SUPERGENERIX

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SUPERGENERIX I

A FILTER OF LARGE SETS

DEFINITIONS

Let G be any group ; a subset X of G is

- *generic* (or *syndedic*) if $G = a_1.X \cup \ldots \cup a_m.X$ for a certain tuple $\underline{a}$ of elements of G

- *supergeneric* if any intersection $b_1.X \cap \ldots \cap b_n.X$ of a finite number of translates of X is generic (and supergeneric !)

- *thick* if any intersection $b_1.X \cap \ldots \cap b_n.X$ is non-empty, i.e. if $\neg X$ is not generic.

BASIC FACTS

1. A set containing a generic, a super-generic, or a thick set, is of the same kind.

2. A left or right translate of a generic, a supergeneric, or a thick set, is equally so.

3. If $a_1.X \cup \ldots \cup a_m.X$ is generic, so is X (does not work for supergeneric !).

4. If X is supergeneric or thick, so is $a_1.X \cap \ldots \cap a_m.X$ (does nor work for generic !)

5. The intersection of a generic set and a thick set is not empty.

6. The intersection of a supergeneric set and a thick set is thick.

7. *Fundamental observation* : the $\cap$tion of a generic set and a supergeneric set is generic.

8. The intersection of two supergeneric sets is supergeneric.

9. The filter of supergenerics is the $\cap$ of all the maximal filters closed by translations.

Proof of 7 **: if $G = a_1.X \cup \ldots \cup a_m.X$, then $a_1.Y \cap \ldots \cap a_m.Y \subseteq a_1.(X \cap Y) \cup \ldots \cup a_m.(X \cap Y)$.**

Proof of 8 **: $a_1.(X \cap Y) \cap \ldots \cap a_m.(X \cap Y) = (a_1.X \cap \ldots \cap a_m.X) \cap (a_1.Y \cap \ldots \cap a_m.Y)$.**

Proof of 9 **: if Φ is a filter of thick sets, by 5 it generates with the supergenerics a non-trivial filter ; if $Y = a_1.X \cap \ldots \cap a_m.X$ is not generic, any non-trivial filter closed under translations containing $\neg Y$ does not contains X .**

COUNTEREXAMPLES

In any infinite group, we can find two complement sets which are both generics, and two complement sets which are both non-generic (and thick !).

Neither the generic sets, nor the thick sets, nor the sets which are generic and thick, form a filter : we can find pairs of sets which are both generic and thick, but whose intersection is neither generic nor thick.

EXAMPLES

If G is finite, X generic $\Leftrightarrow$ $X \neq \emptyset$, X thick $\Leftrightarrow$ X supergeneric $\Leftrightarrow$ $X = G$.

For infinite G , cofinite subsets are super-generic ; but there exist many non-cofinite supergenerics. The complement of a subset of G of stricly smaller cardinal is supergeneric.

H generic subgroup of G $\Leftrightarrow$ H has finite index in G ; G is its only thick subgroup.

If H is a subgroup of infinite index, its complement is supergeneric (NEUMAN 1952).

If H has finite index in G, X is supergeneric in G $\Leftrightarrow$ for each a, $X \cap a.H$ is a translate of a supergeneric in H.

Let f be a surjective homomorphism from G onto H ; if X is supergeneric in G, $f(X)$ is supergeneric in H ; if Y is supergeneric in H, $f^{-1}(Y)$ is supergeneric in G. Same thing for generic or thick.

LEFT AND RIGHT

We have defined genericity, supergenericity and thickness in association with left-translations a.X ; we can consider as well right-translations X.a , and even two-sides-translations a.X.b .

We can also consider these notions corresponding to conjugacy $a.X.a^{-1}$, and even to any action of G on a set E .

If X is left-supergeneric, it is also two-sides-supergeneric.

Proof **: Since all the right-translates $X.b$ are left-super-generic, so is any intersection $a_1.X.b_1 \cap \ldots \cap a_n.X.b_n$.**

Ergo a left-supergeneric is right- and even two-sides-thick, but there are examples of left-supergenerics which are not right-supergeneric.

PRODUCTS

Fubini Assumption **: X is supergeneric in $G \times H$ ⇔ the set of points a whose fiber $X_a = \{ y \,/\, (a,y) \in X \}$ is supergeneric in H is itself supergeneric in G . We expect this assumption to be false in both directions.**

Nevertheless, if the fiber is supergeneric for thickly many a , X is thick.

<u>Question</u> : ***Is it true if we assume only thickness for the fibers ?***

Also, if, for supergenerically many a and some fixed integer n, the fiber contains all the points except n, then X is supergeneric.

But there exist non-generic (but thick !) sets whose each fiber is cofinite.

In the other direction, there exist generic sets with no generic fibers, and thick sets with no thick fibers, but supergeneric examples are wanted.

LOGIC

Stable context **: Γ is a stable group and X is a definable subset of Γ ; in other words the structure $(\Gamma ; \times, X(x))$ is stable.**

Substable context **: G is a subgroup of a stable group Γ (for instance G is linear, $\Gamma = GLn(K)$ with K an acf) and X is the $\cap$section with G of a definable subset of Γ .**

The big idea **: obtain generic constraints in the substable context.**

In stable context, it is known since Neanderthal (Lachlan, Poizat) that :

- left = right = two-sides

- if $X \cup Y$ is generic, then X or Y is generic, so that we have generic types

- X is generic $\Leftrightarrow$ X contains a gen. type

- X is supergeneric $\Leftrightarrow$ X is thick $\Leftrightarrow$ X contains all the generic types

- if Γ is connected, X is supergeneric or $\neg X$ is supergeneric

- in general, to X is associated a definable subgroup Δ of finite index, such that, for each a, $X \cap a.\Delta$ is a translate of a super-generic or cosupergeneric subset of Δ

- Fubini Assumption is valid.

It is known since Cromagnon (Wagner, Newelski) that, in a substable context, the generic types of cl(G) are finitely satisfiable in G ; that, if X is a definable subset of cl(G) , X is generic, or supergeneric, or thick, in cl(G) $\Leftrightarrow$ X $\cap$ G is the same in G .

Consequently, all that we have said assuming stability is valid for substability.

In particular, if cl(G) is connected, X$\cap$G is supergeneric or cosupergeneric in G .

SUPERGENERIX II

UNIFORMITY

We say that X is m-generic if G is the union of m translates of X ; we say that X is *uniformly supergeneric* if, for each n , there is an integer $\gamma(n)$ s.t. any intersection of n translates of X be $\gamma(n)$-generic.

If $\gamma(n)$ is bounded by the constant m , we say that X is m-supergeneric.

<u>Example</u> : the cofinite subsets of an infinite group are 2-supergeneric.

We say that X is *parametrically super-generic* if, for each n , there exists a finite subset A_n of G such that, for any intersection X' of n translates of X , G is the union of the translates of X' by the elements of A_n ; if $X \neq G$, A_n must have at least $n+1$ points.

<u>Example</u> : if $X = G - \{b\}$, we can take $A_n = \{a_0 , \ldots a_n\}$ where the a_i are distinct.

Théorème 8. ***The intersection of two uniformly supergeneric sets is also unifly supergeneric.***

Théorème 9. ***The intersection of two parametrically supergeneric sets is also paramly spg.***

Théorème 10. ***If* G *is commutative, every supergeneric set is parametrically supergeneric.***

<u>Proof of 10</u>. Nous montrons l'existence de A_n par induction sur n . Pour $n = 1$, si $G = a_1.X \cup \ldots \cup a_m.X$, on a aussi $G = a_1.a.X \cup \ldots \cup a_m.a.X$ pour n'importe quel a , puisque G est commutatif, et on pose $A_1 = \{a_1 , \ldots a_m\}$.

Supposons $A_n = \{a_1 , \ldots a_s\}$ déjà construit, et considérons $X' = b_1.X \cap \ldots \cap b_n.X$; l'intersection Y des $a_i.X$ est contenue dans la réunion des $a_i.(X \cap X')$; comme Y est générique, on trouve $c_1 , \ldots c_k$ tel que G soit la réunion des $c_j.Y$, et alors G est la réunion des $c_j.a_i.(X \cap X')$; comme on peut tout translater par un quelconque a , vu la commutativité de G , on peut prendre pour A_{n+1} les produits $c_j.a_i$. <u>End</u>

There exist supergeneric sets which are not parametrically supergeneric. But :

Questions. ***Existence of non-uniformly supergeneric sets ? of uniformly supergeneric set which are not*** **m**-***supergeneric for a certain integer*** **m** ***? of parametrically supergeneric sets whose number of elements in*** $\mathbf{A_n}$ ***cannot be bounded by a linear function of*** **n** ***?***

$X \subseteq G$ is said *largely generic* if, for some n, the n-tuples $\underline{y}$ such that $y_1.X \cup \ldots \cup y_n.X = G$ form a supergeneric subset of G^n.

X is *very largely generic* if to each n is associated $\lambda(n)$ such that the $\lambda(n)$-tuples $\underline{y}$ such that $y_1.X' \cup \ldots \cup y_{\lambda(n)}.X' = G$ for every intersection X' of n translates of X form supergeneric subset of $G^{\lambda(n)}$.

Vlg $\Rightarrow$ param. supergeneric ; the intersection of two vlg is so. But :

Questions. ***Supergeneric $\Leftrightarrow$ largely generic ? Is the intersection of two largely generic sets always largely generic ?***

In a stable or substable context, a supergeneric set is very largely generic with a linear function $\lambda(n)$; if Γ has finite Morley (or Lascar) rank d , $\lambda(n) = (n+1).d + 1$.

<u>Proof in finite Morley rank context</u>. We can assume that Γ is saturated. Consider in it a sequence $e_1, \ldots e_\lambda$ of $\lambda(n) = (n+1).d + 1$ points which are generic and independent over the parameters defining X ; for any $a, b_1, \ldots b_n$ in Γ , one of them, say e_i , remains generic over $\{a, b_1, \ldots b_n\}$, so that the generic point $e_i^{-1}.a$ belongs to $b_1.X \cap \ldots \cap b_n.X$, since this set is supergeneric ; in consequence $\Gamma = e_1.(b_1.X \cap \ldots \cap b_n.X) \cup \ldots \cup e_{\lambda(n)}.(b_1.X \cap \ldots \cap b_n.X)$ and the λ-tuples $\underline{y}$ such that $G = y_1.Y \cup \ldots \cup y_\lambda.Y$ for any intersection Y of n translates of X form a definable subsets of G^λ which contains all its generic types. <u>End</u>

SUPERGENERIX III

SOME CONCRETE SITUATIONS

LINEAR GROUPS

G is a subgroup of $GL_n(K)$, for K acf.

X is a definable subset of $cl(G)$ $\Leftrightarrow$ X is constructible, i.e. a boolean combination of a finite number of Zariski-closed sets.

If $cl(G)$ is connected, X contains a non-empty open set or is contained in a proper closed subset ; in the first case, X is super-generic in $cl(G)$ and $X \cap G$ is supergeneric in G. Otherwise, consider $G \cap cl(G)°$.

SLOW GENERATORS

A generating subset Y of G is *fast* if, for some s, every point in G can be expressed as a product of less than s elements in $Y \cup Y^{-1}$; if not, it is *slow*.

If G is commutative, the complement of a slow generating set is vlg, with $\lambda(n) = n+2$.

Every gen. set of the group of permutations of an infinite set is fast; no countable group is known with this property.

THE INTEGERS

$X \subseteq Z$ is generic $\Leftrightarrow$ $Z = 1{+}X \cup 2{+}X \cup \ldots$ $n{+}X$ for large enough n .

X is supergeneric $\Leftrightarrow$ for each n , there is $\chi(n)$ such that any interval of length $\chi(n)$ contain n consecutive elements all in X .

X is thick $\Leftrightarrow$ for each n , X contains an interval of length n .

We say that Y is m-*spaced* if, for each n , there is only a finite nb of segments of length n containing more than m points in Y .

<u>Théorème</u> 19. ***The complement of a*** **m-*spaced set is*** **(2m+2)-*supergeneric.***

<u>Théorème</u> 20. ***For each*** **m** **,** ***there exists a*** **m-*spaced set whose complement is not*** **m-*generic.***

The set of squares is 2-spaced ; it is a fast generating set since any integer can be written $x^2 + (y+1)^2 - y^2$.

The set of powers of 2 is slow.

<u>Théorème</u> 22. ***The structure formed by the additive group of the integers and a predicate*** Π ***interpreting the*** 2^n ***is superstable with Lascar rank*** ω **.**

The additive group of the rationals equipped with Π ***is*** ω***-stable with Morley rank*** ω **.**

<u>Question</u>. ***Is a supergeneric subset of*** **(Z,+)** ***, which is the intersection with*** **Z** ***of a definable subset of an overgroup of*** **Z** ***of finite Morley rank, necessarily cofinite ?***

<u>Anxious question</u>. ***True and well-known when we embed*** **Z** ***in*** **GLn(K)** ***?***

PRÜFER GROUPS

$P = Z(p^\infty)$ is the divisible closure of Z/pZ, where p is a prime number ; it is the union of a sequence of cyclic groups $P_0 \subset P_1 \subset \ldots \subset P_n \subset \ldots$, P_n having p^n elements ; as a group, it is a strongly minimal structure.

If P is embedded in a stable group, $cl(P)$ is divisible, and therefore connected, and the traces on P of the definable sets are super-generic or cosupergeneric.

X is generic $\Leftrightarrow$ there exists n such that each coset modulo P_n intersect X.

X is supergeneric $\Leftrightarrow$ for each n there exists $\varphi(n)$ such that each coset modulo $P_{\varphi(n)}$ have a point whose full coset modulo P_n be included in X.

X is thick $\Leftrightarrow$ for each n there exists a point whose full coset modulo P_n be included in X.

X is m-*spaced* if , for each n except a finite number, there is no more than m points in each coset modulo P_n .

<u>Théorème</u> 23. *The complement of a* m-*spaced set is* (2m+2)-*supergeneric.*

Any sequence a_0 , a_1 , ... a_n , ... such that, for each n , a_n be in $P_n - P_{n-1}$, and $a_{n+1}{}^p = a_n$ forms a slow and 1-spaced generating set. Such a sequence is unique up to isomorphy.

<u>Théorème</u> 24. ***The structure formed by the Prüfer group and a predicate interpreting this sequence is ω-stable of Morley rank ω.***

<u>Remark</u>. If X is infinite and $(P\,;\times, X(x))$ is of finite RM, X is a fast generating set.

<u>Question</u>. ***Is the intersection with*** **P** ***of a definable subset of an overgroup of*** **P** ***of finite RM necessarily finite or cofinite ?***

This is true and when we embed P in GLn(K) (Gramain-Poizat).

GROUPES LIBRES

L'article se conclut par quelques observations préselayennes sur les groupes libres, et les groupes libres commutatifs, à une infinité de générateurs, dans le langage des groupes augmenté, pour chaque n, d'un symbole $I_n(x_1, \ldots x_n)$ interprétant l'orbite des n-uplets de la suite génératrice.

Il demande si ce sont les seuls modèles positivement existentiellement clos de leur théorie hom-inductive dans ce langage, et demande aussi comment on doit définir généricité et supergénéricité dans un groupe en Logique positive.

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