Chapter 9. Approximation Theory in Homogeneous Banach Spaces

9.1 Background. At various points in the preceding lectures we have encountered (without, however, attempting a systematic discussion) particular Banach spaces of functions defined on spaces which admit a topological group of homeomorphisms on themselves (translations of R^n, rotations of spheres). We wish now to discuss this situation systematically in the context of R^n, and develop a framework within which we can quite economically formulate and prove "direct" and "inverse" theorems concerning approximation by convolution integrals, and by trigonometric and algebraic polynomials. The treatment we shall give is very "concentrated", as we avail ourselves simultaneously of two quite independent sources of unification. The first is the "Tauberian" viewpoint which we have expounded in Chapter 5 of our earlier book ($SHAPIRO_1$), endowed with considerable technical improvements from BOMAN & $SHAPIRO_2$. The second is the point of view of "homogeneous Banach spaces" (so christened in KATZNELSON) whereby approximation in a variety of norms is treated at one stroke (see 9.2, 9.3 for details). This very fruitful viewpoint seems to originate in the writings of Bochner; one can trace the beginnings already in his definition of almost-periodicity, and applications to approximation theory may be found in $BOCHNER_1$, Chapter 1. For the fullest exploitation of the underlying idea one requires the integration of vector-valued functions (a theory to which, once again, Bochner made pioneering contributions). We shall take for granted the elements of vector-valued integration (see e.g. HILLE & PHILLIPS).

Katznelson later developed the Bochner idea systematically, greatly extending its scope. Also, writings of Stechkin and some other Soviet analysts show a clear appreciation of the usefulness of these "homogeneous Banach spaces" as a unifying tool in approximation theory (cf. $STECHKIN_2$ and KUPTSOV).

We shall operate in spaces of functions defined on R^n, and handle simultaneously (i) all L^p norms, $1 \leq p \leq \infty$, as well as (ii) spaces both of periodic and non-periodic functions. So far as we know no treatment embodying feature (ii) has appeared in print, although such a treatment has the practical merit of eliminating a major source of duplication in approximation theory literature. Concerning (i), it

should be remarked that the simultaneous treatment of all L^p norms withir atic framework, with its attendant advantages, also encounters natural li problems of more delicate character, subtle differences between the vari make themselves felt; on this point, see BOMAN & $SHAPIRO_{1,2}$. In these le however, restrict ourselves to the coarser theory wherein these subtleti taken into account.

Finally we remark that some portion, at least, of the theory deve can be carried out more generally, for spaces of functions defined on l. or on homogeneous spaces. However, here we shall consider as carrier spa (and by implication the quotient space T^n thereof) in view of the great the results attainable, and the importance of these cases in application of procedure is different in some details from that of Bochner or that o in the works cited.

It is possible to give a relatively abstract, or a relatively con tion space") definition of the class of spaces with which we shall deal. formulation is given in 9.2 and the more concrete formulation in 9.3.

9.2 Abstract homogeneous Banach spaces

9.2.1 Definition of an AHBS. Concerning functions and measures on R^n, Fourier transforms, we shall maintain the nomenclature of Chapter 7.

Let B denote a (for the time being, "abstract") Banach space, the which shall be denoted by $\| \|_B$, or simply by $\| \|$ when there is no risk c We suppose that for each $u \in R^n$ there is defined a linear operator T_u or the map $u \rightarrow T_u$ is a homomorphism of the additive group R^n, i.e. T_o is t and $T_{u+v} = T_u T_v$. We shall impose three further assumptions, listed belc (H1), (H2), (H3).

(H1) T_u is an isometry of B, i.e. $\|T_u f\| = \|f\|$, all $f \in B$.

(H2) For each $f \in B$, the function Φ_f from R^n to B defined by

$$\Phi_f(u) = T_u f \qquad (1)$$

is continuous.

Let now $\sigma \in M = M(R^n)$ (bounded complex measure on R^n), and $f \in B$. The element $f * \sigma$ of B is defined to be

(2) $$f * \sigma = \int \Phi_f(u)\, d\sigma(u) ,$$

the integral being understood as a Bochner-Stieltjes integral of the (continuous, bounded) B-valued function Φ_f. Observe that $f * (\sigma + \tau) = (f * \sigma) + (f * \tau)$, and a similar distributive law holds with respect to f. From basic properties of the integral we have also

(3) $$\|f * \sigma\|_B \leqslant \|\sigma\|_M \, \|f\|_B .$$

Observe that for every $k \in L^1 = L^1(R^n)$, since k is naturally identified as an element of M, we have a natural interpretation of $f * k$, for $f \in B$.

We now impose our final axiom:

(H3) For every pair σ, $\tau \in M$ and every $f \in B$,

(4) $$f * (\sigma * \tau) = (f * \sigma) * \tau .$$

Of course, in (4), the asterisk between σ and τ stands for convolution in M, whereas the remaining asterisks stand for the operation defined by (2).

9.2.1.1 Definition. An abstract homogeneous Banach space (abbreviated AHBS) on R^n is a Banach space B, together with a *group* $\{T_u\}$ of linear operators on B, such that (H1), (H2) and (H3) are satisfied.

Remark. Later we will want to consider a more particular situation, whereby B is a space of measurable functions on R^n and $\{T_u\}$ is the set of translations; however, we prefer to develop as much as we can of the necessary machinery in this abstract framework. The prototypical examples to keep in mind are $B = L^p(R^n)$ or $B = L^p(T^n)$, $\leqslant p < \infty$, with $(T_u f)(t) = f(t - u)$. In these cases (H2) is the well-known "continuity of translations" property. (H2) is not satisfied in $L^\infty(R^n)$, but it is in the closed subspace $\dot{C}(R^n)$ thereof, consisting of all bounded uniformly continuous functions on R^n. The validity of (H3) requires some discussion; it is a consequence of .3.2.3, together with the fact that the spaces just enumerated satisfy the axioms

for an HBS (see 9.3).

9.2.2 Dilations of measures. Let $C_o = C_o(R^n)$ denote the set of continuous functions on R^n which vanish at infinity. If $\sigma \in M$ and $a > 0$, the map $\varphi \to \int \varphi(a$ [illegible] a continuous linear functional on C_o, and hence the integral equals $\int \varphi(u)\, d$[illegible] for some element $\sigma_{(a)}$ of M. This measure $\sigma_{(a)}$, we call the a-dilation of σ. [illegible] function $k \in L^1(R^n)$, we write $k_{(a)}$ to denote the function defined by

$$k_{(a)}(t) = a^{-n}\, k(a^{-1} t) .$$

This is consistent with the preceding notation if k is interpreted in the na[illegible] way as an element of M.

We leave as exercises the proofs of the relations

(1) $$\|\sigma_{(a)}\|_M = \|\sigma\|_M , \quad a > 0$$

(2) $$(\sigma_{(a)})_{(b)} = \sigma_{(ab)} ,$$

(3) $$(\sigma * \tau)_{(a)} = \sigma_{(a)} * \tau_{(a)} .$$

Although the relation

(4) $$\int \varphi(u)\, d\sigma_{(a)}(u) = \int \varphi(au)\, d\sigma(u)$$

holds a priori only for $\varphi \in C_o$, it is easily shown that it remains valid al[illegible] all bounded continuous functions φ on R^n, in particular for the functions φ [illegible] $t \to e^{ixt}$, for each $x \in \hat{R}^n$; therefore we have

(5) $$\hat{\sigma}_{(a)}(x) = \hat{\sigma}(ax) .$$

Moreover, we leave to the reader the verification that (4) holds even when [illegible] bounded continuous B-valued function.

9.2.2.1 Lemma. Let B be an AHBS, and suppose $\sigma \in M$ and $\int d\sigma = 0$. Then

(1) $$\lim_{a \to 0} \|f * \sigma_{(a)}\| = 0 , \quad \text{all } f \in B .$$

Proof. We have

$$f * \sigma_{(a)} = \int \Phi_f(u)\, d\sigma_{(a)}(u) = \int \Phi_f(au)\, d\sigma(u) = \int [\Phi_f(au) - \Phi_f(0)]\, d\sigma(u) .$$

Hence, denoting by $\tilde{\sigma}$ the total variation of σ,

(2) $$\|f * \sigma_{(a)}\| \leqslant \int \|T_{au} f - f\|\, d\tilde{\sigma}(u)$$

and since the (real-valued) continuous functions Ψ_a,

$$\Psi_a(u) \underset{\text{def.}}{=} \|T_{au} f - f\| ,$$

are uniformly bounded on R^n and satisfy $\lim_{a \to 0} \Psi_a(u) = 0$ for each $u \in R^n$, (1) follows from (2) by dominated convergence. ◇

An immediate corollary is

9.2.2.2. Lemma. Let B be an AHBS, and suppose $k \in L^1$ and $\int k\, dt = 1$. Then,

$$\lim_{a \to 0} \|f - (f * k_{(a)})\| = 0 .$$

Proof. Take for σ in 9.2.2.1 the measure $\delta - k\, dt$, δ = Dirac measure. ◇

9.2.3 Modulus of continuity

9.2.3.1 Definition. Let B be an AHBS and $f \in B$. The modulus of continuity of f is the function

$$\omega(a) = \omega(f;\, a) = \sup_{|u| \leqslant a} \|T_u f - f\| ; \quad a \geqslant 0 .$$

"Modulus of continuity" will be abbreviated m.o.c. Later we shall sometimes speak of the B m.o.c. in cases where a function f is presented in such a manner that there is possible ambiguity concerning the containing Banach space B we have in mind. When necessary, we shall employ such notations as $\omega(f;\, a)_B$, $\omega(f;\, a)_{C(T)}$, etc. to specify the m.c.c. in question.

9.2.3.2 Lemma. Let B be an AHBS, and $f \in B$. The m.c.c. $\omega(a) = \omega(f;\, a)$ is continuous on $[0, \infty)$, non-decreasing, subadditive, and bounded by $2\|f\|$; moreover, $\omega(0) = 0$, and for each $\lambda > 0$

(1) $$\omega(\lambda a) \leqslant (\lambda + 1)\, \omega(a) .$$

Proof. The continuity and boundedness, as well as that $\omega(0) = 0$, is evident (H1) and (H2), and the monotonicity is a consequence of the definition of m.

$$\|T_{u+v} f - f\| \leqslant \|T_{u+v} f - T_u f\| + \|T_u f - f\| = \|T_u(T_v f - f)\| + \|T_u f - f\|$$
$$= \|T_v f - f\| + \|T_u f - f\| .$$

Suppose now $a \geqslant 0$, $b \geqslant 0$. If u, v vary subject to the conditions $|u| \leqslant a$, |v the sum $u + v$ takes on all values in the ball $|t| \leqslant a + b$. Therefore, the al equality implies $\omega(a + b) \leqslant \omega(a) + \omega(b)$, i.e. the subadditivity. By inducti implies $\omega(na) \leqslant n\omega(a)$, $n = 1, 2, \ldots$. Hence, if n is the largest integer n ceeding λ, we have

$$\omega(\lambda a) \leqslant \omega((n + 1)a) \leqslant (n + 1)\, \omega(a) \leqslant (\lambda + 1)\, \omega(a) .$$

9.2.4 Lemma. Let B be an AHBS, $f \in B$, and $k \in L^1$, $\int k\, dt = 1$. Then

(1) $$\|f - (f * k_{(a)})\| \leqslant \omega(a) \int (1 + |u|)\, |k(u)|\, du$$

(observe that the integral is over R^n, and $|u|$ denotes as usual the Euclide of u.)

Proof. Just as in the proof of 9.2.2.2.,

$$\|f - (f * k_{(a)})\| \leqslant \int \|f - T_{au} f\| \cdot |k(u)|\, du \leqslant \int \omega(f;\, a|u|)\, |k(u)|\, du$$
$$\leqslant \omega(a) \int (1 + |u|)\, |k(u)|\, du .$$

9.2.4.1 Corollary. Let B be an AHBS, $f \in B$, $k \in L^1$, and $\int k\, dt = 1$. Then

$$\lim_{a \to 0} f * k_{(a)} = f .$$

Proof. From the preceding proof,

$$\|f - (f * k_{(a)})\| \leqslant \int \omega(f;\, a|u|)\, |k(u)|\, du$$

and the last integral tends to zero as $a \to 0$ by 9.2.3.2 and dominated conv

9.3 Homogeneous Banach spaces of functions on R^n

9.3.1 Definition of a HBS. Throughout this section, B shall denote some Banach space, the elements of which are (Lebesgue) measurable functions on R^n, and T_u shall denote the operator of translation; that is, for $f \in B$,

$$(T_u f)(t) = f(t-u) , \quad u \in R^n .$$

Consider now the following axioms that a space B may satisfy:

(H´1) For each $u \in R^n$, T_u is an isometry of B on itself.

(H´2) "Translation is continuous in B", i.e. for each $f \in B$, the map $u \to T_u f$ is a continuous B-valued function on R^n.

(H´3) The functions in B are uniformly locally integrable, i.e. there is some constant α such that

(1) $$\int_W |f(t-u)| \, dt \leqslant \alpha \, \|f\|_B$$

for every $f \in B$, $u \in R^n$; here W denotes the cube $\{0 \leqslant t_1 \leqslant 1; \ldots; 0 \leqslant t_n \leqslant 1\}$.

Observe that (1) implies that the analogous inequality holds with W replaced by an arbitrary compact set V, provided α is replaced by a suitable constant $\alpha(V)$.

9.3.1.1 Definition. A Banach space B consisting of Lebesgue measurable functions on R^n is a homogeneous Banach space (abbreviated HBS) if (H´1), (H´2) and (H´3) are satisfied.

Exercises. (Note: the notations introduced in these exercises will be used later without further commentary.) a) Prove that $\dot{C} = \dot{C}(R^n)$, the Banach space of bounded uniformly continuous functions on R^n, is a HBS.

b) Prove that $L^p(R^n)$ is a HBS, for $1 \leqslant p < \infty$.

c) Let $L^p(T^n)$ denote the Banach space of measurable functions on R^n which have period 2π in each variable, normed by

$$\|f\| = [(2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(t_1, \ldots, t_n)|^p \, dt_1 \ldots dt_n]^{1/p} .$$

Prove that, for $1 \leqslant p < \infty$, $L^p(T^n)$ is a HBS.

d) Let $S^p = S^p(R^1)$ denote the ("Stepanov") space of measurable f on R^1 such that

$$\|f\|_{S^p} \underset{def}{=} \sup_{u \in R} \left(\int_u^{u+1} |f(t)|^p \, dt \right)^{1/p}$$

is finite. Prove that S^p is a Banach space, satisfying (H´1) and (H´3), but (Note: The study of closed subspaces of S^p which satisfy (H´2) is closely re the theory of almost-periodic functions, cf. BOHR & FØLNER.)

e) Construct a Banach space of measurable functions on R^1 which (H´1) and (H´2) but not (H´3).

9.3.2 Relation of the concepts HBS and AHBS. In this section we shall prov every HBS is an AHBS (relative, of course, to the group of translations). Th obvious, because the axioms (H3) and (H´3) are very different in appearance.

We begin with

9.3.2.1 Lemma. If f is any measurable function on R^n which is uniformly lo integrable, i.e. $\int_W |f(t-u)| \, dt \leqslant C$, where W is the unit cube of R^n, and C dependent of $u \in R^n$, and if $\sigma \in M(R^n)$, then for almost all $t \in R^n$

(1) $$g(t) = \int f(t-u) \, d\sigma(u)$$

exists as a Lebesgue-Stieltjes integral, and g is again uniformly locally in

Proof. Clearly it is enough to give the proof for a positive measure σ. Now $s \in R^n$,

$$|g(t-s)| \leqslant \int |f(t-s-u)| \, d\sigma(u) .$$

$$\int_W |g(t-s)| dt \leqslant \int_W \left(\int |f(t-s-u)| d\sigma(u)\right) dt = \int \left(\int_W |f(t-s-u)| dt\right) d\sigma(u) \leqslant C \int d\sigma .$$

In particular, for f in a HBS, (1) defines a locally integrable functi Observe that (H´3) implies f is a tempered distribution, and one verifies tha the convolution of f and σ in the sense of the theory of distributions. Now, the symbol $f * \sigma$ can also be given meaning, as an element of B in the sense of 9.2.1, i.e. as the vector-valued integral $\int (T_u f)\, d\sigma(u)$, and it is not a priori obvious that the two functions thus described are one and the same. We shall now prove that this is in fact the case.

9.3.2.2 Lemma. Let B be a HBS, and $f \in B$. Suppose $\sigma \in M$, and define

$$g(t) = \int f(t-u)\, d\sigma(u) ,$$

$$h = \int (T_u f)\, d\sigma(u) .$$

The first integral denotes a Lebesgue-Stieltjes integral (existing a.e., by the preceding lemma), or equivalently, a distribution-theoretic convolution; the second, a B-valued integral. Then, $g(t) = h(t)$ except on a set of Lebesgue measure zero.

Proof. We shall, for simplicity, give the proof only for R^1. The general case involves more measure-theoretic sophistication, but can be carried out along the same lines. In the one-dimensional case, we can interpret σ as a function of bounded variation on R, and (by the definition of the Bochner-Stieltjes integral) we have, for suitably selected points

$$-\infty < u_{n,1} < u_{n,2} < \dots < u_{n,N(n)} < \infty ,$$

$$h = \lim_{n \to \infty} \sum_j (T_{u_{n,j}} f)(\sigma(u_{n,j+1}) - \sigma(u_{n,j}))$$

where the convergence is in the norm topology of B; here the summation is from $j = 1$ to $N(n)$. Let now φ be a continuous function vanishing outside a compact interval J. Since B-convergence implies convergence in the $L^1(J)$ topology,

$$\int h(t)\, \varphi(t)\, dt = \lim_{n \to \infty} \sum_j (\sigma(u_{n,j+1}) - \sigma(u_{n,j})) \int (T_{u_{n,j}} f)(t)\, \varphi(t)\, dt$$

$$= \lim_{n \to \infty} \sum_j (\sigma(u_{n,j+1}) - \sigma(u_{n,j})) \int f(t - u_{n,j})\, \varphi(t)\, dt .$$

Now, this relation is valid (in view of the definition of the integral) prov $u_{n,j}$ satisfy the conditions $\lim_{n \to \infty} u_{n,1} = -\infty$, $\lim_{n \to \infty} u_{n,N(n)} = \infty$, and

$$\lim_{n \to \infty} \sup_{1 \leq j < N(n)} (u_{n,j+1} - u_{n,j}) = 0 .$$

We have therefore, writing $\psi(u) = \int f(t-u)\, \varphi(t)\, dt$ (observe that ψ is conti bounded on R)

$$\int h(t)\, \varphi(t)\, dt = \lim_{n \to \infty} \sum_j (\sigma(u_{n,j+1}) - \sigma(u_{n,j}))\, \psi(u_{n,j})$$

whenever $u_{n,j}$ satisfy the preceding conditions, and this implies

$$\int h(t)\, \varphi(t)\, dt = \int \psi(u)\, d\sigma(u) = \int (\int f(t-u)\, \varphi(t)\, dt)\, d\sigma(u) = \int g(t)\, \varphi(t)\, d$$

by Fubini's theorem, and the desired conclusion follows.

Therefore, whenever we have a homogeneous Banach space B, we may inter the "convolution" of an element $f \in B$ with an element $\sigma \in M$, which we shall d by $f * \sigma$, in either of the above two senses without ambiguity. This has impor consequences. First of all, axiom (H3) is satisfied, i.e. 9.2.1 (4) holds, si both sides of the equality $f * (\sigma * \tau) = (f * \sigma) * \tau$ are meaningful (and equa when $*$ is interpreted throughout as convolution of tempered distributions. Se the important inequality 9.2.1 (3) holds. Summarizing, we have

9.3.2.3 Theorem. Let B be an HBS. Then

(i) for $f \in B$, $\sigma \in M$ the Lebesgue-Stieltjes integral

(1) $$g(t) = \int f(t-u)\, d\sigma(u)$$

exists outside a t-set of Lebesgue measure zero, and the function g so defined (apart from correction on a set of Lebesgue measure zero) an element of B, whi satisfies

(2) $$\|g\| \leq \|\sigma\|_M \|f\| .$$

(ii) B is an AHBS (and so, the results proved in 9.2 are applicable, with $f * \sigma$ interpreted as the function g in (1)).

9.3.3 Jackson's theorem in HBS. As a simple but important first application, we observe from 9.2.4.1: if $k \in L^1$, $\int k \, dt = 1$, then $\lim_{a \to 0} f * k_{(a)} = f$, for f in a HBS. This is a rather general form of "approximate identity"; a quantitative version is 9.2.4 (1).

9.3.3.1 Theorem (Jackson's theorem for HBS). Let B be any HBS on R^1, and suppose $f \in B$ has period 2π. Then, for each integer $n \geq 1$, there exists $U \in \mathcal{T}_{n-1}$ satisfying

$$(1) \qquad \|f - U\| \leq A\omega(f; 1/n)$$

where A is an absolute constant.

Proof. Let $k \in L^1(R)$ satisfy (i) $\int |t| \cdot |k(t)| \, dt < \infty$, (ii) $\int k \, dt = 1$ and (iii) $\hat{k}(x) = 0$, $|x| \geq 1$. (Such k obviously exist; we can choose any three times differentiable function K on $\hat{R}$ vanishing outside $[-1, 1]$ and satisfying $K(0) = 1$, and define k to be the inverse Fourier transform of K. Then $k(t) = O(|t|^{-3})$ at ∞, so (i) holds.) Define $U = f * k_{(1/n)}$. Now, the definition of HBS guarantees that $f \in L^1(T)$; also, the Fourier transform of $k_{(1/n)}$ vanishes for $|x| \geq n$, and from this fact it is easy to deduce $U \in \mathcal{T}_{n-1}$ (cf. SHAPIRO$_1$, pp. 48 ff.). (Observe that, by slight abuse of notation, we are here denoting by $\mathcal{T}_{n-1}$ the space of trig. polys. of degree at most $n - 1$ with complex coefficients; we have done this also on several occasions earlier.) Finally, 9.2.4 (1) with $a = 1/n$ yields (1), with

$$A = \int (1 + |u|) \, |k(u)| \, du \, .$$

◇

Remarks. a) This method of proof obviously yields also the analogous result for HBS of functions on R^n which have period 2π in each variable. Moreover, it yields the corresponding result where we assume in place of periodicity that f has its spectrum in some discrete set Δ, and require that U shall be a trigonometric sum whose frequencies are chosen from Δ and have absolute value less than n. As this kind of generalization involves no new idea, we omit the details.

b) Choosing B to be C(T) we obtain of course the classical Jackson for trigonometric polynomial approximation; we have only to observe that the m. in the HBS sense coincides here with the classical m.o.c. In like manner, one tains the L^p versions of Jackson's theorem (cf. TIMAN).

c) The idea of obtaining an approximating trigonometric polynomial convolution in the above manner occurs in BOCHNER$_2$ (we regret that, owing to it peculiar context, this had escaped our notice when SHAPIRO$_1$ was written).

9.3.3.2 Concerning functions of higher smoothness, we have: the Favard-Ahiezer theorem 7.5.2 is valid in HBS, i.e. if $f \in B$ has period 2π and is the r-fold in of some element of B (denoted by $f^{(r)}$) then for a suitable $U \in \mathcal{T}_{n-1}$ we have

$$\|f - U\| \leq \lambda_r \|f^{(r)}\| \, n^{-r}$$

where λ_r is as in 7.5.2. The proof we gave for 7.5.2 applies word for word. (In same way, one can generalize many other results in earlier chapters to HBS; we not deal with all of these generalizations explicitly.)

9.3.3.3 Algebraic polynomials. Suppose now $f \in C(I)$ and ω denotes the (classic m.o.c. of f. The function g: $g(t) = f(\cos t)$ is in $C(T)$, $\omega(g, a) \leq \omega(f, a)$ and t exists $U \in \mathcal{T}_{n-1}$ satisfying $\|g - U\| \leq A\omega(g; 1/n)$. Moreover, since $g(-t) = g(t)$, can assume that U is a "cosine polynomial" so that $U(t) = P(\cos t)$ for a suitabl $P \in \mathcal{P}_{n-1}$. Therefore

$$\|f - P\|_{C(I)} = \|g - U\|_{C(T)} \leq A\omega(g; 1/n) \leq A\omega(f; 1/n)$$

for a suitable absolute constant A, which is Jackson's theorem for $\mathcal{P}_{n-1}$. It is o interest, as shown by Timan (cf. TIMAN) that a more careful analysis leads to a sharper estimate.

9.3.3.4 Theorem (Timan). Let $f \in C(I)$. For each $n \geq 1$ there exists $P \in \mathcal{P}_{n-1}$ suc that

$$(1) \qquad |f(x) - P(x)| \leq A'\omega(f; \delta_n(x)) \, , \text{ where}$$

$$(2) \qquad \delta_n(x) = n^{-2} + n^{-1}(1 - x^2)^{1/2}$$

ınd A′ is an absolute constant.

ːemark. Since $\delta_n(x) \leqslant 2/n$ for $x \in I$, this result is clearly stronger than that in '.3.3.3.

'roof. Let $g(t) = f(\cos t)$, and define $U \in \mathcal{T}_{n-1}$ by $U = g * k_{(1/n)}$, where k is as in he proof of 9.3.3.1, except that now we impose the extra restriction $\int t^2 |k(t)|dt < \infty$. e have

3) $$g(t) - U(t) = \int (g(t) - g(t - (u/n)))\ k(u)\ du\ .$$

ow, $|g(t) - g(t - a)| = |f(\cos t) - f(\cos (t - a))| \leqslant \omega(f;\ |\cos t - \cos(t - a)|)$,

nd since

$$|\cos t - \cos(t - a)| = |\cos t(1 - \cos a) - \sin t \sin a| \leqslant (a^2/2) + |a|\cdot|\sin t|\ ,$$

e obtain from (3), writing $\omega(\ )$ for $\omega(f;\ )$,

$$|g(t) - U(t)| \leqslant \int \omega((u^2/2n^2) + |u|\cdot|\sin t|/n)\ |k(u)|\ du = \int \omega(G(u))\ |k(u)|\ du$$

$$\leqslant \omega(s)\int (s^{-1}G(u) + 1)\ |k(u)|\ du = \omega(s)s^{-1}\int G(u)\ |k(u)|\ du + \omega(s)\int |k(u)|\ du$$

ıere $G(u) = (u^2/2n^2) + |u|\cdot|\sin t|/n$, and s is a positive quantity which is at our isposal. Let us choose

$$s = \int G(u)|k(u)|\ du = (2n^2)^{-1}\int u^2|k(u)|\ du + n^{-1}|\sin t|\int |u|\cdot|k(u)|\ du$$

$$\leqslant A_1(n^{-2} + n^{-1}|\sin t|)$$

ıere A_1 is an absolute constant. Hence, if $U(t) = P(\cos t)$, where $P \in \mathcal{P}_{n-1}$, we have ınally

$$|f(\cos t) - P(\cos t)| \leqslant \omega(s)(1 + \int |k(u)|\ du) \leqslant A_2\ \omega(n^{-2} + n^{-1}|\sin t|)$$

ıd, putting $\cos t = x$, we have (1), where $A' = A_2$ is some absolute constant. ◇

3.4 Applications to Fourier analysis. Jackson's theorem for HBS implies a result out the Fourier partial sums.

3.4.1 Definition. Let B be a HBS whose elements are measurable functions on R of

period 2π, and $n \geqslant 0$ an integer. The Lebesgue constant Λ_n^B is the norm of the (Four partial sum) operator F_n which maps each function onto the partial sum of order n its Fourier series,

$$\Lambda_n^B = \|F_n\|_{B \to B} = \sup_{\substack{f \in B \\ \|f\| = 1}} \|F_n f\|_B\ .$$

If B is C(T), $L^\infty(T)$ or $L^1(T)$, Λ_n^B is the classical Lebesgue constant $\Lambda_n \sim (4/\pi^2)\log n$, discussed in 8.6. For $B = L^2(T)$, $\Lambda_n^B = 1$, and for $B = L^p(T)$, $1 < p <$, $\Lambda_n^B \leqslant A_p < \infty$ for all n (cf. ZYGMUND).

9.3.4.2 Theorem. Let B be a HBS whose elements are measurable functions on R of period 2π. Let $f \in B$, and denote by s_n the partial sum of order n of the Fourier series of f. Then

(1) $$\|f - s_{n-1}\| \leqslant (1 + \Lambda_n^B)\ A\ \omega(f,\ 1/n)$$

where A is an absolute constant (the same as in 9.3.3.1 (1)).

Proof. Let $U \in \mathcal{T}_{n-1}$ satisfy

(2) $$\|f - U\| \leqslant A\omega(1/n)$$

where ω is the B m.o.c. of f; we know such U exists, by 9.3.3.1. Therefore, applyin the operator F_{n-1},

(3) $$\|s_{n-1} - U\| = \|F_{n-1}(f - U)\| \leqslant \Lambda_n^B\ A\omega(1/n)$$

and (2), (3) imply (1). ◇

9.3.4.3 Corollary. Under the hypotheses of 9.3.4.2, if $\lim_{n \to \infty} \Lambda_n^B\ \omega(f;\ 1/n) = 0$, the Fourier series of f converges to f, in the B norm.

In particular, for $B = C(T)$, we get the Dini-Lipschitz theorem: if $\omega(f;\ a) = o(|\log a|^{-1})$ the Fourier series of f converges uniformly to f.

Another criterion for norm-convergence of the Fourier series is in terms of the coefficients (cf. KATZNELSON, p. 52).

9.3.4.4 Theorem. Let B be a HBS whose elements are measurable functions on R of period 2π, and suppose the B norms of the characters $t \to e^{int}$ are bounded. Let $f \in B$, and denote the Fourier coefficients of f by $\{f\hat{}(n)\}_{n=-\infty}^{\infty}$. Suppose

$$(1) \qquad \lim_{\rho \to 1} \overline{\lim_{N \to \infty}} \sum_{N < |n| \leq \rho N} |f\hat{}(n)| = 0 .$$

(In particular, (1) holds if $f\hat{}(n) = O(1/n)$.) Then, the Fourier partial sums s_n of f converge to f in the B norm.

Proof. Choose $k_\rho \in L^1(R)$ so that its Fourier transform $K_\rho = \hat{k}_\rho$ equals one on $[-1, 1]$ and zero for $|x| \geq \rho > 1$, and is bounded by one in between (e.g. a "trapezoid function"). Then, * denoting convolution of tempered distributions on R,

$$(k_\rho)_{(1/N)} * f = s_N + r_N$$

where

$$r_N(t) = \sum_{N < |n| \leq \rho N} f\hat{}(n)\, K_\rho(n/N)\, e^{int} ;$$

hence

$$\|r_N\| \leq A \sum_{N < |n| \leq \rho N} |f\hat{}(n)| \underset{\text{def.}}{=} B_{N,\rho} .$$

(here A is an upper bound for the norms of the characters). We have, therefore

$$(2) \qquad \overline{\lim_{N \to \infty}} \|[(k_\rho)_{(1/N)} * f] - s_N\| \leq \overline{\lim_{N \to \infty}} B_{N,\rho} \underset{\text{def.}}{=} \psi(\rho) ,$$

where, by hypothesis, $\lim_{\rho \to 1} \psi(\rho) = 0$. Now, $(k_\rho)_{(1/N)} * f$ tends to f as $N \to \infty$ by 9.2.4.1. Therefore, from (2), $\overline{\lim_{N \to \infty}} \|f - s_N\| \leq \psi(\rho)$, and since $\rho > 1$ here is arbitrary, $\overline{\lim_{N \to \infty}} \|f - s_N\| = 0$. ◇

Remarks. a) The analogous theorem in several variables is also true, the proof being essentially the same.

b) This theorem is quite sharp; for example, if $\lim_{|n| \to \infty} \lambda_n = \infty$, one can find $f \in C(T)$ such that $|f\hat{}(n)| \leq \lambda_n (1 + |n|)^{-1}$ and yet the Fourier series for f diverges at a point (ZYGMUND, vol. 1, p. 304).

The m.o.c. corresponding to certain Banach norms other than the cla
norm have interesting connections with other topics. Thus, the $L^2(T)$ (or I
is related to absolute convergence of Fourier series (or Fourier transforn
the $L^1(T)$ (or $L^1(R)$) m.o.c. is related to the rate of decay of Fourier coe
(or Fourier transforms).

9.3.4.5 Theorem (S. Bernstein). If $f \in L^2(T)$ and the $L^2(T)$ m.o.c. ω of f

$$\int_0^1 \omega(s)\, s^{-3/2}\, ds < \infty ,$$

(in particular, if $f \in C(T)$ belongs to Lip α in the classical sense, for s
$\alpha > 1/2$) then $\sum |f\hat{}(n)| < \infty$.

Proof. Denote $f\hat{}(n)$ by c_n. Then, for $m \geq 0$, we have (with an easily under
tion)

$$S_m \underset{\text{def.}}{=} \sum_{|n|=2^m+1}^{2^{m+1}} |c_n| \leq 2^{(m+1)/2} \Big(\sum_{|n|=2^m+1}^{2^{m+1}} |c_n|^2 \Big)^{1/2} ,$$

and

$$\sum_{|n|=2^m+1}^{2^{m+1}} |c_n|^2 \leq \sum_{|n| \geq 2^m+1} |c_n|^2 = \|f - s_{2^m}\|^2_{L^2(T)} \leq A\omega(2^{-m})^2 ,$$

where A is an absolute constant, by 9.3.3.1, since in $L^2(T)$ norm the Fourie
gives at least as good approximation to f as any other trig. poly. of the s
gree. Thus,

$$(1) \qquad \sum_{|n| \geq 2} |c_n| = \sum_{m=0}^{\infty} S_m \leq A^{1/2} \sum_{m=0}^{\infty} 2^{(m+1)/2}\, \omega(2^{-m})$$

and since

$$2^{m/2}(2 - \sqrt{2})\, \omega(2^{-m}) \leq \int_{2^{-m}}^{2^{-m+1}} \omega(t)\, t^{-3/2}\, dt$$

we obtain from (1)

$$\sum_{|n| \geq 2} |c_n| \leq A_1 \int_0^2 \omega(t)\, t^{-3/2}\, dt$$

where A_1 is an absolute constant, which implies the assertion. ◇

9.3.4.6 Corollary. *If $f \in C(T)$ and the $C(T)$ m.o.c. ω of f satisfies $\int_0^1 \omega(t)^{1/2}\, t^{-1}\, dt < \infty$, (in particular, if $f \in \mathrm{Lip}\, \epsilon$ for some $\epsilon > 0$), and f is of bounded variation, then $\sum |f\hat{}(n)| < \infty$.*

Proof. If f is of bounded variation, we have $\int_0^{2\pi} |f(t + a) - f(t)|\, dt \leq Ka$ for some constant K (see e.g. SHAPIRO$_1$, p. 40). Hence

$$(1/2\pi) \int_0^{2\pi} |f(t + a) - f(t)|^2\, dt \leq Ka\, \omega(f; a)_{C(T)}$$

which implies

$$\omega(f; a)_{L^2(T)} \leq K_1 [a\omega(f; a)_{C(T)}]^{1/2}$$

and 9.3.4.5 now implies the desired conclusion. ◇

9.3.4.7 The Bernstein theorem above is sharp, e.g. in the sense that *there exists a function $f \in C(T)$ belonging to Lip 1/2, such that $\sum |f\hat{}(n)| = \infty$.*

To prove this, we use the fact that there exists a sequence $\{a_n\}_{n=0}^{\infty}$, where each a_n is 1 or -1, such that

$$(1) \qquad \left|\sum_{k=0}^{n} a_k e^{ikt}\right| \leq A(n + 1)^{1/2}, \quad n = 0, 1, \ldots$$

uniformly in t; here A is an absolute constant. (This was proved by me in 1950, and announced much later in SHAPIRO$_9$. Cf. KATZNELSON, p. 33, KAHANE & SALEM pp. 134 ff., KAHANE; other unimodular sequences would work equally well, cf. ZYGMUND, vol. 1, p. 197 ff., however the example cited is the most elementary.)

Consider now the series $\sum_{k=0}^{\infty} (a_k/(k + 1))\, e^{ikt}$. It follows easily by Abel partial summation, using (1), that the series converges uniformly, hence defines a function $f \in C(T)$. Moreover, denoting by s_n the partial sum of order n of this series,

another partial summation shows $\|f - s_n\|_{C(T)} = O(n^{-1/2})$; hence by an "inverse of approximation, which we shall prove later (cf. also SHAPIRO$_1$, p. 69), $f \in$ Since $|f\hat{}(n)| = 1/(n + 1)$, we have the desired counter-example.

An interesting use of the $L^1(R)$ m.o.c. is

9.3.4.8 Theorem. *For $f \in L^1(R)$,*

$$|\hat{f}(x)| \leq (1/2)\, \omega(f, \pi |x|^{-1})_{L^1(R)} .$$

Proof. It is clearly sufficient to deal with $x > 0$. Now, $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\, e^{-}$ hence

$$\hat{f}(x) = -\int_{-\infty}^{\infty} f(t + (\pi/x))\, e^{-itx}\, dt ,$$

and adding,

$$2\hat{f}(x) = \int_{-\infty}^{\infty} (f(t) - f(t + (\pi/x)))\, e^{-itx}\, dt$$

whence

$$|\hat{f}(x)| \leq (1/2) \int |f(t) - f(t + (\pi/x))|\, dt \leq (1/2)\, \omega(f, \pi/x)_{L^1(R)} .$$

9.3.4.9 Exercises. a) Prove that $A(\hat{R})$ is a HBS.

b) Check which of the axioms for HBS are satisfied by each of the follo spaces of periodic functions: Lip α (with norm $\|f\| = \sup_{t_1, t_2} (|f(t_1) - f(t_2)|/|t_1$ its closed subspace lip α consisting of f such that $f(t + a) - f(t) = o(|a|^{\alpha})$ formly in t; the space V_p of functions whose Fourier coefficients are summable the power p ($1 \leq p \leq 2$), with the obvious norm based on the coefficients. (Not certain fairly well known spaces, the verification of axiom (H´2) presents gre difficulties, e.g. it is not known whether, in the "multiplier" space M_p of fu f on the circle such that $fg \in V_p$ whenever $g \in V_p$ (where $\|f\|_{M_p}$ is defined as norm of the operator $g \rightarrow fg$ from V_p to V_p), translations are continuous. Here a Banach algebra, and the question at hand is a stumbling block to the study o maximal ideal space.)

c) Same question as b) for r times differentiable functions, with usual Banach topology.

d) Prove that, if B is a space of periodic functions which satisfies (H´1) and (H´3), and contains all characters, then (H´2) holds if and only if the trigonometric polynomials are dense in B.

e) Prove that if f belongs to an AHBS and its m.o.c. is o(a), then $T_u f = f$ for all $u \in R^n$; in particular, if f belongs to a HBS and $\omega(f, a) = o(a)$, f is constant.

f) Prove the remark "Observe ... " following the formulation of (H´3).

g) Prove 9.3.2.2 in R^n.

h) Can you prove anything similar to 9.3.3.4 in the $L^p(I)$ metric?

i) Prove 9.3.4.5 without using Jackson´s theorem for L^2.

j) Carry out the omitted estimations in 9.3.4.7. Also, prove that $f \in$ Lip 1/2 without quoting the "inverse theorem".

k) Prove that the $L^1(T)$ m.o.c. of the function $f(t) = e^{i/|t|}$, $|t| \leqslant \pi$ is $O(a^{1/2})$.

l) Prove the analog of 9.3.4.8 for Fourier coefficients. What estimate does this give for the coefficients of the function defined in the preceding exercise? Is this a good estimate?

m) Let W denote any compact subset of $L^1(T)$. Prove that

$$\lim_{|n| \to \infty} \sup_{f \in W} |f^\wedge(n)| = 0 ,$$

(i) using the preceding exercise, and (ii) not using it, but rather 8.2.1.(Hint for (ii): $|f^\wedge(n)| = |(2\pi)^{-1} \int_0^{2\pi} f(t)\, e^{-int}\, dt| = |(2\pi)^{-1} \int_0^{2\pi} (f(t) - U(t))\, e^{-int}\, dt| \leqslant \|f - U\|_{L^1(T)}$, for every $U \in \mathcal{T}_{n-1}$.)

n) Prove that if $f \in L^\infty(T)$ and $f^\wedge(n) = O(1/n)$, the partial sums of the Fourier series of f are uniformly bounded. (Hint: convolve f with a suitable "trapezoid" kernel.)

o) Prove the following form of "Fourier inversion": Suppose K ∈
$k(t) = (1/2\pi) \int K(x)\, e^{itx}\, dx$ belongs to $L^1(R)$ and $\int k\, dt = 1$. Then, for a
we have: if $f_\varepsilon(t)$ denotes $(1/2\pi) \int \hat{f}(x)\, K(\varepsilon x)\, e^{itx}\, dt$, $\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^1}$

p) Deduce from o) that if $f \in L^1(R)$ and $\hat{f} \in L^1(\hat{R})$, then

$$f(t) = (2\pi)^{-1} \int \hat{f}(x)\, e^{itx}\, dx \quad \text{holds a.e.}$$

q) Prove that if B is a HBS on R, containing the function f of p
Fejér sums σ_n formed from f ($\sigma_n = (s_0 + \dots + s_n)/(n+1)$) converge to f
(Hint: establish the formula $\sigma_{n-1}(t) = (f * k_{(1/n)})(t)$ where $k(t) = (2/$

r) Prove the analogous proposition for the Abel means $f(r, t)$, be
formula $f(e^{-a}, t) = (f * k_{(a)})(t)$ with $k(t) = (1/\pi)(1 + t^2)^{-1}$. (Note: th
suggested in q) and r) are not the usual ones; the study of summability
series by means of convolutions on the infinite line is often useful, es
multidimensional problems.)

9.4 Comparison theorems

The material in this section is based on BOMAN & SHAPIRO$_2$.

9.4.1 The notion of σ-modulus

9.4.1.1 Definition. Let B be any AHBS and $f \in B$. For $\sigma \in M = M(R^n)$, the
of f, denoted by $\omega_\sigma(f; a)$ or $\omega_{\sigma,B}(f; a)$ is the function on $[0, \infty)$ defined

(1) $$\omega_\sigma(f; a) = \sup_{0 < b \leqslant a} \|f * \sigma_{(b)}\| .$$

(For the meaning of $\sigma_{(b)}$, see 9.2.2.)

Clearly, for fixed $f \in B$, $\sigma \in M$ the σ-modulus is a non-decreasing
a. Moreover, it satisfies the following relations, whose verification we
the reader.

(2) $$\omega_{\sigma * \tau}(f; a) \leqslant \|\sigma\|_M\, \omega_\tau(f; a)$$

(3) $$\omega_{\sigma+\tau}(f; a) \leqslant \omega_\sigma(f; a) + \omega_\tau(f; a) .$$

9.4.2 A special partition of unity.

9.4.2.1 Lemma. There exists a function Φ on $\hat{R}^n$ which is infinitely differentiable at all points, and such that further

(i) $\Phi(x) > 0$ for $1/2 < |x| < 2$, and Φ vanishes elsewhere,

(ii) $\sum_{j=-\infty}^{\infty} \Phi(2^j x) = 1$ if $x \neq 0$.

Proof. Let $h(a)$ denote a function defined for $0 \leqslant a < \infty$ and equal to one for $0 \leqslant a \leqslant 1$, to zero for $a \geqslant 2$, strictly decreasing on $[1,2]$, and infinitely differentiable. Then $\Phi(x) = h(|x|) - h(2|x|)$ satisfies the requirements. Observe that in the series (ii) at most two terms are different from zero, for each $x \neq 0$. ◇

Φ is the Fourier transform of a certain function φ which is infinitely differentiable, and φ and all its partial derivatives tend to zero at infinity more rapidly than any negative power of $|t|$.

We shall, throughout, write φ_j to denote $\varphi_{(2^j)}$, i.e.

(1) $$\varphi_j(t) = \varphi_{(2^j)}(t) = 2^{-nj}\,\varphi(2^{-j}t), \quad j = 0, \pm 1, \ldots$$

Observe that

(2) $$\hat{\varphi}_j(x) = \Phi(2^j x), \quad j = 0, \pm 1, \ldots$$

We shall also require the relation, for positive integral r,

(3) $$\sum_{j=-r}^{r} \Phi(2^j x) = 1 \quad \text{for} \quad 2^{-r} \leqslant |x| \leqslant 2^r ,$$

which follows from (ii), since for x in this range, and $|j| > r$, $\Phi(2^j x) = 0$.

In the analysis which follows, we shall always suppose the dimension n and the choice of a particular Φ with the properties enumerated in the lemma to have been fixed, and treat as "constants" numbers which depend only on n and Φ.

9.4.3 "Tauberian condition". A continuous function F on $\hat{R}^n$ is said to satisfy the Tauberian condition if, for every x with $|x| = 1$, there exists $c \geqslant 0$ such that

$F(cx) \neq 0$, in other words, if F takes a non-zero value on every closed half-r
(If, in particular, $F(0) \neq 0$, F satisfies the Tauberian condition trivially.)

9.4.3.1 Lemma. If F is continuous on $\hat{R}^n$ and satisfies the Tauberian conditi
$\delta > 0$ is arbitrary, there exist positive numbers $d_1 < d_2 \ldots < d_r$ such that

$$\sum_{j=1}^{r} |F(d_j x)| > 0 , \quad \delta \leqslant |x| \leqslant 1/\delta .$$

Proof. Let S denote $\{x: |x| = 1\}$ and F_c, for $c > 0$, the function on S defined $F_c(x) = F(cx)$. The hypotheses imply $\{F_c\}_{c>0}$ have no common zero on S, i.e. t closed subsets E_c of S defined by $E_c = \{x: F_c(x) = 0\}$ have an empty intersecti Therefore, since S is compact, there is some finite subset $E_{c_1}, \ldots, E_{c_m}$ of th whose intersection is empty. This means that $G(x) = \sum_{i=1}^{m} |F(c_i x)|$ is positiv and hence by continuity remains positive in the spherical shell $b \leqslant |x| \leqslant 1/b$ chosen sufficiently close to 1. Therefore, if k is chosen large enough, $\sum_{j=-k}^{k}$ is positive for $\delta \leqslant |x| \leqslant 1/\delta$, which implies the assertion in the lemma.

9.4.4 Comparison of two moduli.

9.4.4.1 Lemma. Let B be any AHBS, $f \in B$ and $\sigma, \tau \in M(R^n)$. Suppose moreover $\hat{\sigma}$ fies the Tauberian condition, and $\hat{\tau}$ vanishes in neighborhoods of 0 and ∞. Then

(1) $$\omega_\tau(f; a) \leqslant A_1\, \omega_\sigma(f; A_2 a)$$

where A_1, A_2 are positive constants depending only on σ, τ.

Proof. By 9.4.3.1 there exist numbers $d_j > 0$ such that $\sum_{j=1}^{r} |\hat{\sigma}(d_j x)|$ is posi on the support of $\hat{\tau}$. Hence, by the basic ideal theory of $M(R^n)$, τ belongs to th ideal in $M(R^n)$ generated by $\sigma_{(d_1)}, \ldots, \sigma_{(d_r)}$, i.e.

$$\tau = \sum_{j=1}^{r} \rho_j * \sigma_{(d_j)} , \quad \text{where} \quad \rho_j \in M(R^n) .$$

Hence, for $a > 0$

$$\tau_{(a)} = \sum_{j=1}^{r} (\rho_j)_{(a)} * \sigma_{(ad_j)} .$$

Therefore, for $f \in B$,

$$\|f * \tau_{(a)}\| \leqslant \sum_{j=1}^{r} \|\rho_j\|_M \|f * \sigma_{(ad_j)}\| \leqslant \sum_{j=1}^{r} \|\rho_j\|_M \omega_\sigma(f; ad_j) \leqslant A_1 \omega_\sigma(f; A_2 a)$$

and now (1) follows. ◇

We work from now on in the slightly lesser generality of HBS, because we will require that the elements of the space under consideration be tempered distributions.

9.4.4.2 Lemma. Let B be any HBS, $f \in B$ (in particular, f is locally integrable, and moreover is a tempered distribution on R^n), and suppose the support of $\hat{f}$ does not contain the origin. For any $b > 0$ we have

(1) $$\|f\| \leqslant \sum_{j=-\infty}^{\infty} \|f * \varphi_{(2^j b)}\| .$$

Proof. Suppose first $\hat{f}$ has compact support. Then, for a suitably large integer N (depending on b and supp $\hat{f}$)

$$f = \sum_{j=-N}^{N} f * \varphi_{(2^j b)}$$

since the Fourier transform of $\sum_{j=-N}^{N} \varphi_{(2^j b)}$, namely $\sum_{j=-N}^{N} \Phi(2^j bx)$, equals 1 on a neighborhood of supp $\hat{f}$. Hence

$$\|f\| \leqslant \sum_{j=-N}^{N} \|f * \varphi_{(2^j b)}\| \leqslant \sum_{j=-\infty}^{\infty} \|f * \varphi_{(2^j b)}\| .$$

In the general case, choose $k(t) \geqslant 0$ such that $\int k\, dt = 1$ and $\hat{k}$ has compact support; then $f * k_{(a)}$ has a (distributional) Fourier transform whose support is compact and does not contain the origin, so we may apply the above to it and obtain

$$\|f * k_{(a)}\| \leqslant \sum_{j=-\infty}^{\infty} \|f * k_{(a)} * \varphi_{(2^j b)}\| \leqslant \sum_{j=-\infty}^{\infty} \|f * \varphi_{(2^j b)}\| ,$$

and now, letting $a \to 0$ and observing that $\|f * k_{(a)}\| \to \|f\|$ (see 9.2.4.1), tain (1).

For later purposes, it is important to observe that the restriction support of $\hat{f}$ is not really essential; indeed:

If B does not contain the function identically equal to 1, then (1) all $f \in B$. Moreover, in any case, (1) holds with f replaced by $g * \tau$ provid $\tau \in M$ and $\hat{\tau}(0) = 0$.

To see this, observe that (1) holds trivially unless

(2) $$\sum_{j=-\infty}^{\infty} \|f * \varphi_{(2^j b)}\| < \infty .$$

We may therefore suppose that (2) holds. Hence

$$\sum_{j=-\infty}^{\infty} f * \varphi_{(2^j b)}$$

converges (in B) to an element which we denote by h. If ψ is any element of whose Fourier transform vanishes in neighborhoods of 0 and ∞, then

$$h * \psi = \sum_{j=-\infty}^{\infty} (f * \psi) * \varphi_{(2^j b)} = f * \psi ;$$

(cf. the proof of Lemma 9.4.4.2) hence the Fourier transform of $f - h$ is su at the origin, so $f - h$ is a polynomial. Since $f - h$, being an element of B formly locally integrable, it reduces to a constant. Hence, if $\underline{1} \notin B$, $f = \sum$ and (1) holds. If $\underline{1} \in B$, we have in any case, for each $g \in B$ and some const $\lambda = \lambda(g)$,

$$g = \lambda \underline{1} + \sum_{j=-\infty}^{\infty} g * \varphi_{(2^j b)} .$$

Thus, if $\hat{\tau}(0) = 0$,

$$g * \tau = \sum_{j=-\infty}^{\infty} (g * \tau) * \varphi_{(2^j b)}$$

yielding again (1), with $g * \tau$ in place of f. ◇

We come now to the most important lemma.

9.4.4.3 Lemma. Let B be any HBS, $f \in B$ and $\sigma, \tau \in M(R^n)$. Suppose moreover $\hat{\sigma}$ satisfies the Tauberian condition, and $\hat{\tau}$ vanishes in a neighborhood of 0. Then

(1) $$\omega_\tau(f; a) \leq A \int_0^{Ba} \omega_\sigma(f; s)\, ds/s$$

where A, B depend only on σ, τ.

Remark. The difference, small but crucial, between this lemma and 9.4.4.1 is that now $\hat{\tau}$ is not required to have compact support; observe that (1) is, in general, weaker than 9.4.4.1 (1).

Proof of lemma. For any $a > 0$, $f * \tau_{(a)}$ has a Fourier transform whose support does not contain the origin. Hence, by the preceding lemma, for any $b > 0$

(2) $$\|f * \tau_{(a)}\| \leq \sum_{j=-\infty}^{\infty} \|f * \tau_{(a)} * \varphi_{(2^j b)}\| .$$

Suppose now $\hat{\tau}(x) = 0$ for $|x| \leq c$, then $\hat{\tau}_{(a)}(x) = 0$ for $|x| \leq c/a$. Since the Fourier transform of $\varphi_{(2^j b)}$ vanishes outside $\{2^{-j-1} b^{-1} \leq |x| \leq 2^{-j+1} b^{-1}\}$ we see that if we choose

(3) $$b = 2a/c ,$$

$\tau_{(a)} * \varphi_{(2^j b)}$ is zero for $j \geq 0$. Thus, defining b by (3), the summands in (2) with $j \geq 0$ are zero, and we get

(4) $$\|f * \tau_{(a)}\| \leq \sum_{j=-\infty}^{-1} \|f * \tau_{(a)} * \varphi_{(2^j b)}\| \leq \|\tau\|_M \sum_{j=-\infty}^{-1} \|f * \varphi_{(2^j b)}\| .$$

Now, by Lemma 9.4.4.1, taking φdt in place of the measure τ in that lem

$$\|f * \varphi_{(2^j b)}\| \leq A_1\, \omega_\sigma(f; A_2\, 2^j b)$$

where A_1, A_2 depend only on σ and φ (hence effectively only on σ, φ bei fixed once for all). Thus, from (4)

$$\|f * \tau_{(a)}\| \leq A_1 \|\tau\|_M \sum_{j=1}^{\infty} \omega_\sigma(f; A_2\, 2^{-j} b)$$

and observing the inequality

$$(\log 2)\, \omega_\sigma(f; A_2\, 2^{-j} b) \leq \int_{2^{-j}}^{2^{-j+1}} \omega_\sigma(f; A_2\, bs)\, ds/s = \int_{2^{-j}A_2 b}^{2^{-j+1}A_2 b} \omega_\sigma(f;$$

summing on j, and substituting the value of b from (3), we obtain (1), $A = (\log 2)^{-1} A_1 \|\tau\|_M$ and $B = 2A_2/c$.

From this result we deduce easily our first "comparison theorem" moduli.

9.4.4.4. Theorem. Let B be any HBS, $f \in B$, and $\sigma, \tau \in M(R^n)$. Suppose satisfies the Tauberian condition, and $\hat{\sigma}$ divides $\hat{\tau}$ at $x = 0$ (see 7.3.11 terminology). Then

(1) $$\omega_\tau(f; a) \leq A \int_0^{Ba} \omega_\sigma(f; u)\, du/u$$

where A, B depend only on σ, τ.

Proof. For some $\mu \in M$, the measure $\rho = \tau - \mu * \sigma$ has a Fourier transfor vanishes in a neighborhood of 0, hence by 9.4.4.3

(2) $$\omega_\rho(f; a) \leq A_0 \int_0^{B_0 a} \omega_\sigma(f; u)\, du/u$$

where A_0, B_0 depend only on σ and ρ, i.e. only on σ and τ. Now, by 9.4.1

(3) $$\omega_\tau(f; a) \leqslant \|\mu\|_M \cdot \omega_\sigma(f; a) + \omega_\rho(f; a)$$

and

(4) $$\omega_\sigma(f; a) \leqslant \int_a^{ea} \omega_\sigma(f; u)\, du/u \,.$$

Substituting into (3) the estimates for ω_σ and ω_ρ from (4), (2) respectively, gives (1). ◇

By a slight generalization of the arguments leading up to 9.4.4.4 one can prove the more general proposition: if $\tau, \sigma_1, \ldots, \sigma_r \in M(R^n)$, $\sum_{j=1}^{r} |\hat\sigma_j(x)|$ satisfies the Tauberian condition, and $\hat\tau$ belongs locally (at $x = 0$) to the ideal generated by $\hat\sigma_1, \ldots, \hat\sigma_r$, then

(5) $$\omega_\tau(f; a) \leqslant A \int_0^{Ba} \Big(\sum_{j=1}^{r} \omega_{\sigma_j}(f; u)\Big)\, du/u$$

where A, B depend only on $\tau, \sigma_1, \ldots, \sigma_r$. The details are left to the reader.

The point of Theorem 9.4.4.4 is that only a local divisibility condition is imposed; in case $\hat\sigma$ divides $\hat\tau$ (globally), we have in place of (1), and without assuming $\hat\sigma$ satisfies the Tauberian condition,

(6) $$\omega_\tau(f; a) \leqslant A_1\, \omega_\sigma(f; B_1 a)$$

with suitable constants A_1, B_1. This is a consequence of the relatively trivial inequality 9.4.1.1 (2). What is striking is that (1) is nearly as strong as (6), and fully as useful for many applications. We shall discuss later other hypotheses under which (6) holds. First, however, we shall deduce from 9.4.4.4 another comparison theorem, whereby τ is slightly more restricted but in exchange the local divisibility hypothesis can be dropped.

9.4.4.5 Theorem. Suppose $\sigma, \tau \in M(R^n)$ and $\hat\sigma$ satisfies the Tauberian condition. Let P be a positive-homogeneous function on $\hat{R}^n$ of degree r (i.e. $P(cx) = c^r P(x)$ for $c > 0$), where r is a positive real number, $P \in C^\infty(R^n \setminus \{0\})$, and suppose for some $\theta \in M(R^n)$ we have $\hat\tau(x) = \hat\theta(x) P(x)$ for all x in some neighborhood of the origin. Then, for $f \in B$,

(1) $$\omega_\tau(f; a) \leqslant A \int_0^\infty [\min(1, a/u)]^r\, \omega_\sigma(f; u)\, du/u$$

where A depends only on σ, τ.

Proof. We may suppose that $\hat\theta$ has compact support, since this can be achiev multiplication by an element of $B(\hat{R}^n)$ which has compact support and equals neighborhood of 0. We can then write $\tau = \mu + \nu$, where $\mu, \nu \in M(R^n)$, and

(2) $$\hat\mu(x) = P(x)\, \hat\theta(x)\,, \quad \text{all } x \in \hat{R}^n$$

(3) $\hat\nu(x)$ vanishes in a neighborhood of 0.

For, $P\hat\theta$ belongs to $B(\hat{R}^n)$ at each point of $\hat{R}^n$, including infinity (cf. the d in 7.3.11), hence $P\hat\theta$ is an element of $B(\hat{R}^n)$; denoting it by $\hat\mu$, $\hat\nu \underset{\text{def.}}{=} \hat\tau - \hat\mu$ satisfies (3). Now,

(4) $$\omega_\tau(f; a) \leqslant \omega_\mu(f; a) + \omega_\nu(f; a)$$

and, because of (3) and 9.4.4.3

(5) $$\omega_\nu(f; a) \leqslant A_1 \int_0^{B_1 a} \omega_\sigma(f; u)\, du/u$$

for suitable constants A_1, B_1. If A is chosen large enough, the right side ceeds that of (5). Hence, in view of (4), it suffices to prove (1) with ω_μ of ω_τ. We have, for $c > 0$, by the generalization of Lemma 9.4.4.2,

(6) $$\|f * \mu_{(a)}\| \leqslant \sum_{j=-\infty}^{\infty} \|f * \mu_{(a)} * \varphi_{(2^j ca)}\| \,.$$

Since $\hat\mu$ has compact support and $\hat\varphi(x) = 0$ for $|x| \leqslant 1/2$, we can choose c (ind of a) such that $\mu_{(a)} * \varphi_{(2^j ca)} = 0$ for $j \leqslant -1$; making this choice of c, we those summands in (6) with $j < 0$. Now,

$$(\mu_{(a)} * \varphi_{(2^j ca)})^\wedge(x) = \hat\mu(ax)\, \hat\varphi(2^j cax) = P(ax)\, \hat\theta(ax)\, \hat\varphi(2^j cax)$$

$$= P(2^j cax)\, \hat\varphi(2^j cax)\, \hat\theta(ax) (2^j c)^{-r} = (2^j c)^{-r}\, G(2^j cax)\, \hat\theta(ax)$$

where $G = P\hat{\varphi}$ is of class $C^\infty(\hat{R}^n)$ and supported in $\{1/2 \leqslant |x| \leqslant 2\}$. Writing $G = \hat{g}$, with $g \in L^1(R^n)$, we have therefore

$$\|f * \mu_{(a)}\| \leqslant \sum_{j=0}^{\infty} \|f * \mu_{(a)} * \varphi_{(2^j ca)}\| = \sum_{j=0}^{\infty} (2^j c)^{-r} \|f * g_{(2^j ca)} * \theta_{(a)}\|$$

$$\leqslant c^{-r} \|\theta\|_M \sum_{j=0}^{\infty} 2^{-jr} \|f * g_{(2^j ca)}\| .$$

Moreover, by 9.4.4.1 (taking $g\,dt$ in place of the τ in that lemma), for any $b > 0$

$$\|f * g_{(b)}\| \leqslant A_2\, \omega_\sigma(f;\, A_3 b) ,$$

where A_2, A_3 (as A_4, A_5, ... below) depend only on σ, τ. Hence

$$\|f * \mu_{(a)}\| \leqslant c^{-r}\|\theta\|_M A_2 \sum_{j=0}^{\infty} 2^{-jr}\, \omega_\sigma(f;\, A_3\, 2^j ca) = A_4 \sum_{j=0}^{\infty} 2^{-jr}\, \omega_\sigma(f;\, A_5\, 2^j a) .$$

Because of the inequality

$$\int_{2^j}^{2^{j+1}} u^{-r}\, \omega_\sigma(f;\, A_5\, au)\, du/u \geqslant 2^{-r(j+1)}\, \omega_\sigma(f;\, A_5\, 2^j\, a) \log 2 ,$$

we have

$$\|f * \mu_{(a)}\| \leqslant A_6 \int_1^\infty u^{-r}\, \omega_\sigma(f;\, A_5\, au)\, du/u = A_7\, a^r \int_{A_5 a}^\infty u^{-r}\, \omega_\sigma(f;\, u)\, du/u$$

and the right side of (1) exceeds this last expression if A is chosen large enough. This yields the needed estimate for $\omega_\mu(f;\, a)$, and in view of (4), (5) the proof is complete. ◇

9.4.4.6 Refinements of the preceding theorems. Recently Jan Boman has shown that if, in Theorem 9.4.4.4, σ is subjected to the additional requirement to belong to a certain subclass N of $M = M(R^n)$, the conclusion of the theorem can be strengthened to

(1) $$\omega_\tau(f;\, a) \leqslant A\omega_\sigma(f;\, Ba)$$

for certain (other) constants A, B depending only on σ and τ. Using th to refine also the conclusion of Theorem 9.4.4.5, in the case that $\sigma \in$

(2) $$\omega_\tau(f;\, a) \leqslant A\, a^r \int_a^\infty u^{-r}\, \omega_\sigma(f;\, u)\, du/u .$$

Concerning the class N, we shall not give its definition here, but onl it contains every measure which is the sum of a non-null purely atomic an absolutely continuous one. One very special, but none the less inte of Boman's results is that where $\sigma = \delta - k\,dt$, where δ denotes as usua measure (unit point mass at 0), and $k \in L^1(R^n)$, $\int k\,dt = 1$. Here $\hat{\sigma}(x) =$ as $|x| \to \infty$, so $\hat{\sigma}$ satisfies the Tauberian condition. Let us show that w cial choice of σ, if $\tau \in M$ and $\hat{\sigma}$ divides $\hat{\tau}$ at 0, then (1) holds.

Indeed, suppose first $\mu \in M$ and $\hat{\mu}(x)$ vanishes for $|x| \leqslant c$. By 7. exist a number $b > 0$, and $\nu \in M$ such that $\hat{\nu}(x)\, \hat{\sigma}(x) = 1$ for $|x| \geqslant b$. He

$$\hat{\mu}(x)(\hat{\nu}(bx/c)\, \hat{\sigma}(bx/c) - 1) = 0 , \quad x \in \hat{R}^n ,$$

i.e. $\hat{\mu}(x) = \hat{\mu}(x)\, \hat{\nu}(bx/c)\, \hat{\sigma}(bx/c)$, showing that μ is a multiple (in M) o hence

(3) $$\omega_\mu(f;\, a) \leqslant A_1\, \omega_\sigma(f;\, B_1 a)$$

holds, with $A_1 = \|\mu\|_M\, \|\nu\|_M$, $B_1 = b/c$. If now $\hat{\sigma}$ divides $\hat{\tau}$ at 0, then, fo $\hat{\mu} = \hat{\tau} - \hat{\rho}\, \hat{\sigma}$ vanishes in a neighborhood of 0. Since

$$\omega_\tau(f;\, a) \leqslant \|\rho\|_M\, \omega_\sigma(f;\, a) + \omega_\mu(f;\, a) ,$$

we obtain (1) upon substituting into this last inequality the estimate (This proves the asserted proposition. Under the same hypothesis on σ, w deduce the sharpened version of Theorem 9.4.4.5 whereby (2) replaces 9.

Observe that if $\sigma = \delta - k\,dt$, where $\int k\,dt = 1$ and $k(t) \geqslant 0$, then 1 except at $x = 0$, the hypothesis "$\hat{\sigma}$ divides $\hat{\tau}$ at 0" implies "$\hat{\sigma}$ divides in view of 7.3.11.3, and the conclusion of Theorem 9.4.4.4 can be sharpe

$$\|f * \tau_{(a)}\| \leqslant A\, \|f * \sigma_{(a)}\| ;$$

particular (1) holds with B = 1, in view of 9.4.1.1 (2).

The full results of Boman, which reach much deeper than the special case just cussed, are not yet published. He kindly made available to us a preliminary manu-.pt and consented to the inclusion of some of his material (which here was placed the setting of HBS) in the present paragraph.

It should also be mentioned that refinements, of a different character, of orems 9.4.4.4 and 9.4.4.5 are known when the underlying HBS is $L^p(R^n)$ or $L^p(T^n)$, $p < \infty$; see BOMAN & SHAPIRO$_{1,2}$.

5 An "inverse theorem". The preceding comparison theorems have numerous applica-ns; in the case where the underlying HBS is C(T) or $\dot{C}(R)$ (bounded uniformly con-ious functions on R), a number of such applications were given in SHAPIRO$_1$; one .d repeat these applications nearly word for word in the general context of HBS. e this is just a matter of routine, we shall choose only one example, to which ave already referred earlier: the inverse theorem (due to S. Bernstein and de la ée Poussin) for trigonometric approximation. The classical case is that where C(T).

5.1 Theorem. Let B be a HBS consisting of 2π-periodic measurable functions on nd containing $\mathcal{T}_n$, n = 0, 1, ... Suppose $f \in B$ and dist $(f, \mathcal{T}_n) = O(n^{-c})$, where $c < 1$. Then $\omega(f, a) = O(a^c)$, where ω denotes the B-modulus of continuity of f.

f. If τ is the "dipole" measure, which places masses of 1, -1 at the points 0, 1 respectively, $\omega_\tau(f; a)$ is just $\omega(f, a)$. For this measure $\hat{\tau}(x) = 1 - e^{-ix}$, τ satisfies the hypotheses of Theorem 9.4.4.5 with r = 1, P(x) = x. Let now $L^1(R)$, $\hat{k}(x) = 1$ for $|x| \leqslant 1$, and define σ to be the measure $\delta - k\,dt$. Then, σ sfies the Tauberian condition. Let now $a > 0$, and denote by $n = n(a)$ the largest ger not exceeding 1/a. Since $\hat{\sigma}(x)$ vanishes for $|x| \leqslant 1$, we have, denoting by T_n closest element to f in $\mathcal{T}_n$, $T_n * \sigma_{(a)} = 0$; hence

$$\|f * \sigma_{(a)}\| = \|(f - T_n) * \sigma_{(a)}\| \leqslant \|\sigma\|_M \text{ dist } (f, \mathcal{T}_n) = O(a^c) \quad \text{as } a \to 0 .$$

e $\omega_\sigma(f; a) = O(a^c)$; substituting this into 9.4.4.5 (1) (with r = 1) gives

$$\omega_\tau(f; a) \leqslant A \left[\int_0^a A_1 u^{c-1} du + \int_a^\infty (a/u) A_1 u^{c-1} du \right] \leqslant A_2 a^c$$

and the theorem is proved. <

Remark 1. In case c = 1, 9.4.4.5 (1) yields

$$\omega_\tau(f; a) = O(a \log (1/a)) .$$

If we use, as our choice of τ, that measure with $\hat{\tau}(x) = (1 - e^{-ix})^2$, then the hypc theses of 9.4.4.5 are satisfied with r = 2, $P(x) = x^2$; we now get, in case c = 1, the estimate $\omega_\tau(f; a) = O(a)$ from 9.4.4.5 (1). For this τ, however, ω_τ is not a modulus of continuity but rather an analog thereof based upon the second differen (so-called "modulus of smoothness"), and we obtain (in the case B = C(T)) a theor of Zygmund, cf. e.g. LORENTZ$_1$ or NATANSON.

Remark 2. If our hypothesis about f is taken in the more general form

$$\text{dist } (f, \mathcal{T}_n) = O(\psi(n)) ,$$

where $\psi(n)$ is a decreasing function and $\lim \psi(n) = 0$, we can still, as a rule, ge information about $\omega(f, a)$ with the aid of Theorem 9.4.4.5. However, when $\psi(n)$ ten very slowly to 0, roughly when $\psi(n)$ tends to zero like 1/log n or slower, the in-tegral on the right of 9.4.4.5 (1), which arises upon application of the above te nique, is found to diverge; Theorem 9.4.4.5 fails in this case to give any inform tion. This is a weakness of the theorem, insofar as other methods (cf. LORENTZ$_1$, Section 4.4, or TIMAN, Section 6.1) allow a non-trivial conclusion about $\omega(f, a)$ be drawn, no matter how slowly $\psi(n)$ tends to zero.

Just this is the point of Boman's refinement of Theorem 9.4.4.5, discussed 9.4.4.6; the measure σ is of the special type for which 9.4.4.6 (2) holds, and th integral on the right side of that inequality converges always. It can be shown t yield an essentially best possible estimate for $\omega(f, a)$ (corresponding to theorem of A.F. and M.F. Timan when $B = L^p(T)$, $1 \leqslant p < \infty$, and of Stechkin when B = C(T); references in TIMAN); there are a few remarks concerning this matter in BOMAN & SI PIRO$_2$, and it shall be discussed in detail in a forthcoming work of Boman. It can shown that for measures σ not of class N (cf. 9.4.4.6), refinements of the Boman type are not possible, i.e. the hypotheses of Theorems 9.4.4.4 and 9.4.4.5 do not allow conclusions as strong as 9.4.4.6 (1) and 9.4.4.6 (2), respectively.

Exercises. a) Let β_1 denote the measure on R^1 consisting of two point-masses, namely $\beta_1(\{0\}) = 1$, $\beta_1(\{1\}) = -1$, and define $\beta_n = \beta_1^n$ (i.e. n-fold convolution power of β_1). For $B = C(T)$, the associated moduli

$$\omega_i(f, a) = \omega_{\beta_i}(f, a)$$

are the moduli of smoothness (of orders $i = 1, 2, \ldots$) of f (ω_1 is, of course, the usual modulus of continuity). Prove, for $i \geqslant 1$,

$$\omega_i(f, t) \leqslant C_i\, t^i(\|f\|_\infty + \int_t^A s^{-i-1}\, \omega_{i+1}(f, s)\, ds)$$

where A, C_i are constants independent of f.

b) Prove, for $f \in C(T)$, and $i = 1, 2, \ldots$,

$$\omega_i(f, t) \leqslant A_i\, t^i \sum_{0 \leqslant n \leqslant t^{-1}} (n + 1)^{i-1} \operatorname{dist}(f, \mathcal{T}_n)$$

where A_i depends only on i.

c) Let $k \in L^1(R)$, $\int k\, dt = 1$, and suppose there exists $\mu \in M(R)$ such that

$$(*) \qquad (1 - \hat{k}(x))\, \hat{\mu}(x) = x^2$$

for x in a neighborhood of 0. Deduce that if $f \in B$ and $\|f - (f * k_{(a)})\| = o(a^2)$ as $a \to 0$, then $\omega_{\beta_2}(f, a) = o(a^2)$. Deduce from this that (i) in case $B = \dot{C}(R)$, f is constant, (ii) in case $B = L^p(R)$, $1 \leqslant p < \infty$, f is zero. Moreover, characterize those f for which $\|f - (f * k_{(a)})\| = O(a^2)$.

d) Solve the analogous problems when, in place of x^2 on the right hand side of (*), we have x; similarly for $|x|$. (These are special cases of "saturation problems", cf. Appendix I.)

e) Take $B = C(T)$, and let φ denote the function defined in 9.4.2.1. Ver[ify] for any trigonometric polynomial f, that

$$f * \varphi_{(a)} = \tilde{f} * \theta_{(a)}$$

where $\tilde{f}$ is the function conjugate to f (i.e. if $f(t) = \sum_{-n}^{n} a_j e^{ijt}$, then $\tilde{f}(t) = - i \sum_{-n}^{n} (\operatorname{sgn} j)\, a_j e^{ijt}$) and $\hat{\theta}(x) = (i \operatorname{sgn} x)\, \hat{\varphi}(x)$. Use this to deduce Prival[ov's] Theorem, i.e. if $g \in \operatorname{Lip} \alpha$, $0 < \alpha < 1$, then there exists a function in Lip α w[hose] Fourier coefficient of order n is $(i \operatorname{sgn} n)\, \hat{g}(n)$.

f) Let u be a bounded harmonic function in the upper half plane, satisf[ying] for $y \leqslant y_o$,

$$(**) \qquad \sup_x \left| \frac{\partial u}{\partial x}(x, y) \right| \leqslant A(y) .$$

(i) If $\int_0^{y_o} A(y)\, dy < \infty$, show u is continuously extendible to the closed plane, and obtain an estimate for the modulus of continuity of $u(x, 0)$.

(ii) Construct an example where (**) holds with $A(y) = (y \log 1/y)^{-1}$, y[et u] is not continuously extendible to the closed half-plane. (Note: this is an exa[mple] of a problem whereby Theorems 9.4.4.4 and 9.4.4.5 are applicable, cf. SHAPIRO$_1$ Chapter V, but the refined estimates discussed in 9.4.4.6 are not.)

(iii) Carry out a similar analysis for each of the differential operato[rs] $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\, \partial y}$, as well as generalizations to bounded harmonic functions in half-space of R^n.