Space of Finitely Generated Groups

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Let $G$ be a group and $S = (s_1, s_2, \ldots, s_k)$ be an ordered set of (not necessarily distinct) generators of $G$. The pair $(G, S)$ is called a $k$-marked group.

Two $k$-marked groups $(G, S)$ and $(H, T)$ are equivalent if the map $s_i \mapsto t_i$ extends to an isomorphism from $G$ to $H$.

Let $\mathcal{M}_k$ denote the set of $k$-marked groups up to this equivalence.

Each marked group determines a (labeled) Cayley graph $Cay(G, S)$, whose vertex set is $G$ and edges are given by $(g, gs_i)$ with label $i$.

$(G, S)$ and $(H, T)$ are equivalent if and only if $Cay(G, S)$ and $Cay(H, T)$ are isomorphic as labelled graphs.
Let $B(R, Cay(G, S))$ denote the ball of radius $R$ (around the identity) in the graph $Cay(G, S)$.

Define a metric on $\mathcal{M}_k$ by:

$$d((G, S), (H, T)) = 2^{-R}$$

where $R$ is the largest integer such that $B(R, Cay(G, S))$ and $B(R, Cay(H, T))$ are isomorphic as labelled graphs.

$$d((G, S), (H, T)) \leq 2^{-R}$$

$\iff$

$(G, S)$ and $(H, T)$ have the same relations of length $\leq 2R + 1$

$\iff$

$s_i \mapsto t_i$ extends to a bijection

$\phi : B(R, Cay(G, S)) \rightarrow B(R, Cay(H, T))$ such that

$\phi(gh) = \phi(g)\phi(h)$ where $|g|_S + |h|_S \leq R.$
Let $F_k$ be the free group with basis $X = \{x_1, \ldots, x_k\}$.

Given two subsets $A, B \subset F_k$ let

$$m(A, B) = \max\{n \mid A \cap B(n, \text{Cay}(F_k, X)) = B \cap \text{Cay}(n, \text{Cay}(F_k, X))\}$$

This gives a metric on $2^{F_k}$ given by $\rho(A, B) = 2^{-m(A, B)}$ making $2^{F_k}$ compact and totally disconnected.

Sets of the form $O_{A, B} = \{Y \subset F_k \mid A \subset Y, B \cap Y = \emptyset\}$ where $A, B$ are finite subset of $F_k$ form a basis for the topology generated by this metric.

The set $\mathcal{N}(F_k)$ of normal subgroups of $F_k$ is closed in $2^{F_k}$.

For $(G, S) \in \mathcal{M}_k$, let $N_{(G,S)}$ be the kernel of the map $F_k \to G$, $x_i \mapsto s_i$.

$$d((G, S), (H, T)) \leq 2^{-R} \iff \rho(N_{(G,S)}, N_{(H,T)}) \leq 2^{-(2R+1)}$$
The Chabauty Topology

It follows that the map $\mathcal{M}_k \to \mathcal{N}(F_k)$, $(G, S) \mapsto N_{(G,S)}$ is a homeomorphism with inverse $N \mapsto (F_k/N, \overline{X})$ (where $\overline{X}$ is the image of $X$ in $F_k/N$).

So $\mathcal{M}_k$ is a compact and totally disconnected space.

It has isolated points, for example if $G$ is a finite group $(G, S)$ is an isolated point.

The marked groups $(G, S)$, for $G$ a finitely presented group, are dense in $\mathcal{M}_k$:

If $\langle s_1, \ldots, s_k \mid r_1, r_2, r_3, \ldots \rangle$ is an infinitely presented group then the marked groups $\langle s_1 \ldots, s_k \mid r_1, \ldots, r_n \rangle$ converge to $\langle s_1, \ldots, s_k \mid r_1, r_2, r_3, \ldots \rangle$ in $\mathcal{M}_k$. 
Neighbourhood of a finitely presented group

Let $p : F_k \to G$ is a surjective homomorphism. The map

$$p^* : \mathcal{N}(G) \to \mathcal{N}(F_k), \, N \mapsto p^{-1}(N)$$

is easily seen to be injective and continuous.

Also, $L \in \text{Im}(p^*)$ if and only if $\text{Ker}(p) \leq L$.

**Proposition**

$\text{Im}(p^*)$ is open if and only if $\text{Ker}(p)$ is finitely generated as a normal subgroup of $F_k$ (i.e., $G$ is finitely presented)

As a corollary, we have the following:

**Corollary**

If $G$ is finitely presented then $(G, S) \in \mathcal{M}_k$ has a neighbourhood consisting of (marked) quotients of $G$. 
An example $G = \mathbb{Z} \wr \mathbb{Z}$

Given a finitely generated group $G$, if $G$ is a limit of a sequence of groups which cannot be quotients of $G$, then $G$ cannot be finitely presented.

Let $G = \mathbb{Z} \wr \mathbb{Z} = (\bigoplus \mathbb{Z}) \rtimes \mathbb{Z} = \langle s, t \mid [s, s^{t^i}], i \geq 1 \rangle$ let

$G_n = \langle s, t \mid [s, s^{t^i}], 1 \leq i \leq n \rangle$ so that $\lim_{n \to \infty} G_n = G$. Another presentation of $G_n$ is

$\langle s_0, s_1, \ldots, s_n, t \mid [s_i, s_j], s^t_k = s_{k+1}, 0 \leq k \leq n - 1 \rangle$

$H_n = \langle s_0, \ldots, s_n \rangle \leq G_n$ is free abelian. Let

$K_n = \langle s_0, \ldots, s_{n-1} \rangle \leq G_n$ and $L_n = \langle s_1, \ldots, s_n \rangle \leq G_n$. The map $\psi_n : K_n \to L_n, s_i \mapsto s_{i+1}$ gives an isomorphism and the group $G_n$ is the HNN extension corresponding to $(H_n, \psi_n : K_n \to H_n)$.

Since both $K_n$ and $L_n$ are proper subgroups of $H_n$, it follows from Britton’s Lemma that the HNN extension $G_n$ contains a non-abelian free subgroup.

Since $G$ is solvable, it follows that $G$ is not finitely presented.
Dependence on the marking

Proposition

Let $G$ be a group and $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_n)$ be two generating sets of $G$. Then, there are neighborhoods, $U$ of $(G, S)$ in $\mathcal{M}_k$ and $V$ of $(G, T)$ in $\mathcal{M}_n$ and a homeomorphism $\varphi : U \to V$, such that $\varphi(G, S) = (G, T)$ and $\varphi$ preserves isomorphism.

Proof. Let $p : F_k \to G$, $p(x_i) = s_i$ and $q : F_n \to G$, $q(y_j) = t_j$. Let $w_j \in F_k$ such that $p(w_j) = t_j$ and $v_i \in F_n$ such that $q(v_i) = s_i$. Define $\gamma : F_k \to F_n, x_i \mapsto v_i$ and $\delta : F_n \to F_k, y_j \mapsto w_j$. $U = \{ N \leq F_k \mid \delta \gamma(x_i)x_i^{-1} \in N \}$ and $V = \{ H \leq F_n \mid \gamma \delta(y_j)y_j^{-1} \in H \}$ are open subset of $\mathcal{N}(F_k)$ and $\mathcal{N}(F_n)$ respectively. Since $p \circ \delta = q$ and $q \circ \gamma = p$ we have $N_{(G, S)} \in U$ and $N_{(G, T)} \in V$. $\delta^* : U \to V$ and $\gamma^* : V \to U$ are inverse of each other. \qed
Let $\mathcal{P}$ be a property of groups. A group is called \textbf{fully residually} $\mathcal{P}$, if for any distinct elements $g_1, \ldots, g_n$ of $G$, there exists a surjective homomorphism $\phi : G \to H$ onto a group $H$ with property $\mathcal{P}$, such that $\phi(g_1), \ldots, \phi(g_n)$ are distinct.

**Proposition**

Let $G$ be a finitely generated fully residually $\mathcal{P}$ group. Then $G$ is a limit of groups with property $\mathcal{P}$.

In particular, since residually finite groups are fully residually finite, every finitely generated residually finite group is a limit of finite groups.

The converse of this is not true.

The group $\text{Sym}_f(\mathbb{Z}) \rtimes \mathbb{Z}$ is a limit of finite groups but is not residually finite (it contains an infinite simple group).
Let $\mathcal{P}$ be a property of groups. A group $G$ is called **locally embeddable into $\mathcal{P}$ groups** (LE$\mathcal{P}$ in short) if for every finite subset $E \subset G$, there exists a function $\phi : G \to H$ onto a group $H$ with property $\mathcal{P}$, such that $\phi$ is injective on $E$ and for all $g, h \in E$ we have $\phi(gh) = \phi(g)\phi(h)$.

**Proposition**

A finitely generated group is LE$\mathcal{P}$ if and only if it is a limit of groups with property $\mathcal{P}$.

A finitely presented group is fully residually $\mathcal{P}$ if and only if it is LE$\mathcal{P}$.

**Proof.** The first assertion is clear. The second follows from the fact that a finitely presented group has a neighbourhood of quotients.
The set of abelian groups is both open and closed.

The set of nilpotent groups of nilpotency class at most $d$ is both open and closed.

The set of nilpotent groups is open but not closed since free groups are residually finite $p$ for any prime $p$.

The set of solvable groups of class at most $d$ is closed but not open.

The set of solvable groups is neither closed nor open ($\mathbb{Z} \wr \mathbb{Z}$ is a limit of non-solvable groups.).

The set of amenable groups is neither closed nor open.

The set of groups with Kazhdan’s property ($T$) is open. (Shalom, 2000).
Isolated Groups

We say a finitely generated group $G$ is isolated, if some (equivalently every) marking $(G, S)$ is isolated in $\mathcal{M}_k$.

Finite groups and finitely presented simple groups are examples of isolated groups.

A group $G$ is called **finitely discriminable** if there is a finite set $F \subset G \setminus \{1\}$ such that, every non-trivial normal subgroup contains an element of $F$.

Such a subset $F$ is called a finite discriminating subset of $G$.

**Proposition**

A group $G$ is finitely discriminable if and only if the trivial subgroup $\{1\}$ is isolated in $\mathcal{N}(G)$.

**Proof.** $F$ is a finite discriminating subset if and only if $O_{\emptyset,F} = \{\{1\}\}$.  

\boxed{ }
Isolated groups

Theorem (Grigorchuk 2005, Cornulier, Guyot, Pitsch 2007)

A group $G$ is isolated if and only if it is finitely presented and finitely discriminable.

Proof. Let $G$ be finitely presented and finitely discriminable with a finite discriminating set $E$. Suppose $G$ is not isolated and $G_n$ is a sequence of proper homomorphic images of $G$ converging to $G$. If $E \subset B(N, \text{Cay}(G))$ then for large $n$, $B(N, \text{Cay}(G))$ embeds into $B(N, \text{Cay}(G_n))$. So for large $n$, $\text{Ker}(G \to G_n) \cap E$ is empty, contradicting the fact that $E$ is a discriminating set.

Conversely, if $G$ is isolated then clearly it is finitely presented. If $G$ is isolated then $N_{(G,S)}$ must be isolated in $\mathcal{N}(F_k)$ and hence $\{1\}$ must be isolated in $\mathcal{N}(G)$. Therefore $G$ is finitely discriminable.
Isolated groups

**Proposition (CGP, 2007)**

Finitely discriminable groups are dense in $\mathcal{M}_k$.

**Proof.** Suppose $G$ is infinite and for each finite subset $E \subset G \setminus \{1\}$, select a normal subgroup $N_E \triangleleft G$ maximal among normal subgroups intersecting $E$ trivially. The image of $E$ in $G/N_E$ is a finite discriminating set. Since $\bigcap N_E = 1$, we see that the groups $G/N_E$ accumulate to $G$.

**Theorem (CGP, 2007)**

An isolated group has solvable word problem.

**Proof.** Let $\langle X \mid r_1, \ldots, r_m \rangle$ be a finite presentation of $G$ and let $w$ be a word in $F_X$. Also let $E \subset F_X \setminus N$ be a finite discriminating subset. Given $w \in F_X$, enumerate all the consequences of $r_1, \ldots, r_m$ and all the consequences of $w, r_1, \ldots, r_m$. If $w$ appears in the first list then $w = 1$ in $G$, if some element of $E$ appears on the second list then $w \neq 1$ in $G$. 


Isolated groups

Corollary

The class of isolated groups is not dense.

Proof. By a theorem of C.F. Miller III (1981), there exists a non-trivial finitely presented group $G$ such that the only quotient of $G$ with solvable word problem is the trivial group. So, $G$ is not a limit of groups with solvable word problem, in particular is not a limit of isolated groups.

Theorem (CGP, 2007)

Every finitely generated group is a quotient of an isolated group.

There exists an isolated 3-solvable group which is non-Hopfian.

Note that nilpotent groups and 2-solvable groups are residually finite and hence are not isolated unless they are finite.
Open questions:

Is every finitely generated group with solvable word problem a limit of isolated groups?

Is every hyperbolic group a limit of isolated groups?

Is every solvable group a limit of isolated groups?
For a topological space $X$, let $X'$ denote its set of accumulation points.

For any ordinal $\alpha$ define $X^{(\alpha)}$ inductively by

$X^{(0)} = X$, $X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$ and $X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)}$ for $\lambda$ limit ordinal.

If $X$ is a Polish space, for some countable ordinal $\alpha_0$, $X^{(\alpha_0)} = X^{(\alpha)}$ for all $\alpha \geq \alpha_0$.

The set $X^{(\alpha_0)}$ is called the **condensation part of $X$** (or, the perfect kernel of $X$) it will be denoted by $\text{Cond}(X)$.

The least such $\alpha_0$ is called the **Cantor-Bendixson rank** of $X$.

Points in $\text{Cond}(X)$ are called condensation points.

$x \in \text{Cond}(X) \iff$ every neighbourhood of $x$ is uncountable.
The Cantor-Bendixson Rank of an element $x \in X \setminus \text{Cond}(X)$ is $\sup\{\alpha \mid x \in X^{(\alpha)}\}$.

Elements of Cantor-Bendixson rank 0 are isolated points of $X$, elements of rank 1 are points which are not isolated but isolated among non-isolated points etc.

$\text{Cond}(X) = \emptyset \iff X$ is countable, and for a compact metric space $X$, $\text{Cond}(X)$ is homeomorphic to the Cantor set if it is not empty.

Hence for each $k \geq 2$, the condensation part of $\mathcal{M}_k$ is a Cantor set.

A finitely generated group $G$ will be called a condensation group if some (and hence all) marking $(G, S)$ is in the condensation part of the corresponding space $\mathcal{M}_k$.

The Cantor-Bendixson rank of $G$ is the the Cantor-Bendixson rank of some marking $(G, S)$ in the corresponding space $\mathcal{M}_k$. 
Groups with CB rank 0 are isolated groups.

An infinite group $G$ is called just-infinite, if every proper quotient of $G$ is finite.

Finitely presented, residually finite just-infinite groups have CB rank 1.

CB rank of $\mathbb{Z}^n$ is $n$.

Let $G$ be a polycyclic group. The number of infinite factors in a subnormal series is called the Hirsch length of $G$.

**Proposition (Cornulier, 2011)**

If $G$ is a finitely generated nilpotent group. Then the Cantor-Bendixon rank of $G$ is equal to the Hirsch length of $G$. 
Question (Grigorchuk): What is the Cantor-Bendixson rank of $M_k$? Does it depend on $k$?

Theorem (Cornulier, 2011)

For every $\alpha < \omega^\omega$, there exists a finitely presented, 2-generated metabelian-by-finite group $G$ with Cantor-Bendixson rank $\alpha$.

Therefore, the Cantor Bendixson rank of $M_k$ is $\geq \omega^\omega$. 
Condensation Groups

Theorem (Bieri, Cornulier, Guyot, Strebel, 2014)

Every finitely generated group with a normal, non-abelian free subgroup is a condensation group.

Therefore all non-elementary hyperbolic groups are condensation groups.

Theorem (Cornulier, 2011)

Let $G$ and $H$ be finitely generated groups with $H \neq \{1\}$ and $G$ infinite. Then the wreath product $H \wr G = H^G \rtimes G$ is a condensation group.
Groups with a minimal presentation

For a subset $A \subset G$ let $\langle\langle A \rangle\rangle$ denote the normal subgroup generated by $A$.

A presentation $\langle X \mid R \rangle$ is called minimal if for all $r \in R$ we have $r \notin \langle\langle R \setminus \{r\} \rangle\rangle$.

**Proposition**

Let $G = \langle X \mid r_1, r_2, \ldots \rangle$ be an infinite minimal presentation. Then the group determined by this presentation is a condensation group.

**Proof.**

Let $B = B(2^{-N}, (G, X))$ be a ball of radius $2^{-N}$ around $(G, X)$.
Let $A = \{w \in F_k \mid |w| \leq 2N + 1$ and $w = 1$ in $G\}$. Choose $M = M(N) \in \mathbb{N}$ large enough so that $A \subset \langle\langle r_1, r_2, \ldots, r_M \rangle\rangle$. For any subset $U \subset \mathbb{N}$ such that $\{1, 2, \ldots, M\} \subset U$, let $(G_U, X)$ be the group $\langle X \mid r_i, i \in U \rangle$. Clearly all $(G_U, X) \in B$ and since the initial presentation is minimal all of them are distinct marked groups. Hence $B$ is uncountable.
Groups without a minimal presentation

Theorem (Bieri, Cornulier, Guyot, Strebel, 2014)

There exists infinitely presented groups with Cantor-Bendixson rank 1. Moreover, they can be chosen to be nilpotent-by-abelian.

Hence there exists groups without a minimal presentation.
Some Generic Properties

Let $\mathcal{H}_k$ be the closure of all non elementary, hyperbolic groups in $\mathcal{M}_k$.

<table>
<thead>
<tr>
<th>Theorem (Champetier, 2000)</th>
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<tbody>
<tr>
<td>There exists a $G_\delta$ dense subset $Y \subset \mathcal{H}_k$ which consist of groups which are infinite and torsion.</td>
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</table>

Let $\mathcal{H}_{k}^{tf}$ be the closure of all non elementary, torsion-free hyperbolic groups in $\mathcal{M}_k$.

<table>
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<tr>
<td>There exists a $G_\delta$ dense subset $Y \subset \mathcal{H}_{k}^{tf}$ which consist of groups which are</td>
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<tr>
<td>- torsion free, perfect and having no non-trivial finite quotients of exponential growth and are non-amenable</td>
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<tr>
<td>- do not contain non-abelian free groups</td>
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<td>- has Kazhdan’s property ($T$)</td>
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Let $X = \{0, \ldots, d - 1\}$ and let $X^*$ be the set of all finite words over $X$.

$X^*$ is in bijection with the vertices of a rooted $d$-ary tree.

Let $\text{Aut}(X^*)$ be the group of automorphisms of the tree.

Given $g \in \text{Aut}(X^*)$ and $u \in X^*$, the section of $g$ at $u$ is the automorphism $g_u$ uniquely determined by

$$g(uv) = g(u)g_u(v) \quad \text{for all } u, v \in X^*$$

This gives an isomorphism

$$\text{Aut}(X^*) \rightarrow \text{Sym}(X) \ltimes \text{Aut}(X^*)^d$$

$$g \mapsto \pi_g(g_0, \ldots, g_{d-1})$$

Where $\pi_g$ is the permutation given by the action of $g$ on $X$. 
A subgroup $G \leq Aut(X^*)$ is called **self-similar** if for every $g \in G$ and for every $u \in X^*$, $g_u \in G$.

If $G$ is self-similar we have an embedding

$$G \rightarrow Sym(X) \rtimes G_d$$

Let $S = \{s^1, \ldots, s^m\}$ be a list of symbols and let $\pi_1, \ldots, \pi_m \in S_d$. Consider the system

$$
\begin{align*}
    s^1 &= \pi_1(s^1_0, \ldots, s^1_{d-1}) \\
    \vdots & \quad \vdots \\
    s^m &= \pi_m(s^m_0, \ldots, s^m_{d-1})
\end{align*}
$$

where $s^i_j \in S$.

Such a system (called a **wreath recursion**) defines a unique set of $m$ automorphisms of $X^*$.

Since $(s^i)_j = s^i_j$ the group $G = \langle S \rangle$ will be self similar.
A (Mealy) automaton is a tuple $(S, X, \tau, \lambda)$ where

- $S$ is a set called the **set of states**
- $X$ is a set called the **alphabet**
- $\tau : S \times X \rightarrow S$ is a function called the **transition function**
- $\lambda : S \times X \rightarrow X$ is a function called the **output function**

The automaton is called **invertible** if the functions $x \mapsto \lambda(s, x)$ are bijective for all $s \in S$.

Automata usually are given by their Moore diagrams:

$$
\begin{array}{c}
S \xrightarrow{x|\lambda(s, x)} \tau(s, x)
\end{array}
$$
Let $X = \{0, 1\}$ and consider the wreath recursion

\[
\begin{align*}
    a &= (01) \ (b, e) \\
    b &= (a, e) \\
    e &= (e, e)
\end{align*}
\]

The corresponding Moore diagram is:

The group generated by this automaton $\mathcal{B} = \langle a, b \rangle$ is called the Basilica group.
The Grigorchuk group

\[ G = \langle a, b, c, d \rangle \]

\[
\begin{align*}
a &= (01) (e, e) \\
b &= (a, c) \\
c &= (a, d) \\
d &= (e, b)
\end{align*}
\]
The family of Grigorchuk groups

Let $\Omega = \{0, 1, 2\}^\mathbb{N}$.

Let $\sigma : \Omega \to \Omega$ be the shift $\sigma(\omega_1\omega_2\ldots) = \omega_2\omega_3\ldots$

Let
\[
\begin{align*}
\beta(0) &= a & \beta(1) &= a & \beta(2) &= e \\
\zeta(0) &= a & \zeta(1) &= e & \zeta(2) &= a \\
\delta(0) &= e & \delta(1) &= a & \delta(2) &= a
\end{align*}
\]

For each $\omega \in \Omega$ define the automorphisms of the binary tree:

\[
\begin{align*}
a &= (01) & (e, e) \\
b_\omega &= (\beta(\omega_1), b_{\sigma\omega}) \\
c_\omega &= (\gamma(\omega_1), c_{\sigma\omega}) \\
d_\omega &= (\delta(\omega_1), d_{\sigma\omega})
\end{align*}
\]

and let $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$.

The group $G$ is isomorphic to the group $G_{012012\ldots}$.
The family of Grigorchuk groups

We have an embedding

$$G_\omega \to Sym(\{0, 1\}) \ltimes G^2_{\sigma \omega}$$

Let $S_\omega = (a, b_\omega, c_\omega, d_\omega)$ so that $\{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset M_4$.

Let $\Omega_0 \subset \Omega$ be the subset of eventually constant sequences.

Proposition (Grigorchuk, 1984)

The map $\Omega \to \{(G_\omega, S_\omega) \mid \omega \in \Omega\}$ is not closed in $M_4$. Replacing $G_\omega, \omega \in \Omega_0$ with the appropriate limits, one gets a compact subset $\{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega\} \subset M_4$, so that $(\tilde{G}_\omega, \tilde{S}_\omega) = (G_\omega, S_\omega)$ for $\omega \in \Omega \setminus \Omega_0$.

The map $\Omega \to \{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega\}$ given by $\omega \mapsto (\tilde{G}_\omega, \tilde{S}_\omega)$ is a homeomorphism.
Let $G$ be a group and $S$ a finite generating set.

The growth function of $G$ with respect to $S$ is the function

$$\gamma_{(G,S)}(n) = |B(n, \text{Cay}(G, S))|$$

For two monotone functions $f_1, f_2$, let $f_1 \preceq f_2$ if there exists $C > 0$ such that $f_1(n) \leq f_2(Cn)$ for all $n$. Let $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

If $S$ and $T$ are two generating sets of $G$ we have $\gamma_{(G,S)} \sim \gamma_{(G,T)}$.

The growth function $\gamma_G$ of $G$ is the equivalence class of its growth functions.
**Groups of polynomial growth**

$G$ has polynomial growth if $\gamma_G$ is equivalent to a polynomial.

**Theorem (↔ Milnor, Wolf, Bass, Guivarch 68-71, ⇒ Gromov, 81)**

A group has polynomial growth if and only if it is nilpotent by finite.

**Theorem (Shalom, Tao, 2010)**

Let $G$ be a finitely generated group with generating set $S$. If $\gamma_{(G,S)}(n) \leq n^{c(\log \log n)^c}$ for some $n > 1/c$ where $c > 0$ is an absolute constant, then $G$ is nilpotent by finite.
Groups of exponential growth

$G$ has exponential growth if $\gamma_G \sim e^n$.

Clearly non-abelian free groups have exponential growth and since every group is quotient of a free group, every group has at most exponential growth.

A sufficient condition for a group to have exponential growth is that it contains a free semi-group of rank 2.

The Basilica group $B$ contains a free semi-group of rank 2.

Free Burnside groups (of sufficiently large exponent) have exponential growth, and clearly do not contain free semi-groups.

**Theorem (Milnor, Wolf, 1968)**

A solvable group either is nilpotent by finite or contains a free semi-group of rank 2.

Thus, by Tits alternative, a finitely generated linear group has either polynomial growth or exponential growth.
A group $G$ is called **amenable** if there is a finitely additive, invariant probability measure on $2^G$.

That is, there exists

$$\mu : 2^G \to [0, 1]$$

such that $\mu(G) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$ and $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subset G$.

**Theorem (Folner, 1955)**

A countable group $G$ is amenable if and only if there exists a sequence of finite subsets $F_n \subset G$ such that $\lim_{n \to \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$ for all $g \in G$.

**Corollary**

Groups with sub-exponential growth are amenable.
Let $\mathcal{AG}$ denote the class of amenable groups.

**Theorem (Von Neumann, 1929)**

$\mathcal{AG}$ is closed under taking subgroups, quotients, extensions and directed unions.

The free group $F_2$ is not amenable.

Let $\mathcal{NF}$ denote the class of groups which do not contain a non-abelian free subgroup. We have $\mathcal{AG} \subset \mathcal{NF}$.

Von Neumann problem: Is $\mathcal{AG} = \mathcal{NF}$?

Let $\mathcal{EG}$ be the smallest class of group which contains finite and abelian groups and is closed under the taking subgroups, quotients, extensions and directed unions. This class is called the class of **elementary amenable** groups.

Day’s problem: Is $\mathcal{EG} = \mathcal{AG}$?

**Theorem (Chou, 1980)**

*Every torsion group in $\mathcal{EG}$ is locally finite.*

*Every finitely generated group in $\mathcal{EG}$ has either polynomial or exponential growth.*
A group $G$ has **intermediate growth** if $n^d \leq \gamma_G$ for all $d$ and $\gamma_G \sim e^n$.

Question: (Milnor, 1968) Are there groups with intermediate growth?

**Theorem (Grigorchuk, 1983)**

The group $G$ is an infinite 2-group and has intermediate growth.

Thus, $\mathcal{E}G \subsetneq AG$
Subexponentially amenable groups

Let $\mathcal{SG}$ be the smallest class of groups containing all groups of sub-exponential growth and closed under taking subgroups, quotients, extensions and directed unions. This class is called the class of subexponentially amenable groups.

Question (Grigorchuk, 1998): Is $\mathcal{SG} = \mathcal{AG}$?

Theorem (Grigorchuk, Zuk, 2002)

The Basilica group $\mathcal{B}$ is not subexponentially amenable.

Theorem (Bartholdi, Virag, 2005)

The Basilica group $\mathcal{B}$ is amenable.
Conjecture (Stepin): For finitely generated groups, \( \mathcal{EG} \) is dense in \( \mathcal{AG} \).

**Theorem (Lysenok, 1985)**

\( \mathcal{G} \) has the following infinite recursive presentation

\[
\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \theta^i((ad)^4), \theta^i((adacac)^4), i \geq 0 \rangle
\]

where \( \theta : a \mapsto aca, b \mapsto c, c \mapsto d, d \mapsto b \)

\( \theta \) induces an injective map \( \theta : \mathcal{G} \to \mathcal{G} \). Let \( \overline{\mathcal{G}} \) be the HNN extension of \( (\mathcal{G}, \theta) \).

**Theorem (Grigorchuk, 1998, Sapir, Wise, 2002)**

\( \overline{\mathcal{G}} \) is a finitely presented amenable group.

*Every proper quotient of \( \overline{\mathcal{G}} \) is metabelian.*

It follows that \( \overline{\mathcal{G}} \) is an isolated group in \( \mathcal{AG} \setminus \mathcal{EG} \).
Growth of Grigorchuk groups

Recall, $\Omega = \{0, 1, 2\}^\mathbb{N}$ and $\Omega_0$ is the set of eventually constant sequences.

Let $\Omega_{\infty}$ be the set of sequences in which 0, 1, 2 appear infinitely often. Let $\gamma_\omega$ denote the growth function of the group $G_\omega$.

**Theorem (Grigorchuk, 1984)**

If $\omega \in \Omega_{\infty}$ then $G_\omega$ is a 2-group.

If $\omega \in \Omega \setminus \Omega_0$ then $G_\omega$ has intermediate growth.

If $\omega \in \Omega_0$ then $G_\omega$ has exponential growth.

For every $\omega \in \Omega$, $e^{\sqrt{n}} \preceq \gamma_\omega$.

There is an uncountable subset $\Lambda \subset \Omega$ such that the functions $\{\gamma_\omega | \omega \in \Lambda\}$ are incomparable with respect to $\preceq$.

For any function $f$ such that $f \prec e^n$, there exists $\omega \in \Omega \setminus \Omega_0$ for which $\gamma_\omega \not\preceq f$. 

Growth of Grigorchuk groups

Theorem (Grigorchuk, '84, Bartholdi, Sunic, '01, Muchnik, Pak, '01)

If there exists a number $r$ such that every subword of $\omega$ of length $r$ contains all the symbols $\{0, 1, 2\}$ then $\gamma_\omega \leq e^{n^\alpha}$ for some $0 < \alpha < 1$ depending only on $r$.

Theorem (Bartholdi, 1998, 2001, Leonov, 2001)

If $\omega = (012)^\infty \in \Omega$ (i.e., $G_\omega = G$), $e^{n^{\alpha_0}} \leq \gamma_\omega \leq e^{n^{\theta_0}}$, where $\alpha_0 = 0.5157$, $\theta_0 = \log(2)/\log(2/x_0)$ and $x_0$ is the real root of the polynomial $x^3 + x^2 + x - 2$ ($\theta_0 \approx 0.7674$).

Theorem (Erschler, 2004)

If $\omega = (01)^\infty \in \Omega$, then $\exp\left(\frac{n}{\log^{2+\epsilon} n}\right) \leq \gamma_\omega(n) \leq \exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$ for any $\epsilon > 0$. 
Let \((X, \mu)\) be a probability space and \(T : X \rightarrow X\) be a measure preserving a transformation. \(\mu\) is called ergodic with respect to \(T\), if \(T^{-1}(A) = A\) implies \(\mu(A) = 0\) or \(1\), for any measurable \(A \subset X\).

**Theorem (B., Grigorchuk, Vorobets, 2014)**

Suppose \(\mu\) is a Borel probability measure on \(\Omega\) that is invariant and ergodic relative to the shift transformation \(\sigma : \Omega \rightarrow \Omega\).

a) If the measure \(\mu\) is supported on \(\Omega_\infty\), then there exists \(\alpha = \alpha(\mu) < 1\) such that \(\gamma_\omega \leq e^{n\alpha}\) for \(\mu\)-almost all \(\omega \in \Omega\).

b) In the case \(\mu\) is the uniform Bernoulli measure on \(\Omega\), one can take \(\alpha = 0.999\).
Let $f_1, f_2$ be two functions such that $f_1 \prec f_2 \prec e^n$. $G$ is said to have **oscillating growth of type** $(f_1, f_2)$ if $f_1 \not\preceq \gamma_G$ and $\gamma_G \not\preceq f_2$ (i.e., neither $f_1 \preceq \gamma_G$ nor $\gamma_G \preceq f_2$).

**Theorem (B., Grigorchuk, Vorobets, 2014)**

For any $\theta > \theta_0 \approx 0.7674$ and any function $f$ satisfying $e^{n\theta} \prec f(n) \prec e^n$, there exists a dense $G_δ$ subset $\mathcal{Z} \subset \{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset M_4$ such that any group in $\mathcal{Z}$ has oscillating growth of type $\left(e^{n\theta}, f\right)$. 
The first examples of groups with intermediate growth whose growth functions are exactly known are constructed by Bartholdi and Erschler.

**Theorem (Bartholdi, Erschler, 2012)**

There is a group with growth function $\sim e^{n^{\theta}}$ for $\theta$ in a dense subset of $(\theta_0, 1)$.

$(\theta_0 \approx 0.7674)$. 
Finitely presented covers

Let us say a group $G$ covers another group $H$ if $H$ is a homomorphic image of $G$.

Let $\mathcal{P}$ be a property of groups.

Question: Does every finitely generated group with property $\mathcal{P}$ have a finitely presented cover with property $\mathcal{P}$?

**Proposition**

If $G$ is a limit of a sequence $G_n$ and $H$ is a finitely presented cover of $H$, then $H$ is a cover of $G_n$ for large $n$.

We have seen that the group $\mathbb{Z} \wr \mathbb{Z}$ is a limit of groups with free subgroups. Thus, the question has negative answer if $\mathcal{P}$ is solvability or amenability.

The question has a positive answer for the Kazhdan’s property ($T$), by a result of Shalom, 2000.
One of the main questions open regarding groups of intermediate growth is the following:

Question: Is there a finitely presented group of intermediate growth?

Question: What kind of finitely presented covers can a group of intermediate growth have?

A group is called large if it has a finite index subgroup mapping onto a non-abelian free group. Large groups have non-abelian free subgroups.

**Theorem (B., De la Harpe, Grigorchuk, 2013)**

The groups $\mathcal{G}, \mathcal{B}, G_\omega, \omega \in \Omega$ are limits of large groups. Hence any finitely presented cover of these groups is large.
Let us say that a group $G$ **prefers** another group $H$, if for some marking the marked group $(H, T)$ is a limit of a sequence of markings $(G, S_n)$ of $G$.

In other words, the closure of marked groups isomorphic to $G$ contains a marking of $H$.

**Theorem (Bartholdi, Erschler, 2013)**

*The group $G$ prefers the free group $F_3$.***