Definable topological dynamics and real Lie groups

Grzegorz Jagiella

Uniwersytet Wrocławski

Models and Groups, Istanbul
28 March 2014

Topological dynamics

(Ellis, ...)
A **point transitive** G-**flow** is an action of the group G on a compact Hausdorff space X by homeomorphisms such that X contains a dense G-orbit.

Let X be a point-transitive G-flow.
Every $g \in G$ determines a homeomorphism $\pi_g \in X^X$.

Let $E(X) = \mathrm{cl}\{\pi_g : g \in G\}$.
This is a compact subspace of X^X and itself a (point-transitive) G-flow ($(g \cdot f)(x) = f(g^{-1} \cdot x)$).
$(E(X), *)$ is a semigroup (where $*$ is the function composition). It is called the **Ellis semigroup** of the flow (G, X).

Topological dynamics

Important objects associated with $E(X)$:

- Algebraic: minimal (left) ideals,
- Topological: minimal subflows (minimal nonempty closed G-invariant subsets).

$$\text{minimal ideals} = \text{minimal subflows}$$

Properties of minimal ideals

For every minimal subflow I of $E(X)$, $I = \mathrm{cl}(Gp)$ for any $p \in I$

A $p \in E(X)$ such that $\mathrm{cl}(Gp)$ is a minimal subflow of $E(X)$ is called **almost periodic**.

Properties of minimal ideals

For every minimal subflow I of $E(X)$, $I = \mathrm{cl}(Gp)$ for any $p \in I$

A $p \in E(X)$ such that $\mathrm{cl}(Gp)$ is a minimal subflow of $E(X)$ is called **almost periodic**.

A $p \in E(X)$ with $p * p = p$ is called an **idempotent**.
Let I be a minimal subflow of $E(X)$ and let $J(I)$ be the set of idempotents in I.
We have:

$$I = \coprod_{u \in J(I)} u * I,$$

where every $(u * I, *)$ is a group.

Ideal subgroups

The groups $(u * I, *)$ are all isomorphic (even for different I's) and called **ideal subgroups** of $E(X)$.

Their isomorphism class is called the **Ellis group** (of the flow (G, X)).

Definable setting

(Newelski)
Fix a first-order structure M and a sufficiently saturated $\mathcal{C} \succ M$.
Assume that all types over M are definable.
Let G be a group definable in M.

Definable setting

(Newelski)
Fix a first-order structure M and a sufficiently saturated $\mathcal{C} \succ M$.
Assume that all types over M are definable.
Let G be a group definable in M.

There is a category of **definable $G(M)$-flows**: actions of $G(M)$ on the quotient $X(\mathcal{C})/E$ where X is M-definable on which $G(M)$ acts definably and transitively, and E is a G-invariant btde relation.
This category has the universal object $S_G(M)$.
It is the definable equivalent of βG.

Ellis semigroup

The Ellis semigroup of $S_G(M)$ turns out to be isomorphic to $S_G(M)$ itself (this requires definability of types).
The semigroup operation on $S_G(M)$ can be described as follows:

$$p * q = \mathrm{tp}(a \cdot b/M),$$

where
$a \models p$,
$b \models q$, and
$\mathrm{tp}(b/Ma) \supset q$ is the **heir extension**.

o-minimality

Recall that a structure $(M, <, \ldots)$ is *o*-**minimal** if $<$ is dense and linear without endpoints, and every definable subset of M is a finite union of intervals.

o-minimality

Recall that a structure $(M, <, \ldots)$ is *o*-**minimal** if $<$ is dense and linear without endpoints, and every definable subset of M is a finite union of intervals.

Fix $\mathbb{R} = (\mathbb{R}, +, \cdot, <, \ldots)$ and *o*-minimal expansion of reals.
Some properties of $\mathbb{R}$:

- *NIP*,
- cells and cell decomposition theorem,
- topological dimension (coincides with acl-dimension),
- definability of types.

$\mathbb{R}$-definable groups

Let G definable in $\mathbb{R}$.

Proposition (Pillay)

There is a definable atlas of maps making $G(\mathbb{R})$ a definable manifold over $\mathbb{R}$, making the group operations continuous (i.e. $G(\mathbb{R})$ is a real Lie group).

Goal: describe topological dynamics of G.
Two important cases: torsion-free and definably compact.

Topological dynamics: torsion-free case

Let G be torsion-free. This case is straightforward:

Proposition (Conversano, Pillay)

There is a $G(\mathbb{R})$-invariant type p in $S_G(\mathbb{R})$.

That is, $S_G(\mathbb{R})$ has a one-point minimal subflow.

Topological dynamics: definably compact case

Let G be definably compact. Newelski gave a full description of $(G(\mathbb{R}), S_G(\mathbb{R}))$:

- $S_G(\mathbb{R})$ contains the unique minimal flow $\mathrm{Gen}_K(\mathbb{R})$ consisting of all generic types in G.
- The Ellis group of $(G(\mathbb{R}), S_G(\mathbb{R}))$ is isomorphic to $G(\mathbb{R})$.
- An ideal subgroup of $S_G(\mathbb{R})$ is a selector of $S_G(\mathbb{R})/\ker(\mathrm{st})$.
- Detailed description of the semigroup operation.

Compact-torsion-free decomposition

Definition

Let G be definable. We say that G has a **definable compact-torsion-free decomposition** if there is a definable, definably compact $K < G$ and a definable, torsion-free $H < G$ such that $K \cap H = \{e\}$ and $G = KH$.

Compact-torsion-free decomposition

Definition

Let G be definable. We say that G has a **definable compact-torsion-free decomposition** if there is a definable, definably compact $K < G$ and a definable, torsion-free $H < G$ such that $K \cap H = \{e\}$ and $G = KH$.

Proposition (Conversano)

There is a definable, central subgroup $\mathcal{A}(G) < G$ such that $G/\mathcal{A}(G)$ has a definable compact-torsion-free decomposition, and is the maximal quotient with this property.

In particular, definable semisimple groups have this decomposition.

Topological dynamics of $G(\mathbb{R})$

Let $G = KH$ be a definable compact-torsion-free decomposition.
Goal: describe the $G(\mathbb{R})$-flow $S_G(\mathbb{R})$.

We already have a description of $(K(\mathbb{R}), S_K(\mathbb{R}))$ and $(H(\mathbb{R}), S_H(\mathbb{R}))$.

We need to understand the interaction between H and K.

Natural subgroup actions

H acts on the coset space G/H. This quotient can be identified with K.

So we have a group action of $H(\mathbb{R})$ on $K(\mathbb{R})$.

This induces a group action of $H(\mathbb{R})$ on $S_K(\mathbb{R})$ and a semigroup action of $S_H(\mathbb{R})$ on $S_K(\mathbb{R})$.

The action $H(\mathbb{R}) \circlearrowright S_K(\mathbb{R})$ preserves $\mathrm{Gen}_K(\mathbb{R})$.

Minimal subflow

Proposition (J.)

M arbitrary with all types definable.
Let G be an M-definable group. Let K, H be M-definable subgroups of G such that the following conditions hold:

(1) $G = K \cdot H$ and $K \cap H = \{e\}$.

(2) $S_H(M)$ has an $H(M)$-invariant type p.

(3) The flow $(K(M), S_K(M))$ has a minimal subflow I which is invariant under the natural $H(M)$-action on $S_K(M)$.

Then $I * p$ is a minimal subflow of $(G(M), S_G(M))$.

Ellis group

Direct calculation ("coheir arithmetic" + description of $*$ in $S_K(M)$ + properties of nonforking extensions) shows that the semigroup operation on $\mathrm{Gen}_K(\mathbb{R}) * p$ depends only on a definable function $\psi : K(\mathbb{R}) \to K(\mathbb{R})$ induced (in a certain way) by the type p.
In particular, the Ellis group of $S_G(\mathbb{R})$ is isomorphic to the image of ψ.

Ellis group

Direct calculation ("coheir arithmetic" + description of $*$ in $S_K(M)$ + properties of nonforking extensions) shows that the semigroup operation on $\mathrm{Gen}_K(\mathbb{R}) * p$ depends only on a definable function $\psi : K(\mathbb{R}) \to K(\mathbb{R})$ induced (in a certain way) by the type p.
In particular, the Ellis group of $S_G(\mathbb{R})$ is isomorphic to the image of ψ.
Again by calculation: $H(\mathbb{R})$-invariance of p implies that if $x \in \mathrm{im}\psi$ then x normalizes H in G.
On the other hand, it is easy to check that $N_G(H) \cap K(\mathbb{R}) \subset \mathrm{im}\psi$.

Proposition (J.)

The Ellis group of $(G(\mathbb{R}), S_G(\mathbb{R}))$ is isomorphic to $N_G(H) \cap K(\mathbb{R}) \quad (= N_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R}))$.

Ellis group

Example

$G(\mathbb{R}) = SL_n(\mathbb{R})$. Then the Ellis group of $S_G(\mathbb{R})$ is isomorphic to $\mathbb{Z}_2^{n-1}$.

The particular case of $n = 2$ was first done by Gismatullin, Penazzi and Pillay. It is a counterexample to the question by Newelski whether (at least in a "sufficiently tame" setting), the Ellis group of $S_G(M)$ is isomorphic to G/G^{00}.

Generalizations

- Universal covers interpreted in a two-sorted structure.
- Generalizations to elementary extensions more difficult - types are no longer definable and we are forced to work with external types.
- (even for $\mathbb{R}$-definable groups interpreted in elementary extensions)

References

- A. Conversano, *A reduction to the compact case for groups definable in o-minimal structures*, preprint
- J. Gismatullin, D. Penazzi, A. Pillay, *Some model theory of* $SL(2, \mathbb{R})$, preprint
- G. Jagiella, *Definable topological dynamics and real Lie groups*, preprint
- L. Newelski, *Topological dynamics of definable group actions*, J. Symbolic Logic Volume 74, Issue 1 (2009), pp. 50-72