Alternatives for pseudofinite groups.

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Introduction: Alternatives

Notation

Let F_2 be the free group on 2 generators and let M_2 be the free monoid on 2 generators.

Theorem (Tits alternative)

A linear group, i.e. a subgroup of some $GL_n(K)$, with K a field, either contains F_2 or is soluble-by-(locally finite).

S. Black considered the analog of Tits alternative in the class of finite groups. Reading her result in the class of pseudofinite groups, we get:

• Let G be an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank), then either G contains F_2 or G is nilpotent-by-abelian-by-(uniformly locally finite).

Further,

•• let G be an $\aleph_0$-saturated pseudofinite group, then either G contains M_2 or is nilpotent-by-uniformly locally finite.

See [Ould Houcine, Point, Alternatives for pseudofinite groups, 2013].

Preliminaries

Notation

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures, we will denote by $Th(\mathcal{C})$ (respectively by $Th_\forall(\mathcal{C})$) the theory of $\mathcal{C}$ (respectively the universal theory of $\mathcal{C}$), namely the set of sentences (respectively universal sentences) true in all elements of $\mathcal{C}$.

Given a set I, an ultrafilter $\mathcal{U}$ over I and a set of $\mathcal{L}$-structures $(C_i)_{i\in I}$, we denote by $\prod_I^{\mathcal{U}} C_i$ the ultraproduct of the family $(C_i)_{i\in I}$ relative to $\mathcal{U}$.

An $\mathcal{L}$-structure $\mathcal{M}$ is pseudo-$\mathcal{C}$ if $\mathcal{M}$ is a model of $Th(\mathcal{C})$; (i.e. any sentence true in $\mathcal{M}$ is also true in some element of $\mathcal{C}$).

Equivalently, $\mathcal{M}$ is pseudo-$\mathcal{C}$ if it is elementary equivalent to an ultraproduct of elements of $\mathcal{C}$.

Equivalently, $\mathcal{M}$ is pseudo-$\mathcal{C}$ if there is an elementary embedding of $\mathcal{M}$ into an ultraproduct of elements of $\mathcal{C}$.

Pseudofinite groups/fields

Let $\mathcal{F}$ be the class of finite groups.
A group G is pseudofinite (pseudo-$\mathcal{F}$) if G is a model of $Th(\mathcal{F})$.

This implies, in particular, that there are no torsion-free abelian finitely generated pseudofinite groups.
Consider the map: $x \to 2.x$. It is injective but not surjective.

A field K is pseudofinite if K is a model of the theory of the class of finite fields.

Note that here a pseudofinite structure may be finite.

Examples of pseudofinite groups

1. General linear groups over pseudofinite fields $GL_n(K)$, where K is a pseudofinite field).
   More generally any Chevalley group (of twisted or untwisted type e.g. $PSL_n(K)$) over a pseudofinite field is a simple pseudo finite group.
2. Conversely, any pseudofinite infinite simple group is isomorphic to a Chevalley group (of twisted or untwisted type) over a pseudofinite field (U. Felgner, J.S. Wilson, M. Ryten)– Pseudofinite definably simple groups have been studied by P. Urgulu, without CFSG.
3. Unitriangular groups $UT_n(K)$, $n \geq 3$ with K a pseudofinite field.
   (Malcev correspondance): given a group formula ϕ, one can construct a ring formula ϕ^* such that $UT_3(K) \models \phi \leftrightarrow K \models \phi^*$.

Pseudofinite fields

Infinite pseudofinite fields have been characterized algebraically by J. Ax, i.e. such a field is perfect, has one extension of degree n for each $n \in \mathbb{N}$, and is PAC (every absolutely irreducible variety has a point)). The last property is proven using the Lang-Weil estimates (A. Weil for curves and S. Lang for varieties). These Lang-Weil estimates are also used to endow the definable subsets with a dimension and a measure, using the following theorem:

Theorem (Z. Chatzidakis, L. van den Dries, A. Macintyre)

Given a formula $\psi(\bar{x}, \bar{y})$ of the field language, there exists a finite set $D := (\{0, \cdots, \mathit{length}(\bar{x})\} \times \mathbb{Q}^{>0}) \cup \{(0,0)\}$ and constant $C > 0$ such that for every finite field $\mathbb{F}_q$ and every $\bar{a} \in \mathbb{F}_q$,

$$||\psi(\mathbb{F}_q, \bar{a})| - \mu.q^d| \leq C.q^{d-1/2},$$

for some $(d, \mu) \in D$.

Approximability and metric ultraproducts.

• A group G is called approximable by $\mathcal{C}$ if for any finite subset $F \subseteq G$, there exists a group $G_F \in \mathcal{C}$ and an *injective* map $\xi_F : F \to G_F$ such that $\forall g, h \in F$, if $gh \in F$, then $\xi_F(gh) = \xi_F(g)\xi_F(h)$.

When $\mathcal{C}$ is a class of finite groups, then G is called *LEF* (A. Vershik and E. Gordon).

Lemma

A group G is approximable by $\mathcal{C}$ iff G embeds in an ultraproduct of elements of $\mathcal{C}$ iff G is a model of $Th_\forall(\mathcal{C})$.

Approximability and metric ultraproducts.

- Let $(G_i, d_i)_{i\in I}$ be a family of groups endowed with a bi-invariant metric $d_i(.,.) : G_i \times G_i \to \mathbb{R}$ (i.e. invariant by multiplication on the left and the right).

For example the (normalised) Hamming distance d_H on S_n (the symmetric group on n elements):

$$d_H(\sigma, \tau) := \frac{1}{n}|\{j \in n : \tau(j) \neq \sigma(j)\}|.$$

Note that all elements are at distance at most 1.

Let $\mathcal{U}$ be an ultrafilter on I and let $N_d := \bigcap_{n\in\mathbb{N}}\{(g_i) \in \prod_{i\in I} G_i : \ \{i \in I : \ d_i(g_i, 1) \leq \frac{1}{n}\} \in \mathcal{U}\}$. This is a normal subgroup of $\prod_I G_i$.
Then their metric ultraproduct w.r.to $\mathcal{U}$ is $\prod_{i\in I} G_i/N_d$. We have $\prod_{i\in I}^{\mathcal{U}} G_i \twoheadrightarrow \prod_{i\in I} G_i/N_d$.

Metric ultraproducts and sofic groups.

Note that if d_i is bounded, then this group is endowed with a bi-invariant metric with value in $\mathbb{R}$:
$d((g_i).N_d, (h_i).N_d) := st([d_i(g_i, h_i)]_{\mathcal{U}})$.

Let $\mathcal{C}$ be a class of groups (C, d_C) endowed with a bi-invariant metric d_C.
A group G is $\mathcal{C}$-sofic if for any finite $F \subset G$ there exists $\epsilon > 0$ and for all $n \in \mathbb{N} - \{0\}$ there exists $(C, d_C) \in \mathcal{C}$ and an injective map $\xi_F : F \to C \in \mathcal{C}$ such that whenever $g, h, g.h \in F$, $d_C(\xi_F(g.h), \xi_F(g).\xi_F(h)) < \frac{1}{n}$ and for all $g \in F$, $d_C(\xi_F(g), 1) \geq \epsilon$.

Proposition

A group G embeds in a metric ultraproduct of elements of $\mathcal{C}$ iff G is $\mathcal{C}$-sofic.

Approximability and metric ultraproducts.

A. Thom and J.S. Wilson (2014) considered metric ultraproducts of finite non abelian simple groups and showed that their isomorphism type determines whether it is an ultraproduct of alternating groups or finite simple classical groups and in that last case it determines whether it corresponds to a pseudofinite field of characteristic 0 or $p \in \mathcal{P}$.

Back to pseudofinite groups.

1. Pseudofinite groups with a theory satisfying various model-theoretic assumptions like stability, supersimplicity or the NIP property (non independence property) have been studied (D. Macpherson, K. Tent, R. Elwes, E. Jaligot, M. Ryten, C. Milliet,...).
   For instance, C. Milliet showed that a pseudofinite group with a supersimple theory either interprets a pseudofinite field or is solvable-by-finite.
2. G. Sabbagh and A. Khélif investigated finitely generated pseudofinite groups.
3. With Abderezak Ould Houcine, we considered analogs of Tits alternative in the class of pseudofinite groups.

Basic properties

If G is a pseudofinite group, then

1. any definable subgroup H of G is pseudofinite group.
   Proof: We can relativize the quantifiers to H and express satisfaction in H back in G:

   $$H \models \exists x \cdots \phi(x, \cdots) \leftrightarrow G \models \exists x \in H \cdots \phi(x, \cdots).$$

2. Any quotient G/H of G by a definable normal subgroup H is pseudofinite.
   Proof: We can express satisfaction of any atomic formula in the quotient back in G by:

   $$G/H \models t(\bar{x}) = 1 \leftrightarrow G \models t(\bar{x}) \in H.$$

Verbal subgroups

Let $w(x_1, \cdots, x_n)$ be a word in $x_1, \cdots, x_n$. Let $w(G)$ be the subset of the elements $w(\bar{g})$, $\bar{g} \in G^n$.
The verbal subgroup $< w(G) >$ is the subgroup of G generated by $w(G)$.
It is of finite width $\leq d$ if any element of $< w(G) >$ is of the form $w(\bar{g}_1)^{\pm 1}. \cdots .w(\bar{g}_d)^{\pm 1}$, for some $\bar{g}_1, \cdots, \bar{g}_d \in G^{\times n}$.

Note that if $< w(G) >$ is of finite width, then it is definable and if G is $\aleph_0$-saturated and if $< w(G) >$ is definable, then $< w(G) >$ is of finite width.

[N. Nikolov, D. Segal] In the class of finite groups generated by d elements, certain verbal subgroups are definable, e.g. $< [G, G] >$ and $< G^n >$, $n \in \mathbb{N}^*$.

Definability of verbal subgroups in classes of finite groups: the derived subgroup

Theorem (Nikolov, Segal, 2012)

There exists a function f such that for any finite group G generated by d elements:$=y_1, \cdots, y_d$, given a normal subgroup H of G, we have that $< [H, G] >$ is equal to

$$\{\prod_{j=1}^{f(d)} \prod_{i=1}^{d} [x_{ij}, y_i].[z_{ij}, y_i^{-1}] : \ x_{ij}, \ z_{ij} \in H\}.$$

f is $O(d^2)$.

The proof uses CFSG, see discussion in section 1.3 of their Inventiones paper.

Finite Morley rank

A word is concise if whenever $w(G)$ is finite, then the group $< w(G) >$ is finite.

Let G be a group of finite Morley rank and w a concise word, then $< w(G) >$ is of finite width.

[S.I. Ivanov, 1989] Not every word is concise.
$w(x, y) := [[x^{p^n}, y^{p^n}]^n, y^{p^n}]^n$. There exists a torsion-free 2 generator group with cyclic centre; the word takes two values and the non-trivial one generates the centre of G.

Application: Definability of the radical

Let G be a group and let $rad(G)$ be the soluble radical, that is the subgroup generated by all normal soluble subgroups of G.

Theorem (J. Wilson, 2009)

There exists a (universal) formula: $\phi_R(x)$, such that in any finite group G, $rad(G)$ is definable by ϕ_R.

The proof uses CFSG.

Application: Definability of the radical

Let G be a finite group and

$$\sigma(x) := \exists \bar{u} \exists \bar{v} \; (x = \prod_{i=1}^{56} [x^{u_i}, x^{v_i}]).$$

- [Wilson (2006)] G is soluble iff $G \models \forall x \; (\sigma(x) \rightarrow x = 1)$.

- [Gordeev, Grunewald, Kunyavskii, Plotkin (2009)/Flavell] $g \in rad(G)$ iff $\forall x_1, \cdots, x_4 \in G$, the group $< g^{x_1}, \cdots, g^{x_4} >$ is soluble.

- Let $\rho(y, g) :=$

$$\exists \bar{s} \bar{t} \bar{x} \; (y = \prod_{j=1}^{f(4)} \prod_{i=1}^{4} [s_{ij}, g^{x_i}].[t_{ij}, ((g^{x_i})^{-1}]).$$

- Set $\phi_R(g) := \forall y \; (\rho(y, g) \rightarrow (\neg \sigma(y) \vee y = 1)).$

Remarks

- [Wilson (2006)] G is soluble iff $G \models \forall x\ (\sigma(x) \rightarrow x = 1)$.

The proof uses Thompson's description of minimal finite simple groups.

- [Gordeev, Grunewald, Kunyavskii, Plotkin (2009)/Flavell] $g \in rad(G)$ iff $\forall x_1, \cdots, x_4 \in G$, the group $< g^{x_1}, \cdots, g^{x_4} >$ is soluble.

There is a proof without CFSG using more conjugates, see Flavell and al. (2010).
One has the same result with G linear.

- [Baer (1957)-Suzuki] $g \in Fitt(G)$ iff for any $x \in G$, the group $< g, g^x >$ is nilpotent.

Consequences for pseudofinite groups

Recall that a semi-simple group is a group without normal abelian non-trivial subgroups.

Corollary

If G is a pseudofinite group then $rad(G)$ and $G/rad(G)$ are pseudofinite and $G/rad(G)$ is a semi-simple group.

Definability of verbal subgroups in classes of finite groups: non-commutator words

Definition

A word w is d-bounded if there exists a bound in the class of d-generated finite groups of the index $[G :< w(G) >]$.

Theorem (Positive solution of the restricted Burnside problem, (E. Zemanov))

Given d, n, there are only finitely many finite groups generated by d elements of exponent n.

So, the word x^n is d-bounded.

Theorem (Nikolov, Segal, 2012)

Let w be a d-bounded word, then there exists a function $g(w, d)$ such that for any finite group G generated by d elements, the width of $< w(G) >$ is bounded by $g(w, d)$.

Application in the class of pseudofinite groups.

Recall that a group is said to be *uniformly locally finite* if for any $n \geq 0$, there exists $\alpha(n)$ such that any n-generated subgroup of G has cardinality bounded by $\alpha(n)$.

Lemma

A pseudofinite group G of finite exponent is uniformly locally finite.

Proof: Let $< g_1, \cdots, g_k >$ be a k-generated subgroup of G. By definition $G \preceq \prod_I^{\mathcal{U}} G_i$, where G_i is a finite group. Since G is of exponent n, on an element of $\mathcal{U}$, G_i is of exponent n. Let $(g_{ji})_{i \in I}$, $1 \leq j \leq k$, be a representative for g_j and consider the subgroup $< g_{1i}, \cdots, g_{ki} > \subset G_i$ on that element of $\mathcal{U}$. Then by the positive solution of the restricted Burnside problem, there is a bound $N(k, n)$ on the cardinality of each subgroup $< g_{1i}, \cdots, g_{ki} >$. So the subgroup $< g_1, \cdots, g_k >$ embeds into the ultraproduct $\prod_I^{\mathcal{U}} < g_{1i}, \cdots, g_{ki} >$ and so it has cardinality bounded by $N(k, n)$.

Definability of verbal subgroups in pseudofinite groups

Let G be a pseudo-(d-generated finite group).

Proposition

Then,
(1) For any definable subgroup H of G, the subgroup $< [H, G] >$ is definable. In particular the terms of the descending central series of G are 0-definable and of finite width.

(2) The verbal subgroups $< G^n >$, $n \in \mathbb{N}^*$, are 0-definable, of finite width and of finite index.

Finitely generated pseudofinite groups

Question (G. Sabbagh)

Is a pseudofinite finitely generated group finite?

Proposition

Let G be a finitely generated pseudofinite group and suppose that G satisfies one of the following conditions.

1. G is of finite exponent, or
2. (G. Sabbagh, A. Khélif) G is soluble, or
3. G is soluble-by-(finite exponent), or
4. G is pseudo-(finite linear of degree n in characteristic zero), or
5. G is simple.

Then such a group G is finite.

Applications

A group G is *CSA* if for any maximal abelian group A and any $g \in G - A$,

$$A^g \cap A = \{1\}. \ (\star)$$

Note that such A is equal to the centraliser of any of its non-trivial element u (if not one finds $g \in C_G(u) - A$ and so $u \in A^g \cap A$).
Then one expresses $(\star)$ for centralisers that are abelian.
Therefore being *CSA* can be expressed by a universal sentence σ_{CSA}.

Lemma (A. Myasnikov, V. Remeslennikov)

Any finite CSA group is abelian.

One takes a minimal non abelian finite CSA group. Such group is soluble and has a proper normal abelian subgroup, a contradiction.

Corollary

There are no nontrivial torsion-free hyperbolic pseudofinite groups.

In particular, F_2 is not pseudo-finite.

Proof: A torsion-free hyperbolic group G is a CSA-group. Suppose $G \prec \prod_{\mathcal{U}} F_i$, where F_i is a finite group. By Łos' theorem, for a subset of indices i belonging to $\mathcal{U}$, F_i is CSA and so abelian. But there are no infinite abelian finitely generated pseudofinite groups.

Alternatives: the free subsemigroup.

Theorem

Let G be an $\aleph_0$-saturated pseudofinite group. Then either G contains M_2 or G is nilpotent-by-(uniformly locally finite).

Lemma

Let G be an $\aleph_0$-saturated group. Then either G contains F_2, or G satisfies a nontrivial identity (in two variables). In the last case, either G contains M_2, or G satisfies a finite disjunction of nontrivial positive identities in two variables.

Examples of positive identities: set $I_1(x,y) := x.y = y.x$ and $I_{n+1}(x,y) := I_n(x.y, y.x)$.

Milnor identities.

For $a, b \in G$, we let $H_{a,b} =< a^{b^n} | n \in \mathbb{Z} >$ and $H'_{a,b}$ its derived subgroup.

Theorem (J. Rosenblatt (1974))

Let $G \not\supseteq M_2$. Then for any $a, b \in G$, the subgroup $H_{a,b}$ is finitely generated, and G is locally 1-Milnor i.e. $b^{\ell_1}ab^{-\ell_1}...b^{\ell_n}ab^{-\ell_n} = b.a.b^{-1}.b^{m_1}ab^{-m_1}...b^{m_n}ab^{-m_n}$, with $1 < \ell_1 < \cdots < \ell_n$, $1 < m_1 < \cdots < m_n < \ell_n$.

Proof: One expresses that $M_2 \ncong < b, b.a >$.

To a word $t(x,y) := yx^{m_1}y^{-1}...y^{\ell}x^{m_\ell}y^{-\ell}$ with $m_i \in \{0, -1, 1\}$, one associates a polynomial $q_t[X] = \sum_{i=1}^{\ell} m_i.X^i \in \mathbb{Z}[X]$.

We will call an identity of the form $t(x,y) = 1$, with $t(x,y)$ as above, a 1-Milnor identity.

Milnor identities and finite groups.

THEOREM (A. Shalev/ R. Burns, O. Macedońska, Y. Medvedev)

Given a finite number of 1-Milnor identities $t_i = 1$, $i \in I$ and their associated polynomials q_{t_i}, $i \in I$, there exist positive integers $c(q)$ and $e(q)$ only depending on $q := \prod_{i \in I} q_{t_i}$, such that a finite group G satisfying $\bigvee_{i \in I} t_i = 1$, is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$.

One reduces to the soluble case, using a result of Jones that a variety of groups only contains finitely many non abelian finite simple groups. The difficult part is to bound the nilpotency class. Test groups: $C_p wr C_{p^n}$, with C_p the cyclic group of prime order p.

Proof of the theorem: Let $L := \prod_{i\in I}^{\mathcal{U}} F_i$ where F_i is a finite group, and $G \prec L$.
Suppose that G is is $\aleph_0$-saturated and that it does not contain M_2. Then G satisfies a finite disjunction of positive identities. Let q be the corresponding polynomial in the above theorem. So on a subset U of indices belonging to $\mathcal{U}$, the finite group F_i satisfies that same disjunction. Therefore, for $i \in U$, F_i is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$. So, $G^{e(q)}$ is nilpotent of class $\leq c(q)$.
Claim: $L/ < L^{e(q)} >$ is u.l.f.
Take k elements in L: $[\ell_{1i}], \cdots, [\ell_{ki}]$. Let L_k be the subgroup generated by these elements and $L_{ki} :=< \ell_{1i}, \cdots, \ell_{ki} >\subset F_i$, $i \in I$. Each $< {L_{ki}}^{e(q)} >$ is of finite width bounded by $f(k, e(q))$. So

$$L_k/ < L_k^{e(q)} > \hookrightarrow \prod_{i\in I}^{\mathcal{U}} L_{ki}/\langle {L_{ki}}^{e(q)}\rangle.$$

Since $G \prec L$, $< G^{e(q)} >=< L^{e(q)} > \cap G$. So, we have that $G/G^{e(q)} \hookrightarrow L/L^{e(q)}$.

Supra-amenability

Notation

Let G be a group and $S \subset G$.
Let $\gamma_S(n)$ to be the cardinal of the ball $B^{<S>}_{S \cup S^{-1}}(n)$ of radius n in $< S >$ (for the word distance with respect to $S \cup S^{-1}$).

Definition

• A group G is said to be *exponentially bounded* if for any finite subset $S \subseteq G$, and any $b > 1$, there is some $n_0 \in \mathbb{N}$ such that $\gamma_S(n) < b^n$ whenever $n > n_0$.

• A group G is supramenable if for any $A \subset G$, there is a finitely additive measure μ on $\mathcal{P}(G)$ invariant by right translation such that $\mu(A) = 1$.

Alternative: containing M_2 versus supra-amenability

[See the book of S. Wagon on Banach-Tarski paradox] Recall that for any group G, we have the following implications:
G is superamenable $\Rightarrow$ M_2 is not contained in G.
(Straightforward.)
G is exponentially bounded $\Rightarrow$ G is superamenable.

In the class of $\aleph_0$-saturated pseudofinite groups, we have the reverse implications.

Corollary

Let G be an $\aleph_0$-saturated pseudofinite group. Then, T.F.A.E.

(1) G is superamenable.

(2) G is nilpotent-by-(uniformly locally finite).

(3) G is nilpotent-by-(locally finite).

(4) Every finitely generated subgroup of G is nilpotent-by-finite.

Free subgroups, amenability

Definition

A group G is amenable if there is a finitely additive measure μ on $\mathcal{P}(G)$ invariant by right translation such that $\mu(G) = 1$.

The class of amenable groups contains the class of finite groups and of abelian groups (A. Tarski) and is closed under subgroups, homomorphic images, extensions and direct limits.

F_2 is not amenable. Set $F_2 =< a, b >$ and denote all the non trivial elements of F_2 that can be represented as a word of non zero length beginning by a (respectively a^{-1}, b, b^{-1}) by W_a (respectively $W_{a^{-1}}$, W_b, $W_{b^{-1}}$). Note that $F_2 = a^{-1}.W_a$ however

$$F_2 = W_a \dot{\cup} W_b \dot{\cup} W_{a^{-1}} \dot{\cup} W_{b^{-1}} \dot{\cup} \{1\}.$$

Thus, an amenable group does not contain F_2.

Free subgroups, amenability

Another definition of amenability is the following: G is amenable if for every finite subset A of G and every $0 < \epsilon < 1$ there is a finite subset E of G with $|E.A| < (1+\epsilon)|E|$ (Følner).

Let $\sigma_{p,n,f}$ be the following sentence with $(p,n) \in \mathbb{N}^2$ and $f : \mathbb{N}^2 \to \mathbb{N}$:

$$\forall a_1 \cdots \forall a_n \exists y_1 \cdots \exists y_{f(p,n)}$$

$$p.|\{a_i.y_j : 1 \leq i \leq n; 1 \leq j \leq f(p,n)\}| < (p+1).f(p,n).$$

A group G is uniformly amenable if there exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that $G \models \sigma_{p,n,f}$ for any $(p,n) \in \mathbb{N}^2$

(G. Keller, J. Wysoczanski) An $\aleph_0$-saturated group is amenable if and only if it is uniformly amenable.

Free subgroups, amenability

Proposition

The following properties are equivalent.

1. Every $\aleph_0$-saturated pseudofinite group either contains F_2 or it is amenable.
2. Every ultraproduct of finite groups either contains F_2 or it is amenable.
3. Every finitely generated residually finite group satisfying a nontrivial identity is amenable.
4. Every finitely generated residually finite group satisfying a nontrivial identity is uniformly amenable.

Let G be a group and $\mathcal{C}$ a class of groups.

Definition

G is residually-$\mathcal{C}$ if G embeds in a direct product of elements of $\mathcal{C}$.

Lemma

Then the following are equivalent:
(i) Any finitely generated subgroup of G is a direct limit of finitely generated residually-$\mathcal{C}$ groups.
(ii) G is approximable by $\mathcal{C}$.

Let us prove (3) $\Rightarrow$ (1). Let G be an $\aleph_0$-saturated pseudofinite group not containing F_2. So, it satisfies a non-trivial identity $t = 1$. Let $\mathcal{C}$ be the class of finite groups satisfying $t = 1$. By Lemma above, any finitely generated subgroup G_0 of G is a direct limit of finitely generated residually-$\mathcal{C}$ groups L_i. By (3), these groups L_i are amenable and so G_0 is amenable, as well as G.

Free subgroups, alternatives

Definition

A group G is of Prüfer rank $\leq n$ if any subgroup of G can be generated by $\leq n$ elements.

Theorem (S. Black, 1999)

Let G be an $\aleph_0$-saturated pseudo-(finite of bounded Prüfer rank) group.
Then either G contains F_2 or G is nilpotent-by-abelian-by-finite (and so is uniformly amenable).

One uses a result of A. Shalev to reduce to finite soluble groups and then a result of D. Segal on residually finite soluble groups.

Weakly r-bounded

Let G be a group. Then $Soc(G)$ is the group generated by the minimal simple subgroups of G. If G is finite, $Soc(G/rad(G))$ the direct product of its minimal non-abelian simple subgroups. If $\mathcal{C}$ is a class of finite groups and if the Prüfer rank in $\mathcal{C}$ is bounded, then by a result of Shalev, one can bound the index of $Soc(G/rad(G))$ in $G/rad(G)$.

Definition

A class $\mathcal{C}$ of finite groups is *weakly of bounded rank* if the index of $Soc(G/rad(G))$ is bounded and $rad(G)$ has bounded Prüfer rank.

Theorem

Let G be an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank) group. Then either G contains F_2 or G is nilpotent-by-abelian-by-(uniformly locally finite) (and so is uniformly amenable).

Centralizer dimension

Definition

A group G has *finite c-dimension* if there is a bound on the chains of centralizers.
A class $\mathcal{C}$ of finite groups has *bounded c-dimension* if for each element $G \in \mathcal{C}$, the c-dimensions of $rad(G)$ and of the sockel of $G/rad(G)$ are bounded.

(Note that a class of finite groups of bounded Prüfer rank is of bounded c-dimension.)

We use the result of E. Khukhro on groups with finite c-dimension.

Theorem (Khukro, 2009)

If a periodic locally soluble group G has finite c-dimension k, then G is soluble of derived length bounded by a function of k.

Centralizer dimension

Proposition

Let $\mathcal{C}$ be a class of finite groups of bounded c-dimension and suppose G is a pseudo-$\mathcal{C}$ group satisfying a nontrivial identity. Then G is soluble-by-(uniformly locally finite).

Corollary

Let G be an $\aleph_0$-saturated pseudo-(finite of bounded c-dimension) group. Then either G contains F_2 or G is soluble-by-(uniformly locally finite).