Space of Finitely Generated Groups

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Marked Groups

Let G be a group and $S = (s_1, s_2, \ldots, s_k)$ be an ordered set of (not necessarily distinct) generators of G. The pair (G, S) is called a k-**marked group**.

Two k-marked groups (G, S) and (H, T) are equivalent if the map $s_i \mapsto t_i$ extends to an isomorphism from G to H.

Let $\mathcal{M}_k$ denote the set of k-marked groups up to this equivalence.

Each marked group determines a (labeled) Cayley graph $Cay(G, S)$, whose vertex set is G and edges are given by (g, gs_i) with label i.

(G, S) and (H, T) are equivalent if and only if $Cay(G, S)$ and $Cay(H, T)$ are isomorphic as labelled graphs.

Topology on $\mathcal{M}_k$

Let $B(R, Cay(G, S))$ denote the ball of radius R (around the identity) in the graph $Cay(G, S)$.

Define a metric on $\mathcal{M}_k$ by:

$d((G, S), (H, T)) = 2^{-R}$ where R is the largest integer such that $B(R, Cay(G, S))$ and $B(R, Cay(H, T))$ are isomorphic as labelled graphs.

$$d((G, S), (H, T)) \leq 2^{-R}$$

$$\Longleftrightarrow$$

(G, S) and (H, T) have the same relations of length $\leq 2R + 1$

$$\Longleftrightarrow$$

$s_i \mapsto t_i$ extends to a bijection
$\phi : B(R, Cay(G, S)) \to B(R, Cay(H, T))$ such that
$\phi(gh) = \phi(g)\phi(h)$ where $|g|_S + |h|_S \leq R$.

The Chabauty Topology

Let F_k be the free group with basis $X = \{x_1, \ldots, x_k\}$.

Given two subsets $A, B \subset F_k$ let

$$m(A,B) = \max\{n \mid A \cap B(n, Cay(F_k, X)) = B \cap Cay(n, Cay(F_k, X))\}$$

This gives a metric on 2^{F_k} given by $\rho(A,B) = 2^{-m(A,B)}$ making 2^{F_k} compact and totally disconnected.

Sets of the form $O_{A,B} = \{Y \subset F_k \mid A \subset Y, B \cap Y = \emptyset\}$ where A, B are finite subset of F_k form a basis for the topology generated by this metric.

The set $\mathcal{N}(F_k)$ of normal subgroups of F_k is closed in 2^{F_k}.

For $(G,S) \in \mathcal{M}_k$, let $N_{(G,S)}$ be the kernel of the map $F_k \to G$, $x_i \mapsto s_i$.

$$d((G,S),(H,T)) \leq 2^{-R} \iff \rho(N_{(G,S)}, N_{(H,T)}) \leq 2^{-(2R+1)}$$

It follows that the map $\mathcal{M}_k \to \mathcal{N}(F_k)$, $(G,S) \mapsto N_{(G,S)}$ is a homeomorphism with inverse $N \mapsto (F_k/N, \overline{X})$ (where $\overline{X}$ is the image of X in F_k/N).

So $\mathcal{M}_k$ is a compact and totally disconnected space.

It has isolated points, for example if G is a finite group (G,S) is an isolated point.

The marked groups (G,S), for G a finitely presented group, are dense in $\mathcal{M}_k$:

If $\langle s_1,\ldots,s_k \mid r_1,r_2,r_3,\ldots\rangle$ is an infinitely presented group then the marked groups $\langle s_1 \ldots, s_k \mid r_1,\ldots,r_n\rangle$ converge to $\langle s_1,\ldots,s_k \mid r_1,r_2,r_3,\ldots\rangle$ in $\mathcal{M}_k$.

Neighbourhood of a finitely presented group

Let $p : F_k \to G$ is a surjective homomorphism. The map

$$p^* : \mathcal{N}(G) \to \mathcal{N}(F_k), N \mapsto p^{-1}(N)$$

is easily seen to be injective and continuous.

Also, $L \in Im(p^*)$ if and only if $Ker(p) \leq L$.

Proposition

$Im(p^*)$ is open if and only if $Ker(p)$ is finitely generated as a normal subgroup of F_k (i.e., G is finitely presented)

As a corollary, we have the following:

Corollary

If G is finitely presented then $(G, S) \in \mathcal{M}_k$ has a neighbourhood consisting of (marked) quotients of G.

An example $G = \mathbb{Z} \wr \mathbb{Z}$

Given a finitely generated group G, if G is a limit of a sequence of groups which cannot be quotients of G, then G cannot be finitely presented.

Let $G = \mathbb{Z} \wr \mathbb{Z} = (\bigoplus \mathbb{Z}) \rtimes \mathbb{Z} = \left\langle s, t \mid [s, s^{t^i}], i \geq 1 \right\rangle$ let

$G_n = \left\langle s, t \mid [s, s^{t^i}], 1 \leq i \leq n \right\rangle$ so that $\lim_{n\to\infty} G_n = G$. Another

presentation of G_n is

$$\langle s_0, s_1, \ldots, s_n, t \mid [s_i, s_j], s_k^t = s_{k+1}, 0 \leq k \leq n-1 \rangle$$

$H_n = \langle s_0, \ldots, s_n \rangle \leq G_n$ is free abelian. Let

$K_n = \langle s_0, \ldots, s_{n-1} \rangle \leq G_n$ and $L_n = \langle s_1, \ldots, s_n \rangle \leq G_n$. The map $\psi_n : K_n \to L_n, s_i \mapsto s_{i+1}$ gives an isomorphism and the group G_n is the HNN extension corresponding to $(H_n, \psi_n : K_n \to H_n)$.

Since both K_n and L_n are proper subgroups of H_n, it follows from Britton's Lemma that the HNN extension G_n contains a non-abelian free subgroup.

Since G is solvable, it follows that G is not finitely presented.

Dependence on the marking

Proposition

Let G be a group and $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_n)$ be two generating sets of G. Then, there are neighborhoods, U of (G, S) in $\mathcal{M}_k$ and V of (G, T) in $\mathcal{M}_n$ and a homeomorphism $\varphi : U \to V$, such that $\varphi(G, S) = (G, T)$ and φ preserves isomorphism.

Proof. Let $p : F_k \to G, p(x_i) = s_i$ and $q : F_n \to G, q(y_j) = t_j$.
Let $w_j \in F_k$ such that $p(w_j) = t_j$ and $v_i \in F_n$ such that $q(v_i) = s_i$.
Define $\gamma : F_k \to F_n, x_i \mapsto v_i$ and $\delta : F_n \to F_k, y_j \mapsto w_j$.
$U = \{N \trianglelefteq F_k \mid \delta\gamma(x_i)x_i^{-1} \in N\}$ and $V = \{H \trianglelefteq F_n \mid \gamma\delta(y_j)y_j^{-1} \in H\}$ are open subset of $\mathcal{N}(F_k)$ and $\mathcal{N}(F_n)$ respectively.
Since $p \circ \delta = q$ and $q \circ \gamma = p$ we have $N_{(G,S)} \in U$ and $N_{(G,T)} \in V$.
$\delta^* : U \to V$ and $\gamma^* : V \to U$ are inverse of each other. □

Residual properties

Let $\mathcal{P}$ be a property of groups. A group is called **fully residually** $\mathcal{P}$, if for any distinct elements $g_1, \dots, g_n$ of G, there exists a surjective homomorphism $\phi : G \to H$ onto a group H with property $\mathcal{P}$, such that $\phi(g_1), \dots, \phi(g_n)$ are distinct.

Proposition

Let G be a finitely generated fully residually $\mathcal{P}$ group. Then G is a limit of groups with property $\mathcal{P}$.

In particular, since residually finite groups are fully residually finite, every finitely generated residually finite group is a limit of finite groups.

The converse of this is not true.

The group $Sym_f(\mathbb{Z}) \rtimes \mathbb{Z}$ is a limit of finite groups but is not residually finite (it contains and infinite simple group).

LE$\mathcal{P}$ groups

Let $\mathcal{P}$ be a property of groups. A group G is called **locally embeddable into $\mathcal{P}$ groups** (LE$\mathcal{P}$ in short) if for every finite subset $E \subset G$, there exists a function $\phi : G \to H$ onto a group H with property $\mathcal{P}$, such that ϕ is injective on E and for all $g, h \in E$ we have $\phi(gh) = \phi(g)\phi(h)$.

Proposition

A finitely generated group is LE$\mathcal{P}$ if and only if it is a limit of groups with property $\mathcal{P}$.

A finitely presented group is fully residually $\mathcal{P}$ if and only if it is LE$\mathcal{P}$.

Proof. The first assertion is clear. The second follows from the fact that a finitely presented group has a neighbourhood of quotients. □

Open and Closed Properties

- The set of abelian groups is both open and closed.
- The set of nilpotent groups of nilpotency class at most d is both open and closed.
- The set of nilpotent groups is open but not closed since free groups are residually finite p for any prime p.
- The set of solvable groups of class at most d is closed but not open.
- The set of solvable groups is neither closed nor open ($\mathbb{Z} \wr \mathbb{Z}$ is a limit of non-solvable groups.).
- The set of amenable groups is neither closed nor open.
- The set of groups with Kazhdan's property (T) is open. (Shalom, 2000).

Isolated Groups

We say a finitely generated group G is isolated, if some (equivalently every) marking (G, S) is isolated in $\mathcal{M}_k$.

Finite groups and finitely presented simple groups are examples of isolated groups.

A group G is called **finitely discriminable** if there is a finite set $F \subset G \setminus \{1\}$ such that, every non-trivial normal subgroup contains an element of F.

Such a subset F is called a finite discriminating subset of G.

Proposition

A group G is finitely discriminable if and only if the trivial subgroup $\{1\}$ is isolated in $\mathcal{N}(G)$.

Proof. F is a finite discriminating subset if and only if $O_{\emptyset,F} = \{\{1\}\}$. □

Isolated groups

Theorem (Grigorchuk 2005, Cornulier,Guyot,Pitsch 2007)

A group G is isolated if and only if it is finitely presented and finitely discriminable.

Proof. Let G be finitely presented and finitely discriminable with a finite discriminating set E. Suppose G is not isolated and G_n is a sequence of proper homomorphic images of G converging to G. If $E \subset B(N, Cay(G))$ then for large n, $B(N, Cay(G))$ embeds into $B(N, Cay(G_n))$. So for large n, $Ker(G \to G_n) \cap E$ is empty, contradicting the fact that E is a discriminating set.

Conversely, if G is isolated then clearly it is finitely presented. If G is isolated then $N_{(G,S)}$ must be isolated in $\mathcal{N}(F_k)$ and hence $\{1\}$ must be isolated in $\mathcal{N}(G)$. Therefore G is finitely discriminable. $\square$

Proposition (CGP, 2007)

Finitely discriminable groups are dense in $\mathcal{M}_k$.

Proof. Suppose G is infinite and for each finite subset $E \subset G \setminus \{1\}$, select a normal subgroup $N_E \lhd G$ maximal among normal subgroups intersecting E trivially. The image of E in G/N_E is a finite discriminating set. Since $\bigcap N_E = 1$, we see that the groups G/N_E accumulate to G. $\square$

Theorem (CGP, 2007)

An isolated group has solvable word problem.

Proof. Let $< X \mid r_1, \ldots, r_m >$ be a finite presentation of G and let w be a word in F_X. Also let $E \subset F_X \setminus N$ be a finite discriminating subset. Given $w \in F_X$, enumerate all the consequences of $r_1, \ldots, r_m$ and all the consequences of $w, r_1, \ldots, r_m$. If w appears in the first list then $w = 1$ in G, if some element of E appears on the second list then $w \neq 1$ in G.

$\square$

Isolated groups

Corollary

The class of isolated groups is not dense.

Proof. By a theorem of C.F.Miller III (1981), there exists a non-trivial finitely presented group G such that the only quotient of G with solvable word problem is the trivial group. So, G is not a limit of groups with solvable word problem, in particular is not a limit of isolated groups. $\square$

Theorem (CGP, 2007)

Every finitely generated group is a a quotient of an isolated group.

There exists an isolated 3-solvable group which is non-Hopfian.

Note that nilpotent groups and 2-solvable groups are residually finite and hence are not isolated unless they are finite.

Isolated groups

Open questions:

Is every finitely generated group with solvable word problem a limit of isolated groups?

Is every hyperbolic group a limit of isolated groups?

Is every solvable group a limit of isolated groups?

For a topological space X, let X' denote its set of accumulation points.

For any ordinal α define $X^{(\alpha)}$ inductively by $X^{(0)} = X, X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$ and $X^{(\lambda)} = \bigcap_{\beta<\lambda} X^{(\beta)}$ for λ limit ordinal.

If X is a Polish space, for some countable ordinal α_0, $X^{(\alpha_0)} = X^{(\alpha)}$ for all $\alpha \geq \alpha_0$.

The set $X^{(\alpha_0)}$ is called the **condensation part of** X (or, the perfect kernel of X) it will be denoted by $Cond(X)$.

The least such α_0 is called the **Cantor-Bendixson rank** of X.

Points in $Cond(X)$ are called condensation points.

$x \in Cond(X) \iff$ every neighbourhood of x is uncountable.

Cantor-Bendixson rank and condensation

The Cantor-Bendixson Rank of an element $x \in X \setminus Cond(X)$ is $\sup\{\alpha \mid x \in X^{(\alpha)}\}$.

Elements of Cantor-Bendixson rank 0 are isolated points of X, elements of rank 1 are points which are not isolated but isolated among non-isolated points etc.

$Cond(X) = \emptyset \iff X$ is countable, and for a compact metric space X, $Cond(X)$ is homeomorphic to the Cantor set if it is not empty.

Hence for each $k \geq 2$, the condensation part of $\mathcal{M}_k$ is a Cantor set.

A finitely generated group G will be called a **condensation group** if some (and hence all) marking (G, S) is in the condensation part of the corresponding space $\mathcal{M}_k$.

The Cantor-Bendixson rank of G is the the Cantor-Bendixson rank of some marking (G, S) in the corresponding space $\mathcal{M}_k$.

Cantor-Bendixson rank and condensation

Groups with CB rank 0 are isolated groups.

An infinite group G is called **just-infinite**, it every proper quotient of G is finite.

Finitely presented, residually finite just-infinite groups have CB rank 1.

CB rank of $\mathbb{Z}^n$ is n.

Let G be a polycyclic group. The number of infinite factors in a subnormal series is called the Hirsch length of G.

Proposition (Cornulier, 2011)

If G is a finitely generated nilpotent group. Then the Cantor-Bendixon rank of G is equal to the Hirsch length of G.

Cantor-Bendixson rank of $\mathcal{M}_k$

Question (Grigorchuk): What is the Cantor-Bendixson rank of $\mathcal{M}_k$? Does it depend on k?

Theorem (Cornulier, 2011)

For every $\alpha < \omega^\omega$, there exists a finitely presented, 2-generated metabelian-by-finite group G with Cantor-Bendixson rank α.

Therefore, the Cantor Bendixson rank of $\mathcal{M}_k$ is $\geq \omega^\omega$.

Condensation Groups

Theorem (Bieri,Cornulier,Guyot,Strebel, 2014)

Every finitely generated group with a normal, non-abelian free subgroup is a condensation group.

Therefore all non-elementary hyperbolic groups are condensation groups.

Theorem (Cornulier, 2011)

Let G and H be finitely generated groups with $H \neq \{1\}$ and G infinite. Then the wreath product $H \wr G = H^G \rtimes G$ is a condensation group.

Groups with a minimal presentation

For a subset $A \subset G$ let $\langle\langle A \rangle\rangle$ denote the normal subgroup generated by A.

A presentation $\langle X \mid R \rangle$ is called minimal if for all $r \in R$ we have $r \notin \langle\langle R \setminus \{r\} \rangle\rangle$.

Proposition

Let $G = \langle X \mid r_1, r_2, \ldots \rangle$ be an infinite minimal presentation. Then the group determined by this presentation is a condensation group.

Proof.

Let $B = B(2^{-N}, (G, X))$ be a ball of radius 2^{-N} around (G, X). Let $A = \{w \in F_k \mid |w| \leq 2N + 1 \text{ and } w = 1 \text{ in } G\}$. Choose $M = M(N) \in \mathbb{N}$ large enough so that $A \subset \langle\langle r_1, r_2, \ldots, r_M \rangle\rangle$. For any subset $U \subset \mathbb{N}$ such that $\{1, 2, \ldots, M\} \subset U$, let (G_U, X) be the group $\langle X \mid r_i, i \in U \rangle$. Clearly all $(G_U, X) \in B$ and since the initial presentation is minimal all of them are distinct marked groups. Hence B is uncountable.

Groups without a minimal presentation

Theorem (Bieri,Cornulier,Guyot,Strebel, 2014)

There exists infinitely presented groups with Cantor-Bendixson rank 1*. Moreover, they can be chosen to be nilpotent-by-abelian.*

Hence there exists groups without a minimal presentation.

Some Generic Properties

Let $\mathcal{H}_k$ be the closure of all non elementary, hyperbolic groups in $\mathcal{M}_k$.

Theorem (Champetier, 2000)

There exists a G_δ dense subset $Y \subset \mathcal{H}_k$ which consist of groups which are infinite and torsion.

Let $\mathcal{H}_k^{tf}$ be the closure of all non elementary, torsion-free hyperbolic groups in $\mathcal{M}_k$.

Theorem (Champetier, 2000)

There exists a G_δ dense subset $Y \subset \mathcal{H}_k^{tf}$ which consist of groups which are

- *torsion free, perfect and having no non-trivial finite quotients*
- *of exponential growth and are non-amenable*
- *do not contain non-abelian free groups*
- *has Kazhdan's property (T)*

Rooted trees and their automorphisms

Let $X = \{0, \ldots, d-1\}$ and let X^* be the set of all finite words over X.

X^* is in bijection with the vertices of a rooted d-ary tree.

Let $Aut(X^*)$ be the group of automorphisms of the tree.

Given $g \in Aut(X^*)$ and $u \in X^*$, the **section of g at u** is the automorphism g_u uniquely determined by

$$g(uv) = g(u)g_u(v) \text{ for all } u, v \in X^*$$

This gives an isomorphism

$$\begin{gathered} Aut(X^*) \to Sym(X) \ltimes Aut(X^*)^d \\ g \mapsto \pi_g(g_0, \ldots, g_{d-1}) \end{gathered}$$

Where π_g is the permutation given by the action of g on X.

A subgroup $G \leq Aut(X^*)$ is called **self-similar** if for every $g \in G$ and for every $u \in X^*$, $g_u \in G$.

If G is self-similar we have an embedding

$$G \to Sym(X) \ltimes G^d$$

Let $S = \{s^1, \ldots, s^m\}$ be a list of symbols and let $\pi_1, \ldots, \pi_m \in S_d$. Consider the system

$$\begin{array}{rcl} s^1 & = & \pi_1(s^1_0, \ldots, s^1_{d-1}) \\ \vdots & \vdots & \vdots \\ s^m & = & \pi_m(s^m_0, \ldots, s^m_{d-1}) \end{array}$$

where $s^i_j \in S$.

Such a system (called a **wreath recursion**) defines a unique set of m automorphisms of X^*.

Since $(s^i)_j = s^i_j$ the group $G = \langle S \rangle$ will be self similar.

Automata

A (Mealy) automaton is a tuple (S, X, τ, λ) where

S is a set called the **set of states**

X is a set called the **alphabet**

$\tau : S \times X \to S$ is a function called the **transition function**

$\lambda : S \times X \to X$ is a function called the **output function**

The automaton is called **invertible** if the functions $x \mapsto \lambda(s, x)$ are bijective for all $s \in S$.

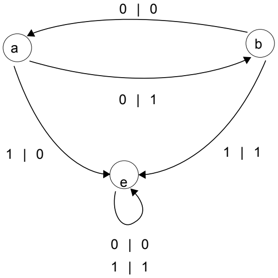
Automata usually are given by their Moore diagrams:

$$s \xrightarrow{x|\lambda(s,x)} \tau(s, x)$$

Let $X = \{0, 1\}$ and consider the wreath recursion

$$\begin{array}{rcll} a & = & (01) & (b, e) \\ b & = & & (a, e) \\ e & = & & (e, e) \end{array}$$

The corresponding Moore diagram is:

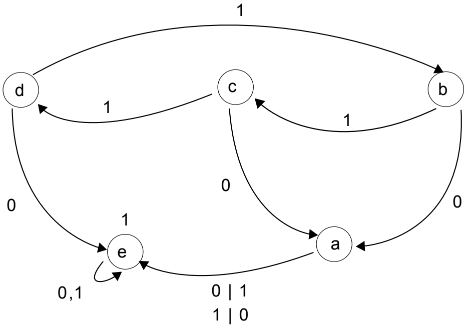


The group generated by this automaton $\mathcal{B} = \langle a, b \rangle$ is called the **Basilica group**.

The Grigorchuk group

$\mathcal{G} = \langle a, b, c, d \rangle$

$$\begin{array}{rclc} a & = & (01) & (e, e) \\ b & = & & (a, c) \\ c & = & & (a, d) \\ d & = & & (e, b) \end{array}$$



The family of Grigorchuk groups

Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$.

Let $\sigma : \Omega \to \Omega$ be the shift $\sigma(\omega_1\omega_2 \dots) = \omega_2\omega_3 \dots$

Let

$$\begin{array}{lll} \beta(0) = a & \beta(1) = a & \beta(2) = e \\ \zeta(0) = a & \zeta(1) = e & \zeta(2) = a \\ \delta(0) = e & \delta(1) = a & \delta(2) = a \end{array}$$

For each $\omega \in \Omega$ define the automorphisms of the binary tree:

$$\begin{array}{rclc} a & = & (01) & (e, e) \\ b_\omega & = & & (\beta(\omega_1), b_{\sigma\omega}) \\ c_\omega & = & & (\gamma(\omega_1), c_{\sigma\omega}) \\ d_\omega & = & & (\delta(\omega_1), d_{\sigma\omega}) \end{array}$$

and let $G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$.

The group $\mathcal{G}$ is isomorphic to the group $G_{012012\dots}$

The family of Grigorchuk groups

We have an embedding

$$G_\omega \to Sym(\{0,1\}) \ltimes G_{\sigma\omega}^2$$

Let $S_\omega = (a, b_\omega, c_\omega, d_\omega)$ so that $\{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$.

let $\Omega_0 \subset \Omega$ be the subset of eventually constant sequences.

Proposition (Grigorchuk, 1984)

$\{(G_\omega, S_\omega) \mid \omega \in \Omega\}$ is not closed in $\mathcal{M}_4$. Replacing $G_\omega, \omega \in \Omega_0$ with the appropriate limits, one gets a compact subset $\{(\widetilde{G}_\omega, \widetilde{S}_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$, so that $(\widetilde{G}_\omega, \widetilde{S}_\omega) = (G_\omega, S_\omega)$ for $\omega \in \Omega \setminus \Omega_0$.

The map $\Omega \to \{(\widetilde{G}_\omega, \widetilde{S}_\omega) \mid \omega \in \Omega\}$ given by $\omega \mapsto (\widetilde{G}_\omega, \widetilde{S}_\omega)$ is a homeomorphism.

Let G be a group and S a finite generating set.

The growth function of G with respect to S is the the function

$$\gamma_{(G,S)}(n) = |B(n, Cay(G,S))|$$

For two monotone functions f_1, f_2, let $f_1 \preceq f_2$ if there exists $C > 0$ such that $f_1(n) \leq f_2(Cn)$ for all n. Let $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

If S and T are two generating sets of G we have $\gamma_{(G,S)} \sim \gamma_{(G,T)}$.

The growth function γ_G of G is the equivalence class of its $\sim$ growth functions.

Groups of polynomial growth

G has polynomial growth if γ_G is equivalent to a polynomial.

Theorem ($\Leftarrow$ Milnor, Wolf, Bass, Guivarch 68-71, $\Rightarrow$ Gromov, 81)

A group has polynomial growth if and only if it is nilpotent by finite.

Theorem (Shalom, Tao, 2010)

Let G be a finitely generated group with generating set S. If $\gamma_{(G,S)}(n) \leq n^{c(\log\log n)^c}$ for some $n > 1/c$ where $c > 0$ is an absolute constant, then G is nilpotent by finite.

Groups of exponential growth

G has exponential growth if $\gamma_G \sim e^n$.

Clearly non-abelian free groups have exponential growth and since every group is quotient of a free group, every group has at most exponential growth.

A sufficient condition for a group to have exponential growth is that it contains a free semi-group of rank 2.

The Basilica group $\mathcal{B}$ contains a free semi-group of rank 2.

Free Burnside groups (of sufficiently large exponent) have exponential growth, and clearly do not contain free semi-groups.

Theorem (Milnor, Wolf, 1968)

A solvable group either is nilpotent by finite or contains a free semi-group of rank 2.

Thus, by Tits alternative, a finitely generated linear group has either polynomial growth or exponential growth.

Amenable groups

A group G is called **amenable** if there is a finitely additive, invariant probability measure on 2^G.

That is, there exists

$$\mu : 2^G \to [0,1]$$

such that $\mu(G) = 1, \mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$ and $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subset G$.

Theorem (Folner, 1955)

A countable group G is amenable if and only if there exists a sequence of finite subsets $F_n \subset G$ such that $\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0$ for all $g \in G$.

Corollary

Groups with sub-exponential growth are amenable.

Amenable groups

Let $\mathcal{AG}$ denote the class of amenable groups.

Theorem (Von Neumann, 1929)

$\mathcal{AG}$ is closed under taking subgroups, quotients, extensions and directed unions.

The free group F_2 is not amenable.

Let $\mathcal{NF}$ denote the class of groups which do not contain a non-abelian free subgroup. We have $\mathcal{AG} \subset \mathcal{NF}$.

Von Neumann problem: Is $\mathcal{AG} = \mathcal{NF}$?

No. Olshanskii 1980, Adian 1982, Olshanskii+Sapir, 2003 , Monod 2013, Lodha, 2014.

Elementary amenable groups

Let $\mathcal{EG}$ be the smallest class of group which contains finite and abelian groups and is closed under the taking subgroups, quotients, extensions and directed unions. This class is called the class of **elementary amenable** groups.

Day's problem: Is $\mathcal{EG} = \mathcal{AG}$?

Theorem (Chou, 1980)

Every torsion group in $\mathcal{EG}$ is locally finite.

Every finitely generated group in $\mathcal{EG}$ has either polynomial or exponential growth.

Groups of intermediate growth

A group G has **intermediate growth** if $n^d \preceq \gamma_G$ for all d and $\gamma_G \not\sim e^n$.

Question: (Milnor, 1968) Are there groups with intermediate growth?

Theorem (Grigorchuk, 1983)

The group $\mathcal{G}$ is an infinite 2-group and has intermediate growth.

Thus, $\mathcal{EG} \subsetneq \mathcal{AG}$

Subexponentially amenable groups

Let $\mathcal{SG}$ be the smallest class of groups containing all groups of sub-exponential growth and closed under taking subgroups, quotients, extensions and directed unions. This class is called the class of **subexponentially amenable groups**.

Question (Grigorchuk, 1998): Is $\mathcal{SG} = \mathcal{AG}$?

Theorem (Grigorchuk, Zuk, 2002)

The Basilica group $\mathcal{B}$ is not subexponentially amenable.

Theorem (Bartholdi, Virag, 2005)

The Basilica group $\mathcal{B}$ is amenable.

Isolated groups in $\mathcal{AG} \setminus \mathcal{EG}$

Conjecture (Stepin): For finitely generated groups, $\mathcal{EG}$ is dense in $\mathcal{AG}$.

Theorem (Lysenok, 1985)

$\mathcal{G}$ has the following infinite recursive presentation

$$\left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \theta^i((ad)^4), \theta^i((adacac)^4), i \geq 0 \right\rangle$$

where $\theta : a \mapsto aca, b \mapsto c, c \mapsto d, d \mapsto b$

θ induces an injective map $\theta : \mathcal{G} \to \mathcal{G}$. Let $\overline{\mathcal{G}}$ be the *HNN* extension of $(\mathcal{G}, \theta)$.

Theorem (Grigorchuk, 1998, Sapir, Wise, 2002)

$\overline{\mathcal{G}}$ is a finitely presented amenable group.

Every proper quotient of $\overline{\mathcal{G}}$ is metabelian.

It follows that $\overline{\mathcal{G}}$ is an isolated group in $\mathcal{AG} \setminus \mathcal{EG}$.

Growth of Grigorchuk groups

Recall, $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ and Ω_0 is the set of eventually constant sequences.

Let Ω_∞ be the set of sequences in which $0, 1, 2$ appear infinitely often. Let γ_ω denote the growth function of the group G_ω.

Theorem (Grigorchuk, 1984)

If $\omega \in \Omega_\infty$ then G_ω is a 2-group.

If $\omega \in \Omega \setminus \Omega_0$ then G_ω has intermediate growth.

If $\omega \in \Omega_0$ then G_ω has exponential growth.

For every $\omega \in \Omega$, $e^{\sqrt{n}} \preceq \gamma_\omega$.

There is an uncountable subset $\Lambda \subset \Omega$ such that the functions $\{\gamma_\omega \mid \omega \in \Lambda\}$ are incomparable with respect to $\preceq$.

For any function f such that $f \prec e^n$, there exists $\omega \in \Omega \setminus \Omega_0$ for which $\gamma_\omega \npreceq f$.

Growth of Grigorchuk groups

Theorem (Grigorchuk, '84, Bartholdi, Sunic, '01, Muchnik, Pak, '01)

If there exists a number r such that every subword of ω of length r contains all the symbols $\{0, 1, 2\}$ then $\gamma_\omega \preceq e^{n^\alpha}$ for some $0 < \alpha < 1$ depending only on r.

Theorem (Bartholdi, 1998, 2001, Leonov, 2001)

If $\omega = (012)^\infty \in \Omega$ (i.e., $G_\omega = \mathcal{G}$), $e^{n^{\alpha_0}} \preceq \gamma_\omega \preceq e^{n^{\theta_0}}$, where $\alpha_0 = 0.5157, \theta_0 = \log(2)/\log(2/x_0)$ and x_0 is the real root of the polynomial $x^3 + x^2 + x - 2$ ($\theta_0 \approx 0.7674$).

Theorem (Erschler, 2004)

If $\omega = (01)^\infty \in \Omega$, then $\exp\left(\frac{n}{\log^{2+\epsilon} n}\right) \preceq \gamma_\omega(n) \preceq \exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$ for any $\epsilon > 0$.

Generic growth of Grigorchuk groups

Let (X, μ) be a probability space and $T : X \to X$ be a measure preserving a transformation. μ is called ergodic with respect to T, if $T^{-1}(A) = A$ implies $\mu(A) = 0$ or 1, for any measurable $A \subset X$.

Theorem (B., Grigorchuk, Vorobets, 2014)

Suppose μ is a Borel probability measure on Ω that is invariant and ergodic relative to the shift transformation $\sigma : \Omega \to \Omega$.

a) *If the measure μ is supported on Ω_∞, then there exists $\alpha = \alpha(\mu) < 1$ such that $\gamma_\omega \preceq e^{n^\alpha}$ for μ-almost all $\omega \in \Omega$.*

b) *In the case μ is the uniform Bernoulli measure on Ω, one can take $\alpha = 0.999$.*

Generic growth of Grigorchuk groups

Let f_1, f_2 be two functions such that $f_1 \prec f_2 \prec e^n$. G is said to have **oscillating growth of type** (f_1, f_2) if $f_1 \not\preceq \gamma_G$ and $\gamma_G \not\preceq f_2$ (i.e., neither $f_1 \preceq \gamma_G$ nor $\gamma_G \preceq f_2$).

Theorem (B., Grigorchuk, Vorobets, 2014)

For any $\theta > \theta_0 \approx 0.7674$ and any function f satisfying $e^{n^\theta} \prec f(n) \prec e^n$, there exists a dense G_δ subset $\mathcal{Z} \subset \{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$ such that any group in $\mathcal{Z}$ has oscillating growth of type $\left(e^{n^\theta}, f\right)$.

Groups of intermediate growth with exact growth functions

The first examples of groups with intermediate growth whose growth functions are exactly known are constructed by Bartholdi and Erschler.

Theorem (Bartholdi, Erschler , 2012)

There is a group with growth function $\sim e^{n^\theta}$ *for* θ *in a dense subset of* $(\theta_0, 1)$.

$(\theta_0 \approx 0.7674)$.

Finitely presented covers

Let us say a group G covers another group H if H is a homomorphic image of G.

Let $\mathcal{P}$ be a property of groups.

Question: Does every finitely generated group with property $\mathcal{P}$ have a finitely presented cover with property $\mathcal{P}$?

Proposition

If G is a limit of a sequence G_n and H is a finitely presented cover of H, then H is a cover of G_n for large n.

We have seen that the group $\mathbb{Z} \wr \mathbb{Z}$ is a limit of groups with free subgroups. Thus, the question has negative answer if $\mathcal{P}$ is solvability or amenability.

The question has a positive answer for the Kazhdan's property (T), by a result of Shalom, 2000.

Finitely presented covers

One of the main questions open regarding groups of intermediate growth is the following:

Question: Is there a finitely presented group of intermediate growth?

Question: What kind of finitely presented covers can a group of intermediate growth have?

A group is called large if it has a finite index subgroup mapping onto a non-abelian free group. Large groups have non-abelian free subgroups.

Theorem (B., De la Harpe, Grigorchuk, 2013)

The groups $\mathcal{G}, \mathcal{B}, G_\omega, \omega \in \Omega$ are limits of large groups. Hence any finitely presented cover of these groups is large.

Closure of the group $\mathcal{G}$

Let us say that a group G **preforms** another group H, if for some marking the marked group (H, T) is a limit of a sequence of markings (G, S_n) of G.

In other words, the closure of marked groups isomorphic to G contains a marking of H.

Theorem (Bartholdi, Erschler, 2013)

The group $\mathcal{G}$ preforms the free group F_3.