary operation $\circ$ of an action which is considered as a family of maps $\Gamma \times B_n \to B_n$, $n \in \omega$.

Adding the continuous *sup*-axioms that the action is linear and unitary, we obtain an axiomatizable class $\mathcal{K}_{GH}$.

Let $\mathcal{K}_{GH}(\mathcal{F})$ be the corresponding class with continuity moduli $\mathcal{F} = \{F_0, F_1, ...\}$ for the corresponding restrictions of $\circ$ on B_n.

Continuation

Choose ε so that the ε-ball (say K) of 1 in G is compact.
Assuming that the continuity moduli $\mathcal{F}$ are fixed
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which in fact say that each $\bigcup x_n K$ has an almost $\frac{1}{n}$-invariant unit vector in $\mathbf{H}$.

The group G from the formulation (without property **(T)** for $\mathcal{F}$-actions) has such an expansion to $\mathbf{H}$.
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Pointed $\mathbb{R}$-trees

Sylvia Carlisle, 2009: The class of poined **R**-trees is axiomatizable in continuous logic in the class of many-sorted metric structures of n-balls of pointed spaces (0 is distinguished)

$$\mathbf{X} = (\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, d), \text{ where } I_{mn} : B_m \to B_n.$$

by the following axioms:

The **approximated midpoint property**: for any $x, y \in \mathbf{X}$ and any $\varepsilon > 0$ there exists $z \in \mathbf{X}$ such that

$$|d(x,z) - d(x,y)/2| \leq \varepsilon \text{ and } |d(y,z) - d(x,y)/2| \leq \varepsilon.$$

$\mathbf{X}$ is 0-**hyperbolic** if for any $x, y, z, w \in \mathbf{X}$ and $\varepsilon > 0$

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$$|d(x,z) - d(x,y)/2| \leq \varepsilon \text{ and } |d(y,z) - d(x,y)/2| \leq \varepsilon.$$

$\mathbf{X}$ is 0-**hyperbolic** if for any $x, y, z, w \in \mathbf{X}$ and $\varepsilon > 0$

$$min((x \cdot y)_w, (y \cdot z)_w) \leq (x \cdot z)_w + \varepsilon \text{ , where}$$

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(w,y) - d(x,y)) \text{ (Gromov product).}$$

Actions on pointed trees

Let us consider the class of continuous metric structures which are unions of continuous groups $(G, \cdot, ^{-1}, d)$ together with many-sorted metric structures of n-balls of pointed $\mathbb{R}$-trees

$$\mathbf{X} = (\{B_n\}_{n\in\omega}, 0, \{I_{mn}\}_{m<n}, d) \text{ , where } I_{mn} : B_m \to B_n,$$

and ternary predicates (describing an action $\circ$ of G on $\mathbf{X}$)

$$\circ_{mn}(g, x, y) : G \times B_m \times B_n \to [0, diam(B_n)] \text{ , } m \leq n \in \omega.$$

$\circ_{mn}(g, x, y)$ is the **length** of the intersection of the segment $[g \circ x, y]$ with B_n.
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Continuity moduli of actions on $\mathbb{R}$-trees

Observation Assume that a group G acts on the tree $\mathbf{X} = \bigcup B_n$ by isometries and the action $\circ$ has a continuity modulus γ_G on G. Let $A(G, \mathbf{X})$ be the corresponding structure in the language as above.
Then the predicates $\circ_{mn}$ have continuity moduli $min(\gamma_G, id)$.

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We say that (G,d) satisfies non-$\mathbf{F}\mathbb{R}(\mathcal{F})$ if it has an isometric $\mathcal{F}$-action on an $\mathbb{R}$-tree (i.e. extends to a structure from $\mathcal{K}_{iso-\mathbb{R}}(\mathcal{F})$) without fixed points.

Observation. *When $A(G,\mathbf{X})$ does not have fixed points there is a rational number q so that for every k there is a natural number $l > k$ and $g \in G$ which takes B_k into B_l and satisfies*

$$sup_{v\in B_k}(q\dot{-}d(g\circ v,v)) = 0.$$

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Polish version of non-**OB**

If G is a Polish group then G is non-**OB** if and only if there is an open symmetric $\emptyset \neq V \subset G$ such that for any finite $F \subset G$ and natural k, $G \neq (FV)^k$
(i.e. there is a real number $\varepsilon_{F,k}$ s.t. some $g \in G$ is ε-distant from $(FV)^k$).

Definition. A metric group G is *uniformly non-***OB** if there is an open symmetric $\emptyset \neq V \subset G$ such that for any natural numbers m and k there is a real number ε s.t. for any m-element $F \subset G$ there is $g \in G$ which is ε-distant from $(FV)^k$.

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Uniform non-**OB**

Theorem. The class of uniformly non-**OB**-groups is bountiful.

In fact for any uniformly non-**OB** group G there is an expansion of G by two continuous unary predicates (imitating functions $d(x, V)$ and $d(x, G \setminus V)$) where uniform non-**OB** is expressible in continuous logic.

Discrete groups

Theorem. The following classes of groups are bountiful (they are reducts of axiomatizable classes in $L_{\omega_1 \omega}$):
(1) The complement of the class of strongly bounded groups;
(2) The class of groups of cofinality $\leq \omega$;
(3) The class of groups which are not Cayley bounded;
(4) The class of all non-trivial free products with amalgamation (or HNN-extensions);
(5) The class of groups having homomorphisms onto $\mathbb{Z}$;
(6) The class of groups which do not have property **FA** (i.e. with actions on simplicial trees without fixed points).

Dfntns. G is *Cayley bounded* if for every $U \subset G$ with $\langle U \rangle = G$, there exists $n \in \omega$ such that $G \subseteq (U \cup U^{-1} \cup \{1\})^n$.
A group is *strongly bounded* if it is Cayley bounded and cannot be presented as the union of a strictly increasing chain $\{H_n : n \in \omega\}$ of proper subgroups (i.e. has *cofinality* $> \omega$).

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Remark.

If G is a Polish group then G is *topologically Cayley bounded* if for every analytic generating subset $U \subset G$ there exists $n \in \omega$ such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$.

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