# Centralizers in infinite locally finite simple groups

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After the classification of finite simple groups in 1980, the applications of the known results in finite simple groups to locally finite simple groups were an active research area.

As is well known the centralizers of elements (involutions) gave a lot of information about the simple groups.

There were two types of questions:

(1) Given a simple group find the structure of the centralizers of elements.

(2) Given the structure of the centralizers of elements in a simple group, determine the simple groups having a subgroup whose centralizer is isomorphic to the given one.

The structure of centralizers of elements in simple groups were studied by many authors.

To study the centralizers of elements in locally finite simple groups in the general case depends on the Kegel sequences of the groups.

It is well known by the constructions of Zaleskii-Serezhkin that there are simple locally finite groups that cannot be written as a union of finite simple groups.

A.E. Zalesskii, V.N. Serezhkin Finite Linear Groups Generated by Reflections. Math. USSR: IZV. 17 (1981), 477-503.

K. Hickin Universal locally finite central extensions of groups. Proc. London Math. Soc. (1986), 53-72.

Question 5.18 posed by O. H. Kegel in Kourovka Notebook (Unsolved problems in Group Theory):

**Question** Let $G$ be an infinite locally finite simple group. Is the centralizer of every element of $G$ infinite?

We proved together with Hartley:

**Theorem 1** *(London Math. Soc. Proc.(1988). In an infinite locally finite simple group, the centralizer of every element is infinite.*

But today I will talk about a simple locally finite group which can be written as a union of finite simple groups.

In this nice situation one can determine the structure of a centralizer of an element and centralizers of finite subgroups in detail.

# Some definitions

## Finitary Symmetric Groups

Let $\Omega$ be an infinite set and $\alpha \in Sym(\Omega)$.

Support of $\alpha$ is the set of elements which are moved by $\alpha$, namely:

$$Supp(\alpha) = \{i \in \Omega : \alpha(i) \neq i \ \ \}$$

Then

$$FSym(\Omega) = \{\alpha \in Sym(\Omega) : \ |Supp(\alpha)| < \infty\}.$$

# Finitary Symmetric Groups

The finitary symmetric group $FSym(\Omega)$ is a locally finite group of cardinality equal to the cardinality of $\Omega$.

Recall that a group is called a **locally finite group** if every finitely generated subgroup is a finite group.

An easy example of an infinite locally finite, simple group of cardinality $\kappa$ is $Alt(\Omega)$ where $|\Omega| = \kappa$.

For $\Omega = \mathbb{N}$, we have

$$FSym(\mathbb{N}) = \bigcup_{n=1}^{\infty} Sym(n)$$

# Finitary Symmetric Groups

We may consider $FSym(\mathbb{N})$ as a union of finite symmetric groups each embedded into the next one by an identity homomorphism:

$$S_2 \leq S_3 \leq S_4 \leq \dots$$

and

$$FSym(\mathbb{N}) = \bigcup_{n=1}^{\infty} S_n$$

# Embeddings

If we take the non-identity embedding of $S_n$ into $S_{k_i n}$ and then into $S_{k_{i+1} k_i n}$

$$S_n \to S_{k_i n} \to S_{k_{i+1} k_i n} \to \dots$$

with respect to the given infinite sequence of positive integers $(k_1, k_2, \dots, k_i, \dots)$ and then we take the direct limit of these groups we have a new infinite locally finite group.

# Embeddings

## Questions

1. What are the structure of such groups?
2. When are they simple locally finite groups?
3. What can we say about the cardinality of the automorphism groups?

The direct limit method is a natural method to produce infinite, simple locally finite groups.

# Construction of Groups of type $S(\xi)$

# Groups of type $S(\xi)$

Let $\alpha$ be the permutation defined by

$$\alpha = \begin{pmatrix} 1 \dots & n \\ i_1 \dots & i_n \end{pmatrix}$$

Then the permutation

$$d^r(\alpha) = \left( \begin{array}{ccc|ccc|c|ccc} 1 & \dots & n & n+1 & \dots & 2n & \dots & (r-1)n+1 & \dots & rn \\ i_1 & \dots & i_n & n+i_1 & \dots & n+i_n & \dots & (r-1)n+i_1 & \dots & (r-1)n+i_n \end{array} \right)$$

is called a **homogeneous $r$-spreading** of the permutation $\alpha$.

**Example.** Consider the permutation

$$\alpha = (12) \in Sym(3).$$

Embed $\alpha$ into $Sym(9)$ by homogenous $3$-spreading.

We obtain

$$d^3(\alpha) = \left( \begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 5 & 4 & 6 & 8 & 7 & 9 \end{array} \right) = (1,2)(4,5)(7,8)$$

## Groups of type $S(\xi)$

Let $\Pi$ be the set of sequences consisting of prime numbers. Let $\xi \in \Pi$ and $\xi = (p_1, p_2, \ldots)$ be a sequence consisting of not necessarily distinct primes $p_i$.
We obtain direct systems from the following embeddings:

$$\{1\} \xrightarrow{d^{p_1}} S_{n_1} \xrightarrow{d^{p_2}} S_{n_2} \xrightarrow{d^{p_3}} S_{n_3} \xrightarrow{d^{p_4}} \ldots$$

$$\{1\} \xrightarrow{d^{p_1}} A_{n_1} \xrightarrow{d^{p_2}} A_{n_2} \xrightarrow{d^{p_3}} A_{n_3} \xrightarrow{d^{p_4}} \ldots$$

where $n_i = n_{i-1}p_i, \quad i = 1, 2, 3\ldots$ and $S_{n_i}$ is the symmetric group on $n_i$ letters and $A_{n_i}$ is the alternating group on $n_i$ letters and $n_0 = 1$.

The direct limit groups are denoted by $S(\xi)$ and $A(\xi)$ respectively.

**Definition.** The embedding $d$ of the transitive permutation group $(G, X)$ into the permutation group $(H, Y)$ is called a **diagonal embedding** if the restriction of $d(G)$ on every orbit of length greater than one is permutation isomorphic to $(G, X)$.

A diagonal embedding is called a **strictly diagonal embedding** if the length of every orbit of the image $d(G)$ on the set $Y$ is greater than one.

## Groups of type $S(\xi)$

The construction of these groups are discussed in Kegel-Wehfritz's book, Locally Finite Groups [2], (1976) and then studied by N. V. Kroshko-V. I. Sushchansky, in

"Direct limits of symmetric and alternating groups with strictly diagonal embeddings, Arch. Math. 71, (1998), [4]".

The same year Zalesskii published the following survey not only for groups but for locally finite dimensional algebras, Lie algebras and so on.

Zaleskii A. E.; Direct limits of finite dimensional algebras and finite groups, Canadian Mathematical conference proceeding, 22, $221 - 239$, (1998).

## Basic properties of groups $S(\xi)$

It is proved that such groups satisfy the followings:

- If the prime $2$ appears infinitely often in the sequence $\xi$, then the direct limit group $S(\xi)$ is a simple non-linear, non-finitary locally finite group.
- If a prime $p$ appears infinitely often, then $S(\xi)$ contains an isomorphic copy of the locally cyclic $p$-group $C_{p^\infty}$.
- The group $FSym(\mathbb{N})$ does not contain $C_{p^\infty}$ for any prime $p$.

# Steinitz numbers

Recall that the formal product $n = 2^{r_2}3^{r_3}5^{r_5}\ldots$ of prime powers with $0 \le r_k \le \infty$ for all $k$ is called a **Steinitz number** (supernatural number).

The set of Steinitz numbers form a partially ordered set with respect to division, namely if $\alpha = 2^{r_2}3^{r_3}5^{r_5}\ldots$ and $\beta = 2^{s_2}3^{s_3}5^{s_5}\ldots$ be two Steinitz numbers, then $\alpha|\beta$ if and only if $r_p \le s_p$ for all prime $p$.

Moreover they form a lattice if we define meet and join as $\alpha \wedge \beta = 2^{min\{r_2,s_2\}}3^{min\{r_3,s_3\}}5^{min\{r_5,s_5\}}\ldots$ and $\alpha \vee \beta = 2^{max\{r_2,s_2\}}3^{max\{r_3,s_3\}}5^{max\{r_5,s_5\}}\ldots$.

For each Steinitz number $\xi$ we can define a strictly diagonal group $S(\xi)$ and for each strictly diagonal group $S(\xi)$ we have a Steinitz number.

Let

$$\Sigma = \{S(\xi) : \ \xi \text{ is a Steinitz number}\}$$

be the set of groups (up to isomorphism) obtained as a direct limit of finite symmetric groups.
On the set $\Sigma$ there is a natural order, namely with respect to being a subgroup.

## Characteristic of $S(\xi)$

For each sequence $\xi$ we define $Char(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}} \ldots$ where $r_{p_i}$ is the number of times that the prime $p_i$ repeat in $\xi$. If it repeats infinitely often, then we write $p_i^{\infty}$.

Therefore for each $\xi \in \Pi$, there corresponds a Steinitz number $Char(\xi)$.
For a group $S(\xi)$ obtained from the sequence $\xi$ we define $Char(S(\xi)) = Char(\xi)$.

**Theorem 2** *(Kuroshko-Sushchansky, 1998)*
*Two groups $S(\xi_1)$ and $S(\xi_2)$ are isomorphic if and only if $Char(S(\xi_1)) = Char(S(\xi_2))$.*

The groups $S(\xi)$ are classified by Kroshko-Sushchansky by using the lattice of Steinitz numbers.

There are uncountably many pairwise non-isomorphic simple locally finite groups of type $S(\xi)$.

Moreover Kroshko-Sushchansky proved that there is a lattice isomorphism between the lattice of groups $S(\xi)$ and the lattice of Steinitz numbers.

Centralizers of elements in groups $S(\xi)$

## Centralizers of elements in finite symmetric groups

Since our groups $S(\xi)$ are obtained as direct limits of finite symmetric groups we remind the centralizers of elements in finite symmetric groups.

## Centralizers of elements in finite symmetric groups

Let $g \in S_n$. The **cycle type** of $g$ is denoted by $t(g) = (r_1, r_2, \ldots, r_n)$ where $r_i$ is the number of cycles of length $i$ in the cycle decomposition of $g$.
Then

$$C_{S_n}(g) \cong \mathop{Dr}_{i=1}^{n} C_i \wr S_{r_i}$$

if $r_i = 0$, then we assume the corresponding factor is $\{1\}$.
**Example.** For $g = (12)(34)\ldots(n-1n)$ with $k$, $2$-cycles where $2k = n$, then the centralizer is

$$C_{S_n}(g) \;\cong\; C_2 \wr S_k$$

# Principal beginning

For the centralizers of elements in $S(\xi)$.

For an element $g \in S(\xi)$ we define the principal beginning as $g_0$ where $g_0 \in S_{n_i}$ and $n_i$ is the smallest in the chain (changing the order is allowed to take the minimum).

## Centralizers of elements in $S(\xi)$

**Theorem 3** *(Güven, Kegel, Kuzucuoğlu [1]) Let* $\xi$ *be an infinite sequence,* $g \in S(\xi)$ *and the type of principal beginning* $g_0 \in S_{n_k}$ *be* $t(g_0) = (r_1, r_2, \ldots, r_{n_k})$. *Then*

$$C_{S(\xi)}(g) \cong \underset{i=1}{\overset{n_k}{Dr}}\, C_i(C_i \bar{\wr}\, S(\xi_i))$$

*where* $Char(\xi_i) = \frac{Char(\xi)}{n_k} r_i$ *for* $i = 1, \ldots, n_k$.
*If* $r_i = 0$, *then we assume that corresponding factor is* $\{1\}$.

## Examples

For the centralizers of elements we have the following examples:

**Example 1.** For $Char(\xi) = 2357\dots$ and $\alpha_0 = (12)$ we have

$$C_{S(\xi)}(\alpha) \cong \langle\alpha\rangle(C_2\bar{\wr}S(\xi_1)\ )$$

where $Char(S(\xi_1)) = \frac{Char(S(\xi))}{2} = 357\dots$.

**Example 2.** Let $Char(\xi) = 5^\infty$ and $g_0 = (12345)$. Then $C_{S(\xi)}(g) \cong \langle g\rangle(C_5\bar{\wr}S(\xi))$

Observe that in this example centralizer of an element contains an isomorphic copy of the original group.

For the centralizers of finite subgroups in $S(\xi)$, we have the following Theorem.

**Theorem 4** *(Güven, Kegel, Kuzucuoğlu [1]) Let $F$ be a finite subgroup of the infinite group $S(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of $F$ such that the action of $F$ on any two orbits in $\Gamma_i$ are equivalent. Let the type of $F$ be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then*

$$C_{S(\xi)}(F) \cong \underset{i=1}{\overset{k}{Dr}} \,(C_{Sym(\Omega_i)}(F|_{\Omega_i})(C_{Sym(\Omega_i)}(F|_{\Omega_i})\,\bar{\wr}\, S(\xi_i)))$$

*where $Char(\xi_i) = \frac{Char(\xi)}{n_{j_i}} r_i$ and $\Omega_i$ is a representative of an orbit in the equivalence class $\Gamma_i$ for $i = 1, \ldots, k$.*

## Automorphisms of $S(\xi)$

One may attach to each infinite sequence of primes $\xi$ a locally finite rooted tree in the following way:

Given a sequence $\xi = (p_1, p_2, \ldots)$. Then we start with a root $v_0$.

This is level zero and consists of the root only.

Then we attach $p_1$ vertices to the root and then to each vertex of the $1^{st}$ level we attach $p_2$ vertices and to each vertex of level number $2$ we attach $p_3$ vertices.

Then we continue in this fashion infinitely many times with respect to the given sequence $\xi$. There are $n_i = p_1 p_2 \ldots p_i$ vertices in level number $i$.

## Automorphisms of $S(\xi)$

On this tree there is a natural distance between the vertices, that is the length of the path connecting the vertices. From this there is a natural topology on the tree.

## Automorphisms of $S(\xi)$

An **end** of the rooted tree is an infinite path without repetition which starts in the root $v_0$.
We will denote by $\partial T$ the set of all ends of the tree $T$ (its boundary).
One can attach a metric on $\partial T$ putting

$$\rho(\gamma_1, \gamma_2) = \frac{1}{(n+1)}$$

where $n$ is the length of the maximum common part of the ends $\gamma_1$ and $\gamma_2$.

# Automorphisms of $S(\xi)$

The topology introduced by the metric $\rho$ is compact and has a base of open sets

$$P_{nv_i} = \{\gamma \in \partial T : \ v_i \in V_n(T) \ \}$$

where $V_n(T)$ is the set of all vertices which has length $n$ to the root $v_0$.

## Automorphisms of $S(\xi)$

The group $S(\xi)$ act as homeomorphisms of the boundary of the tree in the following way:

Let $g$ be a homeomorphism which moves basic open set $P_{nv_i}$ to the basic open set $P_{nv_j}$.

Since on the $i^{th}$ level there are $p_1 \dots p_i$ basic open sets the homeomorphisms defined above forms a group isomorphic to the symmetric group $Sym(p_1 \dots p_i)$.

We may define the vertices of the tree as a set of sequences $(i_0, i_1 \dots i_{n-1}, i_n)$ where $i_k \in \{1, 2, \dots, p_k\}$ and $n \geq 0$. Two vertices are adjacent if and only if they are of the form $(i_0, i_1, \dots i_{n-1})$, $(i_0, i_1 \dots i_{n-1}, i_n)$.

## Automorphisms of $S(\xi)$

**Theorem 5** *(Lavrenyuk-Sushchansky) Automorphism group of* $S(\xi)$ *is locally inner.*

*i.e. For every finitely generated subgroup* $F$ *and for every automorphism* $\alpha$ *of* $S(\xi)$*, there exists an element* $g_\alpha \in S(\xi)$ *such that for any* $x \in F$ *we have* $x^\alpha = x^{g_\alpha}$*.*

# Automorphisms of $S(\xi)$

**Theorem 6** *(Lavrenyuk-Sushchansky) There is a subgroup $K$ in $Aut(S(\xi))$ such that $|K|$ is uncountable and every countable residually finite group can be embedded into $K$.*

# Automorphisms of $S(\xi)$

**Question:** What is the structure of $Aut(S(\xi))$?

Now we construct homogenous finitary symmetric groups $FSym(\kappa)(\xi)$.

Let $\kappa$ be an arbitrary infinite cardinal number.
Let $FSym(\kappa)$ denote the finitary symmetric group and $Alt(\kappa)$ denote the alternating group on the set $\kappa$.

As before, let $\Pi$ be the set of sequences of prime numbers and $\xi \in \Pi$. Then $\xi$ is a sequence of not necessarily distinct primes.

Let $\alpha \in FSym(\kappa)$, $\quad ( \ Alt(\kappa) \ )$. For a natural number $p \in \mathbb{N}$ a permutation $d^p(\alpha) \in FSym(\kappa p)$ defined by
$(\kappa s + i)^{d^p(\alpha)} = \kappa s + i^{\alpha}, \quad i \in \kappa$ and $0 \leq s \leq p-1$ is called **homogeneous** $p$**-spreading** of the permutation $\alpha$.

We divide the ordinal $\kappa p$ to $p$ equal parts and on each part we repeat the permutation diagonally as in the finite case. So if $\alpha = \begin{pmatrix} 1 \ldots & n \\ \dot{i_1} \ldots & \dot{i_n} \end{pmatrix} \in FSym(\kappa)$, then the homogeneous $p-$spreading of the permutation $\alpha$ is

$$d^p(\alpha) = \left( \begin{array}{ccc|ccc|c|ccc} 1 & \ldots & n & \kappa+1 & \ldots & \kappa+n & \ldots & \kappa(p-1)+1 & \ldots & \kappa(p-1)+n \\ i_1 & \ldots & i_n & \kappa+i_1 & \ldots & \kappa+i_n & \ldots & \kappa(p-1)+i_1 & \ldots & \kappa(p-1)+i_n \end{array} \right)$$

with the assumption that the elements in $\kappa p \setminus supp(d^p(\alpha))$ are fixed.

We continue to take the embeddings using homogeneous $p$-spreadings with respect to the given sequence of primes in $\xi$.

From the given sequence of embeddings, we have direct systems and hence direct limit groups $FSym(\kappa)(\xi)$, $(Alt(\kappa)(\xi))$.

Observe that $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are subgroups of $Sym(\kappa\omega)$.

**Lemma 7** *If the characteristics of two homogenous finitary symmetric groups are different (homogenous infinite alternating groups), then the groups are non-isomorphic.*

By the above there exists uncountably many pairwise non-isomorphic simple locally finite groups for each cardinality $\kappa$.

The principal beginning $\alpha_0$ of an element $\alpha \in FSym(\kappa)(\xi)$ is defined to be the smallest positive integer $n_j \in \mathbb{N}$ such that $\alpha_0 \in FSym(\kappa n_j)$ and $\alpha_0$ is not obtained as a sequence of embeddings $d^{p_i}$ for any $p_i \in \xi$.

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**Theorem 8** *(Güven, Kegel, Kuzucuoğlu [1]) Let* $\xi$ *be an infinite sequence. If* $\alpha \in FSym(\kappa)(\xi)$ *with principal beginning* $\alpha_0 \in FSym(\kappa n_i), \quad t(\alpha_0) = (r_1, \ldots, r_n)$, *and* $|supp(\alpha_0)| = n$.

*Then*

$$C_{FSym(\kappa)(\xi)}(\alpha) \cong (\underset{i=1}{\overset{n}{Dr}}\, C_i(C_i \wr S(\xi_i))) \times FSym(\kappa)(\xi')$$

*where* $Char(\xi_i) = \frac{Char(\xi)}{n_i} r_i$ *and* $Char(\xi') = \frac{Char(\xi)}{n_i}$.
*If* $r_i = 0$, *then we assume that the corresponding factor in the direct product is* $\{1\}$.

**Theorem 9** *Let* $\kappa$ *be a fixed infinite cardinal. There is a lattice isomorphism between the lattice of groups* $\Sigma = \{FSym(\kappa)(\xi) \mid \xi \in \Pi \}$ *ordered with respect to being a subgroup and the lattice* $\mathcal{S}$ *of Steinitz numbers ordered with respect to division in Steinitz numbers.*

**Theorem 10** *(Güven, Kegel, Kuzucuoğlu [1]) Let $\xi$ be an infinite sequence of not necessarily distinct primes. Let $F$ be a finite subgroup of $FSym(\kappa)(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of $F$ such that the action of $F$ on any two orbits in $\Gamma_i$ is equivalent. Let the type of $F$ be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then*

$$C_{FSym(\kappa)(\xi)}(F) \cong (\underset{i=1}{\overset{k}{Dr}}\; C_{Sym(\Omega_i)}(F)(C_{Sym(\Omega_i)}(F)\bar{\wr}\, S(\xi_i)) \times FSym(\kappa)(\xi')$$

*where $Char(\xi_i) = \frac{Char(\xi)}{n_{j_i}} r_i$ and $Char(\xi') = \frac{Char(\xi)}{n_{j_1}}$ and $\Omega_i$ is a representative of an orbit in the equivalence class $\Gamma_i$ for $i = 1, \ldots, k$.*

**Theorem 11** *(Güven-Kuzucuoğlu) The automorphism group* $Aut(FSym(\kappa)(\xi))$ *contains a subgroup* $K$ *which is isomorphic to a product of a* $Sym(\kappa)$ *and cartesian product of* $Sym(n_i)$ *for a sequence* $n_i$ *obtained from the given sequence of primes.*

*In particular, the cardinality of* $Aut(FSym(\kappa)(\xi))$ *is* $2^{\kappa}$*.*

Question. Is every automorphism of $FSym(\kappa)(\xi)$ locally inner?

**Theorem 12** *(Kegel, Kuzucuoğlu, [3]) Let* $\kappa$ *be an infinite cardinal. If* $G = \bigcup_{i=1}^{\infty} G_i$, *where* $G_i = FSym(\kappa n_i)$, $(H = \bigcup_{i=1}^{\infty} H_i$, *where* $H_i = Alt(\kappa n_i))$ *is a group of strictly diagonal type and* $\xi = (p_1, p_2, \ldots)$, *then* $G$ *is isomorphic to the homogenous finitary symmetric group* $FSym(\kappa)(\xi)$

*(*$H$ *is isomorphic to homogenous alternating group* $Alt(\kappa)(\xi))$, *where* $n_0 = 1, \ \ n_i = p_1 p_2 \ldots p_i, \ \ i \in \mathbb{N}$.

# References

[1] Güven Ü. B., Kegel O. H., Kuzucuoğlu M.; *Centralizers of subgroups in direct limits of symmetric groups with strictly diagonal embedding*, Comm. in Algebra, **43**) (6) 1-15 (2015) .

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THANK YOU