Definable topological dynamics and real Lie groups

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A **point transitive $G$-flow** is an action of the group $G$ on a compact Hausdorff space $X$ by homeomorphisms such that $X$ contains a dense $G$-orbit.

Let $X$ be a point-transitive $G$-flow. Every $g \in G$ determines a homeomorphism $\pi_g \in X^X$.

Let $E(X) = \text{cl}\{\pi_g : g \in G\}$. This is a compact subspace of $X^X$ and itself a (point-transitive) $G$-flow $((g \cdot f)(x) = f(g^{-1} \cdot x))$. $E(X)$, $*$ is a semigroup (where $*$ is the function composition). It is called the **Ellis semigroup** of the flow $(G, X)$. 
Important objects associated with $E(X)$:

- Algebraic: minimal (left) ideals,
- Topological: minimal subflows (minimal nonempty closed $G$-invariant subsets).

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\text{minimal ideals} = \text{minimal subflows}
\]
Properties of minimal ideals

For every minimal subflow $I$ of $E(X)$, $I = \text{cl}(Gp)$ for any $p \in I$

A $p \in E(X)$ such that $\text{cl}(Gp)$ is a minimal subflow of $E(X)$ is called almost periodic.
Properties of minimal ideals

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A $p \in E(X)$ such that $\text{cl}(Gp)$ is a minimal subflow of $E(X)$ is called **almost periodic**.

A $p \in E(X)$ with $p \ast p = p$ is called an **idempotent**.

Let $I$ be a minimal subflow of $E(X)$ and let $J(I)$ be the set of idempotents in $I$.

We have:

$$I = \bigsqcup_{u \in J(I)} u \ast I,$$

where every $(u \ast I, \ast)$ is a group.
The groups \((u \ast I, \ast)\) are all isomorphic (even for different \(I\)'s) and called \textbf{ideal subgroups} of \(E(X)\).

Their isomorphism class is called the \textbf{Ellis group} (of the flow \((G, X)\)).
(Newelski)
Fix a first-order structure $M$ and a sufficiently saturated $C > M$.
**Assume that all types over $M$ are definable.**
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**Assume that all types over $M$ are definable.**
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There is a category of **definable $G(M)$-flows**: actions of $G(M)$ on the quotient $X(\mathcal{C})/E$ where $X$ is $M$-definable on which $G(M)$ acts definably and transitively, and $E$ is a $G$-invariant btde relation.
This category has the universal object $S_G(M)$.
It is the definable equivalent of $\beta G$. 
The Ellis semigroup of $S_G(M)$ turns out to be isomorphic to $S_G(M)$ itself (this requires definability of types). The semigroup operation on $S_G(M)$ can be described as follows:

$$p * q = tp(a \cdot b/M),$$

where

$a \models p,$

$b \models q,$ and

$tp(b/Ma) \supset q$ is the **heir extension**.
Recall that a structure \((M, <, \ldots)\) is \textit{o-minimal} if \(<\) is dense and linear without endpoints, and every definable subset of \(M\) is a finite union of intervals.
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Fix \(\mathbb{R} = (\mathbb{R}, +, \cdot, <, \ldots)\) and o-minimal expansion of reals. Some properties of \(\mathbb{R}\):

\begin{itemize}
  \item \textit{NIP},
  \item cells and cell decomposition theorem,
  \item topological dimension (coincides with \textit{acl}\-dimension),
  \item definability of types.
\end{itemize}
Let $G$ definable in $\mathbb{R}$.

**Proposition (Pillay)**

*There is a definable atlas of maps making $G(\mathbb{R})$ a definable manifold over $\mathbb{R}$, making the group operations continuous (i.e. $G(\mathbb{R})$ is a real Lie group).*

Goal: describe topological dynamics of $G$.
Two important cases: torsion-free and definably compact.
Let $G$ be torsion-free. This case is straightforward:

**Proposition (Conversano, Pillay)**

There is a $G(\mathbb{R})$-invariant type $p$ in $S_G(\mathbb{R})$.

That is, $S_G(\mathbb{R})$ has a one-point minimal subflow.
Let $G$ be definably compact. Newelski gave a full description of $(G(\mathbb{R}), S_G(\mathbb{R}))$:

- $S_G(\mathbb{R})$ contains the unique minimal flow $\text{Gen}_K(\mathbb{R})$ consisting of all generic types in $G$.
- The Ellis group of $(G(\mathbb{R}), S_G(\mathbb{R}))$ is isomorphic to $G(\mathbb{R})$.
- An ideal subgroup of $S_G(\mathbb{R})$ is a selector of $S_G(\mathbb{R})/\ker(\text{st})$.
- Detailed description of the semigroup operation.
Compact-torsion-free decomposition

Definition

Let $G$ be definable. We say that $G$ has a **definable compact-torsion-free decomposition** if there is a definable, definably compact $K < G$ and a definable, torsion-free $H < G$ such that $K \cap H = \{e\}$ and $G = KH$. 

Proposition (Conversano)

There is a definable, central subgroup $A \subseteq G$ such that $G / \langle A \rangle$ has a definable compact-torsion-free decomposition, and is the maximal quotient with this property. In particular, definable semisimple groups have this decomposition.
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Proposition (Conversano)

*There is a definable, central subgroup $A(G) < G$ such that $G/A(G)$ has a definable compact-torsion-free decomposition, and is the maximal quotient with this property.*

In particular, definable semisimple groups have this decomposition.
Let $G = KH$ be a definable compact-torsion-free decomposition.
Goal: describe the $G(\mathbb{R})$-flow $S_G(\mathbb{R})$.

We already have a description of $(K(\mathbb{R}), S_K(\mathbb{R}))$ and $(H(\mathbb{R}), S_H(\mathbb{R}))$.

We need to understand the interaction between $H$ and $K$. 
Natural subgroup actions

$H$ acts on the coset space $G/H$. This quotient can be identified with $K$.

So we have a group action of $H(\mathbb{R})$ on $K(\mathbb{R})$.

This induces a group action of $H(\mathbb{R})$ on $S_K(\mathbb{R})$ and a semigroup action of $S_H(\mathbb{R})$ on $S_K(\mathbb{R})$.

The action $H(\mathbb{R}) \acts S_K(\mathbb{R})$ preserves $\text{Gen}_K(\mathbb{R})$. 
Proposition (J.)

$M$ arbitrary with all types definable.

Let $G$ be an $M$-definable group. Let $K, H$ be $M$-definable subgroups of $G$ such that the following conditions hold:

1. $G = K \cdot H$ and $K \cap H = \{e\}$.
2. $S_H(M)$ has an $H(M)$-invariant type $p$.
3. The flow $(K(M), S_K(M))$ has a minimal subflow $I$ which is invariant under the natural $H(M)$-action on $S_K(M)$.

Then $I \ast p$ is a minimal subflow of $(G(M), S_G(M))$. 

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Direct calculation ("coheir arithmetic" + description of * in $S_K(M)$ + properties of nonforking extensions) shows that the semigroup operation on $\text{Gen}_K(\mathbb{R}) * p$ depends only on a definable function $\psi : K(\mathbb{R}) \rightarrow K(\mathbb{R})$ induced (in a certain way) by the type $p$. In particular, the Ellis group of $S_G(\mathbb{R})$ is isomorphic to the image of $\psi$. 
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Again by calculation: $H(\mathbb{R})$-invariance of $p$ implies that if $x \in \text{im}\psi$ then $x$ normalizes $H$ in $G$.

On the other hand, it is easy to check that $N_G(H) \cap K(\mathbb{R}) \subset \text{im}\psi$.

**Proposition (J.)**

The Ellis group of $(G(\mathbb{R}), S_G(\mathbb{R}))$ is isomorphic to $N_G(H) \cap K(\mathbb{R})$ ($= N_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$).
Example

$G(\mathbb{R}) = SL_n(\mathbb{R})$. Then the Ellis group of $S_G(\mathbb{R})$ is isomorphic to $\mathbb{Z}_{2^{n-1}}$.

The particular case of $n = 2$ was first done by Gismatullin, Penazzi and Pillay. It is a counterexample to the question by Newelski whether (at least in a “sufficiently tame” setting), the Ellis group of $S_G(M)$ is isomorphic to $G/G^{00}$. 
Generalizations

- Universal covers interpreted in a two-sorted structure.
- Generalizations to elementary extensions more difficult - types are no longer definable and we are forced to work with external types.
- (even for $\mathbb{R}$-definable groups interpreted in elementary extensions)
A. Conversano, *A reduction to the compact case for groups definable in o-minimal structures*, preprint


G. Jagiella, *Definable topological dynamics and real Lie groups*, preprint