# Locally finite profinite rings

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# Polish structures

## Definition (Krupiński)

A Polish structure is a pair $(X, G)$, where $G$ is a Polish group acting faithfully on a set $X$ so that $G_x <_c G$ for every $x \in X$.

## Examples

1. Compact [profinite] structures: $X$ - a compact metric [profinite] space, $G$ - a compact group, the action continuous,
2. [Polish] $G$-spaces: $X$ - a [Polish] space, $G$ - a Polish group, the action is continuous, e.g.:
   - $X$ - a compact metric space, $G = \mathrm{Homeo}(X)$ - the group of all homeomorphisms of $X$ with the compact-open topology,
   - $X$ - a compact metric group, $G = \mathrm{Aut}(X)$ - the group of all topological automorphisms of $X$ with the c-o topology,
3. Borel $G$-spaces: $X$ - a Polish space, $G$ - a Polish group, the action is Borel-measurable.

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# Smallness

## Definition (Krupiński)

A Polish structure $(X, G)$ is small if for every $n \in \omega$ there are only countably many orbits on $X^n$.

## Examples of small Polish structures (Krupiński)

1. $(S^n, \mathrm{Homeo}(S^n))$, $n \in \omega$,
2. $(I^n, \mathrm{Homeo}(I^n))$, $n \in \omega \cup \{\omega\}$,
3. $((S^1)^n, \mathrm{Homeo}((S^1)^n))$, $n \in \omega \cup \{\omega\}$,
4. $(P, \mathrm{Homeo}(P))$, $P$ - the pseudo-arc,
5. $(H, \mathrm{Aut}(H))$, $H$ - a profinite abelian group of finite exponent, $\mathrm{Aut}(H)$ - the group of all topological automorphisms of $H$,
6. $(H, \mathrm{Aut}^0(H))$, $H$ - as above, $\mathrm{Aut}^0(H)$ - the group of all automorphisms of $H$ preserving a distinguished inverse system indexed by $\omega$.

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# $m$-independence

$(X, G)$ - a compact [profinite] structure
$a$ - finite tuple of elements of $X$
$A, B$ - finite subsets of $X$
$o(a/A) := \{g \cdot a : g \in G_A\}$

**Definition (Newelski)**

$a \overset{m}{\perp}_A B \iff o(a/AB) \subseteq_o o(a/A)$,
$a \overset{m}{\not\perp}_A B \iff o(a/AB) \subseteq_{nwd} o(a/A)$.

**Fact (Newelski)**

$\overset{m}{\perp}$ is invariant, symmetric and transitive. If $(X, G)$ is small, then $\overset{m}{\perp}$ satisfies the existence of independent extensions.

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# $nm$-independence

$(X, G)$ - a Polish structure
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$\pi_A : G_A \to o(a/A)$ is defined by $\pi_A(g) = g \cdot a$.

**Definition (Krupiński)**

$a \mathop{\smile\!\!\!\!\!\!\,|}^{nm}_A B \iff \pi_A^{-1}[o(a/AB)] \subseteq_{nm} \pi_A^{-1}[o(a/A)]$,
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$a \mathop{\smile\!\!\!\!\!\!\,|}^{nm}_A B \iff G_{AB} G_{Aa} \subseteq_{nm} G_A$,

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# $\mathcal{NM}$-rank and *nm*-stability

$(X, G)$ - a small Polish structure

**Definition (Krupiński)**

$\mathcal{NM}$ : orbits over finite sets $\to Ord \cup \{\infty\}$

$\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $a \not\downarrow^{nm}_A B$ and $\mathcal{NM}(a/B) \geq \alpha$.

**Definition (Krupiński)**

$\mathcal{NM}(X) = \sup\{\mathcal{NM}(x/\emptyset) : x \in X\}$.

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$(X, G)$ is *nm*-stable if for every $x \in X$, $\mathcal{NM}(x/\emptyset) < \infty$.

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# Polish group structures

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1. A Polish group structure is a Polish structure $(H, G)$ such that $H$ is a group and $G$ acts as a group of automorphisms of $H$.
2. A (topological) $G$-group is a Polish group structure $(H, G)$ such that $H$ is a topological group and the action of $G$ on $H$ is continuous.
3. A Polish [compact] $G$-group is a topological $G$-group $(H, G)$, where $H$ is a Polish [compact] group.

## Main examples of compact $G$-groups

1. $(H, \mathrm{Aut}(H))$, where $H$ is a compact metric group.
2. $(H, \mathrm{Aut}^0(H))$, where $H$ is a profinite group, $\mathrm{Aut}^0(H)$ - the group of all automorphisms of $H$ preserving a distinguished countable inverse system defining $H$.

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# General goal

To understand the structure (or even classify) small compact [or, more generally, Polish] $G$-groups. In other words, we would like to undestand the algebraic/topological consequences of the assumption of smallness in the context of compact [or Polish] $G$-groups.

# Small Polish $G$-groups

**Proposition (Krupiński)**

1. If $(H, G)$ is a small Polish $G$-group, and $S \subseteq H$ is finite, then $\overline{\langle S \rangle}$ is countable, i.e. $\langle S \rangle$ does not have limit points.
2. If $(H, G)$ is a small compact $G$-group, then $H$ is locally finite, and so $H$ is a profinite (so 0-dimensional) group.

# Conjectures on small profinite groups

**Remark (Krupiński)**

If $(H, G)$ is a small compact $G$-group, then $H$ is a profinite group. But $G$ is not necessarily compact, so $(H, G)$ needn't be a small profinite group regarded as profinite structure.

$(H, G)$ - a small profinite group regarded as profinite structure

**Newelski's Conjecture**

$H$ is abelian-by-finite.

**Intermediate Conjectures**

(A) $H$ is solvable-by-finite.

(B) If $H$ is solvable-by-finite, then it is nilpotent-by-finite.

(C) If $H$ is nilpotent-by-finite, then it is abelian-by-finite.

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# Results on small profinite groups

**Theorem (Wagner)**

*Each small, nm-stable profinite group is abelian-by-finite.*

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*Each small, nm-stable profinite group is of finite $\mathcal{NM}$-rank.*

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# The conjectures for small compact $G$-groups

From now on, we consider generalizations of Conjectures (A), (B), (C) to the wider context of small compact $G$-groups. It turns out that in general they are all false.

## Counter-example to (A)

$H$ - any finite non-solvable group
$S_\infty$ - the group of all permutations of $\mathbb{N}$
$S_\infty$ acts on $H^\omega$ by $\sigma\langle h_0, h_1, \ldots\rangle = \langle h_{\sigma(0)}, h_{\sigma(1)}, \ldots\rangle$.

Then $(H^\omega, S_\infty)$ is a small compact $S_\infty$-group, and $H^\omega$ is not solvable-by-finite.

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# Results on small, nm-stable compact $G$-groups

$(H, G)$ - a small compact $G$-group

**Theorem (Krupiński)**

*If $(H, G)$ is nm-stable, then $H$ is nilpotent-by-finite.*

**Conjecture**

If $(H, G)$ is nm-stable, then $H$ is abelian-by-finite.

**Theorem (Krupiński, Wagner)**

*If $\mathcal{NM}(H) \leq \omega$, then $H$ is abelian-by-finite.*

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# Without *nm*-stability

**Question**

What can be said about the structure of small compact $G$-groups without assuming *nm*-stability?

# Open questions on small Polish $G$-groups

**Question**

Is every small, $nm$-stable Polish $G$-group abelian-by-countable?

Even the following question is open.

**Question**

Is every small Polish $G$-group of $\mathcal{NM}$-rank 1 abelian-by-countable?

# Small compact G-rings

**Definition**

Let $G$ be a Polish group. A compact $G$-ring is a Polish structure $(R, G)$, where $R$ is a compact topological ring and $G$ acts on $R$ continuously as a group of automorphisms.

**Goal**

To understand the structure of small compact $G$-rings. Investigate connections with small compact $G$-groups.

**Proposition**

If $(R, G)$ is a small compact $G$-ring, then $R$ is locally finite and profinite.

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# A few classical definitions from ring theory

$R$ - a ring

1. An element $r \in R$ is nilpotent of nilexpenent $n$ if $r^n = 0$ and $n$ is the smallest number with this property.
2. $R$ is nil [of nilexponent $n$] if every element of $R$ is nilpotent [of nilexponent $\leq n$ and there is an element of nilexponent $n$].
3. $R$ is nilpotent of class $n$ if $r_1 \cdot \ldots \cdot r_n = 0$ for all $r_1, \ldots, r_n \in R$ and $n$ is the smallest number with this property.
   $R$ is nilpotent if it is nilpotent of class $n$ for some $n$.
4. $R$ is null if $r_1 \cdot r_2 = 0$ for all $r_1, r_2 \in R$.

# Results on small, *nm*-stable compact *G*-rings

In various contexts, Krupiński has proved certain theorems which allow to deduce some properties of rings from the appropriate properties of groups. As a conclusion of these results and the theorems on small, *nm*-stable compact *G*-groups, we get the following corollary.

**Theorem (Krupiński)**

1. A small, *nm*-stable compact *G*-ring is nilpotent-by-finite.
2. A small, compact *G*-ring of finite $\mathcal{NM}$-rank is null-by-finite.

**Conjecture**

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**Proposition (Krupiński)**

This conjecture is equivalent to the conjecture that each small, *nm*-stable compact *G*-group is abelian-by-finite.

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# The structure of small compact $G$-rings

## Theorem (Krupiński, D.)

Every locally finite profinite ring is (nil of finite nilexponent)-by-(product of complete matrix rings over finite fields with only finitely many factors up to isomorphism). More precisely:

1. The Jacobson radical of a locally finite profinite ring is nil of finite nil exponent.
2. Let $R$ be a topological ring. Then $R$ is semisimple, locally finite profinite ring iff $R$ is a product of complete matrix rings over finite fields with only finitely many factors up to isomorphism.

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# An application to small profinite rings

## Remark

Suppose $(\prod_{i \in I} R_i, G)$ is a small profinite ring. Then only finitely many $R_i$'s are not null rings.

## Comment

Using the above remark, we can easily deduce from the preceding theorem the following result, which was obtained earlier by Wagner and Krupiński:

## Fact

If $R$ is a small profinite ring, then $J(R)$ is nil of finite nilexponent, and has a finite index in $R$.

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