The Open Quantum Brownian Motion and continual measurements

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Abstract

This article is a mathematical analysis of the Open Quantum Brownian Motion. This object was introduced in [11] as the limit of a family of Open Quantum Random Walks on the graph $\mathbb{Z}$. We prove the convergence for the three possible description of this object: the quantum trajectory following a Belavkin equation, the unitary evolution on the Fock space following a quantum Langevin equation, and the Lindbladian evolution. We introduce a very general framework for the continual measurement of non-demolition observables, which is applied to the measurement of the position of the Open Quantum Brownian Motion, and we probe some questions related to the convergence of processes in this context.

1 Introduction

Open Quantum Random Walks (OQW) where introduced in [8] as a quantum generalization of discrete Markov chains. They consists into a particle moving randomly on a discrete graph with transition probability depending on its internal quantum state, and they may model a quantum system subject to dissipation or repeated measurement with control, being used for example as a toy model to study coherence in photosynthetic cells [20]. While OQW are defined on discrete graph and on discrete space, the Open Quantum Brownian Motion (OQBM) was introduced in [11] as a process similar to OQW but modeling a particle moving on $\mathbb{R}$ in continuous time. It was defined as the limit of a family of OQW on $\mathbb{Z}$, with the time scale τ going to zero and the space scale $\delta = \sqrt{\tau}$ in order to get a diffusive limit. The obtained process depends in two operators N and H; in the trivial case with $N = H = 0$ the classical Brownian motion is recovered. The Open Quantum Brownian Motion has been derived from a microscopic physical model in [25] and [24]. A mathematically interesting phenomenon was observed on the OQBM, namely the transition from diffusive to ballistic behavior as the parameters N and H are changed [10] with the appearance of so called spikes in the ballistic regime [28] [12], which where then studied in the context of more general stochastic differential equations [9].

As for OQW, the OQBM has three different descriptions. It can be seen has a Lindblad evolution $\rho_t = \Lambda_S^t(\rho_0)$ on the Hilbert space $\mathcal{H}_G \otimes L^2(\mathbb{R})$, where $\mathcal{H}_G$ represents the internal state of the particle. This evolution admits a unitary dilation $\rho_{tot,t} = \mathfrak{U}_t(\rho_0 \otimes |\Omega\rangle\langle\Omega|)\mathfrak{U}_t^*$ on $\mathcal{H}_G \otimes L^2(\mathbb{R}) \otimes \Phi$, where Φ is the Fock space and $\mathfrak{U}_t$ satisfies a Hudson-Parthasarathy equation. This representation is more complete than the Lindbladian one, since it allows to compute the quantum correlation between the events at two different times. Finally, upon the continual measure of the position of the particle, it admits a quantum trajectories unraveling as the random process $(\varrho_t, X_t)_{t\in\mathbb{R}}$ where ϱ_t is a random state on $\mathcal{H}_G$ and $X_t \in \mathbb{R}$ is a random position. When $\mathcal{H}_G$ is of finite dimension they obey a classical stochastic differential equation, previously known as the Belavkin equation [19] [13].

In the original article on the OQBM [11], most results where derived formally but not rigorously proved. The main purpose of this article is to explicit the mathematical meaning of the statements of [11], pointing out some of the mathematical issues and completing the proofs.

In the second section of this article, we introduce the main concepts needed to define the OQBM (Open Quantum Walks, repeated measurement model, and the Hudson-Parthasarathy calculus) and we prove the convergence of the discrete models for the OQBM to the continuous one in each description: for the unraveled process, we prove a convergence in distribution in the Skorokhod space as a direct consequence of a theorem of Pellegrini [23]. For the unitary dilations, the strong convergence of the unitary operators is proved from a theorem of Attal and Pautrat [7]. This strong convergence allows to prove the strong convergence for the Lindblad operators. A mathematical issue is outlined in the description of the Lindbladian: for an OQW, the evolution projects the states on the set of *diagonal state*, i.e. states the form $\rho = \sum_{x\in\mathcal{V}} \rho(x) \otimes |x\rangle\langle x| \in \mathfrak{S}(\mathcal{H}_G \otimes L^2(\mathcal{V}))$, where $\mathcal{V}$ is the set of vertex of the graph on which the particle is moving. In the continuous case, diagonal operators are replaced by operators in the multiplication form $\int_{\mathbb{R}} \rho(x) d|x\rangle\langle x|$, which cannot be trace class and hence cannot be a state. Hence, the discrete object which converge to the continuous OQBM is actually not an OQW in the strict meaning of the term, though it coincides with an OQW on the set of diagonal states.

In the third section we look into another claim of the article [11], in which the unraveled process $(\varrho_t, X_t)_{t\in[0,T]}$ is obtained from the continual measurement of an observable under the evolution by the unitary operators $\mathfrak{U}_t$. This makes use of the quantum filtering theory [18] [19] [13] and the notion of non-demolition measurement. We introduce rigorously the continual measurement of non-demolition observables in a way which is equivalent to the quantum filtering approach but we believe is more adapted to the SchrÃűdinger picture of the evolution, and we apply it to the case of the OQBM. Finally, we ponder the relation between the convergence of the unitary operators $\mathfrak{U}_t$ and the convergence in distribution of the unraveling, obtaining only an incomplete result which generate a few open questions.

2 The Open Quantum Brownian motion

In this section, we describe the basic objects in quantum mechanics and we introduce the repeated interaction setup and the Belavkin equation, leading to the three descriptions of the Open Quantum Brownian Motion (OQBM): as a Lindblad semigroup, as a stochastic process, and as a unitary evolution following a quantum Langevin equation. We rigorously prove the convergence of the discrete OQBM to the continuous one.

2.1 Von Neumann algebras and quantum states

The basic object in quantum mechanics is a separable Hilbert space $\mathcal{H}$ (all Hilbert spaces are implicitly supposed to be separable in this article). Let us gather some of the notations and definitions we will use:

- The identity operator on $\mathcal{H}$ (respectively $\mathcal{H}_A$ and $\mathbb{C}^n$) is written $I_{\mathcal{H}}$ (respectively I_A and I_n) or simply I when it does not cause confusion. If $\mathcal{H}_A$ and $\mathcal{H}_B$ are two space and A is an operator on $\mathcal{H}_A$, we will still write A the operator $A \otimes I_B$ on $\mathcal{H}_A \otimes \mathcal{H}_B$.

- A vector $v \in \mathcal{H}$ may also be written $|v\rangle$, and the corresponding linear form is written $\langle v|$, so that $|v\rangle \langle v|$ is the projection on v. In the tensor space $\mathcal{H}_A \otimes \mathcal{H}_B$, for any vector $|v\rangle \in \mathcal{H}_B$ and operator A from $\mathcal{H}_A \otimes \mathcal{H}_B$ to some space $\mathcal{H}$ we write $A\,|v\rangle_{\mathcal{H}_B}$ the operator from $\mathcal{H}_A$ to $\mathcal{H}$ defined by $(A\,|v\rangle_{\mathcal{H}_B})(|u\rangle) = A(|u\rangle \otimes |v\rangle)$.

- The algebra of bounded operators on $\mathcal{H}$ is written $\mathcal{B}(\mathcal{H})$, endowed with the operator norm $\|A\|$ (sometimes written $\|A\|_\infty$ to avoid confusion with other norms). The space of compact operators on $\mathcal{H}$ is written $\mathcal{B}_0^\infty(\mathcal{H})$.

- The adjoint of an operator A si written A^*.

- The Schatten space of order p is the space $\mathcal{S}^p(\mathcal{H})$ of bounded operators A such that $\mathrm{Tr}\,(|A|^p) < +\infty$, endowed with the norm $\|A\|_p = \mathrm{Tr}\,(|A|^p)^{1/p}$. In particular, $\mathcal{S}^1(\mathcal{H})$ is the set of trace-class operators.

- The σ-weak (or ultraweak) topology on $\mathcal{B}(\mathcal{H})$ is the topology generated by the seminorms

$$\|A\|_{(u_i)_{i\in\mathbb{N}},(v_i)_{i\in\mathbb{N}}} = \sum_{i\in\mathbb{N}} \langle u_i,\, Av_i\rangle$$

  where the u_i and v_i's are vectors in $\mathcal{H}$ with $\sum_{i\in\mathbb{N}} \|u_i\|^2 + \|v_i\|^2 < +\infty$.

- For a measured space $(\mathcal{X}, \mathcal{F}, \mu)$ we write the corresponding L^p space as $L^p(\mathcal{X}, \mathcal{F}, \mu)$ or when it does not cause confusion $L^p(\mathcal{X}, \mu)$ or even $L^p(\mathcal{X})$.

- For any Banach space B, we write $L^2(\mathcal{X}, B, \mu)$ the set of L^2 function from $\mathcal{X}$ to B, and the Sobolev space of functions $f : \mathbb{R}^n \to B$ with distributional derivatives $f^{(k)} \in L^p$ for $k < l$ is written $W^{l,p}(\mathbb{R}^n, B)$. For $p = 2$ and $B = \mathcal{H}$ a Hilbert space, it is itself a Hilbert space and is written $H^l(\mathbb{R}^n, \mathcal{H})$. It is isomorphic to $\mathcal{H} \otimes H^l(\mathbb{R}^n)$ and injected to a dense subset of $L^2(\mathbb{R}^n, \mathcal{H}) = \mathcal{H} \otimes L^2(\mathbb{R}^n)$. We write $X = M_{x \mapsto x}$ the position operator (defined by $Xf(x) = xf(x)$), and $P = -i\partial_x$ the impulsion operator with domain $H^1(\mathbb{R}, Leb)$.

- On the space $L^2(\mathcal{X}, \mu)$, for any essentially bounded function $f : \mathcal{X} \to \mathbb{C}$ we write M_f the operator of multiplication by f, defined by $M_f g(x) = f(x)g(x)$ for any g such that $fg \in L^2(\mathcal{X}, \mu)$.

- We write $\mathbb{1}_A$ the indicator function of the set A, and $\mathbb{1} = \mathbb{1}_{\mathcal{X}}$.

- We write $\otimes_{alg}$ the algebraic tensor product and $\otimes$ the completed tensor product of Hilbert spaces.

Note that all infinite dimensional Hilbert spaces are isomorphic, hence we can take $\mathcal{H} = L^2(\mathcal{X}, \mathcal{F}, \mu)$ where $(\mathcal{X}, \mathcal{F}, \mu)$ is a standard measured space. The interpretation of the Hilbert space depends on the space $\mathcal{X}$. Let us first remind some facts about standard measured spaces.

2.1.1 Standard measured space

Standard measured spaces form a very large class of measured space; notably, two spaces of special interest in this article are $\mathcal{X} = \mathbb{R}$ with the Lebesgue measure, and $\mathcal{X} = \mathcal{W}([0, +\infty))$ the Wiener space on $[0, +\infty)$ equiped with the Wiener measure (i.e. the space of continuous functions on $[0, +\infty)$ equipped with the measure corresponding to the Brownian motion). Standard measured spaces have many different characterizations, see the chapter on Lebesgue-Rohlin spaces in Bogachev II [15]; let us describe two of them:

Definition 1. *Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a measured space (every measured are nonnegative in this article). It is called a* standard measured space *if it satisfies one of the following equivalent properties:*

1. *There exists a measure ν on $\mathbb{R}$ the form $\nu = \nu_1 + \sum_{i \in \mathbb{N}} c_i \delta_i$ where ν_1 is absolutely continuous with respect to the Lebesgue measure and the c_i are nonnegative numbers such that $(\mathcal{X}, \mathcal{F}, \mu)$ is almost isomorphic to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, that is, there exists sets of full measure $A \subset \mathcal{X}$ and $B \subset \mathbb{R}$ and a measure-preserving isomorphism between (A, μ) and (B, ν).*

2. *There exists a complete metric d on a set of full measure $D \subset \mathcal{X}$ such that $\mathcal{F}|_D$ is the σ-algebra generated by open sets for d and μ is a Radon measure for this topology.*

Note that standard measured spaces are necessarily almost separated (i.e. for almost every $x \neq y \in \mathcal{X}$ then there exists two disjoint measurable sets

$A, B \in \mathcal{F}$ with $x \in A$ and $y \in B$). More importantly, if $\mathcal{F}_1 \subset \mathcal{F}$ is another σ-algebra, the measured space $(\mathcal{X}, \mathcal{F}_1, \mu)$ is standard if and only if $\mathcal{F}_1 = \mathcal{F}$. If $\mathcal{F}_1 \neq \mathcal{F}$, we make $(\mathcal{X}, \mathcal{F}_1, \mu)$ into a standard probability space by quotient:

Definition 2. *For any standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ with a sub-σ-algebra $\mathcal{F}_1$, let $\mathcal{X}/\mathcal{F}_1$ the quotient of $\mathcal{X}$ by the relation: $x \sim y$ if every set $A \in \mathcal{F}_1$ containing x also contains y. There is a surjective map $s_{\mathcal{F}_1} : \mathcal{X} \to \mathcal{X}/\mathcal{F}_1$, we endow $\mathcal{X}/\mathcal{F}_1$ with the image of $\mathcal{F}_1$ by $s_{\mathcal{F}_1}$ and the push-forward measure of μ by $s_{\mathcal{F}_1}$, which we still write $\mathcal{F}_1$ and μ. The space $(\mathcal{X}/\mathcal{F}_1, \mathcal{F}_1, \mu)$ is a standard measured space, called the* quotient *of $(\mathcal{X}, \mathcal{F}, \mu)$ by $\mathcal{F}_1$.*

There exists many different maps $r_{\mathcal{F}_1} : \mathcal{X}_1 \to \mathcal{X}$ such that $s_{\mathcal{F}_1} \circ r_{\mathcal{F}_1} = I_{\mathcal{X}_1}$. Each of them gives an identification of $\mathcal{X}_1$ with a subspace of $\mathcal{X}$, and we have a map $c = r_{ff_1} \circ s_{\mathcal{F}_1} : \mathcal{X} \to \mathcal{X}$ onto this subspace.

An extension *of a standard measured space $(\mathcal{X}_1, \mathcal{F}_1, \mu_1)$ is another standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ with a surjective measurable map $s : \mathcal{X} \to \mathcal{X}_1$ such that the push forward measure $s_*\mu$ of μ by s is μ_1.*

These notions are useful in the description of commutative von Neumann algebras.

2.1.2 Von Neumann algebras

The set of quantum observables of a system is described by a von Neumann algebra on $\mathcal{H}$, i.e. a unital subalgebra of $\mathcal{B}(\mathcal{H})$ which is stable by adjoint and closed for the strong topology. This article does not involve most of the subtleties of von Neumann algebra theory, since we are essentially interested in the simplest cases: the full algebra $\mathcal{B}(\mathcal{H})$, the commutative von Neumann algebras and the tensor products of these. Let us recall a few facts about commutative von Neumann algebras:

1. For any standard probability space $(\mathcal{X}, \mathcal{F}, \mu)$ and any sub-σ-algebra $\mathcal{F}_1 \subset \mathcal{F}$ the space $L^\infty(\mathcal{X}, \mathcal{F}_1, \mu)$ is identified with a commutative von Neumann algebra on $L^2(\mathcal{X}, \mathcal{F}, \mu)$ by $f \in L^\infty(\mathcal{X}, \mathcal{F}_1, \mu) \mapsto M_f$ (the operator of multiplication by f).

2. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a commutative von Neumann algebra. Then there exists a standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$, a sub-$\sigma$-algebra $\mathcal{F}_1 \subset \mathcal{F}$ and a unitary π from $L^2(\mathcal{X}, \mathcal{F}, \mu)$ to $\mathcal{H}$ such that $\mathcal{A} = \pi^* \ L^\infty(\mathcal{X}, \mathcal{F}_1, \mu) \ \pi$. Thus, if we consider the quotient $\mathcal{X}_1 = \mathcal{X}/\mathcal{F}_1$, then $\mathcal{A}$ is isomorphic (as a C^*-algebra) to $L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu)$. The algebra $\mathcal{A}$ is a maximal commutative von Neumann algebra if and only if $\mathcal{F}_1 = \mathcal{F}$ (up to measure-zero sets). It is called "discrete" if $\mathcal{X}_1$ is countable or finite, the σ-algebra $\mathcal{F}_1$ is then called "coarse"[1].

3. Let $\mathcal{A}_1 \subset \mathcal{A}_2$ be two commutative von Neumann algebras on a von Neumann algebra with two isomorphisms of C^*-algebras $\psi_1 : \mathcal{A}_1 \to$

[1] the term "discrete" σ-algebra often refers to the σ-algebra of all subsets of $\mathcal{X}$, so we use coarse to avoid confusion

$L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu_1)$ and $\psi_2 : \mathcal{A}_2 \to L^\infty(\mathcal{X}_2, \mathcal{F}_2, \mu_2)$. Then there exists a measurable map $\eta : \mathcal{X}_2 \to \mathcal{X}_1$ such that μ_1 is absolutely continuous with respect to the push forward measure $\eta_* \mu_2$ and for any $f \in L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu_1)$ we have $\psi_2 \circ \psi_1^{-1}(f) = f \circ \eta$.

See Takesaki's book [27], notably Theorem 8.21 and Lemma 8.22. An application of the las fact is that if U is an isometry of $\mathcal{H}$ with $U\mathcal{A}_1 U^* \subset \mathcal{A}_2$ then its action on $\mathcal{A}_1$ can be implemented by some map η between the underlying spaces $\mathcal{X}_1$ and $\mathcal{X}_2$.

A full study of a non-maximal commutative von Neumann algebra involves direct integrals of Hilbert spaces. We don't need it here, so let us just give a taste of it: if $\mathcal{A} \simeq L^\infty(\mathcal{X}_1, \mathcal{F}_1, \mu)$ then we can decompose $\mathcal{H}$ as $\int_{\mathcal{X}_1}^{\oplus} \mathcal{H}(x) d\mu(x)$ where $x \mapsto \mathcal{H}(x)$ is a measurable field of Hilbert spaces, and the elements of $\mathcal{A}$ are operators the form $\int_{\mathcal{X}}^{\oplus} f(x) I_{\mathcal{H}(x)} d\mu(x)$.

2.1.3 Quantum states

The state of a quantum system with observables in a von Neumann algebra $\mathcal{A}$ is modeled the following way:

Definition 3. *A (normal) state on a von Neumann algebra $\mathcal{M}$ is a linear form ρ on $\mathcal{M}$ which is:*

- *positive, i.e. $\rho(A) \geq 0$ for any positive operator $A \in \mathcal{M}$.*
- *normed, i.e. $\rho(I) = 1$*
- *normal, i.e. continuous for the σ-weak topology (or equivalently for any sequence of mutually orthogonal projections $(p_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$ we have $\sum_i \rho(p_i) = \rho(\sum_i p_i)$).*

The set of states on $\mathcal{M}$ is written $\mathfrak{S}(\mathcal{M})$ or simply $\mathfrak{S}(\mathcal{H})$ if $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Let us consider the two cases of maximal commutative von Neumann algebra and of the full von Neumann algebra:

States on $\mathcal{A} = L^\infty(\mathcal{X}, \mathcal{F}, \mu)$: any state ρ on $\mathcal{A}$ is the form

$$\rho(f) = \int_{\mathcal{X}} f(x) p_\rho(x) d\mu(x)$$

where p is a positive function on $\mathcal{X}$ with $\int_{\mathcal{X}} p(x) d\mu(x) = 1$. Hence the set $\mathfrak{S}(L^\infty(\mathcal{X}, \mu))$ can be identified with the set of probability measures which are absolutely continuous with respect to μ.

States on $\mathcal{B}(\mathcal{H})$: any state ρ on the full algebra is the form

$$\rho(A) = \operatorname{Tr}(AT_\rho)$$

where T_ρ is a positive trace-class operator on $\mathcal{H}$ with $\operatorname{Tr}(T_\rho) = 1$. The same letter will design the state ρ and the operator T_ρ and we identify the set $\mathfrak{S}(\mathcal{H})$ with the set of positive trace-class operators of trace 1.

States on $\mathcal{B}(\mathcal{H}) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu)$: This is the mix of the two previous situations: a state ρ on $\mathcal{B}(\mathcal{H}) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu) \subset \mathcal{B}(\mathcal{H} \otimes L^2(\mathcal{X}, \mathcal{F}, \mu))$ is the form

$$\rho(A \otimes f) = \int_{\mathcal{X}} \mathrm{Tr}\left(AQ_\rho(x)\right) f(x) d\mu(x)$$

where $x \mapsto Q_\rho(x)$ is a measurable function from $\mathcal{X}$ to the set of positive trace-class operators on $\mathcal{H}$ such that $\int_{\mathcal{X}} \mathrm{Tr}\left(Q_\rho(x)\right) d\mu(x) = 1$. We will call $Q_\rho(x)$ the density matrix function.

Remark 1.

1. If $\mathcal{M}_1 \subset \mathcal{M}_2$ are two von Neumann algebras, we may extends states on $\mathcal{M}_1$ to states on $\mathcal{M}_2$, and restrict states on $\mathcal{M}_2$ to states on $\mathcal{M}_1$. In particular, if $\mathcal{M}_1 = L^\infty(\mathcal{X}, \mu)$ and $\mathcal{M}_2 = \mathcal{B}(L^2(\mathcal{X}, \mu))$, a state on $\mathcal{M}_1$ can be extended in many different ways to a state on $\mathcal{M}_2$, notably we can make it a pure state: take $f = \sqrt{p}$ where p is the probability density of the state with respect to μ, and consider the state $|f\rangle \langle f|$ on $\mathcal{M}_2$. We may also be tempted to take the multiplication operator M_p as another extension, but this operator is not trace class in general.

2. Another important example is the case of a bipartite system. If $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and we are given a state ρ on $\mathcal{M}_2 = \mathcal{B}(\mathcal{H})$, its restriction to $\mathcal{M}_1 = \mathcal{B}(\mathcal{H}_A) \otimes \{ I_B \}$ has for density matrix the partial trace of ρ with respect to B, that is $\rho_B = \mathrm{Tr}_B(\rho)$.

3. With $\mathcal{M}_1 = bb(\mathcal{H}_A) \otimes L^\infty(\mathcal{X}, \mathcal{F}, \mu)$ and $mm_2 = \mathcal{B}(hh_A \otimes L^2(\mathcal{X}, \mathcal{F}, \mu))$ the situation is more subtle. A state ρ on $\mathcal{M}_2$ can always be described by a kernel $(x, y) \mapsto K_\rho(x, y)$ from $\mathcal{X}$ to $\mathcal{S}^1(\mathcal{H})$, such that for any function $f \in L^2(\mathcal{X}, \mathcal{H}_A) = \mathcal{H}_A \otimes L^2(\mathcal{X}, \mathcal{F}, \mu)$ we have

$$\rho f(x) = \int_{xx} K_\rho(x, y) f(y) d\mu(y)$$

(where we see ρ as an operator on $\mathcal{H}$). To describe the state $\rho_{\mathcal{M}_1}$ on $\mathcal{M}_1$ it seems natural to take for density matrix function $Q_{\rho_{\mathcal{M}_1}}(x) = K_\rho(x, x)/\mathrm{Tr}\left(K_\rho(x, x)\right)$. Unless K is continuous with respect to some metric, this requires technicalities since the diagonal $\{(x, x) | x \in \mathcal{X}\}$ is possibly of measure zero in $(\mathcal{X} \times \mathcal{X}, \mu \otimes \mu)$. This can be solved by averaging on small rectangles (see Brislawn [16]) or with the notion of virtual continuity (see Vershik et al. [29]).

2.1.4 Measure of an observable

Let A be a self-adjoint operator on $\mathcal{H}$ (which is not necessarily bounded). Assume that the system is in the state ρ. The measurement of A is mathematically described the following way: the von Neumann algebra $\mathcal{A}$ generated by A is commutative, so there exists a unitary $\pi : \mathcal{H} \to L^2(\mathcal{X}, \mu)$ for some standard measured space $(\mathcal{X}, \mathcal{F}, \mu)$ and a measurable function $g : \mathcal{X} \to \mathbb{R}$ such that

$\pi^* A \pi = M_g$. Let ρ be the state on the system, then $\pi^* \rho \pi$ restricts to a state on $L^\infty(\mathcal{X}, \mu)$, that is, a probability measure $\mathbb{P}_\rho$ on $\mathcal{X}$ which is absolutely continuous with respect to μ. This makes $(\mathcal{X}, \mathbb{P}_\rho)$ a probability space. The result of the measurement is then the random variable $\tilde{A}_\rho$ on $(\mathcal{X}, \mathbb{P}_\rho)$ defined by the function g.

Note that for a commuting family of self-adjoint operators $(A_\alpha)_{\alpha \in I}$ we can consider their joint spectral theory: there exists a unitary $U : \mathcal{H} \to L^2(\mathcal{X}, \mu)$ with $U^* A_\alpha U = M_{g_\alpha}$ for a family of functions $(g_\alpha)_{\alpha \in I}$. Thus, we can consider the family of random variables $\tilde{A}_{\alpha,\rho}$ on the same probability space $(\mathcal{X}, p_\rho)$. However, if A and B are not commuting, there is no consistent way to consider jointly $\tilde{A}_\rho$ and $\tilde{B}_\rho$ as random variables on the same probability space.

Now, it is not always possible to describe the quantum mechanical state of ρ after the measurement. In the case where A has only pure point spectrum, it is possible and we do it as follows.

Definition 4 (State after the measurement)**.** *Let A be an observable the form*

$$A = \sum_{a \in sp(A)} a P_a$$

where the P_a are mutually orthogonal projections. Write $\mathcal{A}$ the commutative von Neumann algebra generated by A, it is isomorphic to $L^\infty(sp(A), \sum_a \delta_a)$. We endow $sp(A)$ with the probability $\mathbb{P}_\rho(a) = Tr(\rho P_a)$. The state after the measurement of A is the radom variable $\rho_{|\mathcal{A}}$ on $(sp(A), \mathbb{P})$ defined by

$$\rho_{|\mathcal{A}}(a) = \frac{P_a \rho P_a}{Tr(P_a \rho)}\,.$$

We may also write $\rho_{B|\mathcal{A}} = Tr_B(\rho|\mathcal{A})$, and to shorten notation we will often use the variant calligraphy ϱ for a random density matrix corresponding to a deterministic density matrix ρ.

The action of *forgetting the result of the measurement* consists in discarding the random variable $\tilde{A}_\rho$ and replacing $\rho_{|\mathcal{A}}(a)$ by its expectancy $\rho' = \mathbb{E}(\rho_{|\mathcal{A}})$. The operator $\rho' = \sum_{a\in(A)} P_a \rho P_a$ is in $\mathfrak{S}(\mathcal{H})$. It carries all the information we can get without $\tilde{A}_\rho$, since $\mathbb{E}(\mathrm{Tr}\left(\rho_{|\mathcal{A}} B\right)) = \mathrm{Tr}\left(\rho' B\right)$ for any observable $B \in \mathcal{B}(\mathcal{H})$.

If A has singular spectrum it is no more possible to describe the state after the measurement as a random variable on $\mathfrak{S}(\mathcal{H})$. For example, if me measure the observable $X = M_{x \mapsto x}$ on $L^2(\mathbb{R}, Leb)$ the state of the system after the measurement should correspond to the Dirac measure $\delta_{\tilde{X}_\rho}$ on the algebra $L^\infty(\mathbb{R}, Leb)$, but it is not possible since states on this algebra are absolutely continuous with respect to the Lebesgue measure.

This is not really a physical problem since no real-life measurement is exact, hence we only measure discrete observables in real life. Though, it is always better to have an idealization of the measure of continuous observables, which is the subject of the second section of this article.

2.2 The Belavkin equation and the Open Quantum Brownian Motion

2.2.1 The repeated measurement process

The repeated measurement model relates to many experimental protocols, notably with the experiments of Serge Haroche's team. It describes a process on discrete time, and we will be interested in its continuous-time limit, which was notably studied by Pellegrini [23].

We consider a Hilbert space $\mathcal{H}_G$ describing a system of interest in the state $\rho_0 \in \mathfrak{S}(\mathcal{H}_G)$, and a space modeling a probe $\mathcal{H}_p$ in the fixed pure state $\rho_p = |0\rangle\langle 0|$. In this article the probe space is always $\mathcal{H}_p = \mathbb{C}^2$. Make it evolve according to some unitary V on $\mathcal{H}_G \otimes \mathcal{H}_p$ and measure some observable $A \in \mathcal{B}_{sa}(\mathcal{H}_p)$. Then take a copy of $\mathcal{H}_p$, also in the state $\rho_p = |0\rangle\langle 0|$, and repeat this procedure again and again. What we obtain is a stochastic process $(\varrho_n)_{n\in\mathbb{N}}$ where $\varrho_n \in \mathfrak{S}(\mathcal{H}_G)$ is the state of the system after the n-th measurement, together with another process $(D_n)_{n\in\mathbb{N}}$ where $D_n \in \mathbb{R}$ is the result of the $n+1$-th measurement of A. Since the probe space $\mathcal{H}_p$ is constantly renewed, $(\varrho_n, D_n)_{n\in\mathbb{N}}$ is a Markov process. We can also note that for any n the state ϱ_n deterministically depends in the sequence $(D_k)_{k<n}$, since if P_d is the spectral projection for the eigenvalue d of A, we have

$$\varrho_{n+1} = \frac{\mathrm{Tr}_B\left(P_{D_n} V(\varrho_n \otimes \rho_p) V^* P_{D_n}\right)}{\mathrm{Tr}\left(P_{D_n} V(\varrho_n \otimes \rho_p) V^* P_{D_n}\right)} \ .$$

It is also interesting to study the evolution when the result of the measurement is discarded, that is, the evolution of $\rho_n = \mathbb{E}(\varrho_n)$. We have

$$\rho_{n+1} = \mathrm{Tr}_B\left(VV(\varrho_n \otimes \rho_p)V^*\right) \ .$$

The maps Λ on the set of trace-class operators which are the form $\Lambda(\rho) = \mathrm{Tr}_{\mathcal{H}_p}(V_\tau \rho \otimes \rho_p)V^*)$ are called quantum channels. They can be characterized as the completely positive, trace-preserving and σ-weakly continuous maps on bounded operators (see for example Chapter 6 of [2]). Alternately, they are the maps which are the form

$$\Lambda(\rho) = \sum_{k=1}^{d} K_k \rho K_k^*$$

where the K_k are bounded operators on $\mathcal{H}_A$ with $\sum_{k=1}^{d} K_k^* K_k = I_A$, which are called the Kraus operators for Φ. The objects $(\mathcal{H}_p, V, \rho_p)$ corresponding to Λ is called a Stinespring dilation of the channel (it is not unique).

The evolution of ρ_n is called a quantum dynamical system, and its description as the interaction of the system is called a repeated interaction model [7].

2.2.2 The Belavkin diffusive equation and the Lindblad equation

Now, we want to consider the continuous time limit of this type of process. Thus, we will consider that each step of the process lasts a time $\tau > 0$ and we make τ go to zero with suitable normalization. The case we consider is the following:

1. We take $\mathcal{H}_p = \mathbb{C}^2$ with $\rho_p = |0\rangle \langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

2. The unitary evolution V_τ on $\mathcal{H}_G \otimes \mathcal{H}_p$ is described as follows: fix a self-adjoint bounded operator $H \in \mathcal{B}(\mathcal{H}_G)$ and a bounded operator $N \in \mathcal{B}(\mathcal{H}_G)$ and take

$$V_\tau = \exp\left(-i\tau H + \sqrt{\tau}\begin{pmatrix} 0 & N^* \\ -N & 0 \end{pmatrix}\right) \tag{2.1}$$

$$= I + \sqrt{\tau}\begin{pmatrix} 0 & N^* \\ -N & 0 \end{pmatrix} + \tau\left(-iH - \frac{1}{2}\begin{pmatrix} N^*N & 0 \\ 0 & NN^* \end{pmatrix}\right) + O(\tau^{3/2}) \ . \tag{2.2}$$

3. We measure the observable $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4. The process of states obtained is written $(\varrho_{\tau,n})_{n\in\mathbb{N}}$, and the result of the $n+1$-th measurement is written $D_{\tau,n} \in \{-1, +1\}$. We also define

$$W_{\tau,n} = \sqrt{\tau}\sum_{k=0}^{n-1} D_{\tau,n} \ .$$

The normalization in $\sqrt{\tau}$ to define $W_{\tau,n}$ corresponds to a diffusive limit in physics, where the time scale τ is proportional to the square of the space scale. **In the rest of the article, we will write $\delta = \sqrt{\tau}$ the space scale.**

In this setup, the eigenvectors for the eigenvalues ± 1 of A are

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \ , \tag{2.3}$$

and we have

$$\varrho_{\tau,n+1} = \frac{B_{\tau,D_n}\varrho_{\tau,n}B^*_{\tau,D_n}}{\mathrm{Tr}\left(B_{\tau,D_n}\varrho_{\tau,n}B^*_{\tau,D_n}\right)}$$

where

$$B_{\tau,\pm 1} = \frac{1}{\sqrt{2}}\left(I \pm \delta N + \tau\left(-iH - \frac{1}{2}N^*N\right)\right) + O(\tau^{3/2}) \ . \tag{2.4}$$

The following theorem describes the limit in distribution of this process as $\tau \to 0$. It was proved by Attal And Pellegrini.

Theorem 5 (Theorem 8 of [23])**.** *Assume that $\mathcal{H}_G$ is finite-dimensional. Fix some $T > 0$. Then the process $(\varrho_{\tau,[t/\tau]}, W_{\tau,[t/\tau]})_{0\le t\le T}$ described above converges* *in distribution as $\tau \to 0$ (in the space of bounded functions with the uniform norm) to a process $(\varrho_t, W_t)_{0\leq t\leq T}$ satisfying the following stochastic equation (in the Itô sense):*

$$\begin{cases} d\varrho_t &= \mathcal{L}(\varrho_t)dt + (N\varrho_t + \varrho_t N^* - \varrho_t \mathcal{T}(\varrho_t))dB_t \\ dW_t &= \mathcal{T}(\varrho_t)dt + dB_t \end{cases} \tag{2.5}$$

where (B_t) is a standard Wiener process, $\mathcal{L}$ is the super-operator defined by

$$\mathcal{L}(\rho) = -i[H,\rho] + N\rho N^* - \frac{1}{2}\left(N^*N\rho + \rho N^*N\right) \tag{2.6}$$

and

$$\mathcal{T}(\rho) = Tr\left((N+N^*)\rho\right) \ .$$

This theorem was proved with methods of classical stochastic process, notably the Kurtz-Protter's theorem. In the case where $\mathcal{H}_G$ is not finite-dimensional, it is not clear what rigorous meaning we can give to Equation 2.5; we can show that it is verified in a weak sense, that is: for any continuous martingale g_t with values in $\mathbb{R}$ which is uniformly bounded on $[0,T]$ and any bounded operator $A \in \mathcal{B}(\mathcal{H}_G)$, defining f_t such that $df_t = g_t dB_t$, we have

$$\mathbb{E}(\mathrm{Tr}\,(\varrho_t A)\, f_t) = \mathbb{E}(\mathrm{Tr}\,(\varrho_0 A)\, f_0) + \int_0^t \mathbb{E}(\mathcal{L}(\varrho_s A) f_s)ds + \int_0^t \mathbb{E}(\mathrm{Tr}\,((N\varrho_s + \varrho_s N^* + \varrho_s \mathcal{T}(\rho_t))\, A)\, g_s)ds$$

If we discard the probes before measuring it, the state of the system is the deterministic density matrix

$$\rho_{G,\tau,n} = \mathbb{E}(\varrho_{\tau,n}) \ .$$

It follows a quantum dynamical semigroup, with $\rho_{\tau,n+1} = \Lambda_{G,\tau}(\rho_{\tau,n})$ where

$$\begin{aligned} \phi_\tau(\rho) &= B_{\tau,+1}\rho B^*_{\tau,+1} + B_{\tau,-1}\rho B^*_{\tau,-1} \\ &= \rho + \tau\mathcal{L}(\rho) + O(\tau^{3/2} \end{aligned}$$

where $\mathcal{L}$ is defined in Equation 2.6. Thus $\rho_{G,\tau,[t/\tau]}$ converges to some limit $\rho_{G,t}$ satisfying the so-called Lindblad equation

$$\frac{d}{dt}\rho_{G,t} = \mathcal{L}(\rho_{G,t}) \ .$$

The family of super-operators $\Lambda^t = e^{t\mathcal{L}}$ is called a Lindblad semigroup. Note that $\rho_t = \mathbb{E}(\varrho_t)$, which can be seen both by the above convergence or by using the fact that the term in dB_t in Equation 2.5 is of expectancy zero.

2.2.3 The discrete OQBM

A first way to describe the discrete OQBM is the following: choose a random state $\varrho_{\tau,0} \in \mathfrak{S}(\mathcal{H}_G)$ and a random position $X_{\tau,0} \in \delta\mathbb{Z}$. Then apply the repeated measurement process described in Paragraph 2.2.2 to $\varrho_{\tau,0}$, obtaining a process $(\varrho_{\tau,n}, W_{\tau,n})_{n\in\mathbb{N}}$. We call *trajectory of the discrete OQBM* the process $(\varrho_{\tau,n}, X_{\tau,n})_{n\in\mathbb{N}}$ where $X_{\tau,n} = X_{\tau,0} + W_{\tau,n}$. By Theorem 5 the process $(\varrho_{\tau,[t/\tau]}, X_{\tau,[t/\tau]})_{0\leq t}$ converges as $\tau \to 0$ to a process $(\varrho_t, X_t)_{t\in\mathbb{N}}$ which is solution of Equation 2.5. We call it the *trajectory of the continuous OQBM*. As a direct consequence of Theorem 5 we have the following convergence:

Proposition 6. *For any $T > 0$ the family of processes $(\varrho_{\tau,[t/\tau]}, X_{\tau,[t/\tau]})_{t\in[0,T]}$ converges in distribution as $\tau \to 0$ to a process $(\varrho_t, X_t)_{t\in[0,t]}$ following the following differential equation:*

$$\begin{cases} d\varrho_t &= \mathcal{L}(\varrho_t)dt + (N\varrho_t + \varrho_t N^* - \varrho_t \mathcal{T}(\rho_t))\, dB_t \\ dX_t &= \mathcal{T}(\varrho_t)dt + dB_t \end{cases} \tag{2.7}$$

where B_t is a Wiener process.

This is the trajectorial view on the OQBM; it is not much richer than the Belavkin process, the only difference is in the initial condition which is also random. Something more far-reaching is obtained when we consider the position $X_{\tau,n}$ as a part of the *quantum* description of a bigger system.

More precisely, we consider a particle on the lattice $\delta\mathbb{Z}$, described by the Hilbert space $\mathcal{H}_{\tau,z} = l^2(\delta\mathbb{Z})$. For any $x \in \delta\mathbb{Z}$ we write $|x\rangle \in \mathcal{H}_{\tau,z}$ the sequence with only nonzero component at x and equal to 1. The complete system is now $\mathcal{H}_S = \mathcal{H}_G \otimes \mathcal{H}_{\tau,z}$. The system starts in some state $\rho_S \in \mathfrak{S}(\mathcal{H}_S)$, and we make it evolve so that the particle moves like $W_{\tau,n}$. For this, we consider the translation operator D_τ on $\mathcal{H}_{\tau,z}$ defined by

$$D_\tau = \sum_{x\in\delta\mathbb{Z}} |x+\delta\rangle \langle x| \ .$$

(Note that $D_\tau^* = D_{-\tau}$ is the translation by $-\tau$). The measure of $A \in \mathcal{H}_p$ is transmitted to $\mathcal{H}_{\tau,z}$ by use of the unitary operator R_τ on $\mathcal{H}_{\tau,z} \otimes \mathcal{H}_p$ defined by

$$R_\tau = D_\tau \otimes |+\rangle \langle +| + D_{-\tau} |-\rangle \langle -|$$

(the state $|+\rangle$ and $|-\rangle$ are defined in Equation 2.3). The operator $\mathbb{R}_\tau$ will be called a pointer unitary, see Section 3.1.1.

Now, the discrete OQBM is described as the repeated interaction model with unitary $L_\tau = R_\tau V_\tau$. In other words, we make $\mathcal{H}_G$ and $\mathcal{H}_p$ interact by use of the unitary V_τ, and then we translate the particle according to the result of the measurement of A, by use of the unitary R_τ. This defines a new quantum channel $\Lambda_{S,\tau}$ on $\mathcal{H}_G \otimes \mathcal{H}_{\tau,z}$ defined by

$$\Lambda_{S,\tau}(\rho) = \mathrm{Tr}_{\mathcal{H}_p}\left(W_\tau\left(\rho\otimes\rho_p\right)W_\tau\right) \tag{2.8}$$
$$= \left(B_{+1}\otimes D_\tau\right)\rho\left(B_{+1}\otimes D_\tau\right)^* + \left(B_{+1}\otimes D_\tau\right)\rho\left(B_{+1}\otimes D_\tau\right)^* \ . \tag{2.9}$$

The state of the system is $\rho_{S,\tau,n} = \Lambda^n_{S,\tau}(\rho_S)$. We call "discrete OQBM" the dynamic described by this quantum channel. It is an extension of the repeated interaction model described above, since we have $\mathrm{Tr}_{\mathcal{H}_{\tau,z}}(\Lambda_{S,\tau}(\rho)) = \Lambda_{G,\tau}(\mathrm{Tr}_{\mathcal{H}_{\tau,z}}(\rho))$. In the next paragraph we describe another way to look at the discrete OQBM.

2.2.4 The discrete OQBM as an Open Quantum Walk

Open Quantum Random Walks (OQW) where introduced in [8]. It consists in a quantum particle moving on a graph $G = (\mathcal{V}, E)$, where the set of vertices $\mathcal{V}$ is countable or finite. The internal state of the particle is described by a space $\mathcal{H}_G$ (which is called the "Gyroscope", and have the role of $\mathcal{H}_A$; it is also called the chirality space in the literature). The Hilbert space of the position of the particle is $\mathcal{H}_z = L^2(\mathcal{V}, \nu)$ where ν is the counting measure on $\mathcal{V}$, endowed with the algebra $\mathcal{A}_z = L^\infty(\mathcal{V}, \nu)$. we write $(x \to y)$ an edge oriented from x to y and $\mathring{x} = \{\, y \in \mathcal{V} \mid (x \to y) \in E \,\}$.

An OQW on this space is described the following way: We fix a family of bounded operators $B_{(x\to y)} \in \mathcal{B}(\mathcal{H}_G)$ indexed by edges $(x \to y) \in E$ such that for all $x \in \mathcal{V}$

$$\sum_{y\in\mathring{x}} B^*_{(x\to y)} B_{(x\to y)} = I \ .$$

The OQW is then a Markov process $(\varrho_n, X_n)_{n\in\mathbb{N}}$ with $\varrho_n \in \mathfrak{S}(\mathcal{H}_G)$ and $X_n \in \mathcal{V}$ defined by

$$\varrho_{n+1} = \frac{B_{(X_n\to X_{n+1})}\varrho_n B^*_{(X_n\to X_{n+1})}}{\mathrm{Tr}\left(B_{(X_n\to X_{n+1})}\varrho_n B^*_{(X_n\to X_{n+1})}\right)}$$

and

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \mathrm{Tr}\left((B_{X_n\to X_{n+1})}\varrho_n B^*_{(X_n\to X_{n+1})}\right) \ .$$

The state $\rho_n = \mathbb{E}(\varrho_n)$ evolves according to a quantum channel φ on $\mathcal{B}(\mathcal{H}_G \otimes \mathcal{H}_z)$ defined by the Kraus operators

$$K_{(x\to y)} = B_{(x\to y)} \otimes |y\rangle \langle x| \ .$$

We can see the trajectory $(\varrho_{\tau,n}, X_{\tau,n})_{n\in\mathbb{N}}$ of the OQBM an OQW on the graph $\delta\mathbb{Z}$ with edges on nearest-neighbors, and operators $B_{x\to x\pm\delta} = B_{\pm 1}$.

Note however that the quantum channel $\Lambda_{S,\tau}$ is not equal to the quantum channel φ_τ obtained from the OQW convention. Indeed, $\Lambda_{S,\tau}$ has only two Krauss operators, $K_\pm = B_{\pm 1} \otimes D_{\pm\delta}$, while φ_τ has an infinite number of operators, one for each edge. The relation between the two is the following: let $E_\tau : \mathcal{B}(\mathcal{H}_{\tau,z} \to \mathcal{B}(\mathcal{H}_{\tau,z})$ be the projection on the algebra $\mathcal{A}_{\tau,z} = l^\infty(\delta\mathbb{Z})$ on $\mathcal{H}_{\tau,z}$, that is

$$E_\tau(A) = \sum_{x\in\delta\mathbb{Z}} |x\rangle \langle x| A |x\rangle \langle x| \ .$$

Then we have

$$\varphi_\tau = \Lambda_{S,\tau} \circ E_\tau = E_\tau \circ \Lambda_{S,\tau} \ .$$

Thus, φ_τ and $\Lambda_{S,\tau}$ coincide on the algebra $\mathcal{M}_\tau = \mathcal{B}(\mathcal{H}_G)\otimes\mathcal{A}_{\tau,z}$, as well as their adjoint maps φ_τ^* and $\Lambda_{S,\tau}^*$. The map $\Lambda_{S,\tau}$ is then perfectly good to compute the restriction $\rho_{\mathcal{M}_\tau,n}$ of $\rho_{S,n} = \varphi_\tau^n(\rho)$ to $\mathcal{M}$. This state carries much information on $\varrho_{\tau,n}$ and $X_{\tau,n}$. Indeed, if $x \mapsto Q_{\tau,n}(x)$ is the matrix density function of $\rho_{\mathcal{M}_\tau,n}$ we have

$$\mathbb{P}(X_{\tau,n} = x) = \mathrm{Tr}\left(Q_{\tau,n}(x)\right)$$

and for any operator $A \in \mathcal{B}(\mathcal{H}_G)$ we have

$$\mathbb{E}\left(\mathrm{Tr}\left(A\varrho_{\tau,n}\right)\ \middle|\ X_{\tau,n} = x\right) = \mathrm{Tr}\left(AQ_{\tau,n}(x)\right)\ .$$

However, it does not carry any information about the correlations between $X_{\tau,n}$ and $X_{\tau,m}$ for $m \leq n$, and it does not specifies the exact law of $\varrho_{\tau,n}$.

We choose to study $\Lambda_{S,\tau}$ instead of φ_τ for two reasons. First, any Stinespring dilation of φ_τ necessitate an auxiliary space $\mathcal{H}_p$ of infinite dimension, while for $\Lambda_{S,\tau}$ we can take $\mathcal{H}_p = \mathbb{C}^2$. Second, and more importantly, as τ goes to 0 we want to consider $\mathcal{H}_{\tau,z}$ as converging to $L^2(\mathbb{R})$. The algebra $\mathcal{A}_{\tau,z}$ should then converge to $\mathcal{A}_z = L^\infty(\mathbb{R})$, which is not a discrete algebra, hence there is no projection E_∞ on $\mathcal{A}_z$, and we cannot expect E_τ nor $\varphi_\tau^{[t/\tau]}$ to converge.

In the next paragraph we explain how $\Lambda_{S,\tau}^{[t/\tau]}$ converges as $\tau \to \infty$.

2.2.5 The Lindbladian of the OQBM

A first technical problem is that the space $\mathcal{H}_{\tau,z}$ depends on τ. As $\tau \to 0$, we expect it to look like $L^2(\mathbb{R})$. Rigorously speaking, for each τ there is an isometry of $\mathcal{H}_{\tau,z}$ into a subspace of $L^2(\mathbb{R})$.

$$\begin{array}{rccc} \mathcal{I}_{\delta\mathbb{Z},\mathbb{R}}: & \mathcal{H}_{\tau,z} & \longrightarrow & L^2(\mathbb{R}) \\ & |x\rangle & \mapsto & \frac{1}{\delta}\,\mathbb{1}_{[x\ ,\ x+\delta)} \end{array}$$

The image of this isometry is the space of functions which are constant on each interval $[x\ ,\ x+\delta)$, which we identify with $\mathcal{H}_{\tau,z}$ in the following of the article, and we write $\mathcal{H}_z = L^2(\mathbb{R})$. We define $P_{\delta\mathbb{Z}} = \mathcal{I}_{\delta\mathbb{Z},\mathbb{R}}\mathcal{I}_{\delta\mathbb{Z},\mathbb{R}}^*$ the orthogonal projection on this space. By the Lebesgue differentiation theorem, it strongly converge to the identity as $\delta \to 0$. In this sense, the space $\mathcal{H}_{\tau,z}$ converges to $L^2(\mathbb{R})$ as $\tau \to 0$.

Moreover, the translation operator $D_\tau \in \mathcal{B}(\mathcal{H}_z)$ is transformed into

$$\mathcal{I}_{\delta\mathbb{Z},\mathbb{R}}D_\tau\mathcal{I}_{\delta\mathbb{Z},\mathbb{R}}^* = P_{\delta\mathbb{Z}}e^{-\delta\partial_x}$$

since $e^{-\delta\partial_x} = e^{-iP}$ is the translation operator on $L^2(\mathbb{R})$.

Thus, we can see $\Lambda_{S,\tau}$ as a quantum channel on $\mathcal{H}_G\otimes\mathcal{H}_z$ (identifying it with $\Lambda_{S,\tau}\left(\mathcal{I}_{\delta\mathbb{Z}}\bullet\mathcal{I}_{\delta\mathbb{Z}}^*\right)$). Now we are ready to study the convergence as $\tau \to 0$.

Proposition 7. *There exists a semigroup of quantum channels* $(\Lambda_S^t)_{t\in[0,+\infty)}$ *on* $\mathcal{H}_S = \mathcal{H}_G \otimes \mathcal{H}_z$ *such that for any state* $\rho \in \mathfrak{S}(\mathcal{H}_S)$ *and for any* $t \geq 0$ *the state* $\Lambda_{G,\tau,[t/\tau]}(\rho)$ *converges in* $\mathcal{S}^1(\mathcal{H}_S)$ *to* $\Lambda_S^t(\rho)$.

We do not assume that $\mathcal{H}_G$ is of finite dimension. This proposition, as well as the following theorems of this paragraph, will be proved later in the article. This semigroup is strongly continuous in t, but not continuous for the trace norm. We will see that it has a Lindblad equation, but only valid for sufficiently regular states, that is, Sobolev states, as defined below.

Definition 8. *For any Hilbert space $\mathcal{H}$ and any $k \in \mathbb{N}$ the set $\mathfrak{S}_k(\mathcal{H}, \mathcal{H}_z)$ is the set of states ρ on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_z)$ which admits a kernel $(x, y) \in \mathbb{R}^2 \mapsto K_\rho(x, y) \in \mathcal{S}^1(\mathcal{H})$ which is in the Sobolev space $W^{k,1}(\mathbb{R}^2, \mathcal{S}^1(\mathcal{H}))$. Equivalently, it is the space of states $\rho \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{H}_z)$ such that for any $n \leq k$ the operator $[\rho, |\partial_x|^n]$ is a bounded operator on $\mathcal{H} \otimes W^{2,k}(\mathbb{R})$.*

The set $\mathfrak{S}(\mathcal{H}, \mathcal{A}_z)$ is the set of states ρ on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{A}_z$ which admits a kernel $x \in \mathbb{R} \mapsto K_\rho(x) \in \mathcal{S}^1(\mathcal{H})$ which is in the Sobolev space $W^{k,1}(\mathbb{R}, \mathcal{S}^1(\mathcal{H}))$.

The Lindblad equation is the following.

Theorem 9. *For any initial state $\rho \in \mathcal{S}_2(\mathcal{H}_G, \mathcal{H}_z)$ the state $\rho_S(t) = \Lambda_S^t(\rho)$ is in $\mathcal{S}_2(\mathcal{H}_G, \mathcal{H}_z)$ for all $t > 0$. Moreover, it satisfies the following equation:*

$$\frac{d}{dt}\rho_S(t) = \tilde{\mathcal{L}}(\rho_S(t)) \tag{2.10}$$

where

$$\tilde{\mathcal{L}}(\rho) = -i[\tilde{H}, \rho] + L\rho L^* - \frac{1}{2}\{L^*L, \rho\}$$

with $L = N - \partial_x$ and $\tilde{H} = H - \frac{i}{2}\partial_x(N + N^*)$.

Writing $K_t(x, y)$ the kernel of $\rho_{\mathcal{C}}(t)$ this equation becomes

$$\frac{d}{dt}K_t(x,y) = \mathcal{L}(K_t(x,y)) + \frac{1}{2}\left(\frac{\partial}{\partial_x} + \frac{\partial}{\partial_y}\right)^2 K_t(x,y) - N\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)K_t(x,y) - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)K_t(x,y)N^* \ . \tag{2.11}$$

Equation 2.10 can be formally obtained by writing $e^{-\delta\partial_x} \simeq I - \delta\partial_x + \frac{\tau}{2}\partial_x^2$. Though this can be made rigorous, we will prove it by other methods in Paragraph 2.3.6

Equation 2.11 is obtained from the first equation by using the following formula: if $\rho \in gs_1(\mathcal{H}_G, \mathcal{H}_z)$ then $\partial_x\rho$ and $\rho\partial_x$ are kernel operators, with

$$K_{\rho\partial_x}(x,y) = -\frac{\partial}{\partial y}K_\rho(x,y) \tag{2.12}$$

$$K_{\partial_x\rho}(x,y) = \frac{\partial}{\partial_x}K_\rho(x,y) \ . \tag{2.13}$$

It is obtained with an integration by parts.

In the previous paragraph we explain that the quantum channel φ_τ obtained in the OQW convention does not behave well as $\tau \to 0$. However, the channels φ_τ and $\Lambda_{S,\tau}$ are equal on the algebra $\mathcal{M}_\tau = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}_{\tau,z}$, and the algebra $\mathcal{A}_{\tau,z}$

converges as $\tau \to 0$ to $\mathcal{A}_z = L^\infty(\mathbb{R})$ (indeed $\mathcal{A}_{\tau,z} = P_{\delta\mathbb{Z}}\mathcal{A}_z P_{\delta\mathbb{Z}}$. Thus, studying the restriction of Λ_S^t to $\mathcal{M}$ is a way to study the limit of $\varphi_\tau^{[t/\tau]}$. Note that no density matrix of quantum states is in $\mathcal{M} = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}_z$ (and no non-trivial trace-class state at all are in this space), so we take the dual point of view: we study the evolution of states on $\mathcal{M}$.

Proposition 10. *There exists a semigroup of super-operators $(\Lambda^t_{\mathcal{M}})_{0\leq t}$ on $\mathfrak{S}(\mathcal{M})$ such that for any state $\rho \in \mathfrak{S}(\mathcal{H}_S)$ with restriction $\rho_{\mathcal{M}}$ to $\mathcal{M}$, the restriction to $\mathcal{M}$ of the state $\rho_t = \Lambda_S^t(\rho)$ is $\Lambda^t_{\mathcal{M}}(\rho_{\mathcal{M}})$.*

If a state $\rho_{\mathcal{M}}$ admits a kernel $x \mapsto Q_\rho(x)$ which is in $W^{2,1}(\mathbb{R}, \mathcal{S}^1(\mathcal{H}_G))$ then $\rho_{\mathcal{M},t} = \Lambda^t_{\mathcal{M}}(\rho_{\mathcal{M}})$ also admits a kernel $Q_t \in W^{2,1}(\mathbb{R}, \mathcal{S}^1(\mathcal{H}_G))$ and we have

$$\frac{d}{dt}Q_t(x) = \mathcal{L}(Q_t(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}Q_t(x) - \left(N\frac{\partial}{\partial x}Q_t(x) + \left(\frac{\partial}{\partial x}Q_t(x)\right)N^*\right)\ . \tag{2.14}$$

Equation 2.14 can be obtained from Equation 2.11 if $\rho_{\mathcal{M}}$ admits an extension ρ_S in $\mathfrak{S}_2(\mathcal{H}_G, \mathcal{H}_z)$. Indeed, if K_ρ is the kernel of ρ_S, we have $K_\rho(x,x) = Q_{\mathcal{M}}(x)$ for almost all $x \in \mathbb{R}$ so

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)K(x,x) = \frac{\partial}{\partial_x}Q_{\mathcal{M}}(x)\ .$$

As for the state on $\mathcal{M}_\tau$, the state $\rho_{\mathcal{M},t}$ carries informations about the trajectory of the continuous OQBM $(\varrho_t, X_t)_{0\leq t}$. By the convergence in distribution of the discrete OQBM to the continuous one, for each t the density of X_t with respect to the Lebesgue measure is $p(X_t = x) = \mathrm{Tr}\,(Q_t(x))$ and for any operator $A \in \mathcal{B}(\mathcal{H}_G)$ we have

$$\mathbb{E}(\mathrm{Tr}\,(A\varrho_t)\ \big|\ X_t = x) = \mathrm{Tr}\,(AQ_t(x))\ .$$

2.3 Quantum Stochastic Calculus for the Open Quantum Brownian Motion

The trajectory of the continuous OQBM follow the well-known Belavkin equation; the Lindblad equation for the state $\rho_{S,t} \in \mathfrak{S}(\mathcal{H}_G \otimes \mathcal{H}_z)$ offers a more quantum view on this equation, but it fails to take into account the correlations between the state at different times. A fully quantum view on the OQBM which encompass these correlations will be described by the Quantum Stochastic Calculus on the Fock space. We will briefly introduce this space by approaching it by the repeated interaction process.

2.3.1 Repeated interaction process and the Toy Fock space

In the definition of the repeated interaction process, a new probe space $\mathcal{H}_p$ is introduced at every iteration. The so called Toy Fock space is the Hilbert space $T\Phi$ obtained when considering all these probe spaces at once. Formally,

$T\Phi = \bigotimes_{n\in\mathbb{N}^*} \mathcal{H}_p$. More concretely, it is the Hilbert space which generated by the vectors $\bigotimes_{n\in\mathbb{N}^*} e_n$ where the vectors e_n are unit vectors of $\mathcal{H}_p$ which are all equal to $|0\rangle$ except for a finite number of indices. It has a distinguished unit vector $|\Omega\rangle = \bigotimes_{n\in\mathbb{N}^*} |0\rangle$. For each $n \in \mathbb{N}^*$, the space $T\Phi_d$ contains a copy of $\mathcal{H}_p$ given by the following isometry:

$$\begin{array}{cccc} \mathcal{I}_n : & \mathcal{H}_p & \longrightarrow & T\Phi_d \\ & v & \mapsto & \left(\bigotimes_{k=1}^{n-1} |0\rangle\right) \otimes v \otimes \left(\bigotimes_{k=n+1}^{+\infty} |0\rangle\right) \end{array}$$

The unitary V_τ considered in the repeated interaction process is then replaced by the unitaries $V_{\tau,n} = \mathcal{I}_n V_\tau \mathcal{I}_n^*$, and the evolution from time zero to time n is represented by the unitary

$$U_{\tau,n} = V_{\tau,n} V_{\tau,n-1} \cdots V_{\tau,1} \ .$$

We can obtain the random state ϱ_n by performing the simultaneous measurement of all the observables $A_k = \mathcal{I}_n A \mathcal{I}_n^*$ when in the total state

$$\rho_{tot,\tau,n} = U_{\tau,n} \left(\varrho_{\tau,0} \otimes |\Omega\rangle \langle\Omega|\right) U_{\tau,n}^* \ .$$

The position of the particle is then

$$X_{\tau,n} = X_{\tau,0} + \delta \sum_{k=1}^{n} \tilde{A}_k$$

where $\tilde{A}_k = \pm 1$ is the result of the measurement of A_k.

2.3.2 The Fock space

Before studying the convergence of $T\Phi$ as $\tau \to 0$, let us describe its limit, the Fock space $\Phi = \bigotimes_{t\in\mathbb{R}_+} \mathcal{H}_p$. This space and its interpretation as an infinite tensor product is well known, see Parthasarathy's book [21] for example, or Attal's lecture in the second book of [26], and we refer to these lectures for a more complete introduction to the Fock space. Let us briefly recall two of its descriptions. Here, we only treat the case where $\mathcal{H}_p = \mathbb{C}^2$, but the case where $\mathcal{H}_p = \mathbb{C}^n$ or even $\mathcal{H}_p$ is infinite-dimensional are similar.

The Guichardet interpretation: Let us consider the set $\mathcal{P}$ of increasing sequences of $\mathbb{R}_+$ of finite length (including the empty sequence $(\emptyset)$). We have $\mathcal{P} = \cup_{n\in\mathbb{N}} \mathcal{P}_n$ where $\mathcal{P}_n \subset (\mathbb{R}_+)^n$ is the set of increasing sequence of length n. This set inherits the Lebesgue measure on $(\mathbb{R}_+)^n$ (and $\mathcal{P}_0 = \{ (\emptyset) \}$ has the Dirac measure), so we can endow $\mathcal{P}$ with the sum of these measure, which we write λ. The Fock space in the Guichardet interpretation is $\Phi_G = L^2(\mathcal{P}, \lambda)$.

It can be interpreted as an infinite tensor product. Indeed, if we write $\mathcal{P}_{[s,t]}$ the space of finite sequences in $[s,t]$ and $\Phi_{G,[s,t]} = L^2(\mathcal{P}_{[s,t]}, \lambda)$, we have $\Phi_{G,[s,t]} \otimes \Phi_{G,[t,u]} = \Phi_{G,[s,u]}$. There is a distinguished vector $|\Omega\rangle = \mathbb{1}_{\mathcal{P}_0}$. We identify $\Phi_{G,[s,t]}$ to the subspace $\{ \ | \, \Omega_{[0,s]}\rangle \} \otimes \Phi_{G,[s,t]} \otimes \{ \ | \, \Omega_{[t,+\infty)}\rangle \}$ of Φ_G.

The probabilistic interpretation from the Brownian motion: This interpretation has been introduced by Attal and Meyer [6]. See [4] for more details. We consider the Wiener space $(\mathcal{W}, \mathcal{F})$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ with the Wiener measure μ corresponding to the Browian motion. We then take $\Phi_W = L^2(\mathcal{W}, \mu)$ the space of L^2 random variables on $(\mathcal{W}, \mu)$. There is a distinguished vector $|\Omega\rangle = \mathbb{1}$ (the constant random variable equal to 1). If $\mathcal{W}([s,t])$ is the space of functions from $[s,t]$ to $\mathbb{R}$, we can define $\Phi_{W,[s,t]} = L^2(\mathcal{W}([s,t]), \mu)$, and we have $\Phi_{W,[s,t]} \otimes \Phi_{W,[t,u]} = \Phi_{W,[s,u]}$.

These two interpretation are equivalent: we can construct an unitary $\mathcal{I}_{G,W} : \Phi_G \to \Phi_W$ such that $\mathcal{I}_{G,W}\Phi_{G,[s,t]} = \Phi_{W,[s,t]}$ and $\mathcal{I}\,|\Omega\rangle = |\Omega\rangle$. To describe it, let us write $(W_t)_{t\in\mathbb{R}_+}$ the Brownian motion and dW_t the Itô differential. For any function $f \in L^2(\mathcal{P}_n, \lambda)$, the random variable $X = \mathcal{I}_{G,W} f$ is defined as the successive Itô integrals

$$\mathcal{I}_{G,W} f = X = \int_{0<t_1<t_2<\cdots<t_n<\infty} f(t_1, \cdots, t_n) dW_{t_1} dW_{t_2} ... dW_{t_n}$$

(and if $n = 0$ then $\mathcal{I}_{G,W} f$ is the deterministic variable equal to $f(\emptyset)$).

By the Itô isometry formula, we have

$$\|X\|^2 = \mathbb{E}(|X|^2) = \int_{0<t_1<t_2<\cdots<t_n<\infty} |f(t_1, \cdots, t_n)|^2 dt_1 \cdots dt_n = \|f\|^2 \ .$$

so $\mathcal{I}_{G,W}$ is an isometry, and the chaotic representation property ensure that it is surjective (see [4]).

From now on, we will write Φ the Fock space, and either the Guichardet or the probabilistic interpretation depending on the context. There exists many more probabilistic interpretations, one for each normal martingale. We concentrate on the Brownian interpretation in this article. To complete this picture, we need to approximate the Toy Fock space by the Fock space. This was done by Attal [3] and developed by Attal and Pautrat [7]. Let us first design an isometry of $T\Phi$ into Φ. The idea is the following: for each τ, we have

$$\Phi = \bigotimes_{n\in\mathbb{N}} \Phi_{[\tau n \ , \ \tau(n+1)]}$$

(where the infinite tensor product is taken with respect to $|\Omega_{[\tau n \ , \ \tau(n+1)]}\rangle$ as in the construction of the toy Fock space). Thus, it is sufficient to define an isometry from $\mathcal{H}_p = \mathbb{C}^2$ to $\Phi_{[\tau n \ , \ \tau(n+1)]} = \Phi_{[0,\tau]}$ and to extend it by tensor product to $T\Phi = \otimes_{n\in\mathbb{N}^*}$. We choose the isometry

$$\begin{array}{cccl} \mathcal{I}_{n,\tau} : & \mathcal{H}_p & \longrightarrow & \Phi_{[\tau n \ , \ \tau(n+1)]} \\ & |0\rangle & \mapsto & |\Omega_{[\tau n \ , \ \tau(n+1)]}\rangle \\ & |1\rangle & \mapsto & \frac{1}{\sqrt{\tau}}\left(W_{\tau(n+1)} - W_{\tau n}\right) \ . \end{array}$$

which tensorise to $\mathcal{I}_\tau = \otimes_{n\in\mathbb{N}}\mathcal{I}_{n,\tau} : T\Phi \to \Phi$.

Let us write $P_\tau = \mathcal{I}_\tau \mathcal{I}_\tau^*$ the projection on the image of $\mathcal{I}_\tau$, and $T_\tau\Phi$ this image. Then P_τ strongly converge to the identity on Φ as $\tau \to 0$. In this sense, the Toy Fock space approximate the Fock space, but this is not sufficient; we also need some more precise convergence on operators in $\mathcal{B}(\Phi)$. But first, we need to study the operators in the Fock space.

2.3.3 Quantum Stochastic Calculus on the Fock space

The quantum stochastic calculus is thoroughly described in Parthasarathy [21] and in [4], [7]. We give it a very short introduction geared for this article.

The operators on $\mathcal{H}_p$ are all linear combinations of the four operators $|j\rangle\langle i|$ for $i, j \in \{0,1\}$. In the toy Fock space, they translate as the operators

$$a^i_j(n) = \mathcal{I}_n(|j\rangle\langle i|)\mathcal{I}_n^* \ .$$

Thus, the algebra $\mathcal{B}(T\Phi)$ is generated by the operators $a^i_j(n)$ for $n \in \mathbb{N}^*$ and $i, j \in \{0,1\}$. Using the isometry $\mathcal{I}_\tau$ in the Fock space, we obtain some operators $a^i_j(\tau, n) = \mathcal{I}_\tau a^i_j(n)\mathcal{I}_\tau^*$. Under suitable renormalization, their limit exists: there exists closed operators $a^i_j(t)$ on Φ such that there is strong convergence

$$\tau^{\varepsilon_{j,i}} a^i_j(\tau, [t/\tau]) \underset{\tau\to 0}{\longrightarrow} a^i_j(t)$$

where

$$\tau^{\varepsilon}_{j,i} = \begin{cases} \tau & \text{if } i = j = 0 \\ \sqrt{\tau} & \text{if } (i,j) = (0,1) \text{ or } (i,j) = (1,0) \\ 1 & \text{if } i = j = 1 \end{cases} \quad .$$

The operator $da^0_0(t)$ is just the multiplication by t, while $a^1_0(t)^* = a^0_1(t)$ and $a^1_1(t)$ is self-adjoint (they are respectively the creation, annihilation and number operator on $\Phi_{t]}$. We will write $a^i_j([s,t]) = a^i_j(t) - a^i_j(s)$; we have

$$a^i_j(\tau, n) = \tau^{-\varepsilon j,i} \ P_\tau a^i_j([\tau(n+1) \ , \ \tau n])P_\tau \ .$$

See [3] or [7] for more details on these operators. We will now explain how to integrate with respect to theses operators, in a way parallel to the Itô Stochastic integration. First, we need to define the set of coherent vectors. For any function $u \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we define the coherent vector $\varepsilon(u)$ in the Guichardet interpretation by

$$\varepsilon(u)(t_1, \cdots, t_n) = u(t_1)u(t_2)\cdots u(t_n)$$

(the empty product being considered to be 1). In he probabilistic interpretation, it corresponds to exponential martingale : writing $Y_t = \varepsilon(u\mathbb{1}_{[0,t)})$ it verifies the (classical) SDE

$$dY_t = u(t)Y_t dW_t$$

Thus, writing $H_\infty = \int_0^{+\infty} u(s)dW_s$ and $[H]_\infty = \int_0^{+\infty} |u(s)|^2 ds$ we have

$$\varepsilon(u) = \exp\left(H_\infty - \frac{1}{2}[H]_\infty\right) \ .$$

We have $\|u\|^2 = e^{\|u\|^2_{L^2}}$. Hence, ε is continuous; it is clearly not linear.

An important property is that if $\mathcal{M} \subset L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$, then the vector space $Vect(\varepsilon(\mathcal{M}))$ is dense in Φ. Thus, it is often sufficient to define an object on coherent vectors to fix it.

Now, the objects that we can integrate are the adapted process of operators. We give here a restrictive definition taken from Parthasarathy [21]. A more general definition was produced by Attal and Lindslay [5] , but it is not needed here.

Definition 11 (Adapted process of operators)**.** *A dense subspace $\mathcal{M} \subset L^2(\mathbb{R})$ is called adapted if for any $0 \le s \le t \le \infty$, the space $\mathcal{M}([s,t]) := \{f \in \mathcal{M} | f = \mathbb{1}_{[s,t]} f\}$ is dense in $L^2([s,t])$.*

Consider some Hilbert space $\mathcal{H}_S$. A family of (possibly unbounded) operators $(H_t)_{t\in\mathbb{R}_+}$ on $\mathcal{H}_S \otimes \Phi$ is called adapted if there exists a dense subspace $\mathcal{D}$ and an adapted subspace $\mathcal{M} \subset L^2(\mathbb{R})$ such that for all t the domain of H_t contains $\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})$, and there is an operator $\tilde{H}_t$ on $\mathcal{H}_S \otimes \Phi$ with domain $\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M}([0,t])$ such that $H_t = \tilde{H}_t \otimes I_{\Phi_{[t,+\infty)}}$ on $\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})$.

Now, for an adapted process of operators $(H_t)_{t\in\mathbb{R}_+}$, we want to define the operator

$$\int_0^t H_s da^i_j(s)$$

which would correspond to the limit of

$$\frac{1}{\tau} \sum_{k=0}^{[t/\tau]} H_{k\tau} \left(a^i_j(\tau(k+1)) - a^i_j(\tau k) \right) \ . \tag{2.15}$$

Note that $a^i_j(\tau(k+1)) - a^i_j(\tau k)$ only acts on $\Phi_{[\tau k, \tau(k+1)]}$ so it commutes with $H_{k\tau}$, and the order of the operators in the above formula is not important. The concrete way we define the integral is the following:

Definition 12. *Let $(H_t)_{t\in I}$ be an adapted process of operators on $\mathcal{H}_S \otimes \Phi$, with domain containing $\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})$ where $\mathcal{M}$ is adapted and $\mathcal{D}$ is dense. Let T be an operator on $\mathcal{H}_S \otimes \Phi$. We say that the formula*

$$T = \int_0^t H_t da^i_j(t)$$

is true on $\mathcal{D} \otimes_{alg} \varepsilon(\mathcal{M})$ if for any $a, b \in \mathcal{D}$ and $u, v \in \mathcal{M}$ the following formula is meaningful and true:

$$\langle a \otimes \varepsilon(u) \ , \ T_t b \otimes \varepsilon(v) \rangle = \int_0^t u_j(s) u_i(s) \ \langle a \otimes \varepsilon(u), H_s b \otimes \varepsilon(v) \rangle ds \tag{2.16}$$

where $u_i(s) = 1$ if $i = 0$ and $u_i(s) = u(s)$ if $i = 1$, and by "meaningful" we mean that the integral is absolutely convergent.

If $T_t = \int_0^t H_s da_j^i(s)$ for all t we will write $dT_t = H_t da_j^i(t)$. A more general formula exists to compute Tf for some vector f, see [4]. Note that the existence of an operator $\int_0^t H_t da_j^i(t)$ is not guarantied. If H_t is bounded locally uniformly in t, it is at least possible to define $\int_0^t H_t dt$ on the space generated by $\mathcal{H}_S \otimes \mathcal{D}_B$, where $\mathcal{D}_B$ is the vector space generated by $\varepsilon(L^2(\mathbb{R}) \otimes L^\infty(\mathbb{R}))$. The obtained operator may still be unbounded.

It is easy to check that in the case where H_t is constant on the intervals $t \in [\tau k, \tau(k+1)]$ this formula corresponds to the Riemann sum 2.15. In particular,

$$a_j^i(t) = \int_0^t da_j^i(s) \ .$$

The case of $a_0^0(t) = t$ is simple, the integral being just the integral with respect to dt in the Banach space $\mathcal{B}(\mathcal{H}_S)$.

The case of $a_0^1(t)$ and $a_1^0(t)$ is more subtle, and it actually generalize the Itô integral, as shown by the following proposition.

Proposition 13. *Let $(f_t)_{t\in\mathbb{R}_+}$ be a process of random variables in $L^\infty(\mathcal{W}, \mu)$, adapted in the sense of Itô, and such that $\int_0^t \mathbb{E}(|f_s|^2)ds < \infty$. Let*

$$g = \int_0^t f_s dW_s \ .$$

Consider the operators $H_s = M_{f_s}$ and $T = M_g$ on multiplication by f_s on Φ. Then we have

$$T = \int_0^t H_s(da_0^1(t) + da_1^0(t))$$

on the domain $\varepsilon(L^2(\mathbb{R}))$. Thus, in terms of operators, we can write $dW_t = da_0^1(t) + da_1^0(t)$.

By the chaotic representation property (see [4]), this implies that the commutative von Neumann algebra $\mathcal{A}([0,t]) = L^\infty(\mathcal{W}([0,t]), \mu)$ is generated by the operators $a_0^1(t) + a_1^0(t)$. Note that the observable we measure in the definition of the OQBM is $A = |0\rangle\langle 1| + |1\rangle\langle 0|$, so the observable $A(\tau, n) = \mathcal{I}_\tau A(n)\mathcal{I}_\tau^*$ is

$$A(\tau, n) = \frac{1}{\sqrt{\tau}} P_\tau \tau(a_0^1([\tau n \ , \ \tau(n+1)]) + a_1^0([\tau n \ , \ \tau(n+1)]))P_\tau \ .$$

Thus, the algebra generated by the $A(\tau, k)$ for $k \leq n$ is $P_z L^\infty(\mathcal{W}([0,t], \mu) P_z$, which is why the Brownian representation of Φ is adapted to the study of the OQBM.

The product of two quantum stochastic integrals is itself a quantum stochastic integral under some regularity conditions.

Proposition 14 (Quantum Itô product formula)**.** *Let $(A_t)_{t\in\mathbb{R}_+}$ and $(B_t)_{t\in\mathbb{R}_+}$ be two adapted processes of operators, with domains containing respectively the dense adapted domains $\mathcal{D}_A \otimes_{alg} \varepsilon(\mathcal{M}_A)$ and $\mathcal{D}_B \otimes_{alg} \varepsilon(\mathcal{M}_B)$ in $\mathcal{H}_S \otimes \Phi$. Assume that $(A_t^*)_{t\in\mathbb{R}_+}$ is also an adapted process with domain containing $\mathcal{D}_A \otimes_{alg} \varepsilon(\mathcal{M}_A)$ and that the following integrals are well defined, on $\mathcal{D}_A \otimes_{alg} \varepsilon(\mathcal{M}_A)$ for the first line, $\mathcal{D}_B \otimes_{alg} \varepsilon(\mathcal{M}_B)$ for the second.*

$$T_t = \int_0^t A_s da_j^i(s) \qquad\qquad S_t = \int_0^t A_s^* da_i^j(s)$$

$$U_t = \int_0^t B_s da_l^k(s) \ .$$

Moreover, assume that for all s we have $B_s(\mathcal{D}\otimes\varepsilon(\mathcal{M})) \subset \mathcal{D}_A$ and $U_s(\mathcal{D}\otimes\varepsilon(\mathcal{M})) \subset \mathcal{D}_A$. Then the following formula is satisfied on $\mathcal{D}_B \otimes \varepsilon(\mathcal{M}_B)$:

$$T_tU_t = \int_0^t A_sU_s da_j^i(s) + T_sB_s da_l^k(s) + \delta_{i=l}\delta_{l\neq 0} A_sB_s da_j^k(s) \ .$$

This proposition was proved by Hudson and Parthasarathy, see Proposition 25.26 of Parthasarathy's book [21].

Writing $da_j^i(s)\ da_l^k(s) = \delta_j^j \delta_{k\neq 0}\ da_l^i(s)$, this formula can be written as

$$d(T_tU_t) = T_t dU_t + (dT_t)U_t + (dT_t)(dU_t) \ .$$

Note that in particular, if $A_t = B_t = a_0^1(t) + a_1^0(t)$ we have

$$d(A^2(t)) = 2A(t)dA(t) + dt$$

which is actually the formula $d(W_t^2) = 2W_t dW_t + dt$ for the Brownian motion.

We are now ready to present the theorem of convergence of the repeated interactions of Attal and Pautrat.

2.3.4 Hudson-Parthasarathy equations and Attal-Pautrat convergence

The Attal-Pautrat limit [7] was devised in the context of repeated interaction processes. The idea is to show that $\mathcal{I}_\tau U_{[t/\tau]} \mathcal{I}_\tau^*$ converge to some limit U_t as τ goes to 0, which satisfies a quantum stochastic differential equation. We only present the case which is needed here.

First, we need to describe what will be the limit. It is a family of unitary following the so called quantum Langevin equations (or Hudson-Parthasarathy equations).

Theorem 15. *Let H and N be two bounded operators on $\mathcal{H}_G$ with H self-adjoint. Write*

$$G = -iH - \frac{1}{2}N^*N \ .$$

Then there exists an adapted process of unitary operators U_t on $\mathcal{H}_G \otimes \Phi$ which satisfies the following quantum stochastic equation on $\mathcal{H}_G \otimes_{alg} \varepsilon(L^2(\mathbb{R}))$:

$$dU_t = \big(Gdt + N da_0^1(t) - N^* da_1^0(t)\big)U_t \ . \tag{2.17}$$

The adjoint operator U_t^* satisfies the adjoint equation. With the condition $U_0 = I$, it is unique.

This theorem is proved in [21]; the idea is to make Picard iterations on Equation 2.17 starting from $U_t^0 = I$, applying Formula 2.16 to show that at each iteration the obtained operators are still unitary.

Attal and Pautrat proved the following theorem (expressed in the case which is needed here).

Theorem 16. *Let $(U_{\tau,n})_{n\in\mathbb{N}}$ be a family of operators on $\mathcal{H}_G \otimes T\Phi$ defined as in Paragraph 2.3.2, and write $u_{\tau,n} = \mathcal{I}_\tau U_{\tau,n} \mathcal{I}_\tau^*$ the isometry on $\mathcal{H}_G \otimes \Phi$ corresponding to $U_{\tau,n}$. Then for any $t \geq 0$ the operator $u_{\tau,[t/\tau]}$ converges strongly to the unitary operator U_t solution of the Hudson-Parthasarathy equation of Theorem 15.*

This theorem is proved in [7] in a more general context where there may be some term in $da_1^1(t)$ in the equation and the space $\mathcal{H}_p$ is of arbitrary dimension).

2.3.5 Convergence to the continous OQBM

We are now ready to prove the convergence of the discrete OQBM. We consider the unitary $L_\tau = R_\tau V_\tau$ of the discrete OQBM built in Paragraph 2.2.3. We convert it into an isometry of $\mathcal{H}_G \otimes \mathcal{H}_z \otimes \Phi$: we write $l_{\tau,n} = \mathcal{I}_\tau \mathcal{I}_n L_{\tau,n} \mathcal{I}_n^* \mathcal{I}_\tau^*$, and we define the OQBM isometry $\mathfrak{U}_{\tau,n} = l_{\tau,n} l_{\tau,n-1} \cdots l_{\tau,1}$.

Theorem 17. *For each $t \geq 0$ the operator $gu_{\tau,[t/\tau]}$ converge strongly to some unitary operator $\mathfrak{U}_t$ solution of the equation*

$$d\mathfrak{U}_t = \Big((-iH - \frac{1}{2}N^*N + \frac{1}{2}\partial_x^2 - \partial_x N)dt + (N - \partial_x)da_1^0(t) + (-N^* - \partial_x)da_0^1(t)\Big)\mathfrak{U}_t \tag{2.18}$$

on the set $\mathcal{H}_G \otimes_{alg} H^2(\mathbb{R}) \otimes_{alg} \varepsilon(L^2(\mathbb{R}))$.

Remark 2.
1. This theorem can probably be generalized to cases where N and H depends on the position x, but this would require to extend non-trivially the theorem of Attal and Pautrat, the issue of the non-boundedness of ∂_x being harder to bypass when N and ∂_x are not commuting.
2. Equation 2.18 is a Hudson-Parthasarathy equation of the form of Theorem 15, with N replaced by $\tilde{N} = N - \partial_x$ and H replaced by $\tilde{H} = H - \frac{i}{2}(N^*\partial_x + \partial_x N)$.

3. The operator ∂_x is unbounded, so we cannot directly apply Theorem 15 to show the existence of a solution U_t, neither Theorem 16 to show the convergence. Instead, we will break $\mathfrak{U}_{\tau,n}$ in two parts: one which is solution of a Hudson-Parthasarathy equation with bounded coefficients, and one which is solution of a Hudson-Parthasarathy equation with unbounded coefficients but which is very simple.

We break the proof into a series of lemma. First, let us consider the pointer isomorphism $R_\tau = D_\tau \otimes P_+ + D_{-\tau} \otimes P_-$ defined in Paragraph 2.2.3. We write $R_{\tau,n} = \mathcal{I}_n R_\tau \mathcal{I}_n^*$ the corresponding operator acting on the toy Fock space, and $r_{\tau,n} = (\mathcal{I}_{\delta\mathbb{Z}} \otimes \mathcal{I}_\tau)) R_{\tau,n} (\mathcal{I}_{\delta\mathbb{Z}} \otimes \mathcal{I}_\tau)^*$. Let us consider their product

$$z_{\tau,n} = r_{\tau,n} r_{\tau,n-1} \cdots r_{\tau,1} \ .$$

Note that V_τ is not acting on $\mathcal{H}_z$ and Z_τ is not acting on $\mathcal{H}_G$, so $\mathcal{I}_n^* Z_\tau \mathcal{I}_n$ commutes with $\mathcal{I}_k^* V_\tau \mathcal{I}_k$ for any $n > k$. Thus we have

$$\mathfrak{U}_{\tau,n} = z_{\tau,n} u_{\tau,n} \ .$$

We already know that $u_{\tau,[t/\tau]}$ converges to some operator U_t by Theorem 16. Let us consider the limit of the operator $z_{\tau,n}$.

The pointer process Z_t:

Proposition 18. *For any $t \in \mathbb{R}_+$ the operator $z_{\tau,[t/\tau]}$ strongly converges to a unitary operator Z_t. The process $(Z_t)_{t\in\mathbb{R}_+}$ satisfies the following quantum SDE on the space $H^2(\mathbb{R}) \otimes_{alg} \varepsilon(L^2(\mathbb{R}))$.*

$$dZ_t = \left(\frac{1}{2}\partial_x^2 dt - \partial_x (da_0^1(t) + da_1^0(t)) \right) Z_t \ . \tag{2.19}$$

In the probabilistic representation, Z_t is explicit: for any function $f \in L^2(\mathbb{R})$ and any random variable $A \in L^2(\mathcal{W}, \mu)$ we have

$$(Z_t f A)(x) = f(x - W_t) A \ .$$

Proof. Note that $\mathcal{I}_{\delta\mathbb{Z}}\ D_\tau\ \mathcal{I}_{\delta\mathbb{Z}}^* = e^{-\partial_x} P_{\delta\mathbb{Z}}$, since $e^{-\delta\partial_x}$ is the operator of translation by δ on $L^2(\mathbb{R})$. Moreover,

$$\mathcal{I}_\tau P_\pm(n) \mathcal{I}_\tau^* = \frac{1}{2} \left(a_0^0(\tau,n) + a_1^1(\tau,n) \pm a_0^1(\tau,n) \pm a_1^0(n) \right)$$

so we have

$$r_{\tau,n} = \frac{e^{-\delta\partial_x} + e^{\delta\partial_x}}{2} \left(a_0^0(\tau,n) + a_1^1(\tau,n) \right) P_{\delta\mathbb{Z}} + \frac{e^{-\delta\partial_x} - e^{\delta\partial_x}}{2} \left(a_0^1(\tau,n) + a_1^0(\tau,n) \right) P_{\delta\mathbb{Z}} \ .$$

We want to write $e^{-\delta\partial_x} \simeq I - \delta\partial_x + \frac{1}{2}\delta^2\partial_x^2$. Since ∂_x is unbounded, it cannot be done directly. Let us consider the space $\mathcal{D}_C \subset L^2(\mathbb{R})$ of C-bandlimited functions for $C > 0$. Writing $\mathcal{F}$ the Fourier transform, the space $\mathcal{D}_C$ is defined as

$$\mathcal{D}_C = \{ \, f \in L^2(\mathbb{R}) \ \mid \ \mathcal{F}f \text{ is supported in } [-C, C] \, \} \ .$$

This space is stable by ∂_x and $\bigcup_{C>0}\mathcal{D}_C$ is dense in $L^2(\mathbb{R})$. Restricted to $\mathcal{D}_C$, the operator ∂_x is bounded, so we can expand the exponential. However, the space $\mathcal{D}_C$ is not stable by P_δ, so we introduce

$$\tilde{r}_{\tau,n} = \frac{e^{-\delta\partial_x}+e^{\delta\partial_x}}{2}\left(a_0^0(\tau,n)+a_1^1(\tau,n)\right)+\frac{e^{-\delta\partial_x}-e^{\delta\partial_x}}{2}\left(a_0^1(\tau,n)+a_1^0(\tau,n)\right)$$

so that $r_{\tau,n}=\tilde{r}_{\tau,n}P_\delta$. We also write $\tilde{z}_{\tau,n}=\tilde{r}_{\tau,n}\tilde{r}_{\tau,n-1}\cdots\tilde{r}_{\tau,1}$. Since $P_{\delta\mathbb{Z}}$ commutes with $\tilde{r}_{\tau,k}$ for all k, we have that $z_{\tau,n}=\tilde{z}_{\tau,n}P_{\delta\mathbb{Z}}$. The space $\mathcal{D}_C$ is stable by $\tilde{r}_\tau$, and on this space, since ∂_x is bounded we have

$$\tilde{r}_{\tau,n}=\left(I+\frac{1}{2}\partial_x^2+O(\delta^3)\right)a_0^0(\tau,n)+O(\delta)a_1^1(\tau,n)+\left(-\delta\partial_x+O(\delta^2)\right)\left(a_0^1(\tau,n)+a_1^0(n)\right)$$

With $\delta=\sqrt{\tau}$, this sets us under the hypothesis of Theorem 16, with $K=0$ and $L=-\partial_x$. Thus, $\tilde{z}_{\tau,[t/\tau]}$ converges strongly (on $\mathcal{D}_C$) to a unitary operator Z_t^C which is solution of 2.19. All the Z_t^C's coincide on their common domain of definition, and they are unitary, so we can extend them to $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$. They commute with ∂_x, so they are also unitary for the space $H^2(\mathbb{R})$. Since the $\tilde{r}_{\tau,[t/\tau]}$ are unitary and converge to Z_t strongly on a dense subspace, they converge strongly on the full space. Moreover, $P_{\delta\mathbb{Z}}$ converges strongly to I, so $z_{\tau,[t/\tau]}=\tilde{z}_{\tau,[t/\tau]}P_{\delta\mathbb{Z}}$ also converge strongly to Z_t.

Finally, by the classical Itô formula, for any $\mathcal{C}^2$ function

$$df(x-W_t)=f(x)-\int_0^t\partial_x f(x-W_s)dW_s+\frac{1}{2}\int_0^t\partial_x^2 f(x-W_s)ds\ .$$

Thus, if we write $(\tilde{Z}_t fA)(x)=f(x-W_t)A$ for any $f\in L^2(\mathbb{R})$, the processes $\tilde{Z}_t$ and Z_t follow the same quantum SDE on $\mathcal{C}^2$ functions. Since they have the same initial state $Z_0=I$, this implies that they are equal. □

As a consequence of this proposition, the operators $\mathfrak{U}_{\tau,[t/\tau]}$ converges to $\mathfrak{U}_t:=Z_tU_t$ and the Itô product formula yields the stochastic equation 2.18.

Remark 3. 1. It is also possible to prove Theorem 17 by using the Attal-Pautrat theorem directly on U_t restricted to $\mathcal{H}_G\otimes_{alg}\mathcal{D}_C\otimes_{alg}\Phi$ since $\mathcal{H}_G\otimes_{alg}\mathcal{D}_C$ is stable by H and N. However, the pointer unitary Z_t has its own interest, and may be useful in situations where $\mathcal{D}_C$ is not stable.

2. Note that Z_t does not commute with U_t, we only have the commutation of U_t and $Z_{t,s}:=Z_tZ_s^*$. The formula $\mathfrak{U}_t=Z_tU_t$ is consistent with the construction of the discrete OQBM: we make the system evolve according to the unitary U_t, and we apply the operator Z_t which implements the translation by W_t to the position of the quantum particle.

2.3.6 From the Hudson-Parthasarathy equation to the Lindblad equation

The family of operators $(\mathfrak{U}_t)_{0\le t}$ and of states $\rho_{tot,t}=\mathfrak{U}_t(\rho_S\otimes|\Omega\rangle\langle\Omega|)\mathfrak{U}_t^*$ consists in the most complete description of the OQBM. In this Paragraph, we show how the existence of Λ_S^t and the Lindblad equation 2.11 can be deduced from it.

First, for any state $\rho_S \in \mathfrak{S}(\mathcal{H}_G \otimes \mathcal{H}_z)$, the state

$$\rho_{tot,\tau,t} = \mathfrak{U}_{\tau,[t/\tau]} \left(P_{\delta\mathbb{Z}}\rho_S P_{\delta\mathbb{Z}} \otimes |\Omega\rangle \langle\Omega|\right) \mathfrak{U}_{\tau,[t/\tau]}$$

converges in $\mathcal{S}^1(\mathcal{H}_S \otimes \Phi$ to

$$\rho_{tot,t} = \mathfrak{U}_t(\rho_S \otimes |\Omega\rangle \langle\Omega|)\mathfrak{U}_t^* \ .$$

It is a consequence of Theorem 17 and of the strong convergence of $P_{\delta\mathbb{Z}}$ to $I_{\mathcal{H}_Z}$. In particular, the state $\rho_{S,\tau,t} = \Lambda_\tau^{[t/\tau]}(P_{\delta\mathbb{Z}}\rho_S P_{\delta\mathbb{Z}}) = \mathrm{Tr}_\Phi(\rho_{tot,\tau,t}$ strongly converge to the state

$$\rho_{S,t} = \mathrm{Tr}_\Phi(\rho_{tot,t}) \ .$$

This proves Proposition 7. We can now prove Proposition 9 from the formula

$$\Lambda_S^t(\rho) = \mathrm{Tr}_\Phi\left(\mathfrak{U}_t(\rho \otimes |\Omega\rangle \langle\Omega|)\mathfrak{U}_t^*\right) \ .$$

For this, we are going to use the Hudson-Parthasarathy Equation on $\mathfrak{U}_t$.

Proof of Propositions 9 and 10. The operator $\mathfrak{U}_t$ preserves the space $\mathcal{S}_2(\mathcal{H}_G \otimes \Phi, \mathcal{H}_z)$. To prove it, we use the following characterization of this space:

Lemma 19. *The space $\mathcal{S}_k(\mathcal{H}, \mathcal{H}_z)$ is the space of states $\rho \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{H}_z)$ such that for any $n \leq k$ the operator $[\rho, |\partial_x|^n]$ is bounded on $\mathcal{H} \otimes W^{2,k}(\mathbb{R})$.*

Now, the operator $\mathfrak{U}_t$ commutes with ∂_x (since Z_t and S_t both commute with ∂_x) so for any operator $\rho_{tot} \in \mathcal{S}(\mathcal{H}_G \otimes \phi)$ we have $[\mathfrak{U}_t \rho \mathfrak{U}_t^*, |\partial_x|^n] = \mathfrak{U}_t[\rho, |\partial_x|^n]\mathfrak{U}_t^*$.

Thus, if $\rho \in \mathcal{S}_2(\mathcal{H}_G, \mathcal{H}_z)$ then $\mathfrak{U}_t(\rho \otimes |\Omega\rangle \langle\Omega|)\mathfrak{U}_t^* \in \mathcal{S}_2(\mathcal{H}_G \otimes \Phi, \mathcal{H}_z)$ and so $\rho_S(t) \in \mathcal{S}_2(\mathcal{H}_G, \mathcal{H}_z)$.

To obtain the Lindblad equation we use the Heisenberg representation: for any observable $A \in W^{2,1}(\mathbb{R}^2, \mathcal{B}(\mathcal{H}_G))$ we have

$$\mathrm{Tr}\left(\rho_{S,t} A\right) = \mathrm{Tr}\left(\rho \langle\Omega| \mathfrak{U}_t^*(A \otimes I_\Phi \mathfrak{U}_t |\Omega\rangle\right) \ .$$

Using the Itô formula applied to $\mathfrak{U}_t^* A \mathfrak{U}_t$ on the domain $\mathcal{H}_G \otimes_{alg} H^2(\mathbb{R}) \otimes_{alg} \varepsilon(L^2(\mathbb{R}))$, we obtain that

$$\mathfrak{U}_t^* A \mathfrak{U}_t = A + \int_0^t \mathfrak{U}_s^* \tilde{\mathcal{L}}^*(A) \mathfrak{U}_s ds + R_t$$

where R_t is an integral with respect of terms the form $da_j^i(t)$ with $(i,j) \neq (0,0)$, so that $\langle\Omega| R_t |\Omega\rangle = 0$. Thus

$$\begin{aligned}
\mathrm{Tr}\left(\rho_{S,t} A\right) - \mathrm{Tr}\left(\rho A\right) &= \int_0^t \mathrm{Tr}\left(\mathfrak{U}_s(\rho \otimes |\Omega\rangle \langle\Omega|)\mathfrak{U}_s^* \ \tilde{\mathcal{L}}^*(A)\right) \ ds \\
&= \int_0^t \mathrm{Tr}\left(\tilde{\mathcal{L}}(\rho_{S,t}(s))A\right) ds
\end{aligned}$$

which implies Equation 2.11 by density of $W^{2,1}(\mathbb{R}^2, \mathcal{B}(\mathcal{H}_G))$ in $\mathcal{B}(hh_G \otimes \mathcal{H}_z)$. The equation on the kernel is obtained directly with Equation 2.12. This ends the proof of Proposition 9.

Now, when looking at the restriction to $\mathcal{M}$, we note that $\mathfrak{U}_t^*\mathcal{M}\mathfrak{U}_t \subset \mathcal{M}\otimes\mathcal{B}(\Phi)$ so that for any $A \in \mathcal{M}$ the expectancy $\mathrm{Tr}\left(\Lambda_S^t(\rho)A\right) = \mathrm{Tr}\left((\rho \otimes |\Omega\rangle\langle\Omega|)\mathfrak{U}_t^* A \mathfrak{U}_t\right)$ only depends on the restriction of ρ to $\mathcal{M}$. The rest of Proposition 10 is proved exactly the same as Proposition 9. ☐

Remark 4. We crucially used that $\mathrm{Tr}\left(\rho_S(0) \otimes |\mathbb{1}\rangle\langle\mathbb{1}| H_t da_j^i(t)\right) = 0$ whenever $i \neq j$. We have to be careful with this type of formula, for the following reasons:

1. We may be tempted to write for example $\mathrm{Tr}\left(A da_i^j(t) B da_k^l(t)\right) = \mathrm{Tr}\left(B da_k^l(t) A da_i^j(t)\right)$, which would for example result in

$$\mathrm{Tr}\left(A da_0^1(t) da_1^0(t)\right) = \mathrm{Tr}\left(A\right) dt = \mathrm{Tr}\left(A da_1^0(t) da_0^1(t)\right) = 0$$

which is absurd. Thus, writing the full formula with the integral is advised when using the commutation property of the trace.

2. It seems intuitive that $\mathrm{Tr}\left(\int_0^t H_s da_j^i(s)\right) = 0$ as soon as $i \neq j$. But even when $\int_0^t H_s da_j^i(s)$ is trace-class, it may be of nonzero trace if H_t acts non-trivially on Φ. Thus, it is best to use the Heisenberg representation when computing $\mathrm{Tr}\left(\rho(t)A\right)$ to apply the formula

$$\langle\mathbb{1}| \int_0^t H_s da_j^i(s) |\mathbb{1}\rangle = 0$$

which is valid whenever the integral is meaningful and $(i, j) \neq (0, 0)$.

2.4 Hierarchy of the descriptions of the OQBM

With the OQBM, we have many views on the same object, carrying more or less informations:

a) The state $\rho_{tot,t} = \mathfrak{U}_t(\rho_S \otimes |\Omega\rangle\langle\Omega|)\mathfrak{U}_t^*$ on $\mathcal{H}_G \otimes \mathcal{H}_z \otimes \Phi$ offers the most complete description.

b) The state $\rho_{tot,G,t} = U_t(\rho_G \otimes |\Omega\rangle\langle\Omega|)U_t^* = \mathrm{Tr}_{\mathcal{H}_z}(\rho_{tot,t})$ ignores the position of the particle, though its translation W_t is still registered in Φ.

c) The random state ϱ_t with the random position X_t ignores the quantum aspect of the position, but keeps tracks of the classical correlations between two different times.

d) The state $\rho_{S,t} = \mathrm{Tr}_\Phi(\rho_{tot,t}) = \Lambda_S^t(\rho_S)$ on $\mathcal{B}(\mathcal{H}_S)$ forgets about correlations between different times and the precise distribution of ϱ_t, but conserves a quantum view on the position.

e) The restriction of $\rho_{S,t}$ to $\mathcal{M} = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}_z$ with matrix density function $Q_t(x) = \mathbb{E}(\varrho_t | X_t = x)$: it forgets the correlations between different times and has only the classical information about the position. This is the smallest description where we have a closed equation for the evolution (Equation 2.14) and which allows to compute the law of X_t.

f) The state $\rho_G = \mathrm{Tr}_{\mathcal{H}_z \otimes \Phi}(\rho_{tot,G,t}) = \int_{x\in\mathbb{R}} Q_t(x)dx$ evolves according to the Lindbladian $\mathcal{L}$ and it completely ignores the position X_t.

The descriptions a), c), d), e) are really dealing with the OQBM, while b) and f) are only considering the evolution on $\mathcal{H}_G$. They can be obtained one from another by partial traces, restriction and conditional expectancy according to the following hierarchy:

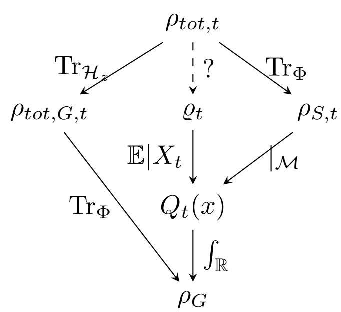

Figure 1: Hierarchy between the descriptions of the Open Quantum Brownian Motion

The way we can obtain (ϱ_t, X_t) directly from $\rho_{tot,t}$ is the subject of the second section of this article.

3 Non-demolition measured evolution applied to the Open Quantum Brownian Motion

In the first section, we described the process $(\varrho_{\tau,n}, X_{\tau,n})_{n\in\mathbb{N}}$ as the result of a succession of unitary evolution by L_τ and measure of the position $X_{\tau,n} \in \delta\mathbb{Z}$. In continuous time this picture is harder to obtain, since the time is continuous the measure and the evolution are happening at the same time. In this section we construct a general framework to deal with simultaneous measurement and evolution, using the crucial idea of non-demolition measurement.

3.1 Evolution and measurement

3.1.1 The quantum state after the measurement of a continuous observable

In Paragraph 2.1.4 we explained that it is not possible to describe the state after the measurement of a non-discrete observable. However, we will need to measure the position $X_t \in \mathbb{R}$ (or the translation $W_t \in \mathbb{R}$), so we need to bypass this problem. The idea is to consider the state after the measurement restricted to some subalgebra of $\mathcal{B}(\mathcal{H})$. The case we consider is the following:

- The space $\mathcal{H}$ is the tensor product of two Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_B$.
- We want to measure a family of mutually commuting operators $(B_\alpha)_{\alpha\in I}$ acting on $\mathcal{H}_B$. Write $\mathcal{A}$ the von Neumann algebra generated by the B_α's.
- We are interested on the state after the measurement on $\mathcal{B}(\mathcal{H}_G)$ only. It will be written $\rho_{G|\mathcal{A}}$.

We will see that concentrating on the state on $\mathcal{H}_G$ and ignoring the full picture on $\mathcal{H}_G \otimes \mathcal{H}_B$ allows us to get a rigorous definition of $\rho_{G|\mathcal{A}}$.

Since the B_α are commuting, we can identify $\mathcal{H}_B$ with $L^2(\mathcal{X},\mu)$ for some standard measured space space $(\mathcal{X},\mathcal{F},\mu)$ such that there exists measurable functions g_α with $B_\alpha = M_{g_\alpha}$. We want to define $\rho_{G|\mathcal{A}}$ as a random variable with values in $\mathfrak{S}(\mathcal{H}_G)$ on the probability space generated by the random variables $g_\alpha = \tilde{B}_{\alpha,\rho}$.

Theorem 20. *Let ρ be a state on $\mathcal{H}_G \otimes L^2(\mathcal{X},\mu)$. Then there exists a measurable map ς from $\mathcal{X}$ to $\mathcal{S}^1(\mathcal{H}_B)$ such that for any $f \in L^\infty(\mathcal{X},\mu)$ and for any observable $A \in \mathcal{B}(\mathcal{H}_G)$ we have*

$$Tr(\rho\ A \otimes M_f) = \int_{\mathcal{X}} Tr(\varsigma(x)A)\, f(x) d\mu(x)\ .$$

It is unique (up to a μ-negligible set), and $\varsigma(x)$ it positive and satisfies

$$Tr(\varsigma(x)) = \frac{d\mathbb{P}_\rho(x)}{d\mu(x)}$$

for μ-ae x. It is called the unnormalized state *on $\mathcal{H}_G$ associated to $(\mathcal{X},\mu)$. Note that its trace depends on the measure μ which is chosen.*

Now, consider a sub-σ-algebra $\mathcal{F}_1 \subset \mathcal{F}$ and let $\mathcal{A} = L^\infty(\mathcal{X},\mathcal{F}_1,\mu)$. Let $\mathbb{P}_\rho$ the probability measure induced by ρ on $\mathcal{X}$. Then there exists a random variable $\rho_{G|\mathcal{A}}$ on $(\mathcal{X},\mathcal{F}_1,\mathbb{P}_\rho)$ with values in $\mathfrak{S}(\mathcal{H}_G)$ such that for any operator $A \in \mathcal{B}(\mathcal{H}_G)$ and any random variable $f \in L^\infty(\mathcal{X},\mathcal{F}_1,\mathbb{P}_\rho)$ we have

$$Tr(\rho A \otimes M_f) = \mathbb{E}_\rho\left(Tr\big(\rho_{G|\mathcal{A}}\, A\big) f\right)$$

where on the right f is seen as a random variable. The random variable $\rho_{G|\mathcal{A}}$ is unique up to a set of probability zero, and for $\mathbb{P}$-almost a $x \in \mathcal{X}$ we have $\rho_{G|\mathcal{A}}(x) =_s sigma(x)/p_\rho(x)$.

We will often write ϱ for $\rho_{G|\mathcal{A}}$ when it does not cause confusion, and we write $\varsigma = u_{\mathcal{X}}(\rho)$ (or $u_{(\mathcal{X},\mu)}(\rho)$ when the measure needs to be precised).

Note that $u_{xx} : \rho \mapsto \varsigma$ is an isometry, contrarily to the map $\rho \mapsto \varrho$.

Proof. The function ς is just the matrix density function of the restriction of ρ to $\mathcal{M} = \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}$, so its existence is just a consequence of the Riesz theorem.

We have $\int_{\mathcal{X}} f(x) \mathrm{Tr}\,(\varsigma(x))\, d\mu(x) = \rho(M_f) = \mathbb{E}_\rho(f)$ so $tr\varsigma(x) = dP_\rho/d\mu$, and so $\mathrm{Tr}\,(\varsigma(x))$ is nonzero $\mathbb{P}_\rho$-almost surely. We now define

$$R(x) = \frac{\varsigma(x)}{\mathrm{Tr}\,(\varsigma(x))}$$

on x such that $\varsigma(x) \neq 0$. It is a random variable on $(\mathcal{X}, \mathbb{P}_\rho)$. Now we take the conditional expectation with respect to the σ-algebra $\mathcal{F}$ generated by the g_α on $\mathcal{X}$:

$$\rho_{G|\mathcal{A}} = \mathbb{E}\left[R \,\middle|\, \mathcal{F}\right].$$

It is easy to show that it fits the requirement of the theorem.

The uniqueness is straightforward. □

Remark 5.

1. With this approach, we clearly separate the quantum superposition, described by a density matrix, and the classical randomness on the probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P}_\rho)$. It is frequent in quantum filtering theory to define ϱ as a state on the commutant of $\mathcal{A}$, which is in general bigger than $\mathcal{B}(\mathcal{H}_B) \otimes L^\infty(\mathcal{X})$, but this does not define ϱ explicitly as a random variable on some probability space.

2. Note that the state $\rho_{G|\mathcal{A}}$ contains more information that $\rho_G = \mathrm{Tr}_{\mathcal{H}_B}(\rho)$ since $\rho_G = \mathbb{E}_{p_\rho d\mu}(\rho_{G|\mathcal{A}})$. Thus, we have three descriptions of the state of the system, containing less and less information: the full state ρ on $\mathcal{H}_G \otimes \mathcal{H}_B$, the random state $\rho_{G|\mathcal{A}}$ and the state ρ_G. We could define a fourth description between $\rho_{G|\mathcal{A}}$ and ρ_G by using the theory of direct integral: if $\mathcal{A}$ is the set of decomposable operators on $\mathcal{H} = \int_{\mathcal{X}}^{\oplus} \mathcal{H}(x) d\mu(x)$ we may consider a random state $\varrho(X)$ on the random Hilbert space $\mathcal{H}(X)$. This level of precision is not needed for our purpose.

As an application of this theorem, we can model the indirect measurement of an observable; it is a framework often called von Neumann measurement of an observable in the literature ([14],[22], [18]). Let us describe the measurement of the observable X on $\mathcal{H}_B = L^2(\mathbb{R}, Leb)$. We couple the system with the pointer of some measurement device, described by $\mathcal{H}_B = L^2(\mathbb{R}, Leb)$. we call $\mathcal{H}_B$ the pointer space (think of it as the needle of a weighting scale or a seismometer). We move the pointer depending on the value of X, which has the effect of applying a unitary Z on $\mathcal{H}_B \otimes \mathcal{H}_B = L^2(\mathbb{R}^2, Leb_2)$ which is defined by

$$(Zf)(x, a) = f(x, a - x).$$

Then, we perform the measurement of the pointer : we measure $A = M_{a \mapsto a}$ on $\mathcal{H}_B$. The result is a random variable $\tilde{A}$ and the state after the measurement is $\rho_{G|\mathcal{A}}$ (where $\mathcal{A}$ is the algebra generated by A). Note that the noise is described by the initial state of the pointer. For example, if the system is in the pure state $f \in L^2(\mathbb{R}, Leb)$ and the pointer in the pure state $g \in L^2(\mathbb{R}, Leb)$, the probability density of $\tilde{A}$ is

$$p(a) = \int_{\mathbb{R}} |f(x)|^2 |g(a - x)|^2 dx = |f|^2 * |g|^2(a)$$

and for any $a \in \mathbb{R}$ the state $\rho G|\mathcal{A}(a)$ is the pure state $|f_a\rangle\langle f_a|$ where

$$f_a(x) = \frac{f(x)g(a - x)}{p(a)}.$$

This really corresponds to a classical noisy measurement : if X is a random variable with density $|f|^2$ and B a random variable with density $|g|^2$ then p is the density of $X+B$ and $|f_a|^2$ is the density of X conditioned to $X+B=a$. Note however that this situation is truly quantum: if we do not perform the measurement, the density matrix of the system is

$$\rho'_G = \mathbb{E}(\tilde{\rho}_G) = \mathrm{Tr}_B(Z\,(\rho_G \otimes \rho_B)\,Z^*)$$

which is of kernel

$$K_{\rho'_G}(x,y) = f(x)\overline{f(y)}\int_{\mathbb{R}} g(a-x)\overline{g(a-y)}da = f(x)\overline{f(y)}C_g(x-y)\,.$$

where

$$C_g(z) = \int_{\mathbb{R}} g(a-z)\overline{g(a)}da.$$

It is no more a pure state.

A more general version of this process is the following:

Definition 21. *Let $\mathcal{H}_G$ be a Hilbert space and $\mathcal{A}$ a commutative von Neumann algebra on $\mathcal{H}_G$, with an isometry $\mathcal{I} : L^2(\mathcal{X},\mu) \to \mathcal{H}_G$ implementing an isomorphism $\mathcal{A} \simeq L^\infty(\mathcal{X},\mu)$. Consider an auxiliary space $\mathcal{H}_B = L^2(\mathcal{Y},\nu)$. A* pointer map *is some measurable function $\psi : \mathcal{X}\times\mathcal{Y} \to \mathcal{X}$ such that for all $x \in \mathcal{X}$ the map $\psi(x,\bullet)$ is a measure-preserving bijection on $\mathcal{Y}$. The pointer unitary Z_ψ on $\mathcal{H}_G \otimes \mathcal{H}_B$ corresponding to ψ is the operator defined as $Z_\psi = \mathcal{I}\tilde{Z}_\psi\mathcal{I}^*$ where $\tilde{Z}_\psi$ is the unitary on $L^2(\mathcal{X}\times\mathcal{Y},\mu\times\nu)$ defined by*

$$(\tilde{Z}_\psi f)(x,y) = f(x,\ \psi(x,y)\ )\,.$$

The indirect measurement corresponding to ψ is the measurement of the algebra $L^\infty(\mathcal{Y},\nu)$ on $\mathcal{H}_B$, resulting in the random value $Y \in \mathcal{Y}$ of the pointer and the random state $\rho_{G|Y} \in \mathfrak{S}(\mathcal{H}_B)$.

This is a little more restrictive than the processes considered by Belavkin [14], in which the unitary Z (written S by Belavkin) is only assumed to commute with elements of $L^\infty(\mathcal{X},\mu)\otimes\{I_B\}$. This restrictive definition has the advantage of making it more explicit.

This definition include the perfect measurement of a discrete observable A: take $\mathcal{X} = \mathcal{Y} = sp(A)$ with μ the counting measure and fix an initial state $a_0 \in \mathcal{Y}$, choose $\rho_B = |\delta_{a_0}\rangle\langle\delta_{a_0}|$ and any pointer function ψ such that $\psi(a,a_0) = a$.

3.1.2 Measurement and evolution

The evolution of a system after the measurement may be impossible to describe. Let us assume that the evolution of the system is described by a unitary U on $\mathcal{H}_G \otimes L^2(\mathcal{X},\mu)$. We may measure the algebra $\mathcal{A} = L^\infty(\mathcal{X},\nu)$ before *or* after applying U, obtaining a random variable $X \in \mathcal{X}$ and random states $\varrho_0 = \rho_{G|\mathcal{A}}$ and $\varrho_1 = (U\rho U^*)_{S|\mathcal{A}}$. However, it is not clear how to describe the measurement before *and* after applying U. There may be two issues there:

1. The state $\rho_{GB|\mathcal{A}}$ is well defined only if $\mathcal{A}$ is discrete, else we only have the partial state $\rho_{G|\mathcal{A}}$. Thus, we cannot define $U\rho_{GB|\mathcal{A}}U^*$.
2. Even if $\mathcal{A}$ is discrete, the measurement before applying U modifies the state of the system, so $(U\rho_{GB|\mathcal{A}}U^*)_{S|\mathcal{A}}$ may not have the same law as $(U\rho U^*)_{S|\mathcal{A}}$.

The restriction to so called non-demolition evolutions allows to bypass these two issues in the general context of measurement under evolution.

Definition 22. *Let $\mathcal{H}_G$ and $\mathcal{H}_B$ be two Hilbert spaces, let $I \subset \mathbb{R}$ be a set of times and $(U_t)_{t\in I}$ be a family of unitary operators on $\mathcal{H}_G \otimes \mathcal{H}_B$ with $U_0 = I$ if $0 \in I$ and let $(\mathcal{A}_t)_{t\in I}$ be a family of commutative von Neumann algebras on $\mathcal{H}_B$. Write $U_{t,s} = U_t U_s^*$ for any $s, t \in I$. We say that the process $(U_t, \mathcal{A}_t)_{t\in I}$ is a $\mathcal{H}_G$-non demolition evolution if for any $s \leq t \in I$ we have*

$$U_{t,s}\mathcal{A}_s U_{t,s}^* \subset I_G \otimes \mathcal{A}_t'$$

where $\mathcal{A}_t'$ is the commutant of $\mathcal{A}_t$.

In most cases the family of algebras will be increasing ($\mathcal{A}_s \subset \mathcal{A}_t$ for $s \leq t$) but we do not require it.

The condition $U_{t,s}\mathcal{A}_s U_{t,s}^* \subset \mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}_t'$ is here to ensure that the measure of $\mathcal{A}_s$ does not disturb the measure of $\mathcal{A}_t$ after evolution, while the condition $U_{t,s}\mathcal{A}_s U_{t,s}^* \subset (I_G \otimes \mathcal{B}(\mathcal{H}_B))$ ensure that the random state at time t is well defined. Let us describe more precisely how the random evolution can be defined.

Let us consider an $\mathcal{H}_G$-non demolition evolution $(U_t, \mathcal{A}_t)_{t\in I}$ and a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_G \otimes \mathcal{H}_B)$. We make the assumption that I is upper bounded[2] by some $T \in I$. We fix some identifications $\mathcal{A}_t \simeq L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$ implemented by some isometries $\mathcal{I}_t : L^2(\mathcal{X}_t, \mathcal{F}_t, \mu_t) \to \mathcal{H}_B$.We want to define a probability space $(\Omega, \mathbb{P})$ with a stochastic process $(X_t, \varrho_t)_{t\in I}$ with $X_t \in \mathcal{X}_t$ and $\varrho_t \in \mathfrak{S}(\mathcal{H}_G)$ obtained by simultaneously measuring $\mathcal{A}_t$ at time t and making evolve the system according to U_t. We construct it as follows.

- Let $\mathcal{A}_t^U$ be the smallest von Neumann algebra containing all the algebras $U_{t,s}\mathcal{A}_s U_{t,s}^*$ for $s \leq t$. It is commutative and contained in $I_G \otimes \mathcal{B}(\mathcal{H}_B)$ by the $\mathcal{H}_G$-non demolition hypothesis. We fix an identification $\mathcal{A}_t^U \simeq L^\infty(\mathcal{X}_t^U, \mathcal{F}_t^U \mu_t^U)$ implemented by an isometry $\mathcal{I}_t^U : L^2(\mathcal{X}^U, \mathcal{F}_t^U, \mu_t^U) \to \mathcal{H}_B$.
- For any $s \leq t$ we have $\mathcal{A}_t \subset \mathcal{A}_t^U$ so there exists a map $\phi_t : \mathcal{X}_t^U \to \mathcal{X}_t$ such that for any $f \in L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$ we have
$$\mathcal{I}_t^U M_{f\circ\phi_t}(\mathcal{I}_t^U)^* = \mathcal{I}_t M_f \mathcal{I}_t^* \ .$$
- For $s \leq t$ we have $U_{t,s}\mathcal{A}_s^U U_{t,s}^* \subset \mathcal{A}_t^U$ so there are maps $\eta_{s,t} : \mathcal{X}_t^U \to \mathcal{X}_s^U$ such that for any $f \in L^\infty(\mathcal{X}_s^U, \mathcal{F}_s, \mu_s^U)$ we have
$$\mathcal{I}_t^U M_{f\circ\eta_{s,t}}(\mathcal{I}_t^U)^* = U_{t,s}^*\mathcal{I}_s^U M_f(\mathcal{I}_s^U)^* U_{t,s}^* \ .$$

[2]this assumption is actually not necessary but it allows to use more concrete notations

- We take for our universe Ω the space $\mathcal{X}_T^U$ with probability $\mathbb{P} = p_T^U d\mu_T^U$ induced by $U_T \rho U_T^*$ and the identification $\mathcal{I}_T^U$. The random variable $X_t \in \mathcal{X}_t$ is then defined as $\phi_t \circ \eta_{t,T}$.
- The random variable ϱ_t is defined as

$$\varrho_t = \left(U_t \rho U_t^*\right)_{G|\mathcal{A}_t^U} (\eta_{t,T}) \ .$$

it is indeed a random variable on $\mathcal{X}_T^U$.

Remark 6. Note that the maps $\eta_{t,s}$ and ϕ_t are defined uniquely only up to a set of measure zero, as well as the random variable $\left(U_t \rho U_t^*\right)_{G|\mathcal{A}_t^U}$. Thus, if I is not countable there is not uniqueness in distribution of the process $(X_t, \varrho_t)_{t\in I}$, only uniqueness in finite-dimensional distributions. For example, when $\mathcal{X}_t = \mathbb{R}$ for all t, the function $t \to X_t$ may be almost surely continuous, but this depends on the $\eta_{t,s}$ and ϕ_t which are chosen.

Definition 23. *Any process $(X_t, \varrho_t)_{t\in I}$ obtained as above is called a measured evolution obtained from the $\mathcal{H}_G$-non demolition evolution $(U_t, \mathcal{A}_t)_{t\in I}$ and the state ρ_0.*

This way of define the stochastic process should seem natural; a first motivation is that X_t has the same law as the result of the measure of $\mathcal{A}_t$ in the state $U_t \rho U_t^*$, indeed for any function $f \in L^\infty(\mathcal{X}_t, \mathcal{F}_t, \mu_t)$ we have

$$\begin{aligned} \mathbb{E}(f) &= \mathbb{E}(f \circ \phi_t \circ \eta_{t,T}) \\ &= \mathrm{Tr}\left(U_T \rho U_T^* \mathcal{I}_t^U M_{f\circ\phi_t\circ\eta_{t,T}} \mathcal{I}_t^*\right) \\ &= \mathrm{Tr}\left(U_t \rho U_t^* \ U_{T,t}^* \ \mathcal{I}_t^U M_{f\circ\phi_t\circ\eta_{t,T}} (\mathcal{I}_t^U)^* U_{T,t}\right) \\ &= \mathrm{Tr}\left(U_t \rho U_t^* \ \mathcal{I}_t M_f \mathcal{I}_t^*\right) \ . \end{aligned}$$

However, this is only the law of X_t at one time, and it does not justifies the joint distribution of the X_t's for $t \in I$. We will use the indirect measurement defined in the previous section to make a more complete and useful argument.

Definition 24. *For each t let us fix an identification $\mathcal{A}_t \simeq L^\infty(\mathcal{X}_t, \mathcal{F}_t, d\mu_t)$. We call an* indirect measurement *of $(\mathcal{A}_t)_{t\in I}$ under the evolution $(U_t)_{t\in I}$ the following type of setup: let $J = \{ t_0, \cdots, t_n \} \subset I$ be a finite subset of I and consider a family of pointer maps $(\psi_k)_{0\leq k\leq n}$ with $\psi_k : (\mathcal{X}, \mathcal{F}_{t_k}) \times \mathcal{Y}_k \to \mathcal{Y}_k$ and a family of states $(\sigma_k)_{0\leq k\leq n}$ on $L^2(\mathcal{Y}_k, \nu_k)$ with corresponding probability density p_k on $\mathcal{Y}_k$. Consider the pointer unitary operators $Z_k = Z_{\psi_k}$ as in Definition 21. Let us perform successive indirect measurement: let $Y_0 \in \mathcal{Y}_{t_0}$ be the result of the measurement of $L^\infty(\mathcal{Y}_0)$ for the state $Z_0\left(U_{t_0} \rho U_{t_0}\right) \otimes \sigma_0 Z_0^*$, and $\varrho_{SB}(t_0)$ the state on $\mathcal{H}_S \otimes \mathcal{H}_B$ after the measurement; then, define Y_1 the result of the measurement of $L^\infty(\mathcal{Y}_1)$ for the state $Z_1\left(U_{t_1,t_0} \varrho_{SB}(t_0) U_{t_1,t_0}^*\right) \otimes \sigma_1 Z_1^*$, and define successively $Y_2, \cdots, Y_n$ the same way. We obtain a random process $(Y_k)_{0\leq k\leq n}$ on the space $\prod_{k=0}^n \mathcal{Y}_k$ and a family of random states $\varrho_{t_k}^Y((Y_l)_{l\leq k}) = Tr_B(\rho_{SB}(t_k))$.*

Note that we can perform this type of indirect measurement even if the property of $\mathcal{H}_G$-non demolition is missing. The non-demolition property makes these indirect measurements to be consistent with the process described above, as follows.

Proposition 25 (Consistency of the unraveling)**.** *Consider any indirect measurement of $(\mathcal{A}_t)_{t\in I}$ under the evolution $(U_t)_{t\in I}$ described as above. Assume that the $\mathcal{H}_G$-non demolition property is satisfied. Consider the random state ϱ_t and the random variables $X_t \in \mathcal{X}_t$ defined above on the universe $\mathcal{X}_{tot}$. Add to this universe a family of random variables $(Y_t^0)_{t\in J}$ with law $p_k d\nu_k$, where p_k is the probability density corresponding to the state σ_k on $L^2(\mathcal{Y}_k, \nu_k)$. Assume that they are mutually independent and independent of $(X_t)_{t\in I}$ and define*

$$\begin{aligned}\tilde{Y}_k &= \psi(X_{t_k}, Y_k^0)\\ \tilde{\varrho}_k^Y &= \mathbb{E}(\varrho_t((X_s)_{s\in I}) \mid (\tilde{Y}_k)_{0\le k\le n})\ .\end{aligned}$$

Then $(\tilde{Y}_k, \tilde{\varrho}_{t_k}^Y)_{0\le k\le n}$ has the same law as $(Y_k, \varrho_{t_k}^Y)_{0\le k\le n}$.

Proof. Let us write

$$W_k = Z_k U_{t_k,t_{k-1}} Z_{k-1} U_{t_{k_1},t_{k-2}} \cdots Z_0 U_{t_0}$$

and let $\mathcal{A}_k^Y = L^\infty(\prod_{l\le k} \mathcal{Y}_k, \bigotimes_{l\le k} \nu_l)$. Then

$$\varrho_{t_k}^Y = (W_k(\rho\otimes\sigma)W_k^*)_{S|\mathcal{A}_k}\ .$$

Moreover, for any function $f \in \mathcal{A}_k^Y$ and operator $A \in \mathcal{B}(\mathcal{H}_S)$ we have

$$\mathbb{E}(\mathrm{Tr}\left(\varrho_{t_k}^Y\right) f(Y_0,\cdots,Y_k)) = \mathrm{Tr}\left(W_k(\rho\otimes\sigma)W_k^* A\otimes M_f\right)\ .$$

Similarly, by the construction of ϱ_t and $\tilde{\varrho}_t^Y$ we have

$$\mathbb{E}(\mathrm{Tr}\left(\tilde{\varrho}_{t_k}^Y\right) f(\tilde{Y}_0,\cdots,\tilde{Y}_k) = \mathrm{Tr}\left((U_{t_k}\rho U_{t_k}^*\otimes\sigma)A\otimes \mathcal{I}_t^U M_g(\mathcal{I}_t^U)^*\right)$$

where $g \in L^\infty(\mathcal{X}_{t_k}^U \times \prod_{l\le k}\mathcal{Y}_k)$ is defined by

$$g(x_{t_k}^U, y_0,\cdots,y_k) = f(\psi_0(\phi_{t_0}\circ\eta_{t_0,t_k}(x_{t_k}^U), y_0),\cdots,\psi_k(\phi_{t_k}(x_{t_k}^U), y_k))\ .$$

(it corresponds to the random variable $f(\tilde{Y}_0,\cdots,\tilde{Y}_k)$).

Now, we have

$$\mathcal{I}_t^U M_g(\mathcal{I}_t^U)^* = U_{t_k} W_k^* f W_k U_{t_k}^*$$

by the definition of the Z_k and $\phi_t, \eta_{t_k,t}$. Thus,

$$\begin{aligned}\mathbb{E}(\mathrm{Tr}\left(\tilde{\varrho}_{t_k}^Y\right) f) &= \mathrm{Tr}\left((U_{t_k}\rho U_{t_k}^*\otimes\sigma)A U_{t_k} W_k^* f W_k U_{t_k}^*\right)\\ &= \mathrm{Tr}\left((W_k\rho\otimes\sigma)U_{t_k}^* A U_{t_k} W_k^* f\right)\ .\end{aligned}$$

Now, by $\mathcal{H}_G$-non demolition, since A is in the commutator of $I_G \otimes \mathcal{B}(\mathcal{H}_B)$ for any $l \leq k$ we have $U^*_{t_k,t_l} A U_{t_k,t_l} \in \mathcal{A}'_{t_l}$ and in particular $U^*_{t_k,t_l} A U_{t_k,t_l}$ commutes with Z^*_l. Thus, we have

$$\begin{aligned} U^*_{t_k} A U_{t_k} W^*_k &= U^*_{t_0}(U^*_{t_0,t_k} A U_{t_k,t_0}) Z_0 \cdots U_{t_{k-2},t_{k-1}} Z^*_{k-1} U_{t_{k-1},t_k} Z^*_k \\ &= U^*_{t_0} Z^*_0 U^*_{t_0,t_k} A U_{t_k,t_1} Z_1 \cdots U_{t_{k-2},t_{k-1}} Z^*_{k-1} U_{t_{k-1},t_k} Z^*_k \end{aligned}$$

and with successive commutations we get

$$U^*_{t_k} A U_{t_k} W^*_k = W^*_k A \ .$$

Thus we have

$$\mathbb{E}(\mathrm{Tr}\left(\tilde{\varrho}^Y_{t_k}\right) f(\tilde{Y}_0, \cdots, \tilde{Y}_k)) = \mathbb{E}(\mathrm{Tr}\left(\varrho^Y_{t_k}\right) f(Y_0, \cdots, Y_k)) \ .$$

This proves the equality in distribution. □

3.1.3 The example of OQW

Open Quantum Random Walks are our first example of measured evolution. Let us consider any OQW $(B_e)_{e \in E}$ on a countable graph $(\mathcal{V}, E)$. It consists in the succession of evolution by the quantum channel $\varphi(\rho) = \sum_{(x \to y) \in E} (B_{(x\to y)} \otimes |y\rangle\langle x|)\rho(B^*_{(x\to y)} \otimes |x\rangle\langle y|)$ and of measure of the algebra $\mathcal{A}_\mathcal{V} = l^\infty(\mathcal{V})$. As such, it does not need the formalism of measured evolution to be defined since $\mathcal{A}_\mathcal{V}$ is discrete, but it can help understanding how measured evolutions work.

Let us construct the auxiliary space $\mathcal{H}_p = l^2(\mathcal{V})$. In the article [8] in which OQW where first defined, it is constructed a unitary U on $\mathcal{H}_G \otimes l^2(\mathcal{V}) \otimes \mathcal{H}_p$ the following way: we fix a point $x_0 \in \mathcal{V}$. For any $x \in \mathcal{V}$ we consider a unitary $V(x)$ such that for all $y \in \mathcal{V}$ we have

$$\langle y|_{\mathcal{H}_p} V(x) |x_0\rangle_{\mathcal{H}_p} = \mathbb{1}_{(x,y)\in\mathbb{E}} B_{(x\to y)} \ .$$

It exists because of the condition $\sum_{y \text{ with } (x\to y)\in E} B^*_e B_e = I$. Write $V(x)_{yz} = \langle y|_{\mathcal{H}_p} V(x) |z\rangle_{\mathcal{H}_p}$. We put

$$U = \sum_{x,y,z \in \mathcal{V}} V(x)_{yz} \otimes |y\rangle\langle x| \otimes |x\rangle\langle z| \ .$$

Consider the Toy Fock space $T\Phi_\mathcal{V} = \bigotimes_{n \in \mathbb{N}^*} \mathcal{H}_p$ with respect to $|x_0\rangle$, and write $|\Omega\rangle = \bigotimes_{n\in\mathbb{N}^*} |x_0\rangle$. We consider the unitary $U(n, n-1) = \mathcal{I}^*_n U \mathcal{I}_n$ on $\mathcal{H}_G \otimes l^2(\mathcal{V}) \otimes T\Phi_\mathcal{V}$ and define $U(n) = U(n, n-1)U(n-1, n-2) \cdots U(2, 1)$. The system $(U(n), \mathcal{A}_z)_{n\in\mathbb{N}}$ is $\mathcal{H}_G$-non demolition, indeed $U(I_G \otimes \mathcal{A}_\mathcal{V})U^* \subset I_G \otimes \mathcal{A}_\mathcal{V} \otimes \mathcal{B}(\mathcal{H}_p)$. More precisely, for any $f = \sum_{x\in\mathcal{V}} f(x) |x\rangle\langle x| \in \mathcal{A}_z$ we have

$$\begin{aligned} UfU^* &= \sum_{x,y,y',z\in\mathcal{V}} f(x) U(x)_{yz} U(x)_{y'z} \otimes |y\rangle\langle y'| \otimes |x\rangle\langle x| \\ &= \sum_{x,y\in\mathcal{V}} f(x) I_G \otimes I \otimes |x\rangle\langle x| \ . \end{aligned}$$

Moreover, we have

$$\mathrm{Tr}_{T\Phi_{\mathcal{V}}}\left(U(n)(\rho\otimes|\Omega\rangle\langle\Omega|)U(n)^*\right)=\varphi^n(\rho)\ ,$$

where φ is the quantum channel defined by the OQW. By Proposition 25 this means that the OQW has the same distribution that the process $(\varrho_n, X_n)_{n\in\mathbb{N}^*}$ given by the measured evolution of $(U(n), \mathcal{A}_{\mathcal{V}})_{n\in\mathbb{N}}$ with initial state $\rho\otimes|\Omega\rangle\langle\Omega|$. Let us just make explicit the algebras $\mathcal{A}_t^U$ and the maps ϕ_t and $\eta_{s,t}$ used in the definition of the measured evolution.

Writing $\mathcal{A}_n = l^\infty(\mathcal{V}^n)$ the algebra generated by the operators $|x_1\rangle\langle x_1|\otimes\cdots\otimes|x_n\rangle\langle x_n|\otimes I$ on $T\Phi_{\mathcal{V}}$ we have

$$\mathcal{A}_n^U=\mathcal{A}_z\otimes\mathcal{A}_n=l^\infty(\mathcal{V}\times\mathcal{V}^n)\ .$$

The operator $\phi_n:\mathcal{V}\times\mathcal{V}^n\to\mathcal{V}$ is simply the projection on the first coordinate, and for $m<n$ the operator $\eta_{m,n}:\mathcal{V}\times\mathcal{V}^n\to\mathcal{V}\times\mathcal{V}^m$ is defined by

$$\eta_{m,n}(x,x_1,\cdots,x_n)=(x_n,x_1,\cdots,x_m)\ .$$

3.2 Application to the Open Quantum Brownian Motion

With the measured evolution setup, we are able to obtain the process $(\varrho_t, X_t)_{0\le t\le T}$ satisfying the diffusive Belavkin equation directly from the unitary $\mathfrak{U}_t$ and no more as the limit of a discrete-time repeated measurement setup. First, we just consider the system $(U_t,\mathcal{A}_t)_{0\le t\le T}$ where $\mathcal{A}_t = L^\infty(\mathcal{W}([0,t])\subset\mathcal{B}(\Phi)$. Second, we apply this to the measured evolution of $(\mathfrak{U}_t,\mathcal{A}_z)_{0\le t\le T}$ where $\mathcal{A}_z=L^\infty(\mathbb{R})\subset\mathcal{B}(\mathcal{H}_z)$.

3.2.1 Measured evolution for the Hudson-Parthasarathy process

In this part we study the measured evolution $(U_t,\mathcal{A}_t)_{0\le t\le T}$ on $\mathcal{H}_G\otimes\Phi$. The setup is quite simple in this case, because $\mathcal{A}_t^U=\mathcal{A}_t$ and $\eta_{s,t}$ is just the map $(w_u)_{0\le u\le t}\to(w_u)_{0\le u\le s}$. This allows to study it in a less contrived way that the measured evolution described above, and the following result is well-known in quantum filtering theory,

Proposition 26. *The system $(U_t,\mathcal{A}_t)_{0\le t\le T}$ is $\mathcal{H}_B$-non demolition. If $\mathcal{H}_G$ is finite-dimensional it admits a measured evolution process $(\varrho_t,(W_s)_{s\le t})_{0\le t\le T}$ corresponding to the initial state $\rho\otimes|\Omega\rangle\langle\Omega|$ which satisfies the diffusive Belavkin equation 2.5.*

Proof. Note that for any s the process of operators $(U_{s,t})_{s\le t\le T}$ satisfies the Hudson-Parthasarathy equation 2.17 and $U_{s,s}=I$. Thus, $U_{s,t}$ does not act on $\Phi_{[0,s]}$, in particular for any $f\in\mathcal{A}_s$ we have $U_{s,t}fU_{s,t}^*=f$. This proves the non-demolition. Since $U_{s,t}\mathcal{A}_sU_{s,t}^*=\mathcal{A}_s$ we have $\mathcal{A}_t^U=\mathcal{A}_t$, we can take ϕ_t the identity map on $\mathcal{W}([0,t])$, and $\eta_{s,t}:\mathcal{W}([0,t])\to\mathcal{W}([0,s])$ is just the restriction to $[0,s]$. Thus, the state ϱ_t satisfies

$$\mathbb{E}_{\mathbb{P}}(\mathrm{Tr}\,(\varrho_tA)\,f((W_u)_{u\le t}))=\mathrm{Tr}\,(U_t(\rho\otimes|\Omega\rangle\langle\Omega|)U_t^*A\otimes f)$$

for any observable $A \in \mathcal{B}(\mathcal{H}_G)$ and function $f \in \mathcal{A}_t$. We study the unnormalized state $\varsigma_t = u_{\mathcal{W}([0,t]}(U_t(\rho \otimes |\Omega\rangle\langle\Omega|)U_t^*$ first. It satisfies

$$\mathbb{E}_\mu(\mathrm{Tr}\,(\varsigma_t A)\, f((W_u)_{u\leq t})) = \mathrm{Tr}\,(U_t(\rho \otimes |\Omega\rangle\langle\Omega|)U_t^* A \otimes f)$$

(where μ is the measure on $\mathcal{W}([0,T])$ under which $(W_t)_{0\leq t\leq T}$ is the Wiener process). We compute the equation for ς_t using the Itô formula. First, we use the Heisenberg representation:

$$\mathrm{Tr}\,(U_t(\rho \otimes |\Omega\rangle\langle\Omega|)U_t^* A \otimes f) = \mathrm{Tr}\,(\rho \langle\Omega| U_t^*(A \otimes f)U_t |\Omega\rangle)\ .$$

Let us write $f_s = \mathbb{E}_\mu(f|\mathcal{F}_s)$ (where $\mathcal{F}_s$ is the σ-algebra generated by $(W_u)_{u\leq s}$). It is a martingale; f_s is bounded for all s since f is bounded, and by the predicable representation theory there exists an adapted process $(g_s)_{s\leq t}$ such that

$$df_s = g_s dW_s$$

or in terms of quantum SDE, $f_s = f_0 + \int_0^s g_s(da_0^1(s) + da_1^0(s)$ on $\varepsilon(L^2(\mathbb{R}))$. We apply the quantum Itô formula two times to the product $U_s^*(A \otimes f_s)U_s$; since we are interested in $\langle\Omega| U_s^*(A \otimes f_s)U_s |\Omega\rangle$ we can ignore the terms which are not in dt. We obtain

$$U_t^*(A \otimes f_t)U_s = A \otimes f_0 + \int_0^t U_s^* \mathcal{L}^*(A) f_s U_s ds + \int_0^t U_s^*(N^*A + AN)U_s g_s ds + R_t\ ,$$

where R_t is a quantum Itô integrals with only terms in $da_0^1(s)$ and $da_1^0(s)$. This implies that

$$\begin{aligned}
\mathbb{E}_\mu(\mathrm{Tr}\,(\varsigma_t A)\, f((W_u)_{u\leq t}) &= f_0 \mathrm{Tr}\,(\rho a) + \int_0^t \mathrm{Tr}\,(U_s(\rho \otimes |\Omega\rangle\langle\Omega|)U_s\,(\mathcal{L}^*(A) f_s + (N^*A + AN)g_s))\, ds \\
&= f_0 \mathrm{Tr}\,(\rho a) + \int_0^t \mathbb{E}_\mu\,(\mathrm{Tr}\,(\mathcal{L}(\varsigma_s)A)\, f_s + \mathrm{Tr}\,((N\varsigma_s + \varsigma_s N^*)A)\, g_s)\ \ ds \\
&= f_0 \mathrm{Tr}\,(\rho a) + \mathbb{E}_\mu\left(\mathrm{Tr}\left(\left(\int_0^t \mathcal{L}(\varsigma_s)ds + \int_0^t (N\varsigma_s + \varsigma_s N^*)dW_s\right) A\right) f\right)
\end{aligned}$$

the last equality being a consequence of the classical Itô formula. This implies that, for $\mathcal{H}_G$ of finite-dimension,

$$d\varsigma_t = \mathcal{L}(\varsigma_t)dt + (N\varsigma_t + \varsigma_t N^*)dW_t\ . \tag{3.20}$$

It is now time to go back to $\varrho_t = \varsigma_t/\mathrm{Tr}\,(\varsigma_t)$, and to compute the measure $\mathbb{P}$ with $d\mathbb{P} = \mathrm{Tr}\,(\varsigma_t)\, d\mu$. First, note that equation 3.20 has linear coefficients, so ς_t is bounded in $L^2(\mathcal{W}([0,T])$. Write $p_t = \mathrm{Tr}\,(\varsigma_t)$. Since $\mathrm{Tr}\,(\mathcal{L}(A)) = 0$ for any operator A, conditioned in $p_t \neq 0$ we have

$$dp_t = \mathrm{Tr}\,(N\varsigma_t + \varsigma_t N^*)\, dW_t = p_t \mathcal{T}(\varrho_t) dW_t\ .$$

Thus, p_t is the exponential martingale

$$p_t = \exp \int_0^t \mathcal{T}(\varrho_t)ds - \frac{1}{2}\int_0^T \mathcal{T}(\varrho_t)^2 ds\ .$$

Note that $\mathbb{E}_\mu(p_T) = 1$ by definition of ς_t, so it is indeed a martingale. By the Girsanov theorem, under the distribution $p_T d\mu$ there exists a Wiener process B_t defined by

$$B_0 = 0 \tag{3.21}$$
$$dB_t = -\mathcal{T}(\varrho_t)dt + dW_t \ . \tag{3.22}$$

This is the second line of Equation 2.5. To compute the equation for ϱ_t, note that

$$d\frac{1}{p_t} = d\exp - \int_0^t \mathcal{T}(\varrho_t)ds + \frac{1}{2}\int_0^T \mathcal{T}(\varrho_t)^2 ds = \frac{1}{p_t}\left(\mathcal{T}(\varrho_t)^2 dt - \mathcal{T}(\varrho_t)dW_t\right)$$

so with $\varrho_t = \varsigma_t p_t^{-1}$ the Itô formula yields the first line of Equation 2.5. □

This derivation can be extended to more general Hudson-Parthasarathy equations, and has also been studied in the case where the state on Φ is not $|\Omega\rangle\langle\Omega|$ but a more complex, single-photon state, with a resulting non-markovian Belavkin equation

Gough12.

3.2.2 The measured evolution applied to the Open Quantum Brownian Motion

The measured evolution of $(gu_t, \mathcal{A}_z)_{0\le t\le T}$ is a little more subtle than the one of $(U_t, \mathcal{A}_t)_{0\le t\le T}$, but it can be reduced to this last one by using the formula $\mathfrak{U}_t = Z_t U_t$.

Theorem 27. *Assume that $\mathcal{H}_G$ is finite-dimensional, and let us fix some $T > 0$. Then the system$(U_t, \mathcal{A}_z)_{t\in[0,T]}$ is $\mathcal{H}_G$-non-demolition, and it admits a measured evolution $(\varrho_t, X_t)_{t\in[0,T]}$ which is almost surely continuous in time. It satisfies Equation 2.7.*

Proof. For any $f \in \mathcal{A}_z$ and any $s \le t$ we have

$$\begin{aligned}\mathfrak{U}_{s,t} f \mathfrak{U}_{s,t} &= Z_{s,t} U_{s,t} f U_{s,t}^* Z_{s,t}^* \\ &= Z_{s,t} f Z_{s,t}^*\end{aligned}$$

which is the operator of multiplication by the function $\tilde{f}_{s,t}(x, (w_u)_{u\le T}) = f(x - w_s + w_t)$. Hence the system $(\mathfrak{U}_t, \mathcal{A}_z)_{0\le t\le T}$ is $\mathcal{H}_G$-non demolition, and we have $\mathcal{A}_t^{\mathfrak{U}} = \mathcal{A}_z \otimes \mathcal{A}_t = L^\infty(\mathbb{R} \times \mathcal{W}([0,t]), Leb \otimes \mu)$. We choose the map $\phi_t : \mathbb{R} \times \mathcal{W}([0,t]) \to \mathbb{R}$ as the projection on the first coordinate, and for $s \le t$ we take the map $\eta_{s,t} : \mathbb{R} \times \mathcal{W}([0,t]) \to \mathbb{R} \times \mathcal{W}([0,s])$ defined by

$$\eta_{s,t}(x, (w_u)_{0\le u\le t}) = (x - w_t + w_s, (w_u)_{0\le u\le s}) \ .$$

Let $(\varrho_t, X_t)_{0\le t\le T}$ be the random measured process corresponding to these maps. Write $h = (\mathfrak{U}_t(\rho \otimes |\Omega\rangle\langle\Omega|)\mathfrak{U}_t^*)_{G|\mathcal{A}_t^{\mathfrak{U}}}$ (it is a random variable on $\mathbb{R}\otimes\mathcal{W}([0,t])$), then ϱ_t is the random variable on $\mathbb{R}\otimes\mathcal{W}([0,T])$ defined by

$$\varrho_t(x, (w_u)_{0\le u\le T}) = h(x - w_T + w_t, (w_u)_{0\le u\le T}) \ .$$

For any $x \in \mathbb{R}$ consider the random variable on $\mathcal{W}([0,T])$ obtained by conditioning ϱ_t to $X_0 = x$. This random variable is $\nu_t(x) = \varrho_t(x + w_t, (w_u)_{0\le u\le T})$. By definition of Z_t it is actually equal to

$$(U_t(\nu_0(x) \otimes |\Omega\rangle\langle\Omega|)U_t^*)_{G|\mathcal{A}_t} \ .$$

Thus, $(\nu_t(x), W_t)_{0\le t\le T}$ is the random evolution corresponding to the measured evolution of $(U_t, \mathcal{A}_t)_{0\le t\le T}$ with initial state $\nu_0(x)$, and by the definition of η_t we have $X_t = X_T - W_T + W_t = X_0 + W_T$ so Proposition 26 yields Equation 2.7. □

3.3 Towards general convergence theorems for measured evolution

The convergence of $\rho_{t,\tau} = \Lambda_\tau^{[t/\tau]}(\rho)$ to $\rho_t = \Lambda_S^t(\rho)$ was obtained directly from the strong convergence of $\mathfrak{U}_{\tau,t}$ to $\mathfrak{U}_t$. On the contrary, the convergence in distribution of $(\varrho_{\tau,t}, X_{\tau,t})_{O\le t\le T}$ to $(\varrho_t, X_t)_{0\le t\le T}$ was shown as a consequence of Pellegrini's theorem 17, which was proved by classical probabilistic methods without any reference to the operators U_t on the Fock space and on the measured evolution.

A natural question is: can we prove the convergence in distribution of a family of processes $(\varrho_{\tau,t}, X_{\tau,t})_{0\le t\le T}$ coming from a measured evolution $(U_{t,\tau}, \mathcal{A})_{0\le t\le T}$ just from the strong convergence of $U_{\tau,t}$ to some operator U_t ?

This question turns out to be rather difficult, since the algebra $\mathcal{A}_t^{U_\tau}$ also depends in $(U_{\tau,t})_{0\le t\le T}$. In what follows we present some results in this direction.
A first result can be obtained when there is no evolution and we are only considering one measurement.

Proposition 28. *Let $\rho_1\rho_2 \in \mathfrak{S}(\mathcal{H}_G \otimes L^2(\mathcal{X}, \mu))$ be two states and let $\mathcal{A} = L^\infty(\mathcal{X}, \mu)$. Define the random variables $\varrho_i = (\rho_i)_{G|\mathcal{A}}$ on $(\mathcal{X}, \mathbb{P}_i)$ where $d\mathbb{P}_i = p_i d\mu$ defined as in Theorem 20. Then*

$$\|p_1 - p_2\|_{L^1(\mathcal{X},\mu)} \le \|\rho_1 - \rho_2\|_{\mathcal{S}^1(\mathcal{H}_G\otimes L^2(\mathcal{X})} \tag{3.23}$$

$$E_{\mathbb{P}_1}(\|\varrho_1 - \varrho_2\|_{\mathcal{S}^1(\mathcal{H}_G}) \le 2\|\rho_1 - \rho_2\|_{\mathcal{S}^1(\mathcal{H}_G\otimes L^2(\mathcal{X})} \ . \tag{3.24}$$

Proof. Write $h_i = u_{\mathcal{X}}(\rho_i)$ the unnormalized states corresponding to ρ_i. Then $p_i(x) = \mathrm{Tr}\,(h_i(x))$ for μ-almost every $x \in \mathcal{X}$ so

$$\|p_1 - p_2\|_{L^1(\mathcal{X},\mu} \leq \int_{\mathcal{X}} \mathrm{Tr}\,(|h_1(x) - h_2(x)|)\, d\mu(x) \leq \|\rho_1 - \rho_2\|_{\mathcal{S}^1}$$

the last inequality being a consequence of the fact that h_i is the restriction to $\mathcal{B}(\mathcal{H}_G) \otimes \mathcal{A}$ of the state ρ_i. Thus,

$$\begin{aligned} E_{\mathbb{P}_1}(\|\varrho_1 - \varrho_2\|_{\mathcal{S}^1(\mathcal{H}_G}) &= \int_{\mathcal{X}} \mathrm{Tr}\,(|\varrho_1(x) - \varrho_2(x)|)\, p_1 d\mu(x) \\ &\leq \int \mathcal{X} \mathrm{Tr}\,(|p_1(x)\varrho_1(x) - p_2(x)\varrho_2(x)|)\, d\mu(x) + \int_{\mathcal{X}} \mathrm{Tr}\,(|(p_1(x) - p_2(x))\varrho_2(x)|)\, d\mu(x) \\ &\leq 2\|\rho_1 - \rho_2\|_{\mathcal{S}^1}\ . \end{aligned}$$

□

As a consequence we have the following:

Corollary 29. *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of states on $\mathcal{H}_G \otimes L^2(\mathcal{X}, \mu)$ converging in $\mathcal{S}^1(\mathcal{H}_G \otimes \mathcal{H}_B)$ to some state ρ. Consider the sequence of random variables $\varrho_n = \rho_{G|\mathcal{A}}$ defined as in Theorem 20. Then ϱ_n converges to ϱ in distribution and in $L^1(\mathcal{X}, \mathcal{S}^1(\mathcal{H}_S), p_\rho d\mu)$.*

Note that it would make no sense to ask that ϱ_n converge to ϱ in probability or almost surely since they are attached to different probability measures on $\mathcal{X}$. The convergence in $L^1(\mathcal{X}, \mathcal{S}^1(\mathcal{H}_S), p_\rho d\mu)$ is already a little strange from a probabilistic point of view though it is mathematically meaningful: the random state $\tilde{\rho_n}$ is $L^1(\mathcal{X}, \mathcal{S}^1(\mathcal{H}_S), p_\rho d\mu)$ since it is bounded in $\mathcal{S}^1(\mathcal{H}_S)$ and $p_\rho d\mu$ is a probability measure.

A really useful result would involve some dependency in the algebra $\mathcal{A}$, in order to generalize Theorem 5 to other measured evolutions. We were only able to obtain the following partial result, in which the convergence of the result of the measurement is obtained, but not the convergence of the random state.

Proposition 30. *Let $\mathcal{X} = \mathbb{R}^d$ with Borelian algebra $\mathcal{F}$ and a radon measure μ. For each $n \in \mathbb{N}$ let $\mathcal{F}_n$ be a coarse sub-σ-algebra of $\mathcal{F}$. Assume $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n and write $\mathcal{X}_n = \mathbb{R}^d/\mathcal{F}_n$. identified with subsets of $\mathbb{R}^d$ such that $\mathcal{X}_n \subset \mathcal{X}_{n+1} \subset \mathcal{X}$. We fix some time set $I = [0, T]$ upper-bounded by some $T \in \mathbb{R}$ and some finite set $I_n \subset I$ with $I_n \subset I_{n+1}$.*

Consider some Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_C$ and write $\mathcal{H}_B = L^2(\mathcal{X}, \mathcal{F}, \mu) \otimes \mathcal{H}_C$. Consider $\mathcal{A} = L^\infty(\mathcal{X}, \mathcal{F}, \mu)$ and let $(U_t, \mathcal{A})_{t\in I}$ be an $\mathcal{H}_G$-non demolition measured evolution and ρ a state on $\mathcal{H}_G \otimes \mathcal{H}_B$. We write

$(X_t)_{t\in I} \in \mathcal{X}^I$ and $(\varrho_t)_{t\in I}$ the random variables obtained by measuring $\mathcal{A}$ under the evolution.

For each $n \in \mathbb{N}$ fix a closed subspace $\mathcal{H}_{n,C} \subset \mathcal{H}_C$ with $\mathcal{H}_{n,C} \subset \mathcal{H}_{n+1,C}$. Write $\mathcal{H}_n = L^2(\mathcal{X}, \mathcal{F}_n, Leb) \otimes \mathcal{H}_{n,C}$ and let P_n the orthogonal projection on $\mathcal{H}_n$. Note that P_n commutes with every elements of $\mathcal{A}$, we define $\mathcal{A}_n = P_n\mathcal{A}$ and $\mathcal{X}_n = \mathbb{R}^d/\mathcal{F}_n$. Consider a process of unitary operators $(U_{n,t})_{t\in I_n}$ on $\mathcal{H}_G \otimes \mathcal{H}_n$ (that we may see as partial isometries on $\mathcal{H}_G \otimes \mathcal{H}_B$), and a state ρ^n on $\mathcal{H}_G \otimes \mathcal{H}_n$ (that we may see as a state on $\mathcal{H}_G \otimes \mathcal{H}_B$). Assume that $(U_{n,t}, \mathcal{A}_n)_{t\in I_n}$ is $\mathcal{H}_G$-non demolition for all t. Define the process $(X_{n,t})_{t\in I_n}$ with values in $\mathcal{X}_n$ and $(\varrho_{n,t})_{t\in I_n}$ obtained by the measured evolution of $\mathcal{A}_n$ under the evolution $U_{n,t}$ with initial state ρ^n. We still write $t \in I \to X_{n,t}$ the extension of $t \in I_n \to X_{nt}$ to I by linear interpolation, and the same for $\varrho_{n,t}$.

We make the following assumptions:

Assumption 1. Writing $I_n = \{t_{1,n}, \cdots, t_{k_n,n}\}$ (in increasing order) we assume that

$$l_n = max\,\{t_{i+1,n} - t_{i,n} | 1 \leq i \leq k_n\}$$

converges to 0 as $n \to \infty$.

Assumption 2. For any $x \in \mathbb{R}^d$ write

$$C_{\mathcal{F}_n}(x) = \bigcap_{A\in\mathcal{F}_n,\ x\in A} A\ .$$

Then we assume that

$$\lim_{n\to\infty} sup_{x\in\mathbb{R}^d}\ diam(C_{\mathcal{F}_n}(x)) = 0\ .$$

Assumption 3. The sequence of processes $(X_{n,t})_{t\in I}$ is tight for the topology of the uniform convergence on the set of continuous functions on I, and $(X_t)_{t\in I}$ is almost surely continuous.

Assumption 4. The sequence of projections $(P_n)_{n\in\mathbb{N}}$ strongly converges to the identity and the state ρ^n converges to ρ in $\mathcal{B}^1$ as $n \to 0$ and for all sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \in I_n$ converging to some $t \in I$ the operator U_{n,t_n} strongly converge to U_t on $\mathcal{H}_G \otimes \mathcal{H}_B$.

Then $(X_{n,t})_{t\in I}$ converges in distribution (in the topology of uniform convergence) to $(X_t)_{t\in I}$.

In the case of the OQBM, we choose a sequence τ_n such that $\delta_n/\delta_{n+1} \in \mathbb{N}$. We have $\mathcal{X} = \mathbb{R}$ and $\mathcal{X}_n = \delta_n\mathbb{Z}$, the algebra $\mathcal{F}_n$ being generated by the sets $[\delta k, \delta(k+1)\ )$ and we take $\mathcal{H}_C = \Phi$ and $\mathcal{H}_{C,n} = T_{\tau_n}\Phi$.

Upon proving the tightness assumption 3, this theorem together with Theorem 27 provides an alternative proof of the convergence of $(X_{n,[t/\tau]})_{t\in[0,T]}$ to a process solution of 2.7. However, it is very incomplete since we do not prove the convergence of $\varrho_{n,[t/\tau]}$.

Note that Assumption 3 depends on the maps $\eta_{s,t}$ and ϕ_t chosen in the construction of the process, which are only defined up to a set of measure zero.

Proof. We separate the dependency on I_n and $\mathcal{A}_n$ on the one hand and on $U_{n,t}$ on the other hand. For $k \leq n$ and any $t \in nI_k$ we write

$$X_{k,n,t} = C_{ff_k}(X_{n,t}$$

and we consider the σ-algebra $\gg_{k,n}$ generated by $(X_{k,n,t})_{t\in I_k}$ and define

$$\varrho_{k,n,t} = \mathbb{E}(\varrho_{n,t}| \gg_{k,n}) \ .$$

Then $(\varrho_{k,n,t}, X_{k,n,t})_{t\in I_k}$ is a measured evolution corresponding to the system $(U_{n,t}, \mathcal{A}_k)_{t\in I_k}$. We also write

$$X_{k,\infty,t} = C_{\mathcal{F}_k}(X_t)$$

and $\gg_k$ the corresponding σ-algebra, and $\varrho_{k,\infty,t} = \mathbb{E}(\varrho_t| \gg_k)$, so that $(\varrho_{k,\infty,t}, X_{k,\infty,t})_{t\in I_k}$ is a measured evolution corresponding to the system $(U_t, \mathcal{A}_k)_{t\in I_k}$. We extend all these functions to I by linear interpolation.

We prove the convergence in distribution of $(X_{n,t})_{t\in I}$. Let f be a bounded Lipschitz function on the space $\mathcal{D}$ of continuous functions from $[0,T]$ to $\mathcal{X}$. We want to show that $\mathbb{E}(f((X_{n,t})_{t\in[0,T]}))$ converges to $\mathbb{E}(f((X_t)_{t\in[0,T]}))$ as $n \to \infty$.

We fix $\varepsilon > 0$. For any k sufficiently large, we have $diam(C_{\mathcal{F}_k}(x)) \leq \varepsilon$ for all $x \in \mathcal{X}$. By the tightness assumption, with probability higher than $1-\varepsilon$ there is $C > 0$ such that for any n sufficiently large we have $\|X_{n,t} - X_{n,s}\| \leq \varepsilon$ if $|t-s| \leq C$. Since $d_k \to 0$ as $n \to \infty$ this implies that for all n and k large enough we have $\|X_{k,n,t} - X_{n,t}\| \leq 2\varepsilon$ for all t. Writing $M = \max|f|$ and L the Lipschitz constant for f, this means that there is $K \in \mathbb{N}$ such that for any $n, k \geq K$,

$$|\mathbb{E}(f((X_{k,n,t})_{t\in I})) - \mathbb{E}(f((X_{n,t})_{t\in I}))| \leq \varepsilon M + 51-\varepsilon)2\varepsilon L \ . \quad (3.25)$$

The crucial point is that this bound is uniform in n. The same reasoning shows that for any k large enough we have

$$|\mathbb{E}(f((X_{k,\infty,t})_{t\in I})) - \mathbb{E}(f((X_t)_{t\in I}))| \leq \varepsilon M + (1-\varepsilon)2\varepsilon L \ . \quad (3.26)$$

Thus we can fix some k such that the two above quantities are less than ε, and compare $(X_{k,\infty,t})_{t\in I_k}$ and $(X_{k,n,t})_{t\in I_k}$. They are both measurement of discrete algebras on a discrete set of times, so we can actually describe them as indirect measurement.

We write $I_{k,n} = \{ t_1, \cdots, t_m \}$ with $t_0 < t_1 < \cdots < t_m$. For $1 \leq l \leq m$ consider some copies $\mathcal{Y}_l$ of $\mathcal{X}_k$ and write $\mathcal{H}_l = L^2(\mathcal{Y}_l, \nu)$ with ν the counting measure. We fix $a \in \mathcal{X}_k$ and define the state $\sigma_l = |a\rangle\langle a|$ on $\mathcal{H}_l$. We consider a pointer map $\psi : \mathcal{X}_k \times \mathcal{X}_k \to \mathcal{X}_k$ such that $\psi(x, a) = x$ for all $x \in \mathcal{X}_k$, and we define the pointer unitaries Z_l on $\mathcal{H}_B \otimes \mathcal{H}_l$ as in Definition 21. Write $\mathcal{H}_Y = \bigotimes_{0 \leq l \leq m} \mathcal{H}_l$ and $\sigma = \sigma_0 \otimes \cdots \otimes \sigma_m$ and:

$$\begin{aligned} W &= Z_m U_{t_m, t_{m-1}} Z_{m-1} U_{t_{m-1}, t_{m-2}} Z_{m-2} \cdots Z_0 U_{t_0} \ . \\ W_n &= Z_m U_{n, t_m, t_{m-1}} Z_{m-1} U_{n, t_{m-1}, t_{m-2}} Z_{m-2} \cdots Z_0 U_{n, t_0} \ . \end{aligned}$$

Consider the states $\rho_W = W(\rho \otimes \sigma)W^*$ and $\rho_{W_n} = W_n(\rho^n \otimes \sigma)W_n^*$. Now, write $\mathcal{A}_Y = L^\infty(\prod_{0 \leq l \leq m} \mathcal{Y}_l, \nu)$, then the result of the measurement of $\mathcal{A}_Y$ in the state ρ_W is a process $(Y_t)_{t \in I_k}$. By Proposition 25 it has the same law as $(X_{k,\infty,t})_{t \in I_k}$. Likewise, the result of the measurement of $\mathcal{A}_Y$ in the state ρ_{W_n} is a process $(Y_{n,t})_{t \in I_k}$ with same law as $(X_{k,n,t})_{t \in I_{k,n_1}}$. Now by Assumption 4 the operator W_n converges strongly to W as $n_2 \to n$ and ρ^k converges to ρ in $\mathcal{B}^1$ so ρ_{W_k} converges to ρ_W in $\mathcal{B}^1$, so by Proposition 28 the process $(Y_{n,t})_{t \in I_k}$ converges in distribution to the process $(Y_t)_{t \in I_k}$.

This implies that for n large enough,

$$|\mathbb{E}(f((X_{k,\infty,t})_{t \in I})) - \mathbb{E}(f((X_{k,n,t})_{t \in I}))| \leq \varepsilon \ .$$

Since the k was already fixed large enough, this implies that

$$|\mathbb{E}(f((X_{n,t})_{t \in I})) - \mathbb{E}(f((X_t)_{t \in I}))| \leq 3\varepsilon$$

thus proving the convergence in distribution of $(X_{n,t})_{t \in I}$. □

The key point is the estimate 3.25 which is uniform in n. Such a uniform estimate could not be obtained for $\varrho_{k,n,t}$. Indeed, even if the σ-algebra $\gg_{k,n}$ is very close to the full σ-algebra for k large enough, this does not implies that $\varrho_{k,n,t} = \mathbb{E}(\varrho_{n,t} | \gg_{k,n})$ is close to $\varrho_{n,t}$ for k large enough uniformly in n.

The hypothesis that $\mathcal{A}_n$ is coarse and I_n is finite is actually not necessary. To go without it, we may use coarse subalgebras $\mathcal{A}_{k,n}$ of $\mathcal{A}_n$ and finite subsets $I_{n,k} \subset I_n$ and look at the measured evolutions of $(U_{n,t}, \mathcal{A}_{k,n})_{t \in I_{n,k}}$.

3.4 Open questions and prospects

Three questions are left open in Theorem 30:

1. What additional assumption would ensure the convergence in distribution of $(\varrho_{n,t})_{t\in I}$?
2. On what condition does an $\mathcal{H}_G$-non-demolition system $(U_t, \mathcal{A}_t)_{0\leq t\leq T}$ admit a measured evolution process $(\varrho_t, X_t)_{0\leq t\leq T}$ which is almost surely continuous in time? It is the case for U_t defined by the Hudson-Parthasarathy equation and $\mathcal{A}_t = L^\infty(\mathcal{W}([0,t]),\mu)$, but it is not the case when $\mathcal{A}_t$ is the algebra generated by the $a_1^1(s)$ for $s \leq t$. Such a condition should involve both $\mathcal{A}_t$ and U_t.
3. Considering a family of $\mathcal{H}_G$-non demolition systems $(U_{\tau,t}, \mathcal{A}_\tau)_{t\in I_\tau}$ with measured evolutions $(\varrho_{t,\tau}, X_{t,\tau})_{0\leq t\leq T}$. Is there any condition on the unitaries and algebras to ensure the tightness of the family of processes in the space of continuous functions?

Some questions concern the OQBM more specifically. Notably,

4. In the trajectories of the Open Quantum Brownian Motion, there is no back-action of the position X_t on the state ϱ_t, which satisfies a closed equation. Thus, it is of weak interest in the context of quantum control, where we would want x to represent some control function which depends on the history of the trajectory. What if N and H depends on the position x ? We may expect that under some regularity condition on the functions $x \mapsto N(x)$ and $x \mapsto H(x)$ (for example, Schwartz functions), there exists an *inhomogeneous OQBM*, whose unitary $\mathfrak{U}_t$ is solution of the equation
$$d\mathfrak{U}_t = \Big((-iM_H - \frac{1}{2}M_{N^*N} + \frac{1}{2}\partial_x^2 - \partial_x M_N)dt + (M_N - \partial_x)da_1^0(t) + (-M_N^* - \partial_x)da_0^1(t)\Big)\mathfrak{U}_t$$
where M_N is the operator on $\mathcal{H}_G \otimes \mathcal{H}_z = L^2(\mathbb{R}, \mathcal{H}_G)$ defined by $M_N f(x) = N(x)f(x)$. This idea was raised in the original article on the OQBM, [11]. Formally, everything works the same way as the homogeneous OQBM, the equation for the measured evolution being expected to be the form
$$\begin{cases} d\varrho_t &= \mathcal{L}_{X_t}(\varrho_t)dt + (N(X_t)\varrho_t + \varrho_t N(X_t)^* - \varrho_t \mathcal{T}_{X_t}(\rho_t))\, dB_t \\ dX_t &= \mathcal{T}_{X_t}(\varrho_t)dt + dB_t \end{cases}$$
However, proving the existence of $\mathfrak{U}_t$ is far more complex than for the homogeneous OQBM, since the operators ∂_x and M_N are no more commuting, and the space of bandlimited functions $\mathcal{D}_C$ is no more preserved.
5. The generalization of the homogeneous OQBM to higher dimensions is straightforward. Going further, we may study an inhomogeneous OQBM on a manifold. With an Einstein manifold for example, this may provide a semiclassical model for a relativistic quantum particle, in the spirit of the relativistic Brownian motion [17][1].

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