Generically n-transitive permutation groups

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Universität Münster

Models and Groups, İstanbul 1
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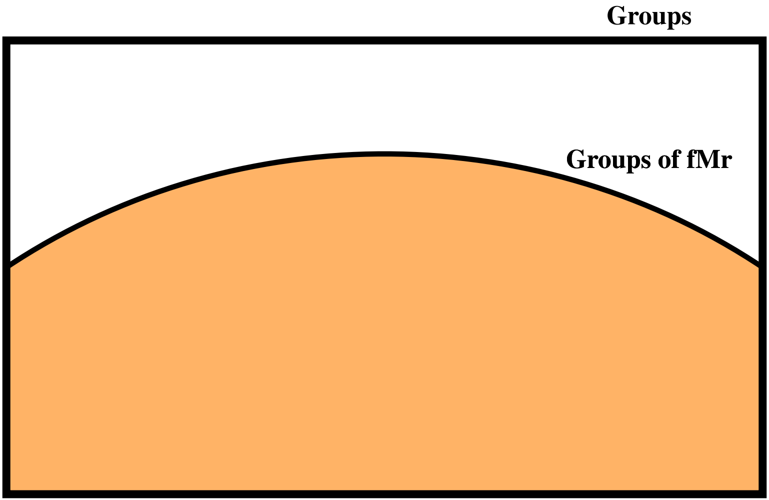
Groups of finite Morley rank (fMr)



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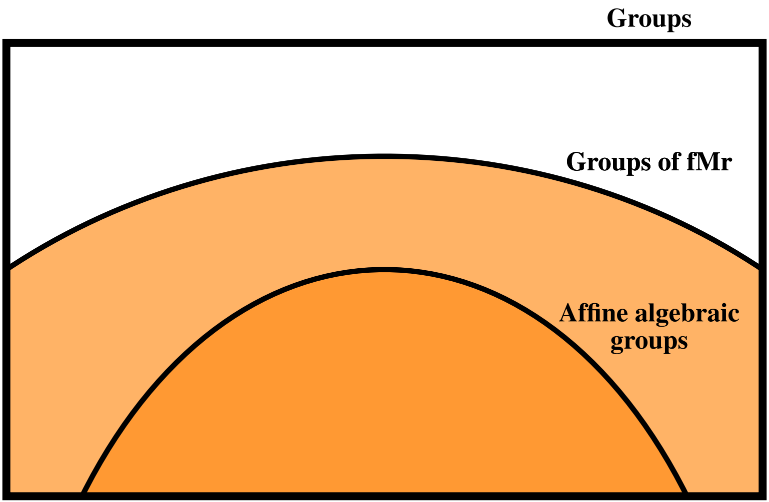

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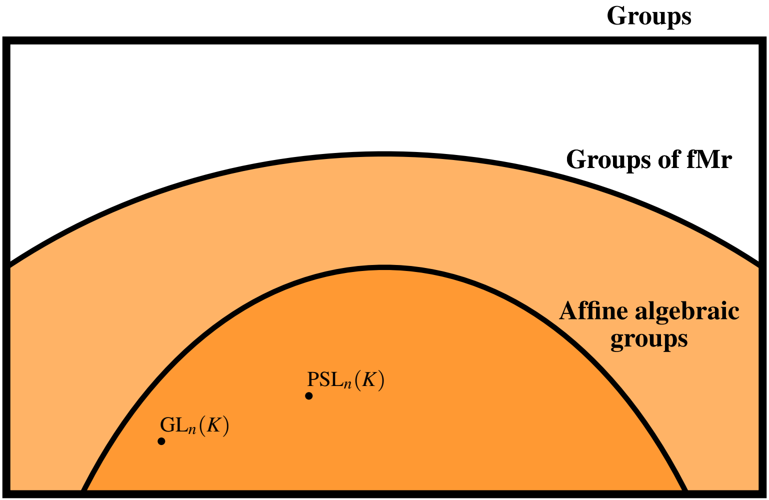

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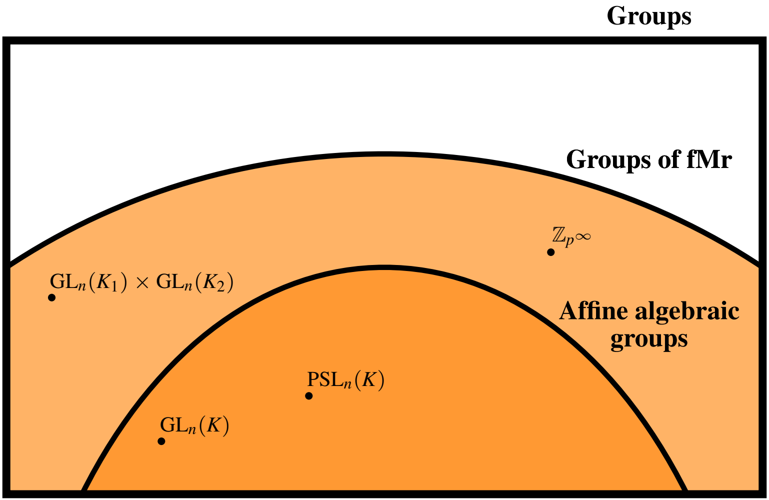

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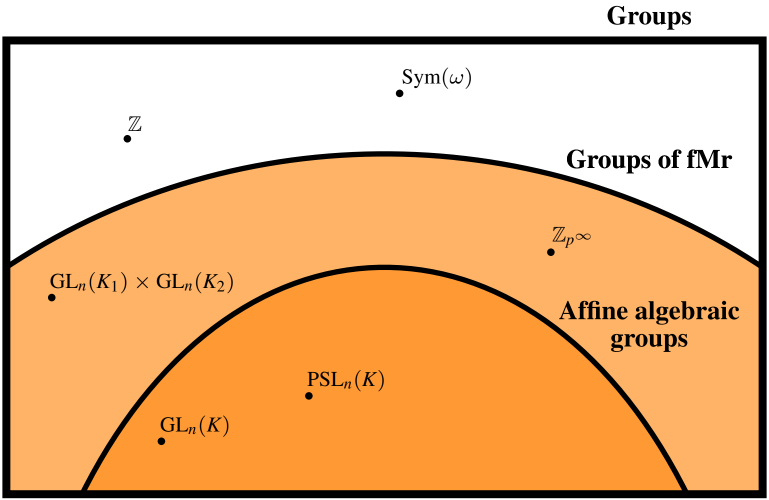

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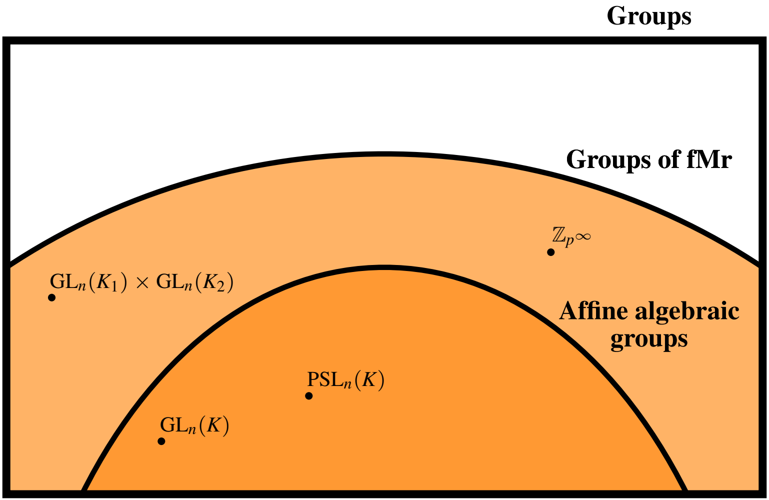

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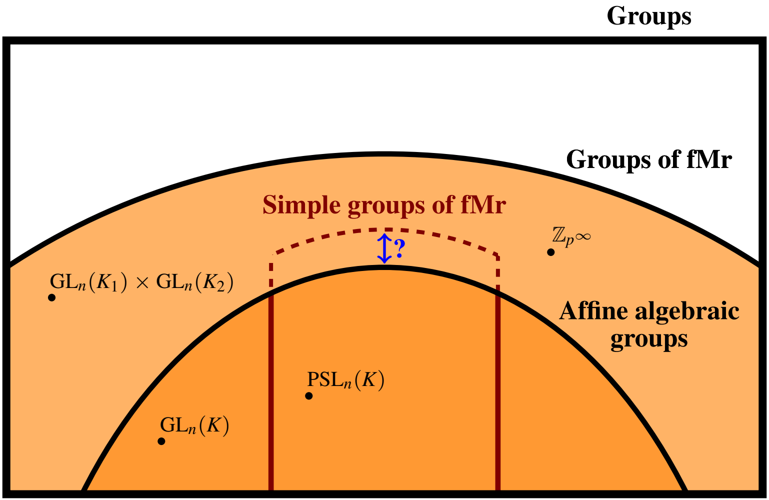

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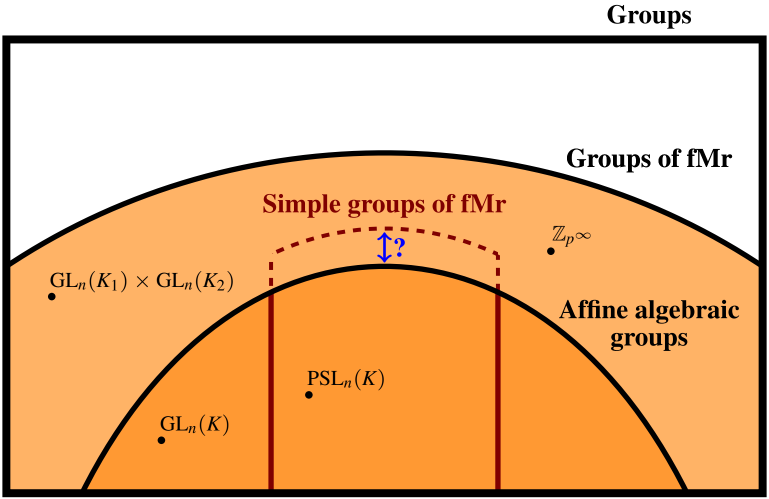

Algebraicity Conjecture:

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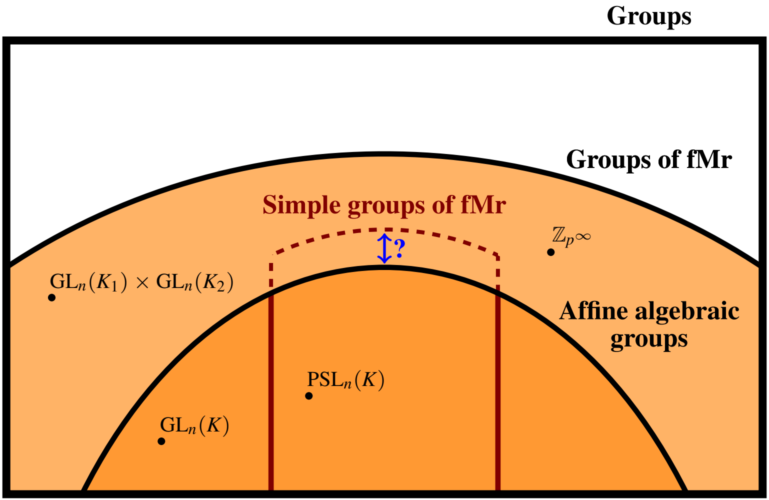

Algebraicity Conjecture: the gap, $\updownarrow$, does not exist.

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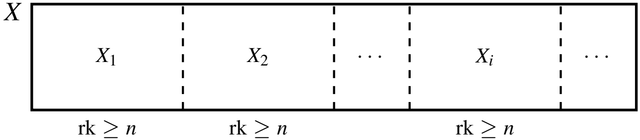

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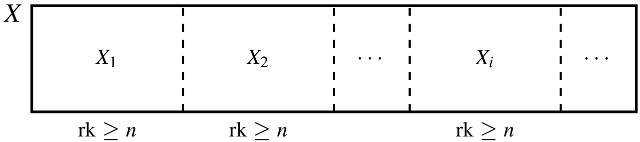

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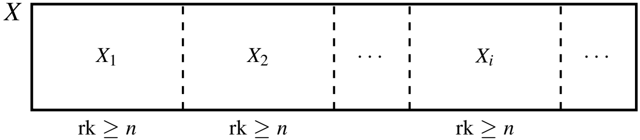

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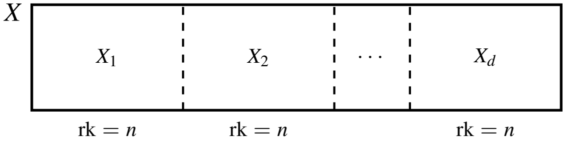

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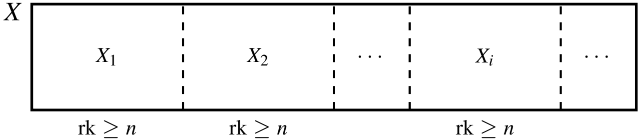

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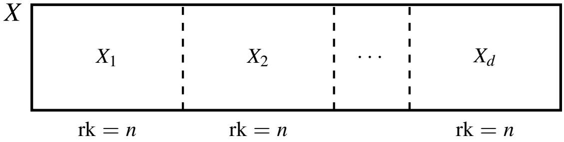

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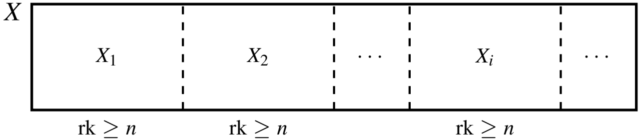

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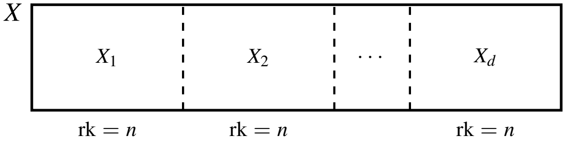

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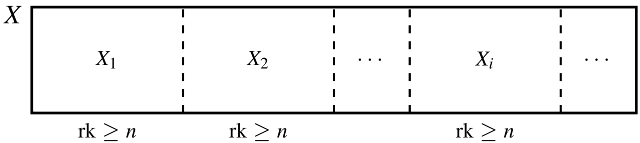

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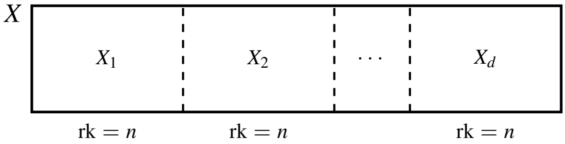

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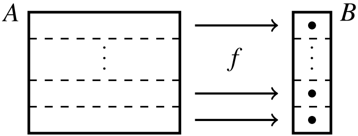

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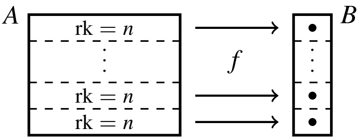

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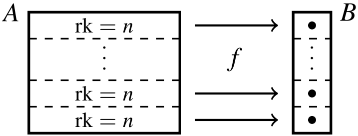

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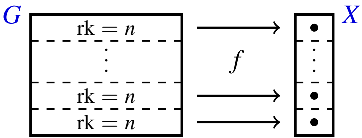

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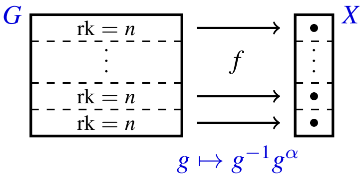

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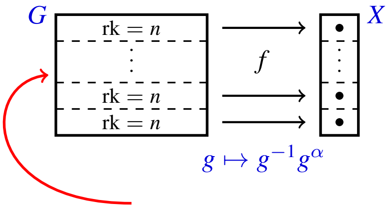

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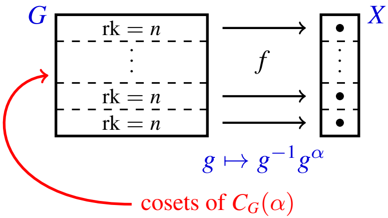

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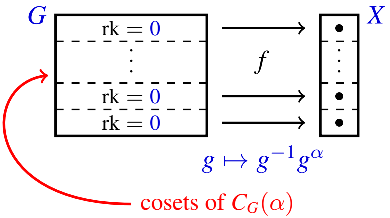

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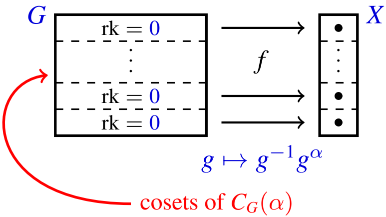

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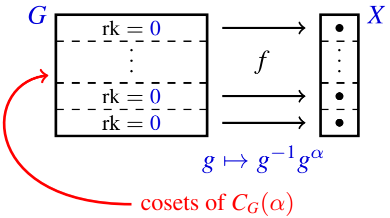

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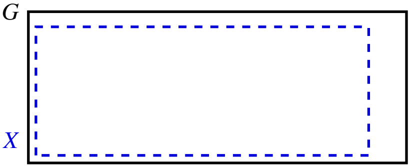

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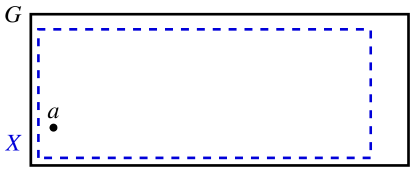

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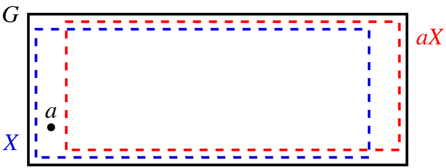

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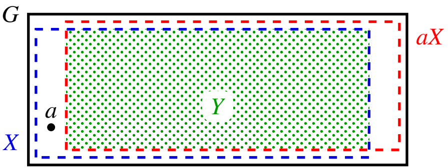

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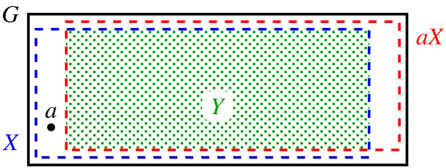

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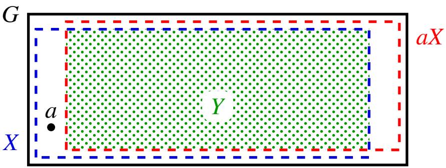

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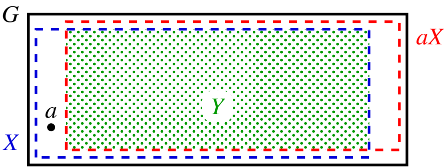

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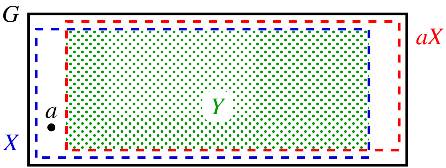

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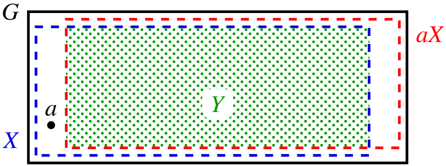

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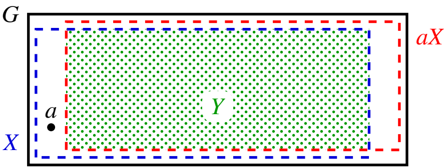

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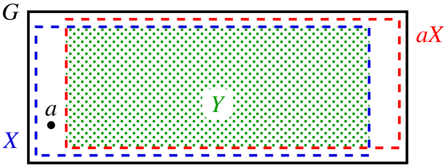

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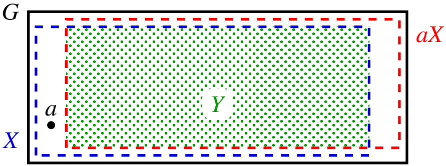

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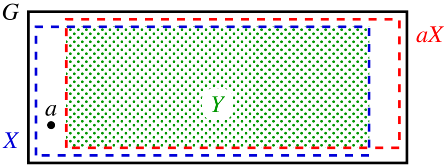

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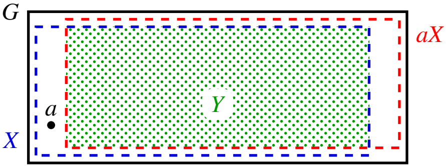

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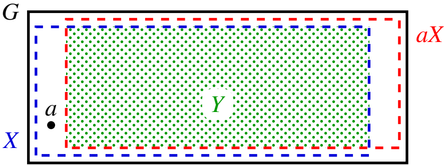

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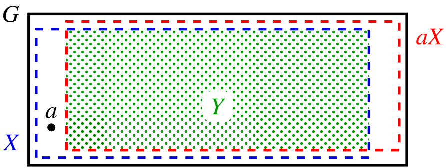

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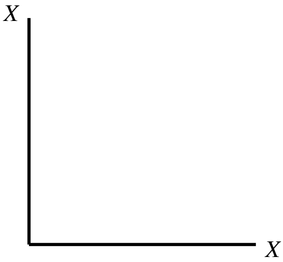

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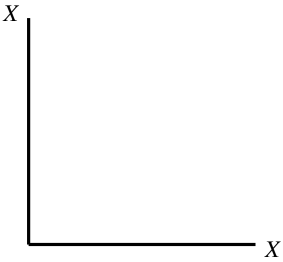

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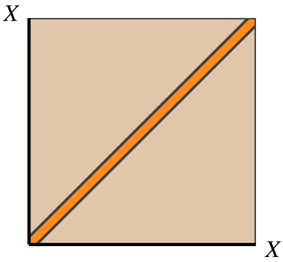

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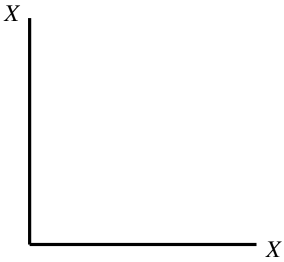

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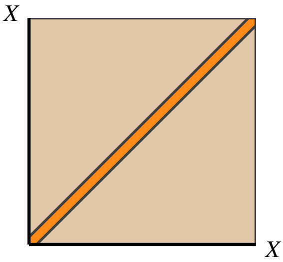

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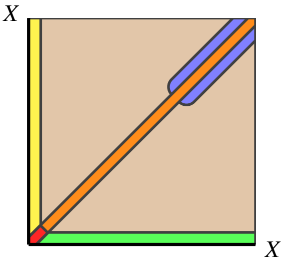

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- $G_1 \times G_2 \curvearrowright X_1 \times X_2$ is generically t-transitive
- $\mathcal{O}$ is (more-or-less) $\mathcal{O}_1 \times \mathcal{O}_2$

Generic t-transitivity

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Example: $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$

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Theorem

1. *(Popov '07) Let G be an infinite simple algebraic group over an alg. closed field of characteristic* 0*. Then* $\mathrm{gtd}(G)$ *is given by*

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$\boldsymbol{A}_n$	$\boldsymbol{B}_n, n \geq 3$	$\boldsymbol{C}_n, n \geq 2$	$\boldsymbol{D}_n, n \geq 4$	$\boldsymbol{E}_6$	$\boldsymbol{E}_7$	$\boldsymbol{E}_8$	$\boldsymbol{F}_4$	$\boldsymbol{G}_2$
$n+2$	3	3	3	4	3	2	2	2

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2. *Let G be an infinite solvable group of fMr. Then $\mathrm{gtd}(G) \leq 2$.*

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Problem (BC '08)

Show that the above table is valid in arbitrary characteristic.

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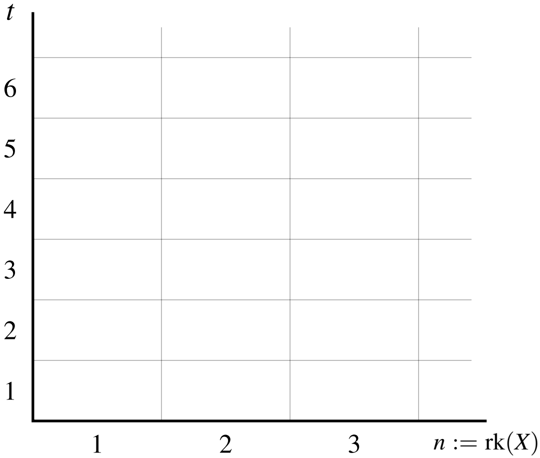

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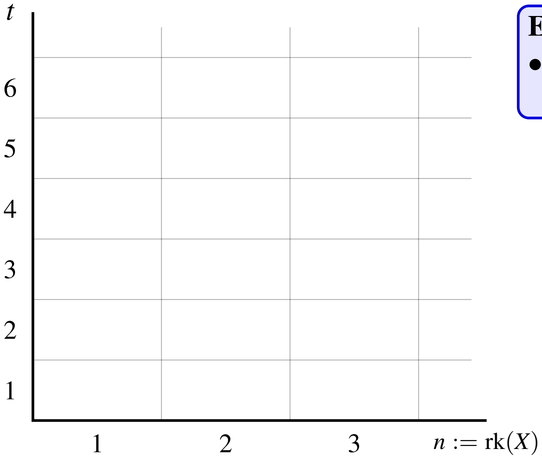

Extra Assumptions

- $G \curvearrowright X$ is transitive

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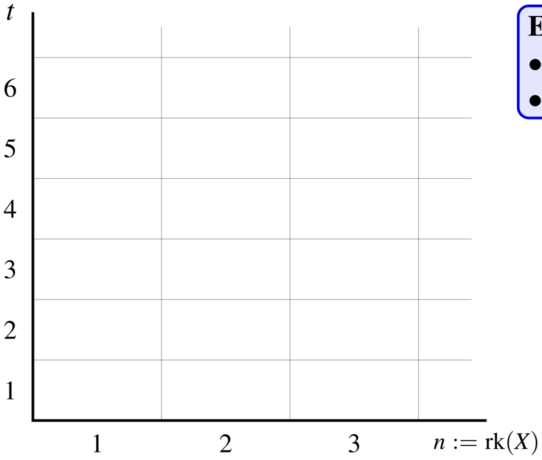

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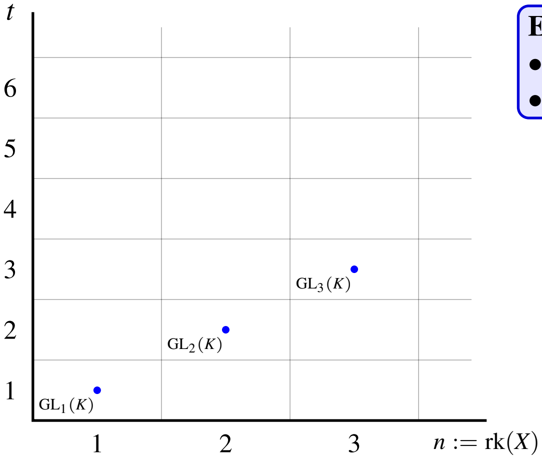

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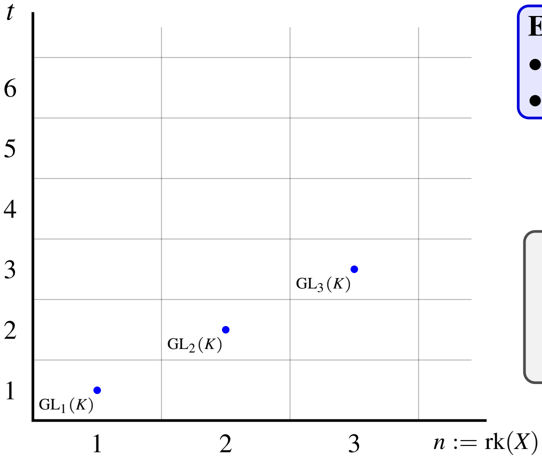

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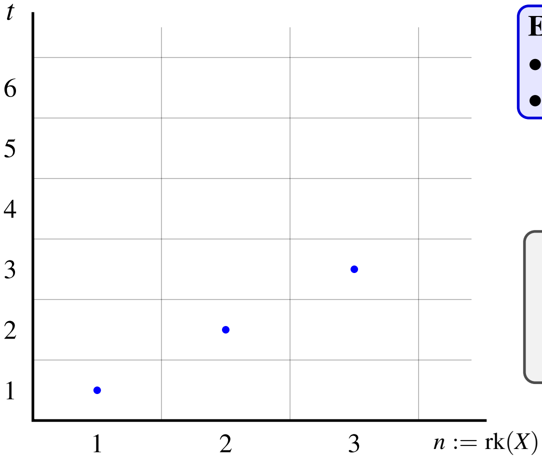

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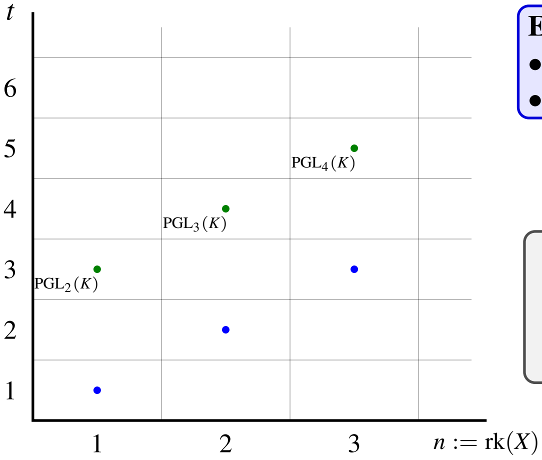

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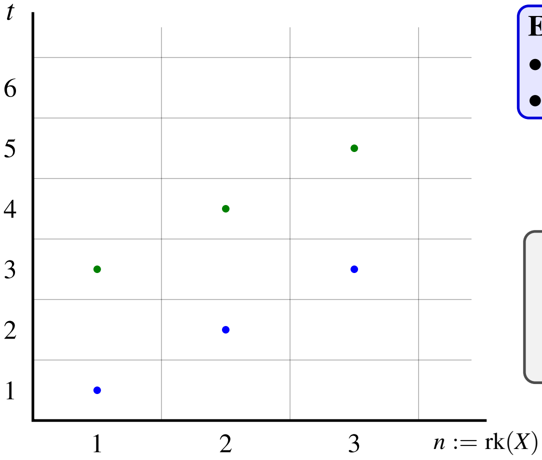

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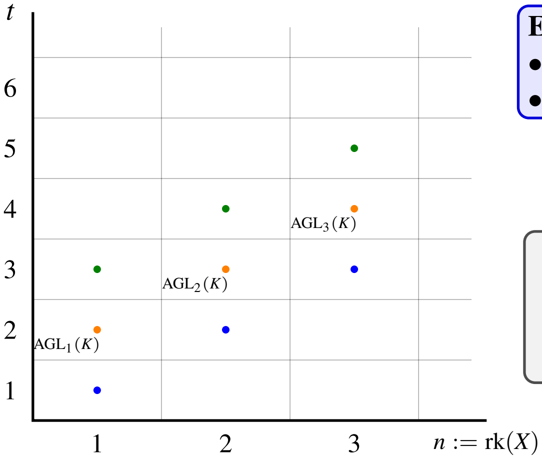

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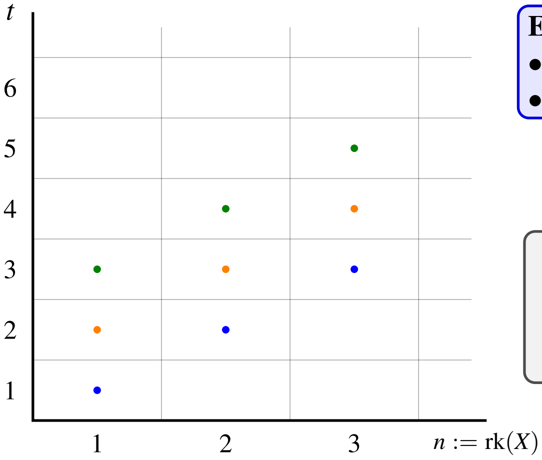

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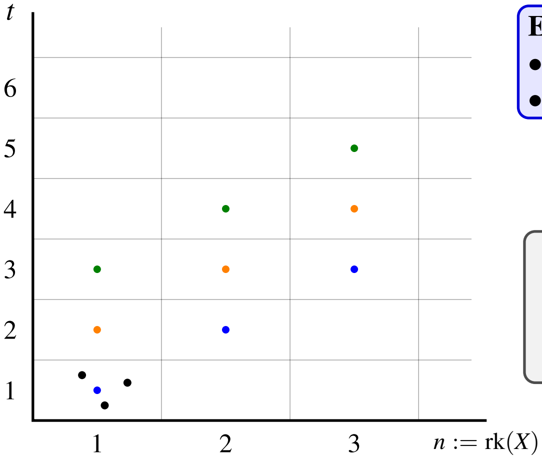

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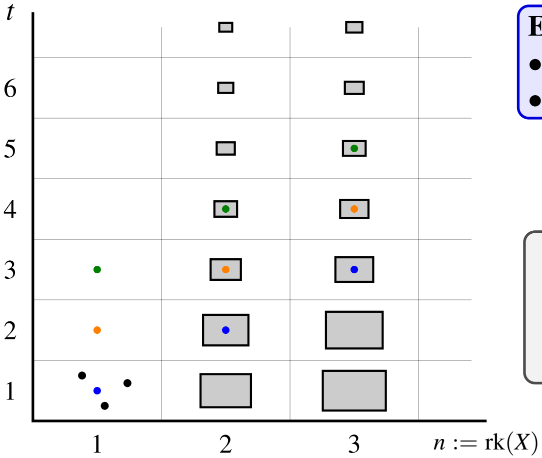

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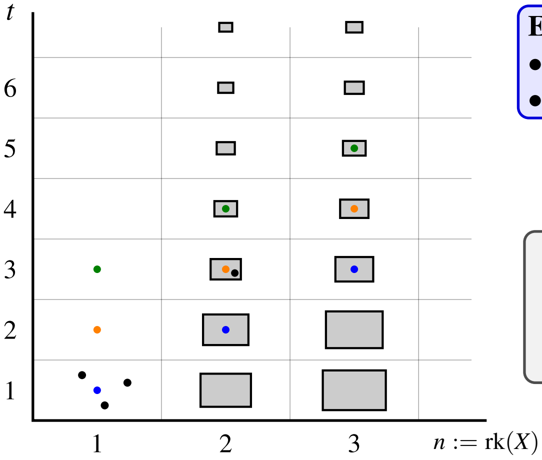

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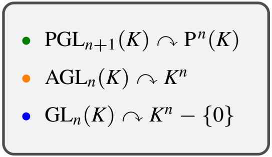

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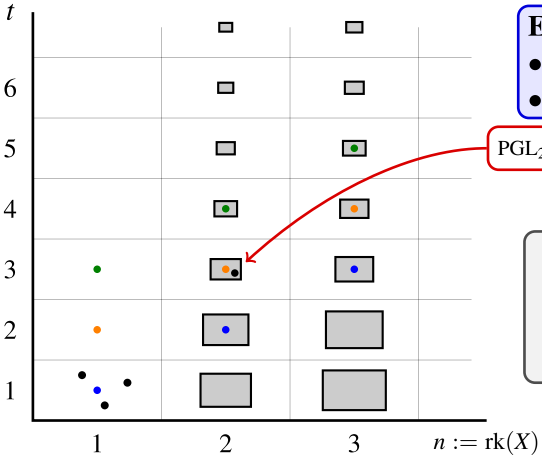

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$\mathrm{PGL}_2(K) \times \mathrm{PGL}_2(L) \curvearrowright \mathrm{P}^1(K) \times \mathrm{P}^1(L)$

- $\mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$
- $\mathrm{AGL}_n(K) \curvearrowright K^n$
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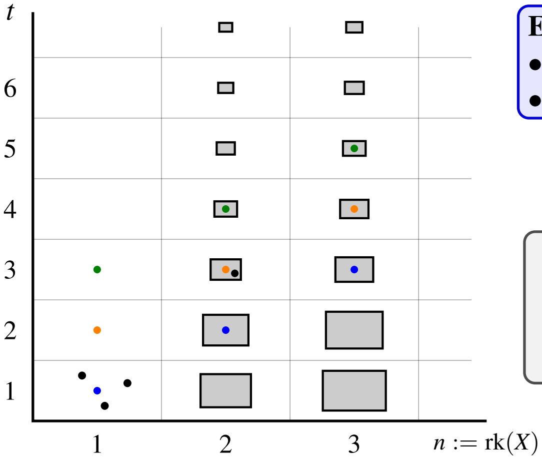
$G \curvearrowright X$ is generically t-transitive



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- $G \curvearrowright X$ is transitive
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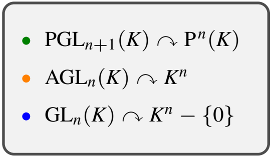

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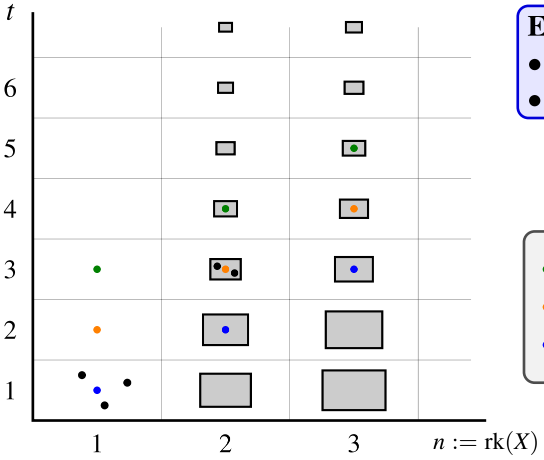

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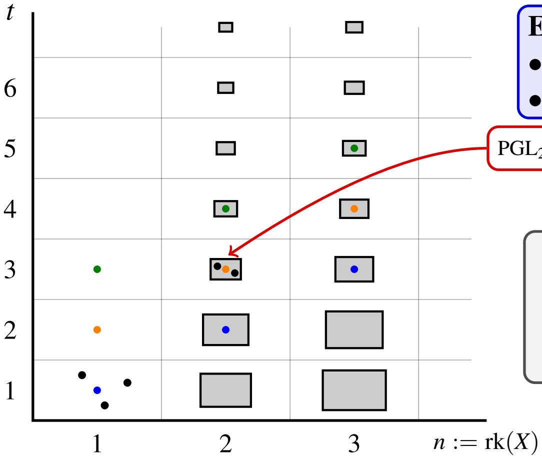

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$\mathrm{PGL}_2(L) \curvearrowright \mathrm{P}^1(L)$ with $\mathrm{rk}(L) = 2$!

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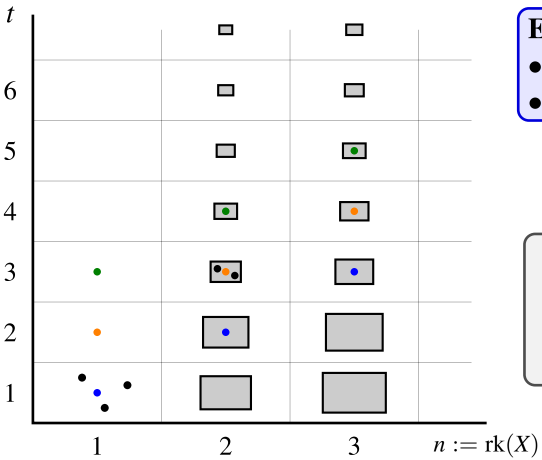

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The Problem (BC ’08)

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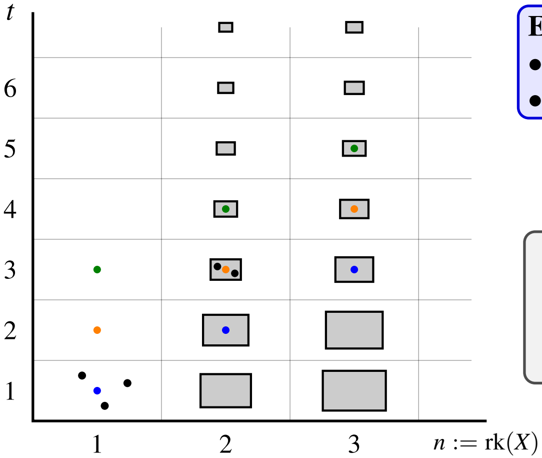

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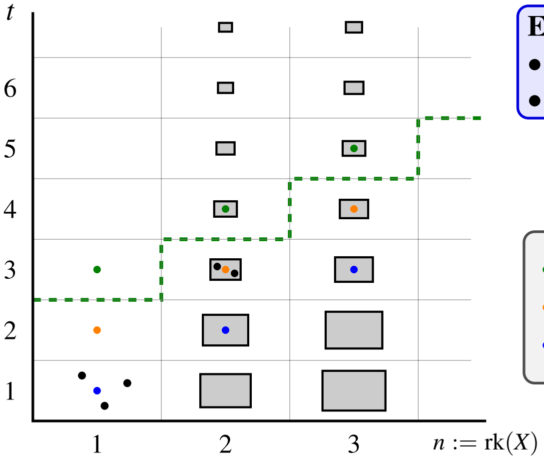

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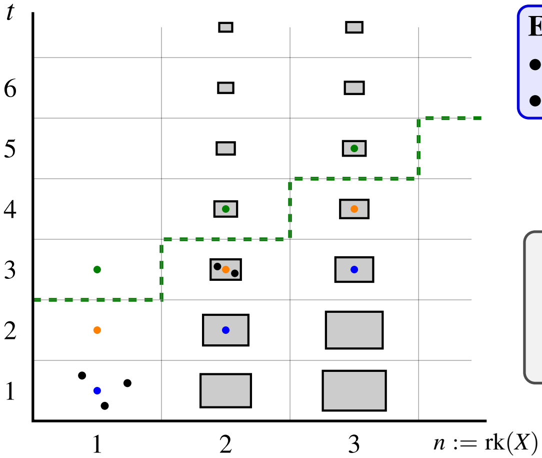

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Show that $t \geq n+2 \implies G \curvearrowright X \cong \mathrm{PGL}_{n+1}(K) \curvearrowright \mathrm{P}^n(K)$

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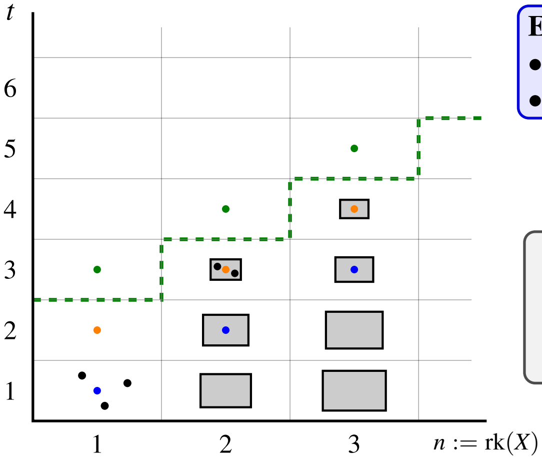

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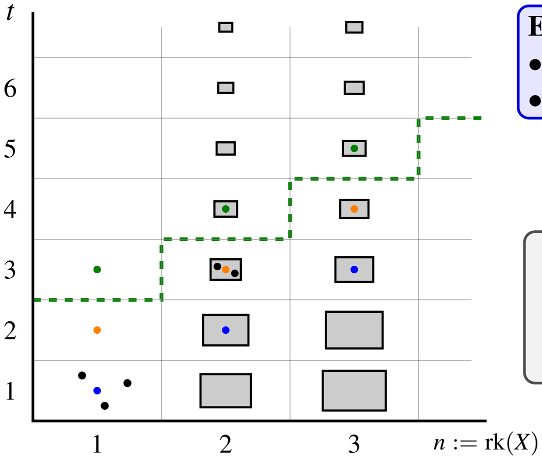

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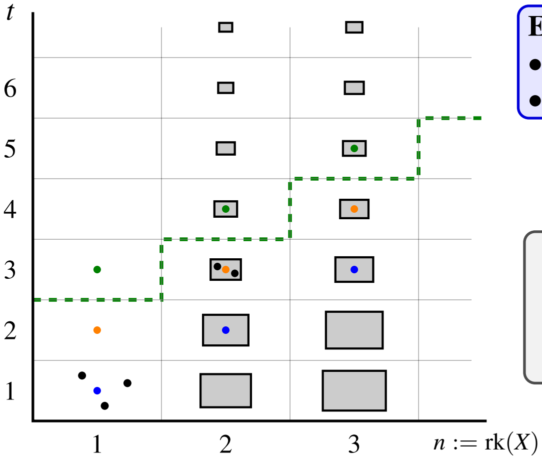

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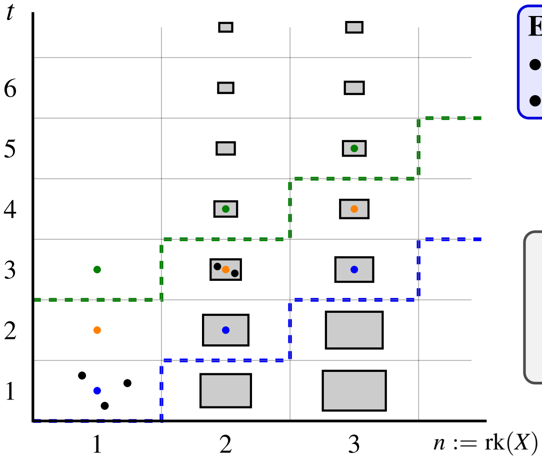

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The Rank Two Problem: bounding t

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Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\mathrm{rk}(X) = 2$. Show that $t \leq 4$.

- Assume $t = 5$;

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- Assume $t = 5$; let $(1, 2, 3, 4, 5) \in \mathcal{O} \subset X^5$
- The orbit of $G_{1,2,3,4}$ on 5 has rank 2 and degree 1

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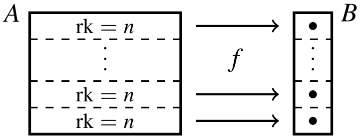
scratch



$$\Longrightarrow \qquad \mathrm{rk}\, A = \mathrm{rk}\, B + n$$

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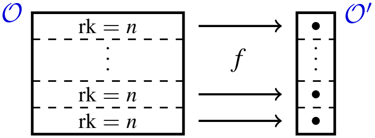

$(x_1, x_2, x_3, x_4, x_5) \mapsto x_1$

$\Longrightarrow \quad \mathrm{rk}\, \mathcal{O} \;=\; \mathrm{rk}\, \mathcal{O}' + n$

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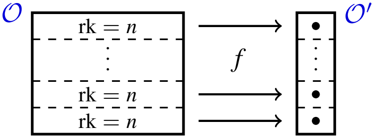

$\Longrightarrow \quad \mathrm{rk}\, X^5 = \mathrm{rk}\, \mathcal{O}' + n$

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The Rank Two Problem: bounding t

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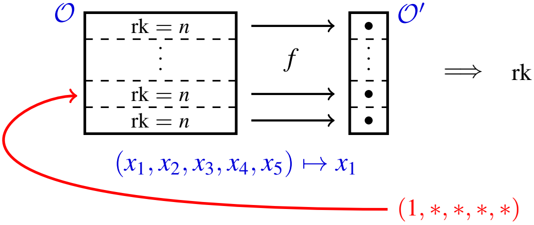

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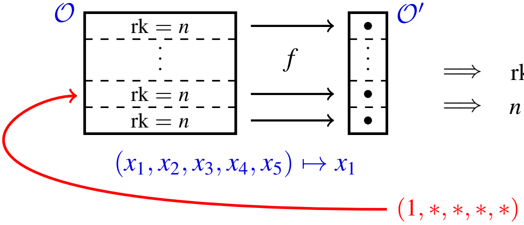

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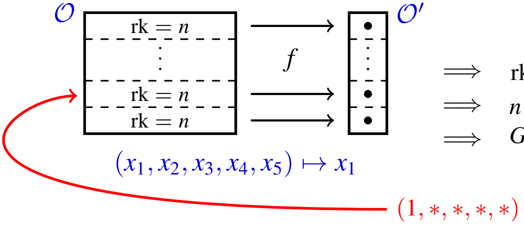

$\implies \mathrm{rk}\, X^5 = \mathrm{rk}\, \mathcal{O}' + n$

$\implies n = \mathrm{rk}\, X^4$

$\implies G_1 \curvearrowright X$ is gen. 4-transitive

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Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\operatorname{rk}(X) = 2$. Show that $t \leq 4$.

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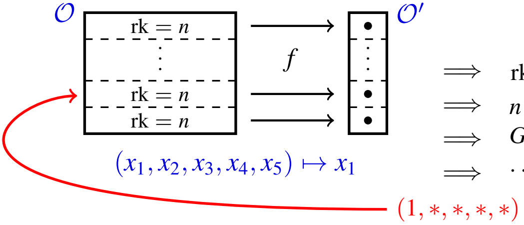
scratch



$$\Longrightarrow \quad \operatorname{rk} X^5 = \operatorname{rk} \mathcal{O}' + n$$
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$$\Longrightarrow \quad \cdots \Longrightarrow G_{1,2,3,4} \curvearrowright X \text{ is gen. 1-transitive}$$

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$$\{g \in G : \ g \text{ normalizes } \{1, 2, 3, 4, 5\}\}$$

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The Rank Two Problem: bounding t

Rank Two Problem (Sharp version)

Let $G = G^\circ$. Suppose $G \curvearrowright X$ is transitive and generically sharply t-transitive with $\mathrm{rk}(X) = 2$. Show that $t \leq 4$.

- Assume $t = 5$; let $(1, 2, 3, 4, 5) \in \mathcal{O} \subset X^5$
- The orbit of $G_{1,2,3,4}$ on 5 has rank 2 and degree 1

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Fact

If S_n acts definably and faithfully on a connected solvable group of rank r, then $n \leq r + 1$ unless $n = 6$ and $r = 4$.

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The Rank Two Problem: building $(\mathcal{P}, \mathcal{L})$

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Want to build a projective plane.

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Want to build a projective plane. Set $\mathcal{P} := X$.

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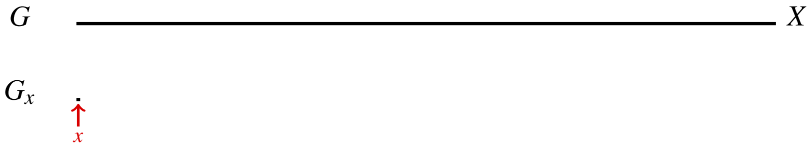

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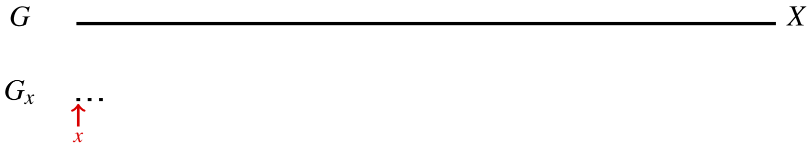

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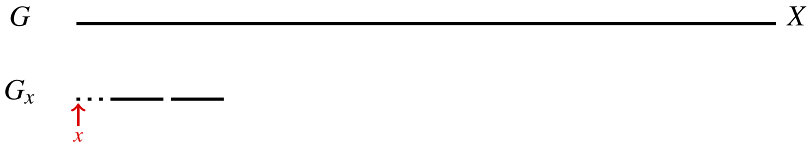

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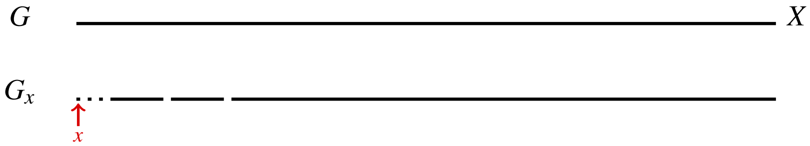

The Rank Two Problem: building $(\mathcal{P}, \mathcal{L})$

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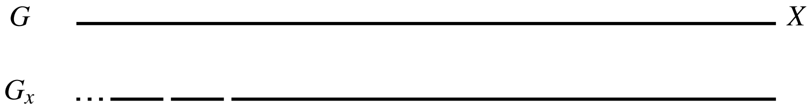

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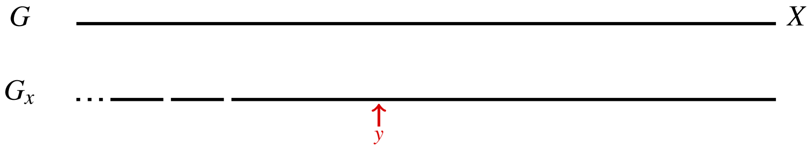

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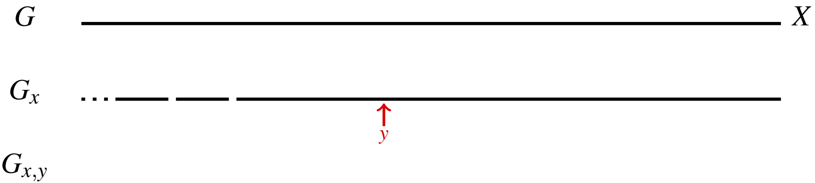

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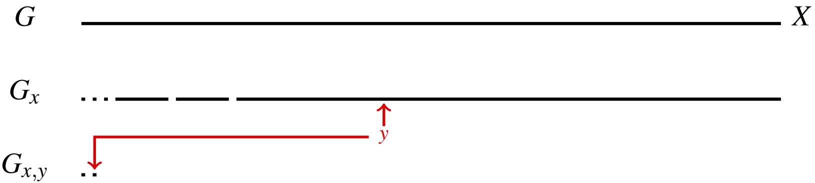

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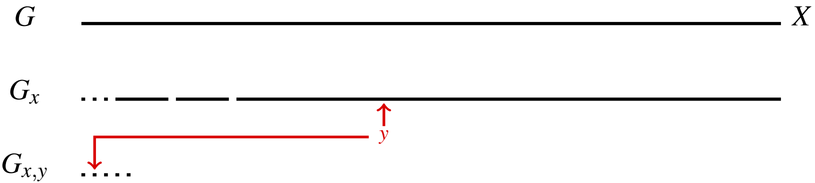

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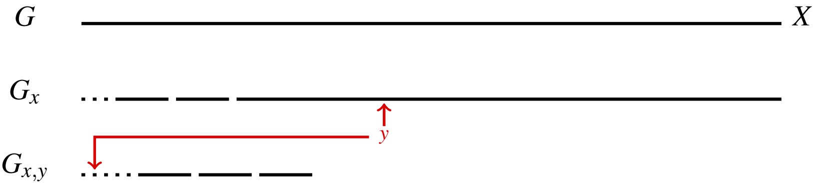

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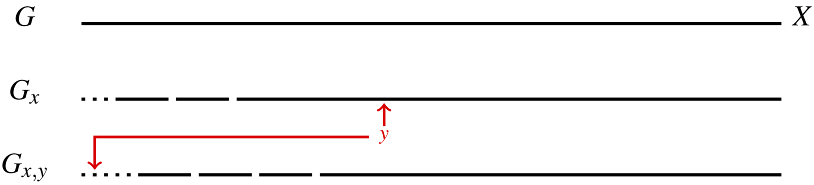

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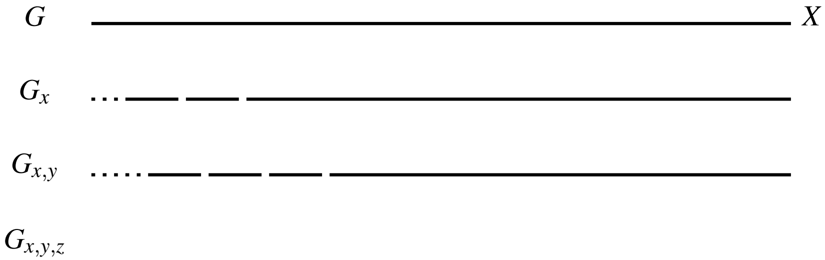

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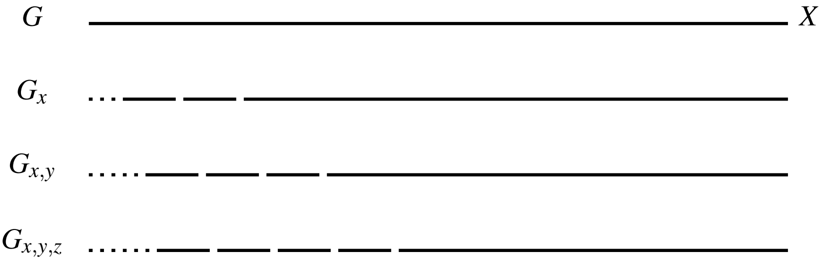

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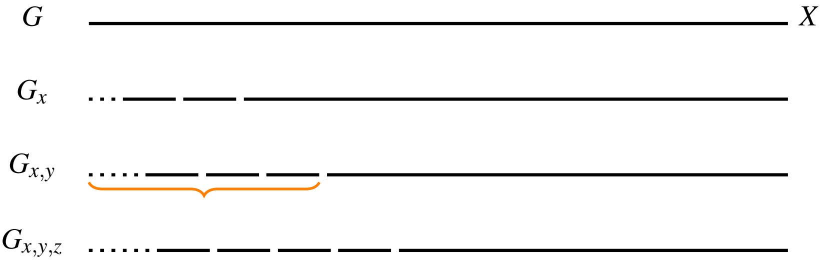

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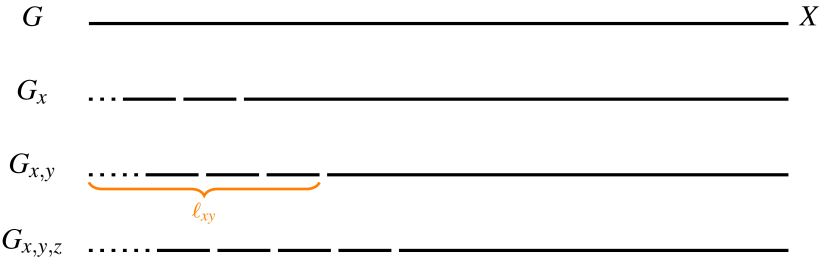

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Want to build a projective plane. Set $\mathcal{P} := X$. How should we define $\mathcal{L}$?

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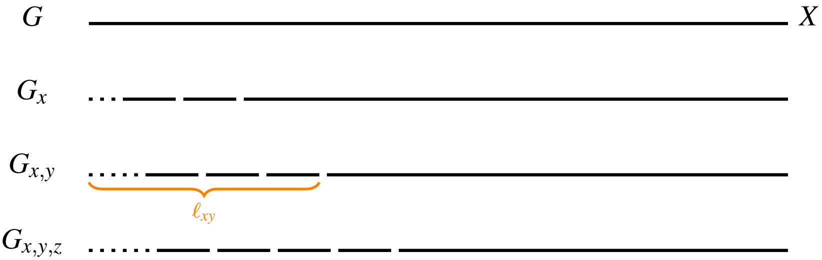

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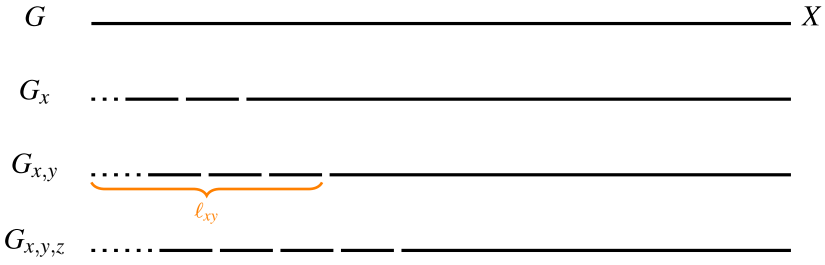

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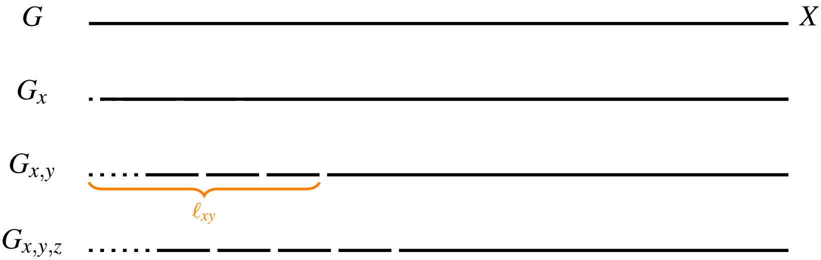

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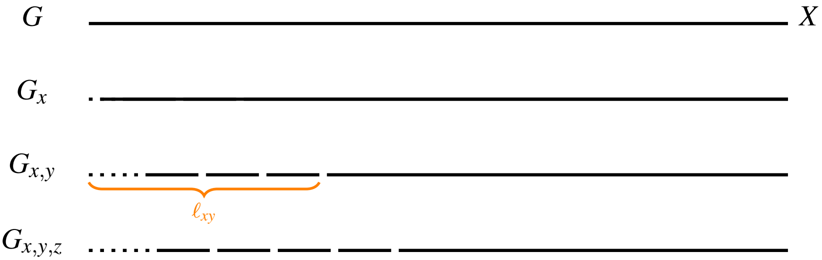

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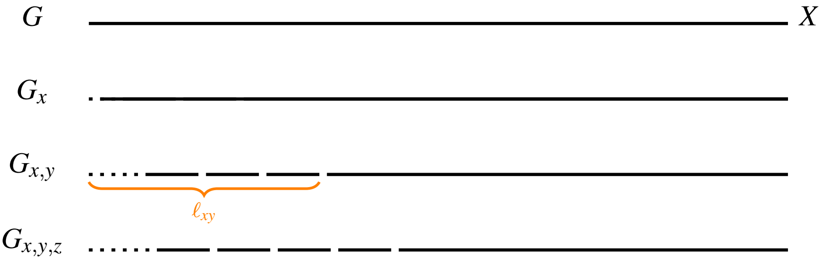

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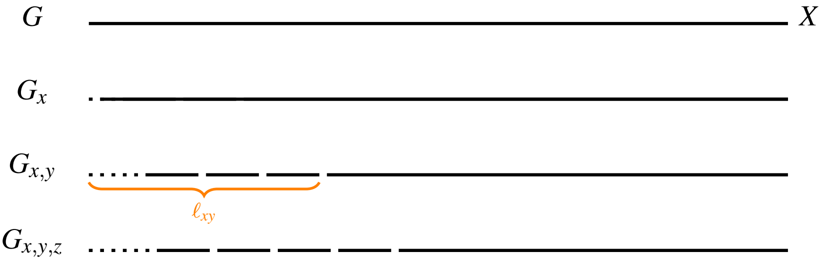

The Rank Two Problem: properties of $(\mathcal{P}, \mathcal{L})$

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The Rank Two Problem: properties of $(\mathcal{P}, \mathcal{L})$

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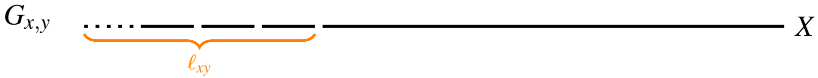

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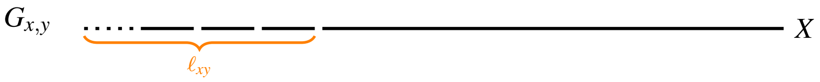

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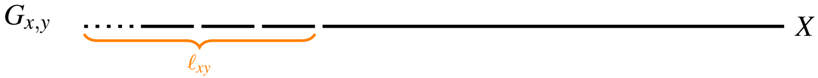

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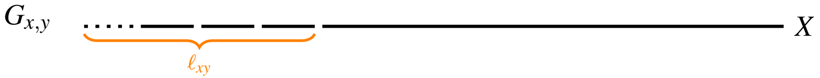

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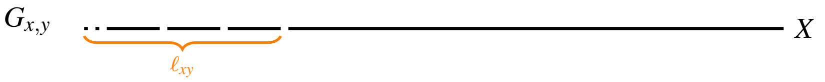

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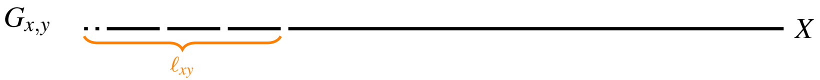

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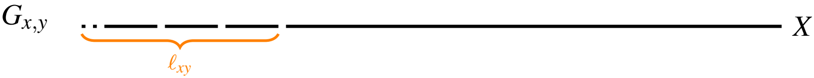

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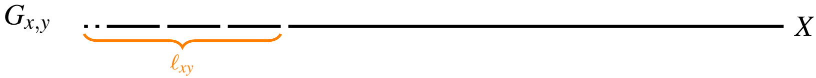

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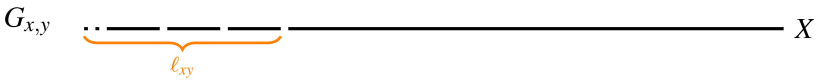

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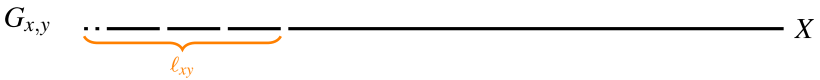

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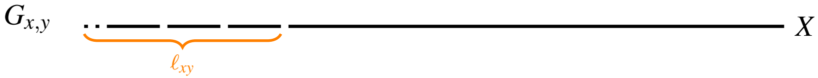

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2. $G_{1,2} \curvearrowright X$ is gen. sharply 2-transitive, so $G_{1,2}$ is connected of rank 4.

Fact

Let G be a connected group of fMr.

1. $\mathrm{rk}\, G = 1 \implies$ *G is abelian*
2. $\mathrm{rk}\, G = 2 \implies$ *G is solvable*
3. $\mathrm{rk}\, G = 3$ *and G simple* $\implies$ *G is* $\mathrm{PSL}_2(K)$ *or a simple bad group*
4. $\mathrm{rk}\, G = 4$ *and G simple* $\implies$ *G is a simple bad group*

- Gen. sharply 2-transitive groups have elements of order 2
- Bad groups do not; they have no definable automorphism of order 2

<u>Thank You</u>

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PERMUTATION GROUPS OF FINITE MORLEY RANK

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Introduction

Groups of finite Morley rank made their first appearance in model theory as *binding groups*, which are the key ingredient in Zilber's ladder theorem and in Poizat's explanation of the Picard-Vessiot theory. These are not just groups, but in fact permutation groups acting on important definable sets. When they are finite, they are connected with the model theoretic notion of algebraic closure. But the more interesting ones tend to be infinite, and connected.

Many problems in finite permutation group theory became tractable only after the classification of the finite simple groups. The theory of permutation groups of finite Morley rank is not very highly developed, and while we do not have anything like a full classification of the simple groups of finite Morley rank in hand, as a result of recent progress we do have some useful classification results as well as some useful structural information that can be obtained without going through an explicit classification. So it seems like a good time to review the situation in the theory of permutation groups of finite Morley rank and to lay out some natural problems and their possible connections with the body of research that has grown