Finding definable envelopes 'around' nilpotent or solvable subgroups

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A **definable subgroup** of $(G, \times, ^{-1}, 1)$: a subgroup H of G such that

$$H = \{g \in G : \varphi(g)\}.$$

Examples.

- $C_G(a) = \{g \in G : xa = ax\}$
- $C_G(a_1, \dots, a_n) = \{g \in G : xa_1 = a_1x \wedge \cdots \wedge xa_n = a_nx\}$
- $Z(G) = \{g \in G : (\forall x) gx = xg\}$
- $Z_2(G) = \{g \in G : (\forall x)(\forall y)[x, g]y = y[x, g]\}$

Counter examples.

- $C_G(A)$, $C_G^n(A) = \left\{x \in \bigcap_{k<n} N_G\left(C_G^k(A)\right) : [x, A] \subset C_G^{n-1}(A)\right\}$.
- $\langle a \rangle$, $\langle D \rangle$ where D is definable, $G' = \left\langle \{[g, h] : g \in G, h \in G\} \right\rangle$
- $FC(G) = \{g \in G : g^G \text{ is finite }\} = \bigcup_{n=1}^{\infty} \{g \in G : | g^G | \leqslant n\}$
- $Fit(G) = \{g \in G : \langle g^G \rangle \text{ is nilp.}\} = \bigcup_{n=1}^{\infty} \{g \in G : \langle g^G \rangle \text{is n-nilp.}\}$

Remark (Ould Houcine). If $Fit(G)$ is nilpotent, then it is definable.

0. Aim of the talk

Problem 1. In G, let A be a subgroup which is abelian (resp. nilpotent, or soluble). Is there a **definable** subgroup of G which **almost** contains A and is **close to being** abelian (resp. nilpotent, soluble)?

Answer. No in general: take an infinite G such that

- $Z(G) = G' \simeq \mathbf{Z}/3\mathbf{Z}$
- $g^3 = 1$ for every $g \in G$.
  - Then every definable abelian subgroup of G is finite (Plotkin).
  - G is unstable, but supersimple of rank 1, and $\aleph_0$-categorical.

Problem 2. Is $Fit(G)$ definable? Is the soluble radical $R(G)$ (generated by all normal solvable subgroups of G) definable?

0. Content of the talk

1. G is stable or has dcc
2. G has a simple theory
3. G does not have the independence property

1. Stable groups, groups with dcc

Definition (folklore ?). G has the **descending chain condition on centralisers (dcc)**, if for all subsets $A_1, A_2, \ldots$ of G, every descending chain $C_G(A_1) \geqslant C_G(A_2) \geqslant \ldots$ is finite.

Remark. Assume G has the **dcc**.

- Any $C_G(A)$ is definable.
- If $A \leqslant G$ is abelian, $Z(C_G(A))$ is a def. abelian envelope of A.

Examples of groups with dcc.

- abelian groups
- torsion-free hyperbolic groups
- linear groups over fields
- finitely generated nilpotent groups
- stable groups.

Fact (Poizat). If G is **stable** and $H \leqslant G$ is n-nilpotent/n-soluble, H has a definable n-nilpotent/n-soluble envelope.

Ingredients of the proof.

- A stable group has dcc.
- If G is stable and $H \lhd G$ definable, then G/H is stable.

Fact (Altınel, Baginsky). If G has **dcc** and $H \leq G$ is n-nilpotent, H has a definable n-nilpotent envelope which is normalised by $N_G(H)$.

An ingredient of the proof. $C^n_G(A)$ is definable for any A and n.

Theorem (Wagner). If G has dcc, then $Fit(G)$ is definable and nilpotent.

Remark. Known for groups of finite RM (Nesin).

Fact (Baudish). *If G is superstable, $R(G)$ is definable and solvable.*

Remark. Known for groups of finite RM (Belegradek), and groups of finite U-rank (Baldwin-Pillay).

2. G has a simple theory

Definition (Shelah). X is a definable subset of G, $\phi(x, y)$ a formula, k a natural number. The $D(\ . \ , \phi, k)$-**rank** of X :

- $D(X, \phi, k) \geq 0$ if $X \neq \emptyset$,
- $D(X, \phi, k) \geq n+1$ if there are infinitely k-disjoint sets defined by $\phi(x, a_1), \phi(x, a_2), \ldots$ with $D(X_i \cap X, \phi, k) \geq n$.

Definition (Shelah). G is **simple** if $D(G, \phi, k) < \aleph_0$ for all ϕ, k.

Definition (Shelah). X is a definable subset of G, $\phi(x, y)$ a formula. The ϕ-**Cantor-Bendixson rank** of X :

- $CB(X, \phi) \geq 0$ if $X \neq \emptyset$,
- $CB(X, \phi) \geq n+1$ if there are infinitely many 2-disjoint ϕ-sets $X_1, X_2, \ldots$ with $CB(X_i \cap X, \phi) \geq n$.

Definition (Shelah). G is **stable** if $CB(G, \phi) < \aleph_0$ for every ϕ.

Remark. $D(X, \phi, k) \leq CB(X, \phi)$.

Examples of groups with a simple theory.

- stable groups
- pseudofinite simple groups

Question. Does a group G with a simple theory has the dcc ?

No, but:

Wagner's Chain Condition. Let $\phi(x, y)$ be a formula. There is some n such that every descending chain of subgroups $G_1, G_2, \dots$ defined by $\phi(x, a_1), \phi(x, a_2), \dots$ has no more than n elements, up to finite index, *ie* such that G_n/G_m is finite whenever $m \geq n$.

2. G has a simple theory

Proposition (abelian case). If $A \leqslant G$ is abelian, then A has a definable envelope which is finite-by-abelian (ie FC).

An ingredient of the proof. $FC(G)$ is definable.

Remark. A similar result by Elwes, Jaligot, Macpherson and Ryten.

Theorem (nilpotent case). If $N \leqslant G$ is nilpotent of class n, then there is a definable subgroup E which is virtually 'nilpotent of class $2n$' normalised by $N_G(N)$ and finitely many translates of which cover N.

Questions.

- Is the bound $2n$ optimal?
- Does N have a definable nilpotent envelope?

Theorem (soluble case). If $S \leqslant G$ is soluble of derived length ℓ, then S has a definable soluble envelope F which is virtually 'soluble of derived length 2ℓ', normalised by $N_G(N)$.

2. G has a simple theory

In a stable theory	Analogue in a simple theory
dcc	dcc up to finite index
abelian groups	FC-groups (eg finite, abelian, finite-by-abelian)
$C_G(H)$	$FC_G(H)= \{g \in G : g^H \text{ is finite}\}$ (Haimo, 1953)
$Z(H)$	$FC(G) = FC_G(G)$
$Z_{n+1}(G)$	$FC_{n+1}(G)$ $\Big(FC_{n+1}(G)/FC_n(G) = FC(G/FC_n(G)\Big)$
n-nilpotent	n-FC-nilpotent $\big(FC_n(G) = G$, Haimo, eg finite, nilp.$\big)$
n-soluble	n-FC-soluble (Duguid, McLain, 1956)
	$G_0 = G \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$ with $G_i \trianglelefteq G$
	and G_i/G_{i+1} FC (eg finite, soluble, virtually-soluble)

Theorem. If $N \leqslant G$ is FC-nilpotent of class n, then it is contained in a definable FC-nilpotent group of class n.

Fact (adapted from Wagner). *A n-FC-nilpotent definable subgroup of G is virtually 'm-nilpotent', with $m \leq 2n$.*

2. G has a simple theory

Theorem (nilpotent case). If $N \leqslant G$ is nilpotent of class n, then there is a definable subgroup E which is virtually 'nilpotent of class $2n$' normalised by $N_G(N)$ and finitely many translates of which cover N.

Theorem. If G is supersimple of finite SU-rank, then $Fit(G)$ is definable and nilpotent.

Theorem (Elwes, Jaligot, Macpherson, Ryten 2010). If G is supersimple of finite SU-rank such that T^{eq} eliminates $\exists^\infty$. Then $R(G)$ is definable and soluble.

Theorem. If G is supersimple, then there is a finite chain of definable subgroups $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ such that every H_{i+1}/H_i is either virtually FC or virtually simple modulo a finite FC-centre.

Corollary. If G is supersimple, then $R(G)$ is definable and soluble, and the FC soluble radical of G is definable and virtually-soluble.

3. G does not have the independence property

Let $f(x,y)$ be a formula.

Definition. We say that $f(x,y)$ **shatters** n (in G) if there is a subset A of size n such that any subset of A is of the form $f(A,b)$ for some $b \in G$, or equivalently if for all natural number n, we can find elements $a_1, a_2, \dots, a_n$ and $(b_I)_{I \subset \{1,\dots,n\}}$ in G such that

$$f(a_i, b_I) \text{ holds in } G \text{ if and only if } i \in I.$$

Definition (Shelah). $f(x,y)$ has the **independence property in G** if it shatters every natural number n.

Definition (Shelah). G **does not have the independence property** if no formula has the independence property in G.

Counter example. In $(\mathbf{Z}, +, \times)$, the formula 'x divides y' has the independence property. Take $a_1, \dots, a_n$ to be the first n prime numbers, and $b_I = \prod_{i \in I} a_i$ for any $I \subset \{1, \dots, n\}$.

3. G does not have the independence property

Baldwin-Saxl Chain Condition. Let $f(x, y)$ be a formula. Let $G_1, G_2, G_3, \ldots$ be subgroups of G defined respectively by the formulas $f(x, a_1), f(x, a_2), f(x, a_3), \ldots$ (ie **uniformly** definable). There is a natural number n (depending only on f) such that for every **finite** subset I of ω, there is a finite subset $I_n \subset I$ of size n such that

$$\bigcap_{i \in I} G_i = \bigcap_{i \in I_n} G_i.$$

Theorem (Shelah). If G has an infinite abelian subgroup (and is ω-saturated), then G has one which is definable.

Theorem (Aldama). If $A \leqslant G$ is abelian, then there is an abelian definable subgroup of G which contains A.

Theorem (Aldama). If $N \leqslant G$ is n-nilpotent, then there is an n-nilpotent definable subgroup of G which contains N.

Theorem (Aldama). If $S \leqslant G$ is ℓ-solvable and **normal**, then there is an ℓ-solvable normal definable subgroup of G which contains S.

In the nilpotent case, the key Lemma is : if $A \leqslant G$ is in the center of $B \leqslant G$, there is a definable $Z \geqslant A$ which is in the centre of a definable $H \geqslant B$.

Question. If $A \leqslant G$ is abelian and normalised by $N_A \leqslant G$, is there a definable abelian $H \geqslant A$ normalised by a definable $N \geqslant N_A$.

I don't know.

3. G does not have the independence property

Key Lemma. H a definable subgroup of G, A a subgroup of H, and N_A a subgroup of G which normalises A. Then there is an ω-definable subgroup K of H, and an ω-definable subgroup N of G such that

1. $A \leqslant K$,
2. $N_A \leqslant N$,
3. N normalises K.

(ω-**definable** means: intersection of countably many definable sets.)

Morever, K is the intersection of conjugates of H, and N is the intersection of uniformly definable sets.

Corollary. If $S \leqslant G$ is 2-soluble, then it is contained in an ω-definable 2-soluble subgroup.

Corollary. If $S \leqslant G$ and $H \leqslant G$ definable, then the S-core $\bigcap_{g \in S} H^g$ is ω-definable.