On Sharply 2-transitive Groups

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Background

A group G acts *n-transitively* on a set X if for any two n-tuples $(x_1, \dots x_n), (y_1, \dots y_n)$ of distinct elements in X there is some $g \in G$ with $x_i^g = y_i, i = 1, \dots n$.
The group G acts *sharply* n-transitively on X if this $g \in G$ is unique.

A sharply 1-transitive action is called *regular*.

If G acts sharply n-transitively on X, then for any $x \in X$ the stabilizer G_x acts sharply $n-1$-transitively on $X \setminus \{x\}$.

Theorem (Jordan, Tits, M. Hall)

There are no sharply n-transitive groups for $n \geq 4$.

Examples

Examples

Let K be a field.

- $\mathrm{AGL}(1,K)$ acts sharply 2-transitively on (the affine line of) K via $x \mapsto ax+b$.
- $\mathrm{PGL}(2,K)$ acts sharply 3-transitively on the projective line of K via $x \mapsto \frac{ax+b}{cx+d}$.

What do we need from $(K,+,\cdot)$ to make $\mathrm{AGL}(1,K)$ into a sharply 2-transitive group?

The normal subgroup $N = \{x \mapsto x+a \colon a \in K\} \cong K_+$ acts regularly on K.
$G_0 = \{x \mapsto ax \colon a \in K^*\} \cong K^*$ acts regularly on $K \setminus \{0\}$, So

$$\mathrm{AGL}(1,K) \cong N \rtimes G_0 \cong K_+ \rtimes K^*.$$

The fact that G_0 acts on N by conjugation is exactly the (left) distributive law of K.

Near-fields

Suppose G acts sharply 2-transitive on X. If G contains a regular normal subgroup N, we can identify $X = N$. Since G_x acts regularly on $X \setminus \{x\}$, we see that the action of G_x on N by automorphisms of N is transitive on $N \setminus \{1\}$ and left-distributive and G splits as

$$G \cong N \rtimes G_x.$$

Definition (Dickson)

A near-field $(D, +, \cdot)$ is a structure where

- $(D, +, 0)$ *is a group (not assumed to be abelian);*
- $(D^*, \cdot)$ *is a group,*
- *multiplication is left-distributive.*

So a near-field is 'the same' as a split sharply 2-transitive group.

Involutions

For a sharply 2-transitive group G on X consider the set

$$J = \text{involutions in } G.$$

Clearly, $J \setminus \{1\}$ is a conjugacy class.

For fixed $a \in J \setminus \{1\}$ the set $J^2 = \{ab \colon b \in J\}$ is regular on X.

Theorem (B.H. Neumann)

A sharply 2-transitive group G splits if and only if J^2 is an (abelian) subgroup of G.

Thus, addition in near-fields is commutative.

Near-domains and sharply 2-transitive groups

If J^2 is *not* a subgroup, then we obtain a *near-domain* $(D, +, \cdot)$.

A sharply 2-transitive group G is 'the same' as a near domain D (Karzel).

The near-domain D associated to a sharply 2-transitive group G is a near-field if and only if G contains a regular normal subgroup, the additive group of D.

Near-fields

Question

Does every sharply 2-transitive group contain a regular abelian normal subgroup?

or, equivalently,

Is every near-domain a near-field?

Yes, in the finite case:

Theorem (Zassenhaus)

Any finite sharply 2-transitive group arises from a near-field. Except for finitely many exceptions, these are just finite fields, with a possibly twisted multiplication.

Proper finite near-fields exist!

More on involutions

Let G act sharply 2-transitively on X. Then:

Either no involution fixes a point or every involution fixes a (unique) point.

Suppose that involutions have a fixed point. Then there is a bijection $J \to X$. Hence the set J^2 forms a unique conjugacy class.

We put $\mathrm{char}\, G = 0$ if elements in J^2 have infinite order, and $\mathrm{char}\, G = p$ if elements in J^2 have order p.

If K is a field of characteristic 2, then involutions in $\mathrm{AGL}(1, K)$ have no fixed points. So we say that $\mathrm{char}\, G = 2$ if involutions have no fixed points (and J^2 may not be a conjugacy class).

Partial results

The sharply 2-transitive groups split in the following settings:

Zassenhaus: (1936) Finite groups;

Tits: (1952) Lie groups;

Kerby: (1974) Groups of characteristic 3;

T.: (2000) Groups definable in *o*-minimal structures;

Mayr: (2005) Groups in which point stabilizers have exponent 3 or 6;

Glaubermann, Mann, Segev: (2012) Locally linear groups.

There are many more partial results, also in terms of near-domains.

New examples

Naive idea: If (y_1, y_2) is not yet in the G-orbit of (x_1, x_2), add a free generator:

Theorem (Rips-Segev-T./ T.- Ziegler)

Let G be a group acting on a set X so that two-point stabilizers are trivial. Suppose that G contains an involution t and that involutions in G are conjugate and fixed point free.
Then G and its action on X can be extended to a sharply 2-transitive group

$$G_1 = G *_{\langle t \rangle} ((\langle t \rangle \times F(S)) * F(R))$$

acting on $X_1 \supset X$ where $F(R), F(S)$ are free groups of rank $|R|, |S| = \max\{|G|, \aleph_0\}$.

New examples

Special Case

$$G_1 = (C_2 \times F(S)) * F(R)$$

acts sharply 2-transitviely in characteristic 2 (on some set).

Note that G_1 does not contain any abelian normal subgroup and $\mathrm{char}\, G_1 = 2$.

By adding an involution if necessary and iterated HNN-extensions we obtain

Corollary

Any group G can be extended to a sharply 2-transitive group G_1 of characteristic 2 without abelian normal subgroup acting (on some appropriate set).

The construction

A *partial action* of $G_1 = G *_{\langle t \rangle} ((\langle t \rangle \times F(S)) * F(R))$ on $Y \supset X$ is given by an action of G on Y and partial actions of $r \in R, s \in S$ such that s commutes with the involution t wherever s is defined.

A partial action of G_1 on Y is *good* if the following holds:

- If $g \in G_1$ fixes some pair (x_1, x_2), then $g = 1$.
- If $g \in G_1$ flips some pair (x_1, x_2), then g is conjugate to t.
- The involution t fixes no point of y.

Extend a good partial action of G_1 on Y inductively in two ways to a total and sharply 2-transitive action on some $X_1 \supset Y$:

- If $s \in S$ is not defined on $x \in Y$, put $Y_1 = Y \cup x'G$ for some new element $x' \notin Y$ and put $x^s = x', (x^t)^s = (x')^t$.
  Similarly for $r \in R$.
- If (y_1, y_2) is not yet in the orbit of (x_1, x_2), add a fresh generator s from R or S and put $x_i^s = y_i, i = 1, 2$.

Other characteristics

Note that in $G_1 = G *_{\langle t \rangle} ((\langle t \rangle \times F(S)) * F(R))$ products of two involutions have infinite order.

We hope to extend the construction to characteristic 0.

Why is it harder to deal with fixed points?
There is a bijection $J \to X$, so the involutions act transitively on the involutions. For $x, y \in X$, the fixpoint of the involution switching x, y is definable, so the set X has an inherent structure.....

Open problems:

- Do non-split sharply 2-transitive groups exist in other characteristics?
- Are there non-split sharply 2-transitive groups of finite Morley rank?
- Can one classify the sharply 2-transitive groups? (Moufang sets....)
- Are there near-domains which allow sharply 3-transitive groups?