$\aleph_0$-categorical models and Roelcke precompact groups

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The continuous functions $f : G \to \mathbb{R}$ that factor through a compactification of G are exactly the Roelcke uniformly continuous functions (i.e. functions uniformly continuous with respect to both the left and right uniformities of G). They form an algebra, $\mathrm{UC}(G)$.

Banach representations of compact G-spaces

If X is a compact G-space and V is a Banach space, a representation of X on V is given by a pair

$$\alpha : X \to V^*,$$
$$h : G \to \mathrm{Iso}(V),$$

where h is a continuous homomorphism and α is a weak*-continuous G-map with respect to the dual action $G \times V^* \to V^*$, $(g\phi)(v) = \phi(h(g)^{-1}(v))$.

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If $\mathcal{K}$ is a class of Banach spaces, the G-space X is said $\mathcal{K}$-approximable if the family of its representations on Banach spaces $V \in \mathcal{K}$ separates points of X.

The dynamical hierarchy after Glasner and Megrelishvili

Theorem

Good dynamical properties of a continuous function $f : G \to \mathbb{R}$ correspond to good classes $\mathcal{K}$ and the possibility of factoring f through a $\mathcal{K}$-approximable compactification X, as follows.

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$$AP(G) \subset WAP(G) \subset Asp(G) \subset Tame_u(G) \subset UC(G)$$

Roelcke precompact Polish groups after Ben Yaacov and Tsankov

G is Roelcke precompact if for every open $U \subset G$ there is a finite $F \subset G$ such that $UFU = G$.

Examples: S_∞, Aut$(\mathbb{Q}, <)$, Aut(RG), Homeo(2^ω), Iso$(\mathbb{U}_1)$, Aut(μ), Aut$^*(\mu)$, $\mathcal{U}(H)$, Homeo$_+([0,1])$, etc.

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Theorem

Equivalently, M is an $\aleph_0$-categorical structure.

Functions as formulas

Proposition

Take $f \in UC(G)$ and define $\varphi : G^2 \to \mathbb{R}$ by $\varphi(h, g) = f(h^{-1}g)$. Then φ extends to an invariant continuous function $\varphi : M^2 \to \mathbb{R}$.

If G is Roelcke precompact, then $\varphi(x, y)$ is a $\emptyset$-definable predicate on M (and so $f = \varphi_1 = \varphi(1, \cdot)$ is an M-definable predicate on M). Moreover, dynamical properties of f correspond to classical model-theoretic properties of φ, as follows.

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The group $\mathrm{Homeo}_+([0,1])$, for which it is known that $\mathrm{WAP}(G)$ is trivial but $\mathrm{Tame}_u(G) = \mathrm{UC}(G)$, offers an example of a completely unstable NIP structure.

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Strongly uniformly continuous functions

A continuous $f : G \to \mathbb{R}$ is called strongly uniformly continuous (SUC) if it factors through a compactification X such that, for all $x \in X$, the map

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We have $\mathrm{Asp}(G) \subset \mathrm{SUC}(G)$. Glasner and Megrelishvili showed that $\mathrm{SUC}(\mathrm{Homeo}_+([0,1]))$ is trivial.

$\mathrm{WAP}(G) = \mathrm{Asp}(G) = \mathrm{SUC}(G)$

Theorem (I.)

If M is $\aleph_0$-categorical and $f \in SUC(M)$, then the associated formula is stable.

Corollary

Let G be a Roelcke precompact Polish group. Then $WAP(G) = Asp(G) = SUC(G)$.

An easy crucial example

Consider the function f on $G = \mathrm{Aut}(\mathbb{Q}, <)$ given by
$$f(g) = \begin{cases} 1 & \text{if } 0 \leq g(0) \\ 0 & \text{otherwise} \end{cases}.$$
Thus, $\varphi(g, h) = f(g^{-1}h) = 1$ means $g(0) \leq h(0)$.

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Contradiction.

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Theorem (Ben Yaacov, I., Tsankov)

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Corollary

For the automorphism group of Hrushovski's pseudoplane we have $Hilb(G) \subsetneq WAP(G) = UC(G)$.

Thank you.

References

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