Some Notes in Diophantine Geometry over Free Groups

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1 Introduction

These notes are meant to be used as a complement for the tutorial "Diophantine Geometry over Free Groups" in the workshop "Models and Groups, İstanbul II" . They are written in a relaxed fashion, mostly aiming to build some intuition around the first steps of Sela's work on Tarski's problem. We include in the introduction some historical remarks.

The subject has been given much attention after Sela proved that the first-order theory of non-Abelian free groups (i.e. the axioms that live in the intersection of the above mentioned first-order theories) is complete. This answers in the affirmative a long standing question that was posed around 1946 by Tarski:

Question 1 (Tarski, 1946)**:** *Do non-Abelian free groups share the same common first-order theory?*

The main purpose of these notes is to analyze the notions and techniques that appear in the first steps of Sela's solution to Tarski's problem. Let us mention that the proof culminates in a series of papers [Sel01],[Sel03],[Sel05a],[Sel04],[Sel05b],[Sel06a] and [Sel06b], that have not been totally absorbed by the mathematical community, despite the fact that they were available since 2001. In brief the proof splits in two parts: first Sela proves that the $\forall\exists$ first-order theories of any two non-Abelian free groups coincide, and then he proves that each first order theory eliminates quantifiers down to boolean combinations of $\forall\exists$ first-order formulas. His methods are purely geometric and a heavy use of the theory of group actions on real trees is made throughout his papers.

Our goal for these notes will be to give the ideas around the proof of the following intermediate result to Tarski's problem:

Theorem 1: *Let $m, n > 1$. Then $Th_{\forall\exists}(\mathbb{F}_n) = Th_{\forall\exists}(\mathbb{F}_m)$*

Note that although this theorem has been first claimed in [Sac73], a complete proof appeared much later (in Sela's work).

The tutorial will be structured as follows: We will first define limit groups using the Bestvina-Paulin method (see [Bes88],[Pau88])and record how one can describe the solution set (in a free group) of a system of equations using them. Limit groups play an important role in all steps of Sela's solution and we will see that one naturally sees them as objects of geometry rather than algebra.

We will then move to the technique of "formal solutions". This technique lies behind the main idea of the proof of Sela. Before stating the prototypical theorem, let us recall that a retraction from a group G to a subgroup H is an epimorphism that is the identity on H.

Theorem 2 (Merzlyakov [Mer66])**:** *Let $\Sigma(\bar{x}, \bar{y})$ be a finite set of words in $\langle \bar{x}, \bar{y} \rangle$. Let $\mathbb{F}$ be a non-Abelian free group. Suppose $\mathbb{F} \models \forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1)$. Then there exists a retraction $r : G_\Sigma \twoheadrightarrow \langle \bar{x} \rangle$, where $G_\Sigma := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$.*

We note that Merzlyakov used this theorem in order to prove that the positive first-order theories of non-Abelian free groups coincide. Let us briefly justify the term "formal solutions": the image of $\bar{y}$ under r of the previous theorem is a tuple of words in $\bar{x}$, say $\bar{w}(\bar{x})$, and it can be easily checked that $\mathbb{F} \models \forall \bar{x} (\Sigma(\bar{x}, \bar{w}(\bar{x})) = 1)$. Thus, the retraction can be thought of as a formal (uniform) way of assigning to each $\bar{a}$ in $\mathbb{F}$, a $\bar{b}$ in $\mathbb{F}$ (i.e. substituting $\bar{x}$ in $\bar{w}(\bar{x})$ by $\bar{a}$), that witnesses the truthfulness of Σ.

Geometry suggests some natural generalizations of the above theorem and this will lead us to the definitions of "towers" and "test sequences" on them. Our feeling is that these notions will be central to the understanding of the class of definable sets in non-Abelian free groups and thus we will try to build some intuition around them.

As noted above, Merzlyakov's theorem lies behind the main idea of all existing proofs to the Tarski's problem. Generalizing it to the case where the universal variables are bounded by a system of equations is a hard task and depends on the geometric structure of the system of equations. Unfortunately, the generalization of Merzlyakov's theorem to an arbitrary variety, that bounds the universal variables, is not possible. We have to restrict ourselves to varieties that their corresponding group has a certain structure. In particular, if a group $G_R := \langle \bar{x} \mid R(\bar{x}) \rangle$ has the structure of a "tower", then the following statement (up to some tuning) is true:

Statement 1: *Let $\Sigma(\bar{x}, \bar{y})$ be a finite set of words in $\langle \bar{x}, \bar{y} \rangle$. Let $\mathbb{F}$ be a non-Abelian free group. Suppose $\mathbb{F} \models \forall \bar{x} (R(\bar{x}) = 1 \to \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1))$. Then there exists a retract $r : G_\Sigma \twoheadrightarrow G_R$, where $G_\Sigma := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$.*

Finally, the addition of inequalities to the sentences above, i.e. sentences of the form $\forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1 \wedge \Psi(\bar{x}, \bar{y}) \neq 1)$ require new machinery and ideas in order to be shown that their truthfulness does not depend on a particular non-Abelian free group. This machinery includes the generalization of Merzlyakov's theorem as stated above, but also requires the development of more delicate tools. We will finish this tutorial by giving the extra ideas needed for completing the proof of Theorem 1 .

Our exposition will be based on the following papers [Bes01],[Gui08],[Sel01],[Sel03],[Sel04] that can be found in the references, and also includes some work in progress with Chloé Perin.

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2 Group actions on trees

2.1 Bass-Serre Theory

Bass-Serre theory gives a structure theorem for groups acting on (simplicial) trees, i.e. contractible 1 or 0 dimensional CW-complexes. It describes a group (that acts on a tree) as a series of *amalgamated free products* and *HNN extensions*. The mathematical notion that contains these instructions is called a graph of groups. For a complete treatment we refer the reader to [Ser83].

We start with the definition of a graph.

Definition 2.1: *A graph $\Lambda(V,E)$ is a collection of data that consists of two sets V (the set of vertices) and E (the set of edges) together with three maps:*

- *an involution $^{-} : E \to E$, where $\bar{e}$ is called the inverse of e;*
- *$\alpha : E \to V$, where $\alpha(e)$ is called the initial vertex of e; and*
- *$\tau : E \to V$, where $\tau(e)$ is called the terminal vertex of e.*

so that $\bar{e} \neq e$, and $\alpha(e) = \tau(\bar{e})$ for every $e \in E$.

An *orientation* of a graph $\Lambda(V,E)$ is a choice of one edge in the couple $(e,\bar{e})$ for every $e \in E$. We denote an oriented graph by $\Lambda^+(V,E)$.

For our purposes simplicial trees can also be viewed as combinatorial objects: a *tree* is a connected graph without a circuit.

Definition 2.2 (Graph of Groups)**:** *A graph of groups $\mathcal{G} := (\Lambda(V,E), \{G_u\}_{u\in V}, \{G_e\}_{e\in E}, \{f_e\}_{e\in E})$ consists of the following data:*

- *a connected graph $\Lambda(V,E)$;*
- *a family of groups $\{G_u\}_{u\in V}$, i.e. a group is attached to each vertex of the graph;*
- *a family of groups $\{G_e\}_{e\in E}$, i.e. a group is attached to each edge of the graph. Moreover, $G_e = G_{\bar{e}}$;*
- *a collection of injective morphisms $\{f_e : G_e \to G_{\tau(e)} \mid e \in E\}$, i.e. each edge group comes equipped with two embeddings to the incident vertex groups.*

The fundamental group of a graph of groups is defined as follows.

Definition 2.3: *Let $\mathcal{G} := (\Lambda(V,E), \{G_u\}_{u\in V}, \{G_e\}_{e\in E}, \{f_e\}_{e\in E})$ be a graph of groups. Let T be a maximal subtree of $\Lambda(V,E)$. Then the fundamental group, $\pi_1(\mathcal{G},T)$, of $\mathcal{G}$ with respect to T is the group given by the following presentation:*

$$\langle \{G_u\}_{u\in V}, \{t_e\}_{e\in E} \mid t_e^{-1} = t_{\bar{e}} \text{ for } e \in E, t_e = 1 \text{ for } e \in T, f_e(a) = t_e f_{\bar{e}}(a) t_{\bar{e}} \text{ for } e \in E \ a \in G_e \rangle$$

Remark 2.4: *It is not hard to see that the fundamental group of a graph of groups does not depend on the choice of the maximal subtree up to isomorphism (see [Ser83, Proposition 20, p.44]).*

In order to give the main theorem of Bass-Serre theory we need the following definition.

Definition 2.5: *Let G be a group acting on a simplicial tree T without inversions, denote by Λ the corresponding quotient graph and by p the quotient map $T \to \Lambda$. A Bass-Serre presentation for the action of G on T is a triple $(T^1, T^0, \{\gamma_e\}_{e\in E(T^1)\setminus E(T^0)})$ consisting of*

- *a subtree T^1 of T which contains exactly one edge of $p^{-1}(e)$ for each edge e of Λ;*
- *a subtree T^0 of T^1 which is mapped injectively by p onto a maximal subtree of Λ;*
- *a collection of elements of G, $\{\gamma_e\}_{e\in E(T^1)\setminus E(T^0)}$, such that if $e = (u,v)$ with $v \in T^1 \setminus T^0$, then $\gamma_e \cdot v$ belongs to T^0.*

Theorem 2.6: *Suppose G acts on a simplicial tree T without inversions (i.e. $g \cdot e \neq \bar{e}$ for all $g \in G$ and $e \in E$). Let $(T^1, T^0, \{\gamma_e\})$ be a Bass-Serre presentation for the action. Let $\mathcal{G} := (\Lambda(V,E), \{G_u\}_{u \in V}, \{G_e\}_{e \in E}, \{f_e\}_{e \in E})$ be the following graph of groups:*

- *$\Lambda(V,E)$ is the quotient graph given by $p : T \to \Lambda$;*
- *if u is a vertex in T^0, then $G_{p(u)} = Stab_G(u)$;*
- *if e is an edge in T^1, then $G_{p(e)} = Stab_G(e)$;*
- *if e is an edge in T^1, then $f_{p(e)} : G_{p(e)} \to G_{\tau(p(e))}$ is given by the identity if $e \in T^0$ and by conjugation by γ_e if not.*

Then G is isomorphic to $\pi_1(\mathcal{G})$.

Part of the motivation for proving Theorem 2.6 was the following result.

Proposition 2.7: *A group is free if and only if it acts freely on a tree.*

The above proposition has a significant corollary that was hard to prove using combinatorial methods.

Theorem 2.8 (Nielsen-Schreier)**:** *A subgroup of a free group is free.*

Among splittings of groups we will distinguish those with some special type vertex groups called *surface type vertex groups.*

We first recall that the fundamental group of a compact surface, Σ, with boundary is a free group. Each boundary component of Σ has cyclic fundamental group, and gives rise in $\pi_1(\Sigma)$ to a conjugacy class of cyclic subgroups: we call these *maximal boundary subgroups.*

Definition 2.9: *Let G be a group acting on a tree T without inversions and $(T_1, T_0, \{\gamma_e\})$ be a Bass-Serre presentation for this action. Then a vertex $v \in T^0$ is called a surface type vertex if the following conditions hold:*

- $\mathrm{Stab}_G(v) = \pi_1(\Sigma)$ *for a connected compact surface Σ with non-empty boundary;*
- *For every edge $e \in T_1$ adjacent to v,* $\mathrm{Stab}_G(e)$ *embeds onto a maximal boundary subgroup of $\pi_1(\Sigma)$, and this induces a one-to-one correspondence between the set of edges (in T^1) adjacent to v and the set of boundary components of Σ.*

2.2 Real trees

Real trees (or $\mathbb{R}$-trees) generalize simplicial trees and occur naturally in mathematics (see [Bes01]).

Definition 2.10: *A real tree is a geodesic metric space in which for any two points there is a unique arc that connects them.*

Note that the assumption that there is a unique arc (and not just a unique geodesic) is essential since $\mathbb{R}^2$ with the usual metric has unique geodesics but of course would not fit our intuition for a real tree. Also the assumption that the metric space is geodesic is not made redundant by the uniqueness of arcs assumption since a "curved" line in $\mathbb{R}^2$ that inherits its metric from $\mathbb{R}^2$ has the unique arc property but is not a geodesic space since we cannot realize the distance between at least two points in it.

When we say that a group G acts on an real tree T we will always mean an action by isometries. An element g in G either fixes a point (in which case it is called *elliptic*) or there is a unique g-invariant line in which g acts by translations (in which case it is called *hyperbolic*). Actually in the latter case the translation length is the infimum of $d_T(x, g \cdot x)$ which is always reached.

Moreover, an action $G \curvearrowright T$ of a group G on a real tree T is called *non-trivial* if there is no globally fixed point and *minimal* if there is no proper G-invariant subtree. The minimality assumption is quite natural since we can (equivariantly) glue complicated real trees to any given action, which will blur the properties we want to deduce for G. As a matter of fact whenever we have a non trivial action of a finitely generated group G on a real tree, there exists a minimal non-degenerate subtree for this action and we left for the reader to check that it is the union of axes of all hyperbolic elements of G. Lastly, an action is called *free* if for any $x \in T$ and any non trivial $g \in G$ we have that $g \cdot x \neq x$.

One could ask if there is an analogue of Bass-Serre theory for group actions on real trees. If we restrict ourselves to group actions satisfying some tameness conditions the answer is positive.

Before we give some examples of (non-simplicial) actions on real trees we record a result of more general nature (see [Gui08, Lemma 1.14]). Note that a subtree Y of T *spans* T if every finite segment of T is covered by finitely many translates of Y.

Lemma 2.11: *Let $G := \langle g_1, \ldots, g_k \rangle$ be a (f.g.) group that acts minimally on a real tree T. Then the action has finite support, i.e. the convex hull of finitely many points of T spans T.*

Proof. Choose an arbitrary point $* \in T$ and let Y_0 be the convex hull of $\{*, g_1 \cdot *, \ldots, g_k \cdot *\}$. Then $g_i \cdot Y_0 \cap Y_0$ is non empty for any $i \leq k$ and $G.Y_0$ is obviously G-invariant. Thus, the result follows by the minimality of the action. □

Let us continue by recalling some families of group actions on real trees that will turn out to be the building blocks for the general analysis.

Example 2.12 (Action of axial type)**:** Let $\mathbb{Z}^2 := \langle z_1, z_2 \rangle$ act on the real line by translations where $tr(z_1), tr(z_2)$ are linearly independent.

We say that a (f.g.) group G acts on real tree T by an *action of axial type* if T is isometric to the real line and G acts with dense orbits, i.e. $\overline{G.x} = T$ for every $x \in T$.

The next type of action is more interesting. It was discovered, by Morgan and Shalen (see [MS91]), that for any (closed) surface Σ, apart from finitely many exceptions (i.e. $\mathbb{P}, 2\mathbb{P}, 3\mathbb{P}$), $\pi_1(\Sigma)$ admits a free action on a real tree. These actions come naturally from measured foliations on the surface, that in turn were defined by Thurston who used them to compactify the Teichmüller space of a surface and led to the classification of surface homeomorphisms.

Before giving an example of a surface type action let us quickly explain the above mentioned notions. A *foliation* of (co-dimension 1) of a 2-manifold is a decomposition of the manifold by a family of subsets $\mathcal{L} := \{\mathfrak{l}_\alpha\}_\alpha$ called the leaves of the foliation. Furthermore, for each point in the manifold we can find a chart (U, ϕ), so that the connected components of $U \cap \mathfrak{l}_\alpha$ are mapped by ϕ to horizontal lines. Naturally, we also ask that the transition maps respect horizontal lines. Note that, by definition, each leaf is a 1-manifold.

A *measured foliation* is a foliation for which the transition maps respect the distance on the y co-ordinate, i.e. $\phi_{ij}(x, y) = (f_{ij}(x, y), y + c_{ij})$. In this case one can "measure" embedded arcs in the surface, by defining $\mu(\gamma)$ for γ an embedded arc to be the total variation of γ in the y co-ordinate.

Remark 2.13:

- *Note that not all surfaces admit foliations, as a matter of fact a surface admits a (co-dimension* 1*) foliation if and only if its Euler characteristic is* 0*. On the other hand if we allow finitely many "singular" points, i.e. points for which the chart maps do not map the leaves to horizontal lines, but rather to a k-prong saddles, for* $k \geq 3$*, then we can indeed find such foliations which we call* singular*.*

- *A (singular) foliation for which each leaf that does not contain singular points is dense in the surface is called* arational*.*

Example 2.14 (Action of surface type)**:** Let $(\mathcal{L}, \mu)$ be an arational measured (singular) foliation of a surface Σ with (possibly empty) boundary. We consider the "lift" $(\tilde{\mathcal{L}}, \tilde{\mu})$ of this foliation to the universal covering $\tilde{\Sigma}$ of Σ. Then the leaf space after identifying leaves of distance 0 with respect to the pseudometric $d(\tilde{\mathfrak{l}}_1, \tilde{\mathfrak{l}}_2) := inf\{\tilde{\mu}(\gamma) \mid \gamma$ an arc from $\tilde{\mathfrak{l}}_1$ to $\tilde{\mathfrak{l}}_2\}$ is a real tree endowed with a natural action by $\pi_1(\Sigma)$.

We say that a group G acts on a real tree T by a *surface type action*, if G is isomorphic to the fundamental group $\pi_1(\Sigma)$ of a surface Σ and T is the dual tree to an arational measured foliation of Σ as described in Example 2.14.

We note in passing that axial and surface type actions have the mixing property of J. Morgan [].

Definition 2.15: *Suppose G acts on a real tree T. Let Y be a non-degenerate subtree of T. Then Y has the mixing property if for any two segments $I, J \subset Y$ there exists a finite cover, $J_1, \ldots, J_k$, of J and $g_1, \ldots, g_k \in G$ such that $g_i J_i \subseteq I$ for $i \leq k$.*

It is also not hard to generalize the already mentioned result of Morgan and Shalen to surfaces with boundary as follows.

Fact 2.16: *Suppose G acts on a real tree T by a surface type action. Then the action is "almost free", i.e. only elements that belong to subgroups that correspond to the boundary components fix points in T and segment stabilizers are trivial.*

In particular when Σ has empty boundary the action is free (see [MS91]).

In analogy of the characterization of free actions on simplicial trees Lyndon had first posed the problem of understanding free actions on $\mathbb{R}$-trees (in different terminology but the equivalence was shown in []). After the discovery of free actions of surface groups it was naturally conjectured that if G acts freely on a real tree, then it is a free product of surface groups and free abelian groups. Rips confirmed the conjecture (unpublished) and made the first steps towards the understanding of group actions on real trees. We will see that in many cases one can understand such actions by "decomposing" them in simpler components and "glue" these components equivariantly in order to obtain the original action.

We will use the notion of a graph of actions in order to glue real trees equivariantly. As noted before, this notion will be useful in neatly stating the output of Rips' machine in the next subsection. We follow the exposition in [Gui08, Section 1.3].

Definition 2.17 (Graph of actions)**:** *A graph of actions* $(G \curvearrowright T, \{Y_u\}_{u \in V(T)}, \{p_e\}_{e \in E(T)})$ *consists of the following data:*

- *A simplicial type action $G \curvearrowright T$;*

- *for each vertex u in T a real tree Y_u;*

- *for each edge e in T, an attaching point p_e in $Y_{\tau(e)}$.*

Moreover:

1. *G acts on $R := \{\coprod Y_u : u \in V(T)\}$ so that $q : R \to V(T)$ with $q(Y_u) = u$ is G-equivariant;*

2. *for every $g \in G$ and $e \in E(T)$, $p_{g \cdot e} = g \cdot p_e$.*

To a graph of actions $\mathcal{A} := (G \curvearrowright T, \{Y_u\}_{u\in V(T)}, \{p_e\}_{e\in E(T)})$ we can assign an $\mathbb{R}$-tree $Y_{\mathcal{A}}$ endowed with a G-action. Roughly speaking this tree will be $\coprod_{u\in V(T)} Y_u/\sim$, where the equivalence relation $\sim$ identifies p_e with $p_{\bar{e}}$ for every $e \in E(T^+)$. We say that a real G-tree Y *decomposes as a graph of actions $\mathcal{A}$*, if there is an equivariant isometry between Y and $Y_{\mathcal{A}}$.

Assume a real G-tree Y decomposes as a graph of actions. Then a useful property is that Y is covered by $(Y_u)_{u\in V(T)}$ and moreover these trees intersect "transversally".

Definition 2.18: *Let Y be an $\mathbb{R}$-tree and $(Y_i)_{i\in I}$ be a family of subtrees that cover Y. Then we call this covering a transverse covering if the following conditions hold:*

- *for every $i \in I$, Y_i is a closed subtree;*

- *for every $i, j \in I$ with $i \neq j$, $Y_i \cap Y_j$ is either empty or a point;*

- *every segment in Y is covered by finitely many Y_i's.*

The next lemma is by no means hard to prove (see [Gui04, Lemma 4.7]).

Lemma 2.19: *Let $\mathcal{A} := (G \curvearrowright T, \{Y_u\}_{u\in V(T)}, \{p_e\}_{e\in E(T)})$ be a graph of actions. Suppose $G \curvearrowright Y$ decomposes as the graph of actions $\mathcal{A}$. Then $(Y_u)_{u\in V(T)}$ is a transverse covering of Y.*

2.3 Rips' machine

Group actions on real trees played a significant role in Sela's approach to the Tarski problem. The first important result in analyzing these actions came from Rips (unpublished) when he proved that if a group acts freely on a real tree then it is a free product of surface groups and free abelian groups (see [GLP94]).

Requiring an action to be free is a rather extreme condition. One could still get a structure theorem, known as Rips' machine, by imposing some milder conditions. Recall that an action of a group on a real tree is called *super-stable* if for any arc I with non-trivial (pointwise) stabilizer and J a subarc of I we have that $\mathrm{Stab}_G(I) = \mathrm{Stab}_G(J)$.

Theorem 2.20 (Rips' Machine)**:** *Let G be a finitely presented torsion-free group. Suppose G acts non-trivially on an $\mathbb{R}$-tree Y. Moreover, assume that the action is minimal, super-stable and tripod stabilizers are trivial.*

Then $G \curvearrowright Y$ decomposes as a graph of actions $\mathcal{A} := (G \curvearrowright T, \{Y_u\}_{u\in V(T)}, \{p_e\}_{e\in E(T)})$ where each of the vertex actions, $Stab_G(u) \curvearrowright Y_u$, is of either simplicial or surface or axial or exotic type.

Remark 2.21: *Actions of exotic type have been discovered by Levitt (see []). Since the existence of an exotic type component implies that our group splits as a non trivial free product we will almost always be able to exclude them.*

Also note that exotic type components in Rips' machine decomposition have the mixing property.

3 Limit Groups

3.1 The space of projectivised equivariant pseudometrics

In this subsection we show how to construct geometrically a group equipped with a natural action on a real tree satisfying Rips' machine assumptions. The construction is credited to M.Bestvina and F.Paulin independently.

We fix a finitely generated group G and we consider the set of non-trivial equivariant pseudometrics $d : G \times G \to \mathbb{R}^{\geq 0}$, denoted by $\mathcal{ED}(G)$. We equip $\mathcal{ED}(G)$ with the compact-open topology (where G is given the discrete topology). Note that convergence in this topology is given by:

$$(d_i)_{i<\omega} \to d \;\text{ if and only if }\; d_i(1,g) \to d(1,g) \;\text{ (in } \mathbb{R}) \;\text{ for any }\; g \in G$$

Is not hard to see that $\mathbb{R}^+$ acts cocompactly on $\mathcal{ED}(G)$ by rescaling, thus the space of *projectivised equivariant pseudometrics* on G is compact.

We also note that any based G-space $(X, *)$ (i.e. a metric space with a distinguished point equipped with an action of G by isometries) gives rise to an equivariant pseudometric on G as follows: $d(g,h) = d_X(g \cdot *, h \cdot *)$. We say that a sequence of G-spaces $(X_i, *_i)_{i<\omega}$ converges to a G-space $(X, *)$, if the corresponding pseudometrics induced by $(X_i, *_i)$ converge to the pseudometric induced by $(X, *)$ in $\mathcal{PED}(G)$.

A morphism $h : G \to H$ where H is a finitely generated group induces an action of G on X_H (the Cayley graph of H) in the obvious way, thus making X_H a G-space. We have:

Lemma 3.1 (Bestvina-Paulin Method)**:** *Let $(h_n)_{n<\omega} : G \to \mathbb{F}$ be a sequence of non-trivial morphisms. Then for each $n < \omega$ there exists a base point $*_n$ in $X_{\mathbb{F}}$ such that the sequence of G-spaces $(X_{\mathbb{F}}, *_n)_{n<\omega}$ has a convergent subsequence to a real G-tree $(T, *)$, where the action of G on T is non trivial.*

Proof. We only give the idea and leave the proof for the reader. Choose the base points $*_n$ so that they minimize the following function $D_n(x) = \Sigma_{s \in S} d_{\mathbb{F}}(x, h_n(s) \cdot x)$ where S is a fixed finite generating set for G. Then choose the rescaling factors to be $D_n(*_n)$. $\square$

Moreover the following proposition is true.

Proposition 3.2: *Assume $G \curvearrowright^{\lambda} T$ is obtained as in Lemma 3.1 and $L := G/Ker\lambda$, where $Ker\lambda := \{g \in G | \lambda(g,x) = x \text{ for all } x \in T\}$, then L acts on T as follows:*

(i) tripod stabilizers are trivial;

(ii) arc stabilizers are Abelian;

(iii) the action is super-stable.

Moreover, given the situation of Lemma 3.1 one can approximate every point in the limiting tree T by a sequence of points of the converging subsequence.

Lemma 3.3: *Assume $(X_{\mathbb{F}}, *_n)_{n<\omega}$ converges to $(T, *)$ as in Lemma 3.1. Then for any $x, y \in T$, the following hold:*

- *there exists a sequence $(x_n)_{n<\omega}$ such that $\hat{d}_n(x_n, g \cdot *_n) \to d_T(x, g \cdot *)$ for any $g \in G$, where $\hat{d}_n$ denotes the rescaled metric of $X_{\mathbb{F}}$, we call such a sequence an approximating sequence;*
- *if $(x_n)_{n<\omega}, (x'_n)_{n<\omega}$ are two approximating sequences for $x \in T$, then $\hat{d}_n(x_n, x'_n) \to 0$;*
- *if $(x_n)_{n<\omega}$ is an approximating sequence for x, then $(g \cdot x_n)_{n<\omega}$ is an approximating sequence for $g \cdot x$;*
- *if $(x_n)_{n<\omega}, (y_n)_{n<\omega}$ are approximating sequences for x, y respectively, then $\hat{d}_n(x_n, y_n) \to d_T(x, y)$.*

Remark 3.4:

- *Is not hard to see that if $h_n(g) = 1$ for all but finitely many n's in the above sequence of morphisms, then $g \in Ker\lambda$. Actually, when the limit tree is not a line, then $Ker\lambda$ is exactly the set of g's that are eventually killed by $(h_n)_{n<\omega}$.*
- *the minimal subtree obtained by the Bestvina-Paulin method is isometric to a line if and only if for all but finitely many n, $h_n(G)$ is cyclic.*

Definition 3.5: *A group L is called a limit group if it is obtained as the quotient of a finitely generated group by the kernel of the action on a real tree obtained as in Lemma 3.1.*

Fact 3.6: *Let L be a limit group. Then:*

- *L is torsion-free;*
- *L is CSA;*
- *L is either abelian or acts non-trivially on a simplicial tree with cyclic edge stabilizers;*
- *L is finitely presented.*

4 Makanin-Razborov Diagrams for non-Abelian free groups

In this section we will develop the tools in order to describe "varieties" in non-Abelian free groups, i.e. solution sets of sytems of equations $\Sigma(\bar{x}) = 1$ in a non-Abelian free group $\mathbb{F}$. Our point of view will be group theoretic, so equivalently we are aiming to describe $Hom(G_\Sigma, \mathbb{F})$ where G_Σ is the the group presented as $\langle \bar{x} \mid \Sigma(\bar{x}) \rangle$.

We will see that a special class of groups, the class of limit groups play a significant role in this description.

For the rest of this section we fix a non-Abelian free group $\mathbb{F}$.

4.1 Equational Noetherianity

We start this subsection by describing $Hom(G, \mathbb{F})$ in some particular cases.

- (Free Groups) Let G be a free group of rank n. Then $Hom(G, \mathbb{F}) \cong \mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}$ n-times.
- (Free Abelian Groups) Let G be a free abelian group of rank n and consider the projection to the first factor $p : \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}$. Then any morphism from G to $\mathbb{F}$ factors through p after precomposing by an automorphism of $\mathbb{Z}^n$, i.e. for any $h : G \to \mathbb{F}$ there exists $\alpha \in GL_n(\mathbb{Z})$ such that $h \circ \alpha = h' \circ p$ for some $h' : \mathbb{Z} \to \mathbb{F}$. In this case $Hom(G, \mathbb{F})$ is described by the one step "resolution" $\mathbb{Z}^n \twoheadrightarrow \mathbb{Z}$ and the parametrization $GL_n(\mathbb{Z}) \times \mathbb{F}$.
- (Closed Surface Groups) Let Σ be an orientable closed surface of genus g. Then there exists an epimorphism $p : \pi_1(\Sigma) \twoheadrightarrow \mathbb{F}_g$ such that any morphism from $\pi_1(\Sigma)$ to $\mathbb{F}$ factors through p after precomposing by an automorphism of $\pi_1(\Sigma)$ (see [] and []).

  In the case of a non-orientable (closed) surface there exist finitely many epimorphisms $p_i : \pi_1(\Sigma) \twoheadrightarrow \mathbb{F}_g$ for $i \leq k$ such that any morphism from $\pi_1(\Sigma)$ to $\mathbb{F}$ factors through one of the p_i after precomposing by an automorphism of $\pi_1(\Sigma)$.

  In both cases $Hom(G, \mathbb{F})$ is described by a one step "resolution" (of finite width) and the parametrization $Aut(\pi_1(\Sigma)) \times \mathbb{F}^g$.

We seek such a description for any (f.g.) group G. We will see that for the first step as well as for proving the finite length of our "diagram" the equational Noetherianity of non-Abelian free groups is important.

Lemma 4.1 (Guba)**:** *A non-Abelian free group is equationally Noetherian, i.e. any system of equations is equivalent to a finite subsystem.*

Proof. $\mathbb{F}_2$ embedds in $SL_2(\mathbb{Z})$. □

A more algebraic definition of limit groups can be given as follows. We recall that a sequence of morphisms between two groups $(h_n)_{n<\omega} : G \to H$ is *convergent* if for any element $g \in G$, there exists a natural number n_g such that either $h_n(g) = 1$ for all $n > n_g$ or $h_n(g) \neq 1$ for all $n > n_g$. To a convergent sequence one can naturally assign its *stable kernel* $\underleftarrow{\mathrm{Ker}} h_n := \{g \in G | \ h_n(g) \text{ is eventually trivial}\}$.

Lemma 4.2: *A (f.g) group L is a limit group if and only if there exists a convergent sequence of morphisms $(h_n)_{n<\omega} : L \to \mathbb{F}$ with $\underleftarrow{\mathrm{Ker}} h_n = \{1\}$*

Now the following theorem is immediate. Recall that a group G is called ω-residually free if for any finite subset $X \subseteq G$ there exists a morphism to a free group $h : G \to \mathbb{F}$ such that $h \restriction X$ is injective. Similarly a group G is called residually free if for any non-trivial element $g \in G$ there is a morphism to a free group $h : G \to \mathbb{F}$ with $h(g) \neq 1$.

Theorem 4.3 (Sela)**:** *A finitely generated group is a limit group if and only if it is ω-residually free.*

Note that the strong result of the finite presentability of limit groups could have been used in order to prove Theorem 4.3. In fact, this is how Sela originaly proves it.

Remark 4.4: *Any group G has a residually free quotient $q : G \to RF(G)$ such that any morphism from G to $\mathbb{F}$ factors through q. Thus when considering $Hom(G, \mathbb{F})$ we may always assume that G is residually free.*

Lemma 4.5 (Finite Length)**:** *Any sequence of proper epimorphisms of residually free groups is finite.*

Proof. Use the fact that if $q : R_1 \twoheadrightarrow R_2$ is proper then there exists a morphism $h : R_2 \to \mathbb{F}$ that does not kill a non-trivial element that is necessarily killed by q together with the equational Noetherianity of non-Abelian free groups. $\square$

Now we are ready to make the first step of the Makanin-Razborov diagram that will eventually give a description of $Hom(G, \mathbb{F})$ for an arbitrary group G.

Theorem 4.6: *Let G be a (f.g.) group which is not a limit group. Then there exist finitely many (proper) epimorphisms $\{q_i : G \to L_i\}_{i \leq k}$ such that each L_i is a limit group and each morphism from G to $\mathbb{F}$ factors through one the q_i's.*

Proof. Since G is not a limit group, there exists a finite subset $X \subset G$ such that any morphism from G to $\mathbb{F}$ kills some element of X. Thus we get the following "factor set" $\{q_i : G \twoheadrightarrow G/\langle\langle x \rangle\rangle \mid x \in X\}$. By Remark 4.4 and Lemma 4.5 we may assume that $G/\langle\langle x \rangle\rangle$ are limit groups. $\square$

Theorem 4.6 reduces the understanding of $Hom(G, \mathbb{F})$ for a general (f.g.) group G to the case where G is a limit group.

The next theorem (which completes the construction of the Makanin-Razborov diagram) is hard to prove. The main dificulty lies on the "shortening argument" a strong tool whose explanation is postponed until the next subsection.

Theorem 4.7: *Let G be a (limit) group which is not free. Then there exist finitely many proper quotients $\{q_i : G \twoheadrightarrow L_i\}_{i \leq k}$ such that each L_i is a limit group and any morphism from G to $\mathbb{F}$ factors through some q_i after precomposing by an automorphism of G, i.e. for any $h : G \to \mathbb{F}$ there exists $i \leq k$ and $\alpha \in Aut(G)$ such that $h \circ \alpha = h' \circ q_i$ for some $h' : L_i \to \mathbb{F}$.*

Now we are ready to describe $Hom(G, \mathbb{F})$ for G a (f.g.) group. Following Theorem 4.7 and Lemma 4.5 we can assign to $Hom(G, \mathbb{F})$ a set of finite sequences of (proper) epimorphisms, called *resolutions*, that all start with G and end in a free group such that any morphism $h : G \to \mathbb{F}$ "factors through some resolution".

Consider the following resolution:

$$G \twoheadrightarrow L_1 \twoheadrightarrow L_2 \twoheadrightarrow \ldots \twoheadrightarrow L_k \twoheadrightarrow \mathbb{F}_n$$

Then we say that $h : G \to \mathbb{F}$ factors *through the resolution* if there exists an automorphism α of G, a sequence of automorphisms $(\alpha_i \in Aut(L_i))_{i \leq k}$ and a morphism from $\mathbb{F}_n$ to $\mathbb{F}$ such that $h = h' \circ ... \circ ...$

In particular $Hom(G, \mathbb{F})$ is described by finitely many resolution each parametrized by a group of the form $Aut(G) \times Aut(L_1) \times \ldots \times Aut(L_k) \times \mathbb{F}^n$.

4.2 The Shortening Argument

The shortening argument is a deep geometric tool first used by Rips and Sela where they proved that the group of modular automorphisms (they call it the group of internal automorphisms) of a torsion-free freely indecomposable hyperbolic group has finite index in the full automorphism group. There are many variations of the shortening argument which is part of its power. In these notes we will present two of the variations and quickly sketch their proof.

We start with a few definitions.

Definition 4.8: *Let G be a finitely generated group, with a fixed finite generating set S, and $h : G \to \mathbb{F}$. Then:*

- *the length of h is defined as $\mathfrak{l}(h) := \Sigma_{s\in S}\{d_X(1, h(s)\cdot 1)\}$, where d_X denotes the usual metric in the Cayley graph X of $\mathbb{F}$;*
- *h is called short if $\mathfrak{l}(h) \leq \mathfrak{l}(Conj(\gamma)\circ h\circ\sigma)$ for any $\gamma \in \mathbb{F}$ and any $\sigma \in Aut(G)$.*

Theorem 4.9: *Let G be a f.p. group and $(h_n)_{n<\omega} : G \to \mathbb{F}$ be a sequence of non-trivial short morphisms. Then either G acts on a (simplicial) tree with trivial edge stabilizers or the limit action induced as in Lemma 3.1 is not faithful.*

Sketch. Assume, for the sake of contradiction, that the induced action on the limiting real tree is faithful. Thus, we can analyze it using Rips' machine (see figure 1). Moreover we may assume that no exotic components exist, since otherwise the group splits as a non trivial free product and this induces an action on a (simplicial) tree with trivial edge stabilizers.

Note that the choice of short morphisms implies that the base point in the limit is approximated by the sequence of trivial elements in $X_{\mathbb{F}}$. The hard part of the shortening argument is to show that we can "shorten" simultaneously all segments of the form $[*, s\cdot *]$ (for s an element in the generating set of G) using automorphisms of the stabilizers of the components that these segments intersect. We will supress somehow that in the case of simplicial components the "shortening" happens in the approximating sequence and not in the limiting real tree.

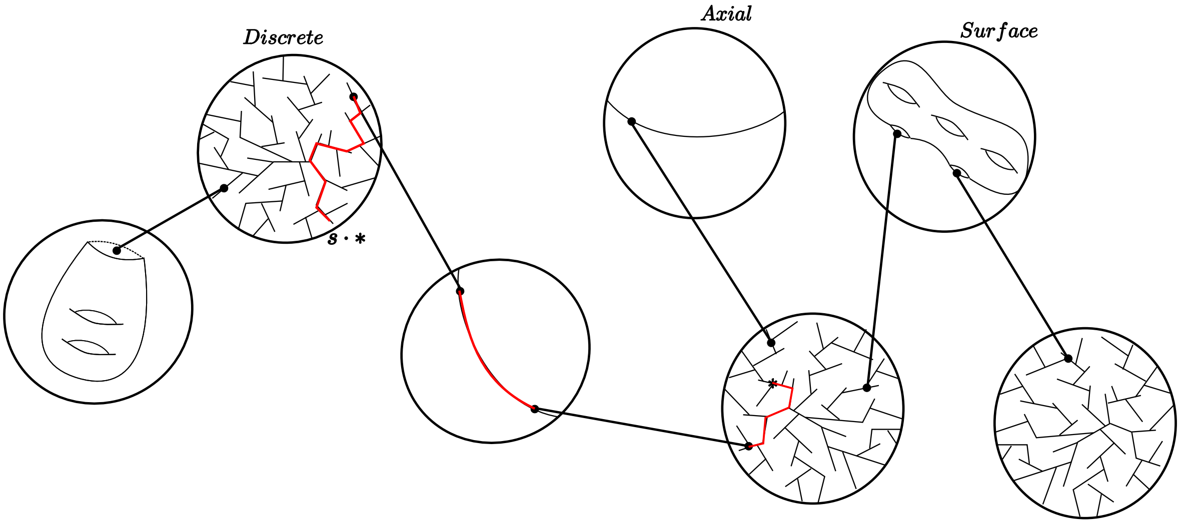

Figure 1: Rips' Decomposition

This would imply that in the approximating sequence $d_X(1, h_n(\sigma((s)))\cdot 1) < d_X(1, h_n(s)\cdot 1)$ for almost every $n < \omega$ contradicting the shortness of h_n. $\square$

We will also consider morphisms that are short but only relative to some (f.g.) subgroup of a f.g. group.

Definition 4.10: *Let G be a (f.g.) group and H a (f.g.) subgroup. Then $h : G \to \mathbb{F}$ is called short with respect to H if $\mathfrak{l}(h) \leq \mathfrak{l}(Conj(\gamma) \circ h \circ \sigma)$ for any $\gamma \in C_{\mathbb{F}}(h(H))$ and any $\sigma \in Aut_H(G)$.*

Similarly to Theorem 4.9 we have:

Theorem 4.11: *Let G be a finitely presented group freely indecomposable with respect to a finitely generated subgroup H. Suppose $(h_n)_{n<\omega} : G \to \mathbb{F}$ is a sequence of non-trivial short morphisms with respect to H, converging to an action of G on a real tree as in Lemma 3.1 where H fixes a point. Then the action is not faithful.*

5 Towers

Towers or more precisely groups that have the structure of an ω-residually free tower (in Sela's terminology), were introduced in [Sel01, Definition 6.1]. We will see in the next section that they provide examples of groups for which Merzlyakov's theorem (see [Mer66]) naturally generalizes (up to some fine tuning). In practice, towers appear as completions of well-structured MR resolutions of limit groups (see Definitions 1.11 and 1.12 in [Sel03]).

We start by defining the main building blocks of a tower, namely free abelian flats and surface flats.

Definition 5.1 (Free abelian flat)**:** *Let G be a group and H be a subgroup of G. Then G has the structure of a free abelian flat over H, if G is the amalgamated free product $H *_A (A \oplus \mathbb{Z})$ where A is a maximal abelian subgroup of H.*

Before moving to the definition of a hyperbolic floor we recall that if H is a subgroup of a group G then a morphism $r : G \to H$ is called a *retraction* if r is the identity on H.

Definition 5.2 (Hyperbolic floor)**:** *Let G be a group and H be a subgroup of G. Then G has the structure of a hyperbolic floor over H, if G acts minimally on a tree T and the action admits a Bass-Serre presentation $(T^1, T^0, \{\gamma_e\})$ such that:*

- *the set of vertices of T^0 is partitioned in two sets, V_1 and V_2, where all the vertices in V_1 are surface type vertices;*
- *T^1 is bipartite between V_1 and $V(T^1) \setminus V_1$;*
- *H is the free product of the stabilizers of vertices in V_2;*
- *either there exists a retraction $r : G \to H$ that, for every $v \in V_1$, sends $\mathrm{Stab}_G(v)$ to a non-Abelian image or H is cyclic and there exists a retraction $r' : G * \mathbb{Z} \to H * \mathbb{Z}$ which, for every $v \in V_1$, sends $\mathrm{Stab}_G(v)$ to a non-Abelian image.*

If a group has the structure of a hyperbolic floor (over some subgroup), and the corresponding Bass-Serre presentation contains just one surface type vertex then we call the hyperbolic floor *a surface flat.*

We use surface and free abelian flats in order to define towers.

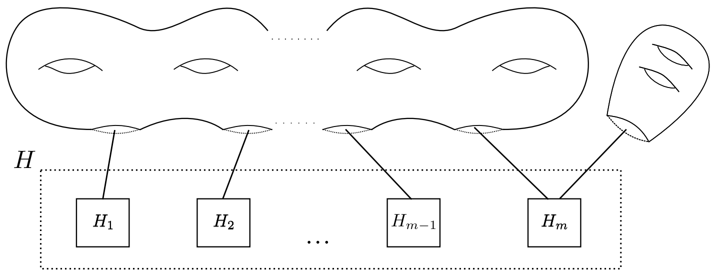

Figure 2: A graph of groups corresponding to a hyperbolic floor

Definition 5.3: *A group G has the structure of a tower (of height m) over a subgroup H if there exists a sequence $G = G^m > G^{m-1} > \ldots > G^0 = H$, where for each i, $0 \leq i < m$, one of the following holds:*

(i) G^{i+1} is the free product of G^i with either a free group or with the fundamental group of a closed surface of Euler characteristic at most -2;

(ii) G^{i+1} has the structure of a surface flat over G^i;

(iii) G^{i+1} has the structure of a free abelian flat over G^i.

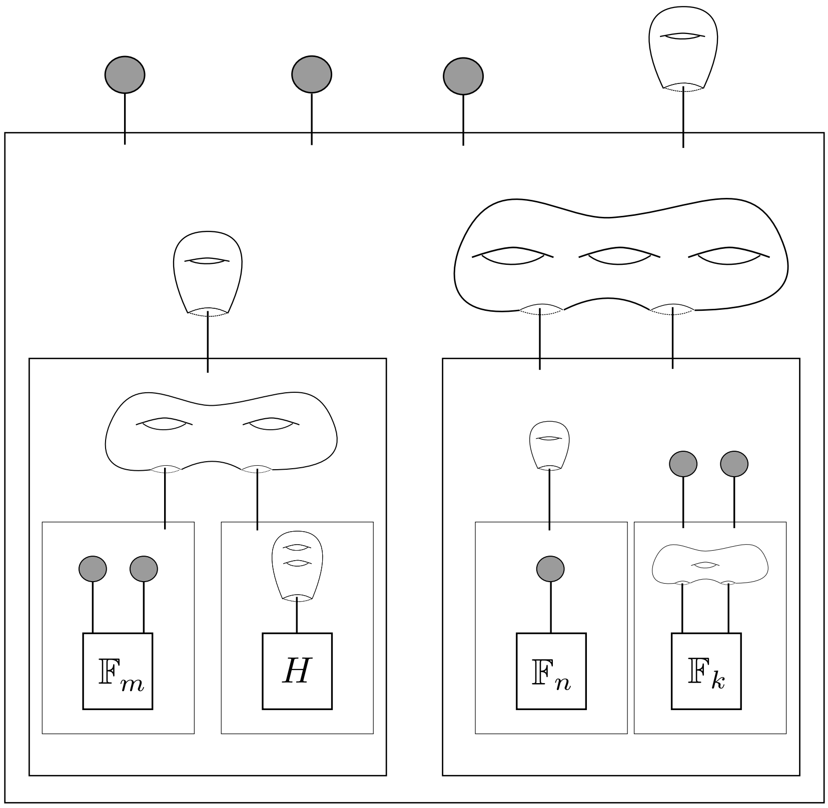

Figure 3: A tower over H.

One can see that a group that has the structure of a tower over the trivial subgroup (or even over a limit group) is a limit group, but not every limit group admits the structure of a tower.

Following the (non-trivial) observation above, it would have been very convenient, as we shall see in the sequel, if every "irreducible variety" in a non-Abelian free group, could be assigned a "co-ordinate" group which admits the structure of a tower. The next theorem of Sela says that every "irreducible variety" can be split in finitely many subsets where each one of them is the projection of an "irreducible variety" whose co-ordinate group has the structure of a tower.

Theorem 5.4: *Let $L := \langle \bar{x} \mid \Sigma(\bar{x}) \rangle$ be a limit group. Then there exists finitely many groups $T_1(\bar{x}, \bar{y}), \ldots, T_k(\bar{x}, \bar{y})$ (generated by $\bar{x}, \bar{y}$) that have the structure of a tower, such that:*

- *for any $i \leq k$ either L or a quotient of L embedds in T_i;*
- *for any morphism $h : L \to \mathbb{F}$, there exists $i \leq k$ such that h extends to a morpshism from T_i to $\mathbb{F}$.*

Note that a group G is said to have the structure of a *hyperbolic tower* if no abelian flat occurs in its "construction".

The following remarkable theorems hold for hyperbolic towers.

Theorem 5.5 (Sela)**:** *A (f.g.) group G is a model of the theory of the free group if and only if it has the structure of a (non-Abelian) hypebolic tower over $\{1\}$.*

Theorem 5.6 (Perin)**:** *Let H be a torsion free hyperbolic group and Γ be an elementary subgroup. Then H admits the structure of a hyperbolic tower over Γ.*

6 Merzlyakov' s Theorem and Generalizations

Merzlyakov proved the following theorem in order to prove that the positive theories of non-Abelian free groups are equal.

Theorem 6.1: *Let $\Sigma(\bar{x}, \bar{y}) = 1$ be a system of equations over $\mathbb{F} := \langle \bar{a} \rangle$. Suppose $\mathbb{F} \models \forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1)$. Then there exists a retraction $r : G_\Sigma \twoheadrightarrow \langle \bar{x} \rangle * \mathbb{F}$, where $G_\Sigma := \langle \bar{x}, \bar{y}, \bar{a} \mid \Sigma(\bar{x}, \bar{y}, \bar{a}) \rangle$.*

For a slightly extended version of Merzlyakov's theorem we first need to define the notion of a test sequence.

Definition 6.2: *A sequence of tuples, $(b_1(n), \ldots, b_k(n))_{n<\omega}$, in $\mathbb{F}$ is called a test sequence if the tuple $(b_1(n), \ldots, b_k(n))$ satisfies the small cancellation property $C'(1/n)$ in $\mathbb{F}$, for $n < \omega$.*

Certainly one can follow Merzlyakov's proof for the following extended version of his theorem. We have chosen to give a geometric proof, on the expense of simplicity, merely because the geometric methods suggest some natural generalizations.

Theorem 6.3: *Let $\Sigma(\bar{x}, \bar{y}) = 1$ be a system of equations over $\mathbb{F} := \langle \bar{a} \rangle$. Let $(\bar{b}_n)_{n<\omega}$ be a test sequence of tuples in $\mathbb{F}$ such that for each n there exists a tuple $\bar{c}_n$ with $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n, \bar{a}) = 1$. Then there exists a retract $r : G_\Sigma \twoheadrightarrow \langle \bar{x} \rangle * \mathbb{F}$, where $G_\Sigma := \langle \bar{x}, \bar{y}, \bar{a} \mid \Sigma(\bar{x}, \bar{y}, \bar{a}) \rangle$.*

Proof. We fix a basis for $\mathbb{F}$ and let $\bar{x}, \bar{y}, \bar{a}$ be the fixed generating set for G_Σ. Since for each n there exists a morphism $g_n : G_\Sigma \to \mathbb{F}$ given by $\bar{x} \mapsto \bar{b}_n$, $\bar{y} \mapsto \bar{c}_n$, $\bar{a} \mapsto \bar{a}$, we choose g_n to be short with respect to $\langle \bar{x}, \bar{a} \rangle$.

By Lemma 3.1 a subsequence of $(X_{\mathbb{F}}, 1)$ converges to an $\mathbb{R}$-tree $(T, *)$ endowed with an action $G_\Sigma \curvearrowright^\lambda T$. Note that since $g_n(G_\Sigma)$ is non-Abelian for every $n < \omega$, we have that T is not a line.

We consider the limit group $L := G_\Sigma / Ker\lambda$, and let $\eta : G_\Sigma \twoheadrightarrow L$ be the canonical quotient map. We note that, by the properties of the test sequence, η is injective on $\langle \bar{x}, \bar{a} \rangle$. Moreover, since T is not a line we have that there is a sequence of morphisms $(h_n)_{n<\omega} : L \to \mathbb{F}$ such that $g_n = h_n \circ \eta$ for all but finitely many $n < \omega$. We argue that we may assume that L is freely indecomposable with respect to $\eta(\langle \bar{x}, \bar{a} \rangle)$ (still denoted $\langle \bar{x}, \bar{a} \rangle$). If not, then we continue with the smallest free factor of L containing $\langle \bar{x}, \bar{a} \rangle$ and the restriction of $(h_n)_{n<\omega}$ on this free factor (after been made short with respect to $\langle \bar{x}, \bar{a} \rangle$). Lemma 4.5 ensures us that after finitely many steps we get what we wanted.

It is immediate that the sequence of morphisms, $(h_n)_{n<\omega}$, is short with respect to $\langle \bar{x}, \bar{a} \rangle$, and induces a faithful action of L on the real tree T (which is not a line). This action can be analyzed using Rips' machine and by Theorem 4.11 $\langle \bar{x}, \bar{a} \rangle$ does not fix any point. Thus, let $\bar{x} = x_1, \ldots, x_m, x_{m+1}, \ldots, x_l$ where x_i moves $*$ if and only if $i \leq m$ (note that at least x_1 moves $*$, since the tuple $\bar{a}$ fixes it). We first prove that the minimal subtree T_{min} that $\langle \bar{x}, \bar{a} \rangle$ acts on lies in the discrete part of T. The following claim will prove very useful.

Claim: Let $1 \leq i \leq k$ and $I \subseteq [*, x_i \cdot *]$ be an arc. Then for any $g \in L \setminus \{1\}$ and any $j \leq k$ we have that gI intersects $[*, x_j \cdot *]$ trivially.
Proof of Claim: Suppose, for the sake of contradiction, that $gI \cap [*, x_j \cdot *]$ is non-trivial for some $j \leq m$. Without loss of generality we may assume that $gI \subseteq [*, x_j \cdot *]$. Let $I = [a, b]$ and suppose $(a_n)_{n<\omega}$, $(b_n)_{n<\omega}$ be sequences approximating a, b respectively and γ be an element of G_Σ such that $\eta(\gamma) = g$. The segment $[a_n, b_n]$ in $X_{\mathbb{F}*\mathbb{F}_{k_n}}$ contains a word, w, which is a subword of $h_n(x_i)$ and similarly the segment $[h_n(\gamma)a_n, h_n(\gamma)b_n]$ contains the same word, w, and is a subword of $h_n(x_j)$. Since g is non-trivial, $h_n(\gamma)$ acts freely for arbitrarily large n, thus w is a piece. But this contradicts the small cancellation hypothesis $C'(1/n)$ of the test sequence.

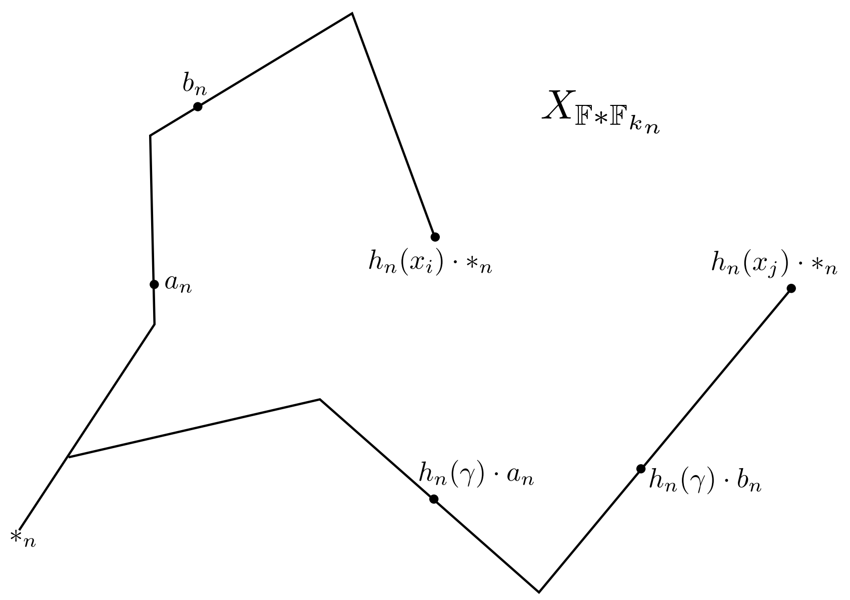

Figure 4: Part of the Cayley graph of $\mathbb{F}$

Now, since T_{min} is covered by translates of the convex hull of $\{*, x_1 \cdot *, \ldots, x_m \cdot *\}$ we have that T_{min} intersects the components of the real tree T having the mixing property only in finitely many points, thus T_{min} is on the discrete part of T.

We continue by proving that T is discrete by showing that no component with the mixing property may exist. Suppose, for the sake of contradiction, that an exotic type component exists this induces a non-trivial free splitting of $L = L_1 * L_2$ where $\langle \bar{x} \rangle \leq L_1$, contradicting the fact that L is freely indecomposable with respect to $\langle \bar{x} \rangle$. If an axial or a surface type component exists, then $[*, y_i \cdot *]$ for some $y_i \in \bar{y}$ would intersect it non-trivially (since $L.*$ spans T). But then, by standard arguments we could shorten the length of $h_n(y_i)$ keeping $\bar{x}$ fixed, contradicting the assumption that h_n is short with respect to $\bar{x}$.

So we are left with a discrete action of L on T. Note that by our previous Claim all edges in T_{min} are trivially stabilized. Using the shortening argument once again we can prove that all edges in T are trivially stabilized. If not, then as before some segment of the form $[*, y_i \cdot *]$ would contain a non-trivially stabilized edge and by standard arguments we can shorten the length of $h_n(y_i)$ fixing $\bar{x}$, contradicting the assumption that h_n is short with respect to $\bar{x}$.

Finally, we claim that for each y_i we have that $y_i \cdot * = w_i(x_1, \ldots, x_m) \cdot *$ for some word $w_i(x_1, \ldots, x_m) \in \langle x_1, \ldots, x_m \rangle$. If not, then we have that there is some $g \in L$ such that $[*, g \cdot *]$ is a proper subarc of $[*, x_i \cdot *]$ for some $i \leq m$. This can be easily shown that contradicts the small cancellation hypothesis of the test sequence $(h_n(\bar{x}))_{n<\omega}$. Thus, L inherits a splitting of the form $\langle x_1, \ldots, x_m \rangle * \mathrm{Stab}(*)$.

We continue in the same manner with $\mathrm{Stab}(*)$, after finitely many steps, we obtain a free decomposition of L as $\langle x_1, \ldots, x_l \rangle * L_0$, with $\langle \bar{a} \rangle \leq L_0$, and morphisms from L_0 to $\langle \bar{a} \rangle$ fixing $\bar{a}$. So we have a retract from L to $\langle \bar{x} \rangle * \langle \bar{a} \rangle$. This gives us a retract from G_Σ to $\langle \bar{x} \rangle * \mathbb{F}$, as we wanted. $\square$

Theorem 6.4: *Let $\Sigma(\bar{x}, \bar{y}) = 1$ be a system of equations over $\emptyset$. Let $(\bar{b}_n)_{n<\omega}$ be a test sequence of tuples in $\mathbb{F}$ such that for each n there exists a tuple $\bar{c}_n$ with $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$. Then there exists a retract $r : G_\Sigma \twoheadrightarrow \langle \bar{x} \rangle$.*

Proof. As in the proof above we choose a sequence of morphisms, $(g_n)_{n<\omega}$, which is short with respect to $\langle \bar{x} \rangle$. And we consider the induced action of G_Σ on a real tree T. In this case the only obstacle to directly applying the proof of Theorem 6.3 is that we may have that the real tree T is isometric to a line. Note that $\bar{x}$ should be a singleton, say x (if not the image $g_n(G_\Sigma)$ is non-Abelian thus T cannot be isometric to a line). We consider the quotient group, L, of G_Σ by the stable kernel $\underleftarrow{Ker} g_n$. We note that there exists a sequence of morphisms $(h_n)_{n<\omega} : L \to \mathbb{F}_{k_n}$ such that $g_n = h_n \circ \eta$ for all but finitely many $n < \omega$, where η is the canonical quotient map from G_Σ to L.

It is easy to see that L is abelian and $(h_n)_{n<\omega}$ is short with respect to $\eta(\langle x \rangle) = \langle x \rangle$ (since η is injective on $\langle x \rangle$).

We consider the limit action λ of L, induced by the sequence $(h_n)_{n<\omega}$, on a real tree T' (which is again isometric to a line). We may assume that x does not fix a point. If not, then x fixes the whole line, and L is the direct product $Ker\lambda \oplus \mathbb{Z}^l$ for some $l < \omega$ with $\langle x \rangle \leq Ker\lambda$. We can continue with $Ker\lambda$, by Lemma 4.5, in finitely many steps we have that x cannot fix a point.

Finally, we claim that the translation length of $\eta(y_i)$ for any $y_i \in \bar{y}$ is a multiple of the translation length of x. If not, then we have $g \in L$ such that $0 < tr(g) < tr(x)$. Thus, we can find a proper non-trivial subarc, I, of $[*, x \cdot *]$ such that $gI \subset [*, x \cdot *]$, contradicting the small cancellation hypothesis of the test sequence $(h_n(x))_{n<\omega}$. This, shows that L acts discretely on the line and $L = \langle x \rangle$ as we wanted.

The rest of the proof follows the proof of Theorem 6.3. $\square$

Merzlyakov's theorem extends to the case where the universal variables are "bounded" by a variety that corresponds to group that has the structure of a hyperbolic tower.

Theorem 6.5: *Let $\Sigma(\bar{x}, \bar{y}) = 1$ be a system of equations over $\mathbb{F} := \langle \bar{a} \rangle$ and $G_T := \langle \bar{x}, \bar{a} \mid T(\bar{x}, \bar{a}) \rangle$ be a group that has the structure of a hyperbolic tower over $\mathbb{F}$. Suppose $\mathbb{F} \models \forall \bar{x}(T(\bar{x}, \bar{a}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1))$. Then there exists a retraction $r : G_\Sigma \twoheadrightarrow G_T$.*

Finally the following generalization allow us to consider inequalities as well.

Theorem 6.6: *Let $\Sigma(\bar{x}, \bar{y}) = 1$ be a system of equations over $\mathbb{F} := \langle \bar{a} \rangle$ and $\Psi(\bar{x}, \bar{y}, \bar{a})$ a set of words in $\langle \bar{x}, \bar{y}, \bar{a} \rangle$. Let $G_T := \langle \bar{x}, \bar{a} \mid T(\bar{x}, \bar{a}) \rangle$ be a group that has the structure of a hyperbolic tower over $\mathbb{F}$. Suppose $\mathbb{F} \models \forall \bar{x}(T(\bar{x}, \bar{a}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1) \land \Psi(\bar{x}, \bar{y}, \bar{a}) \neq 1)$. Then there exists a retraction $r : G_\Sigma \twoheadrightarrow G_T$ such that for each $\psi(\bar{x}, \bar{y}, \bar{a}) \in \Psi(\bar{x}, \bar{y}, \bar{a})$, $r(\psi)$ is not trivial in G_T.*

7 The equivalence of the $\forall\exists$ theories

In this section we will give the strategy for proving that the truthfulness of a $\forall\exists$ sentence (over parameters) does not depend on the rank of a non-Abelian free group. Since the technicalities of the above mentioned result are considerably hard we will restrict ourselves to the special case where every tower that appears in the procedure is hyperbolic and of "minimal rank", i.e. it is a tower over some "parameter" free group $\mathbb{F}$ and as a group it admits no epimorphism to $\mathbb{F} * \mathbb{F}'$ where $\mathbb{F}'$ is a non trivial free group.

We first define a notion of complexity for towers.

Definition 7.1: *Let T be a group that has the structure of a hyperbolic tower over $\mathbb{F}$. Let $\Sigma_1, \ldots, \Sigma_m$ be the surfaces that appear in the surface flats of the tower and let k be the number of the abelian flats of the tower. Then:*

$$Complx(T) := ((genus(\Sigma_1), \chi(\Sigma_1)), \ldots, (genus(\Sigma_m), \chi(\Sigma_m)), k)$$

where the couples are arranged in decreasing lexicographical order (assuming $(genus(\Sigma_1), \chi(\Sigma_1)) \geq \ldots \geq (genus(\Sigma_m), \chi(\Sigma_m))$).

Step 1 We start with a sentence which is true in a non-Abelian free group: $\mathbb{F} \models \forall \bar{x} \exists \bar{y}(\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1 \land \Psi(\bar{x}, \bar{y}, \bar{a}) \neq 1)$. Using Theorem 6.6, we obtain a "formal solution" $\bar{w}(\bar{x}, \bar{a})$ that validates (independently of $\mathbb{F}$) the sentence everywhere but at the varieties defined by $\psi_i(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$, where ψ_i belongs to Ψ (see figure 5).

Step 2 We continue with each variety $\psi_i(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$ separately. Our goal is to find a "formal solution" that will validate our true sentence in these sets. We use Theorem 5.4 to cover $\psi_i(\bar{x}, \bar{w}(\bar{x}, \bar{a}), \bar{a}) = 1$ with finitely many towers $T_1(\bar{x}, \bar{z}, \bar{a}), \ldots, T_k(\bar{x}, \bar{z}, \bar{a})$. We obviously have:

$$\mathbb{F} \models \forall \bar{x} \bar{z}(T_i(\bar{x}, \bar{z}, \bar{a}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1 \land \Psi(\bar{x}, \bar{y}, \bar{a}) \neq 1))$$

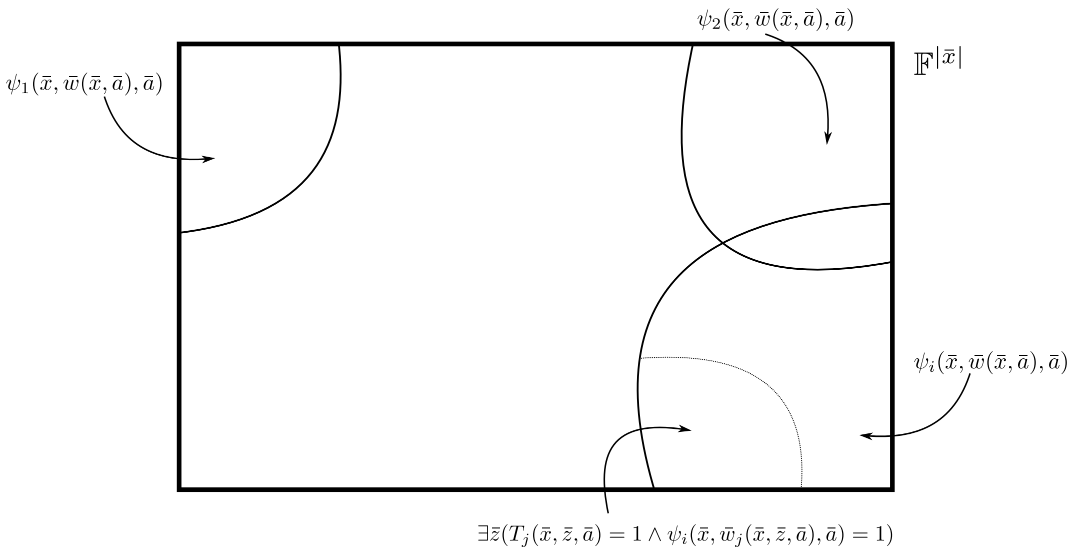

Figure 5: The decomposition of $\mathbb{F}^{|x|}$

We use Theorem 6.6 once again to obtain a "formal solution" $\bar{w_i}(\bar{x}, \bar{z}, \bar{a})$ that validates the sentence in the variety defined by the tower T_i, apart from a union of proper subvarieties defined by the intersection of $T_i = 1$ with $\psi_j(\bar{x}, \bar{w}_i(\bar{x}, \bar{z}, \bar{a}), \bar{a}) = 1$.

Step 3 Iterate Step 2, until it stops?

The following theorem of Sela proves the termination of the above procedure.

Theorem 7.2: *Let $T(\bar{x}, \bar{a})$ be a minimal rank limit group that has the structure of a tower over $\mathbb{F} := \langle \bar{a} \rangle$. Let $Q(\bar{x}, \bar{a})$ be a proper quotient of T, which is a restricted limit group over $\mathbb{F}$. Then Q admits a covering set of finitely many towers $T_1, \ldots, T_k$ (each over $\mathbb{F}$), so that for each $i \leq k$, $Complx(T_i) < Complx(T)$.*

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