Stability and Countable Categoricity in Nonassociative Rings

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In this talk,

- groups will have theories in the language $\mathscr{L}_g = \{x, ^{-1}, 1\}$
- rings will have theories in the language $\mathscr{L}_r = \{+, -, x, 0, 1\}$

Stable groups and rings

A theory T is **stable** if for some infinite λ, for any $A \subset M \models T$ with $|A| \leq \lambda$, $|S_1(A)| \leq |A|$.

More useful for us:

• No order property. There is no formula $\phi(\overline{x}, \overline{y})$ and no elements $\overline{a}_n, \overline{b}_n \in M$ for $n < \omega$ such that $M \models \phi(\overline{a}_n, \overline{b}_m)$ if and only if $n < m$.

Example of an unstable group: $G = Sym(\omega)$.

G has the order property:

Let $\tau_n = (1\ n)$ and $\sigma_n = (1\ 2\ \dots\ n)$ and let $\phi(x_1, y_1, x_2, y_2)$ be $y_1[x_1^{-1}, x_2]y_2 = 1$. The tuples (σ_n, τ_n) are linearly ordered by ϕ, because if $m < n$, then

$$\tau_m[\sigma_m^{-1}, \sigma_n]\tau_n = (1\ m)(1\ n\ m)(1\ n) = 1$$

so $G \models \phi(\sigma_m, \tau_m, \sigma_n, \tau_n)$, but

$$\tau_n[\sigma_n^{-1}, \sigma_m]\tau_m = (1\ n)(1\ m+1\ m)(1\ m) = (1\ n)(m\ m+1) \neq 1$$

so $G \not\models \phi(\sigma_n, \tau_n, \sigma_m, \tau_m)$.

(it also has the independence property, which is a little easier to show)

Examples of stable groups:

- finite groups
- abelian groups
- algebraic groups defined over an algebraically closed field
- algebraic groups defined over a separably closed field
- differential algebraic groups defined over a differentially closed field
- [Sela 2006] free groups
- [Sela 2006] torsion-free hyperbolic groups

Countably categoricity

A theory T is **countably categorical** ($\aleph_0$-categorical) if T has only one countable model up to isomorphism.

Consequences of $\aleph_0$-categoricity:

• The automorphism group has finitely many orbits on G^n (resp. R^n) for each n

• If S is fixed setwise by all automorphisms that fix a set A, then S is definable over A.

• Definable sets A generate subgroups (subrings) in only finitely many steps (hence definable). If A is finite, the number of steps is uniformly bounded in terms of $|A|$.

• An $\aleph_0$-categorical group has finite exponent. An $\aleph_0$-categorical ring has finite characteristic.

Examples of stable *and* $\aleph_0$*-categorical* groups:

- finite groups
- abelian groups *of finite exponent*
- ~~algebraic groups defined over an algebraically closed field~~
- ~~algebraic groups defined over a separably closed field~~
- ~~differential algebraic groups defined over a differentially closed field~~
- ~~[Sela 2006] free groups~~
- ~~[Sela 2006] torsion-free hyperbolic groups~~

Groups

Theorem (Felgner (1978), Baur, Cherlin, Macintyre (1979))

If G is a stable and $\aleph_0$-categorical group, then G is nilpotent by finite, i.e., there is a nilpotent subgroup H in G with $[G : H] < \aleph_0$.

Nilpotent: $\exists n \forall g_1, g_2, \ldots, g_n \in G$, $[[[[g_1, g_2], g_3], \ldots, g_n] = 1$.

Conjecture (Baur, Cherlin, Macintyre (1979))

If G is a stable and $\aleph_0$-categorical group, then G is abelian by finite, i.e., there is a abelian subgroup H in G with $[G : H] < \aleph_0$.

Rings

Rings do not need to be commutative, do not need to have 1. (under this notion, an ideal is a subring).

Theorem (Baldwin, Rose (1977))

If R is a stable and $\aleph_0$-categorical ring, then R is nilpotent by finite, i.e., there is a nilpotent ideal I in R with $[R : I] < \aleph_0$.

Nilpotent: $\exists n \forall r_1, r_2, \ldots, r_n \in R,\ r_1 r_2 r_3 \ldots r_n = 0$.

Conjecture (Baldwin, Rose (1977))

If R is a stable and $\aleph_0$-categorical ring, then R is null by finite, i.e., there is a null ideal I in R with $[R : I] < \aleph_0$.

Null: $\forall x, y \in R,\ xy = 0$

Theorem

If G is an X and $\aleph_0$-categorical group, then G is nilpotent by finite. If R is an X and $\aleph_0$-categorical ring, then R is nilpotent by finite.

Here X can be:

- NSOP (groups is Macpherson (1988), rings is Krupiński (2011))
- NIP for rings or NIP with fsg for groups (Krupiński (2012))
- supersimple (Evans, Wagner (2000), Krupiński, Wagner (2006)) (weaker conclusion: finite by nilpotent by finite)
- generically stable (Dobrowolski, Krupiński (2013))

Krupiński showed a general framework for switching from rings to groups in obtaining these kinds of results.

Nonassociative rings

Nearest nonassociative alternative to associative rings?

A ring R is **alternative** if for all $x, y \in R$, $x(xy) = (xx)y$ and $(yx)x = y(xx)$.

Consequences:

- For all $x_1, x_2, x_3 \in R$, $[x_1, x_2, x_3] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$.
- (Artin) Any two elements generate an associative subring.

Model theory of alternative rings

Rose already thought of extending his and Baldwin's results to alternative rings (his PhD thesis, 1978). Every result about stability or $\aleph_0$-categoricity extended except:

Conjecture

A stable, $\aleph_0$-categorical alternative ring is nilpotent by finite.

Theorem (B. 2012, unpublished)

A stable, $\aleph_0$-categorical alternative ring is nilpotent by finite.

Ingredients: Connectedness

The **connected component** G^0 of a group G is the intersection of all definable subgroups of finite index. G is connected if $G = G^0$.

Since G^0 is preserved by all automorphisms, $\aleph_0$-categoricity tells you G^0 is $\emptyset$-definable. Throw in stability and you get that G^0 itself is finite index (and normal).

The **connected component** R^0 of an associative ring R is the intersection of all definable ideals of finite index. R is connected if $R = R^0$.

Use? 1) If G is nilpotent by finite, G^0 is nilpotent. 2) Nontrivial connected groups are infinite. 3) Connectedness is preserved under definable homomorphism, e.g. quotient by a definable normal subgroup. But R is also a group under addition, so what if we takes its group connected component?

Sketch of ring proof in associative case

Proving R stable + $\aleph_0$-categorical $\Rightarrow$ R nilpotent by finite.

• Replace R with R^0.

• Fact: If I is an ideal of R and both I and R/I are nilpotent, then R is nilpotent.

• Perform induction on the number of $\emptyset$-definable subgroups (not *ideals*) of R

• Base case: R has no $\emptyset$-definable ideals. In particular, $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$. Consider rR minimal nonzero and perform multiplication on this principal ideal:

$$(rs)(rR) = (rsr)R = r(s(rR)) \subseteq rR$$

By minimality, either $rsr = 0$ or $(rs)(rR) = rR$. Jacobson radical arguments handle the first. The second implies rR is basically an infinite field, contradiction since no infinite $\aleph_0$-categorical fields.

What goes wrong in the nonassociative case?

$$(rs)(rR) \stackrel{?}{=} (rsr)R \stackrel{\checkmark}{=} r(s(rR)) \subseteq rR$$

Also do not have:

Fact: If I is an ideal of R and both I and R/I are nilpotent, then R is nilpotent.

So you can't do an induction.

Nonetheless, for alternative rings, if I is the ideal generated by $\mathrm{Ann}_\ell(R) \cup \mathrm{Ann}_r(R)$, then I is nilpotent. Furthermore, R is nilpotent iff R/I is nilpotent. So we can reduce to the case where $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$.

Sketch of proof in alternative case

• Replace R with R^0.

• Reduce: $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$.

• Rose: $R/J(R)$ is finite, where $J(R)$ is Jacobson radical. Since R connected and $J(R)$ definable, $R = J(R)$.

• McCrimmon: In alternative rings, $J(R)$ is nil, i.e. $\forall x \in J(R)\, \exists n,\ x^n = 0$.

So reduced to R nil, connected, and $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$.

Base case

Lemma (B. 2012)

Let R be a connected, stable, $\aleph_0$-categorical alternative ring. If R is a nil ring and either $Ann_\ell(R) = 0$ or $Ann_r(R) = 0$, then $R = 0$.

Proof?

Note: only place in this proof where alternativity appeared is the fact that a finite nil alternative ring is nilpotent. This lemma should hold for a far larger class of nonassociative rings.

Ingredients for more general nonassociativity

• Replace R with R^0. ✓

• Reduce: $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$. *Works in many less associative rings.* ✓

• Rose: $R/J(R)$ is finite, where $J(R)$ is Jacobson radical. Since R connected, $R = J(R)$. *Hard, but the arguments in the lemma seem to apply here as well.*

• McCrimmon: In alternative rings, $J(R)$ is nil, i.e. $\forall x \in J(R)\, \exists n,\ x^n = 0$. *$J(R)$ seems like overkill. May simply want $Nil(R)$, the nilradical.*

So reduced to R nil, connected, and $\mathrm{Ann}_\ell(R) = \mathrm{Ann}_r(R) = 0$. Finish with the lemma, which needs (nil & finite) $\Rightarrow$ nilpotent.

Conjecture

Let R be a stable, $\aleph_0$-categorical ring with very mild associativity assumptions. Then R is nilpotent by finite.

What is mild?

R is a **generalized standard ring** if $\forall x, y, z, w \in R$

- $[x, y, x] = 0$
- $[x^2, y, x] = 0$
- $[x, y, zw] + [z, y, xw] + [w, y, xz] = [x, [z, w, y]] + [x, z, [y, w]]$

Many of the arguments have been pushed through.

Thank you.