Entanglement of Bipartite Quantum Systems driven by Repeated Interactions*

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Abstract

We consider a non-interacting bipartite quantum system $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ undergoing repeated quantum interactions with an environment modeled by a chain of independent quantum systems interacting one after the other with the bipartite system. The interactions are made so that the pieces of environment interact first with $\mathcal{H}_S^A$ and then with $\mathcal{H}_S^B$. Even though the bipartite systems are not interacting, the interactions with the environment create an entanglement. We show that, in the limit of short interaction times, the environment creates an effective interaction Hamiltonian between the two systems. This interaction Hamiltonian is explicitly computed and we show that it keeps track of the order of the successive interactions with $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$. Particular physical models are studied, where the evolution of the entanglement can be explicitly computed. We also show the property of return of equilibrium and thermalization for a family of examples.

1 Introduction

Initially introduced in [2] in order to justify the quantum Langevin equations, Quantum Repeated Interaction models are currently a very active line of research. They have found

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various applications: quantum trajectories [11, 12, 13, 3, 4], thermalization of quantum systems [5, 1]. Moreover several famous physical experiments, such as the ones performed by S. Haroche's team, correspond exactly to Quantum Repeated Interaction schemes ([6, 7]).

Repeated Quantum Interactions are particular discrete time evolutions of Open Quantum Systems where the typical situation is the one of a quantum system $\mathcal{H}_S$ in contact with an infinite chain of quantum systems $\bigotimes_k \mathcal{H}_k$. Each quantum system $\mathcal{H}_k$ interacts with $\mathcal{H}_S$ one after the other during a time duration h. More concretely, $\mathcal{H}_1$ interacts with $\mathcal{H}_S$ during a time duration h and then stops interacting, the second quantum system $\mathcal{H}_2$ then interacts with $\mathcal{H}_S$ and so on. The continuous time limit, when h goes to zero, has been studied in detail in [2]. Remarkably, it has been shown that such discrete time models, under suitable renormalization, converge to the quantum Langevin equations, that is, quantum stochastic differential equations.

In this article, we concentrate on the following particular situation. We consider that the system $\mathcal{H}_S$ is composed of two quantum systems $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ which do not interact together. This "uncoupled" system undergoes Quantum Repeated Interactions as follows. Each piece $\mathcal{H}_k$ of the environment interacts first with $\mathcal{H}_S^A$ during a time duration h without interacting with $\mathcal{H}_S^B$ and then interacts with $\mathcal{H}_S^B$ without interacting anymore with $\mathcal{H}_S^A$. For example, in the spirit of the experiments driven by Haroche et al (cf [6, 7]), the bipartite system can been thought of as two isolated cavities with a magnetic field trapping several photons in each cavities. A chain of two-level systems (such as Rydberg atoms in some particular state, as in the experiment) are passing through the cavities, one after the other, creating this way an entanglement in between the photons of each cavities.

Our work is motivated by entanglement considerations. While the systems $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ are not initially entangled and while there is no direct interaction between them, our special scheme of Quantum Repeated Interactions creates naturally entanglement. More precisely, we show that this scheme of interaction, in the continuous-time limit, is equivalent to a usual Quantum Repeated Interaction model where, actually, $\mathcal{H}_S^A$ interacts with $\mathcal{H}_S^B$. In other words, our special scheme of Quantum Repeated Interactions creates spontaneously an effective interaction Hamiltonian between $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$. We explicitly compute the associated interaction Hamiltonian.

The article is structured as follows. In Section 2, the bipartite Repeated Quantum Interaction model is described in details. In Section 3, we focus on the continuous-time limit, that is, when the time interaction between the systems $\mathcal{H}_k$ and $\mathcal{H}_S = \mathcal{H}_S^A \otimes \mathcal{H}_S^B$ goes to zero. More precisely, we derive the quantum stochastic differential equation representing the limit evolution. This allows to identify the effective coupling Hamiltonian. Section 4 is devoted to the study of the evolution of the entanglement between $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ in the physical example of the spontaneous emission of a photon. In Section 5, we derive the Lindblad generator of the limit evolution in the case of a thermal environment, represented by a Gibbs state. We then study the property of return to equilibrium, that is, the asymptotic convergence for all initial state toward an invariant state.

2 Description of the Bipartite Model

This section is devoted to the presentation of the model. As announced, we consider a quantum system $\mathcal{H}_S = \mathcal{H}_S^A \otimes \mathcal{H}_S^B$, where $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ do not interact together. This means that the free evolution of $\mathcal{H}_S$ is given by

$$H^A \otimes I + I \otimes H^B,$$

where H^A and H^B are the free Hamiltonians of $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$. This system is coupled to an environment made of an infinite chain of identical and independent systems :

$$T\Phi = \bigotimes_{k\in\mathbb{N}^*} \mathcal{H}_k,$$

where $\mathcal{H}_k = \mathcal{H}$ for all k.

The interaction between $\mathcal{H}_S$ and the infinite chain is described by a model of Quantum Repeated Interactions, that is, the copies of $\mathcal{H}$ interact ones after the others with $\mathcal{H}_S$ and then stop interacting. A single interaction between a copy of $\mathcal{H}$ and $\mathcal{H}_S = \mathcal{H}_S^A \otimes \mathcal{H}_S^B$ is described by a particular mechanism, the interaction is divided into two parts: the system $\mathcal{H}$ interacts first with $\mathcal{H}_S^A$ during a time h without interacting with $\mathcal{H}_S^B$, then the system $\mathcal{H}$ interacts with $\mathcal{H}_S^B$ during a time h without interacting with $\mathcal{H}_S^A$.

In terms of Hamiltonians, the evolution of the coupled system $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{H}$ can be described as follows. For the first interaction, we consider an Hamiltonian of the form

$$H_{\text{tot}}^A = H^A \otimes I \otimes I + I \otimes I \otimes H^R + \lambda\, H_I^A, \tag{1}$$

where H_R represents the free Hamiltonian of $\mathcal{H}$, the operator H_I^A represents the interaction Hamiltonian between $\mathcal{H}$ and $\mathcal{H}_S^A$ (this operator acts as the identity operator on $\mathcal{H}_S^B$) and λ is a coupling constant. In a similar way, the second evolution is described by

$$H_{\text{tot}}^B = I \otimes H^B \otimes I + I \otimes I \otimes H^R + \lambda'\, H_I^B, \tag{2}$$

where this time H_I^B acts non-trivially only on $\mathcal{H}$ and $\mathcal{H}_S^B$ and acts as the identity operator on $\mathcal{H}_S^A$. Again λ' represents also the coupling constant of the second interaction.

Each of the operators H_{tot}^A and H_{tot}^B give rise to a unitary evolution during the time interval h:

$$U^A \;=\; e^{-ihH_{\text{tot}}^A}, \qquad U^B = e^{-ihH_{\text{tot}}^B}. \tag{3}$$

Since the space $\mathcal{H}$ interacts first with $\mathcal{H}_S^A$ and then $\mathcal{H}_S^B$, the resulting evolution is then

$$U = U^B U^A. \tag{4}$$

Let us stress that, in more general setup, the interaction between $\mathcal{H}$ and $\mathcal{H}_S$ should have been given by an Hamiltonian of the form

$$\widetilde{H}_{\text{tot}} = H^A \otimes I \otimes I + I \otimes H^B \otimes I + I \otimes I \otimes H^R + \widetilde{\lambda}\, \widetilde{H}_I\,,$$

where $\widetilde{H}_I$ would have been a general interaction Hamiltonian. This would have given rise to a usual unitary evolution of the form

$$\widetilde{U} = e^{-i2h\widetilde{H}_{\mathrm{tot}}}. \tag{5}$$

In the specific model considered in this article, since H_I^A and H_I^B do not commute, we cannot directly put the unitary (4) under the form (5), at least not in a natural way! Though, we shall prove that, in the continuous-time limit, our model with $U = U^B U^A$ is equivalent to a general model with some explicit effective interaction between $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$.

Let us make precise now the form of the interaction Hamiltonians involved in (1) and (2). We assume in this work that all the Hilbert spaces involved in the model, that is, the spaces $\mathcal{H}$, $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ are finite dimensional. For a reason which will appear clearer in the article, we choose the dimension of $\mathcal{H}$ to be of the form $N+1$, for some $N \in \mathbb{N}^*$. We consider an orthonormal basis of $\mathcal{H}$, denoted by $\{e_0, e_1, \dots, e_N\}$, made of eigenvectors of H^R and where the vector e_0 is the ground state of $\mathcal{H}^R$.

Consider the associated canonical operators a_j^i defined by

$$a_j^i e_k = \delta_{ik} e_j \, ,$$

for all i, j and k in $\{0, \dots, N\}$. With this notation, we have

$$H^R = \sum_{j=0}^{N} \lambda_j \, a_j^j \, ,$$

where the λ_j's are the eigenvalues of H^R.

As interaction Hamiltonians we shall only consider operators of the form

$$\begin{aligned} H_I^A &= \sum_{j=1}^{N} V_j \otimes I \otimes a_j^0 + V_j^* \otimes I \otimes a_0^j \, , \\ H_I^B &= \sum_{j=1}^{N} I \otimes W_j \otimes a_j^0 + I \otimes W_j^* \otimes a_0^j \, , \end{aligned}$$

where the V_j's are operators on $\mathcal{H}_S^A$ and the W_j's are operators on $\mathcal{H}_S^B$.

As usual in the Schrödinger picture, the evolutions of states (density matrices here) on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{H}$ are given by

$$\rho \longmapsto U\rho \, U^* \, ,$$

where we recall that U takes the particular form $U = U^B U^A$ in our context.

Now, we are in the position to describe the whole interaction between $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ and the chain $\bigotimes_k \mathcal{H}_k$, with $\mathcal{H}_k = \mathcal{H} = \mathbb{C}^{N+1}$. The scheme is as follows. The first copy $\mathcal{H}_1$ interacts with $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ during a time $2h$ while the rest of the chain remains isolated. Then, the first copy disappears and the second copy comes to interact and so on... Before making precise the evolution, we need to introduce a notation for the operators acting only on $\mathcal{H}_n$ and being the identity operator on the rest of the whole space. If A is an operator on $\mathcal{H}$, we extend it as an operator on $\bigotimes_k \mathcal{H}_k$ but acting non-trivially only on $\mathcal{H}_n$ by putting

$$A(n) = \bigotimes_{k=1}^{n-1} I \otimes A \otimes \bigotimes_{k>n+1} I\,.$$

On $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \bigotimes_k \mathcal{H}_k$ we consider the family of unitary operators $(U_n)_{n\in\mathbb{N}^*}$, where U_n acts as U on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ and the n-th copy of $\mathcal{H}$ and as the identity on the rest of the chain. The operator U_n represents actually the interaction between $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$ and $\mathcal{H}_n$. More precisely, the operator U_n is defined as $U_n = U_n^B U_n^A$, where $U_n^A = e^{-ihH^A_{\mathrm{tot},n}}$ with

$$H^A_{\mathrm{tot},n} = H^A \otimes I \otimes I + I \otimes I \otimes H_R(n) + \lambda \sum_{j=1}^{N} V_j \otimes I \otimes a_j^0(n) + V_j^* \otimes I \otimes a_0^j(n)\,, \tag{6}$$

and the corresponding description for U_n^B.

The whole evolution is finally described by a family of unitary operators $(V_n)_{n\in\mathbb{N}^*}$ which are given by

$$V_n = U_n U_{n-1} \dots U_1\,. \tag{7}$$

As a consequence, if the initial state of $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \bigotimes_k \mathcal{H}_k$ where the density matrix ρ_0, then the state after n interactions is

$$V_n\, \rho_0\, V_n^*\,.$$

Now that the discrete-time evolution is clearly described, we shall investigate its continuous-time limit.

3 Effective Interaction Hamiltonian

This section is devoted to derive the continuous time limit of our special scheme of repeated interactions, i.e. the limit when the time parameter h goes to 0. In order to obtain a relevant limit, the authors of [2] have shown that the total Hamiltonian has to be properly rescaled in terms of h. In particular, it is crucial to strengthen the interaction in order to see its effect at the limit. More precisely, translated in our context, the total Hamiltonians have to be of the following form:

$$H^A_{\mathrm{tot}} = H^A \otimes I \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{i=1}^{N} \left(V_j \otimes I \otimes a_j^0 + V_j^* \otimes I \otimes a_0^j\right)\,, \tag{8}$$

$$H^B_{\mathrm{tot}} = I \otimes H^B \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^{N} I \left(\otimes W_j \otimes a_j^0 + I \otimes W_j^* \otimes a_0^j\right)\,. \tag{9}$$

Let us stress that in the above expressions the coupling constants appearing in (1) and (2) have been replaced by $1/\sqrt{h}$. We denote by $\lfloor \cdot \rfloor$ the floor function. One can show that the operators $(V_{\lfloor t/h \rfloor})_t$ defined in (7) converge to a family of operators $(U_t)_t$ satisfying a particular quantum stochastic differential equation.

More precisely, in [2], it is shown that one can embed the space $T\Phi$ into some appropriate Fock space Φ; the discrete time interaction, described by $(V_{\lfloor t/h \rfloor})_t$, appears naturally as an approximation of a continuous one described by a family of unitary operators $(U_t)_t$ acting on Φ; the family (U_t) is the solution of a particular *quantum stochastic differential equation* describing continuous-time interaction between small system $\mathcal{H}_S$ and the quantum field Φ. In our context, the complete description of the Fock space Φ and the details of the convergence result are not necessary. Nevertheless the "created" interaction Hamiltonian appears naturally in the expression of the limit (U_t). We shall prove the following result by exhibiting only the essential points allowing to apply the theorems of [2].

Theorem 3.1. *When the interaction time h goes to 0, the family $(V_{\lfloor t/h \rfloor})_t$ converges strongly to a family of unitary operators (U_t) which is the solution of the quantum stochastic differential equation*

$$
\begin{aligned}
dU_t = \Bigg[&- i(H^A \otimes I + I \otimes H^B + 2\lambda_0\, I \otimes I) \\
&- \frac{1}{2} \sum_j V_j^* V_j \otimes I + I \otimes W_j^* W_j + 2 V_j \otimes W_j^* \Bigg] U_t\, dt \\
&- i \sum_{i=1}^{N} (V_j \otimes I + I \otimes W_j) U_t\, da_j^0(t) + (V_j^* \otimes I + I \otimes W_j^*) U_t\, da_0^j(t)\,. \qquad (10)
\end{aligned}
$$

Remark. Note that in the expression (10) the terms $(a_j^0(t))$ and $(a_0^j(t))$ are quantum noises. They are particular operators on the limit Fock space Φ. The exact definition of these operators is not needed here and we refer to [10] for complete references.

Proof. In order to prove this result we shall apply the Theorem 13 of [2]. The essential step is to identify the relevant terms when expanding

$$U = U^B U^A = e^{-ihH_{\text{tot}}^B}\, e^{-ihH_{\text{tot}}^A}\,,$$

in terms of h. More precisely, on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B \otimes \mathcal{H}$, one can decompose U as

$$U = \sum_{i,j} U_j^i(h) \otimes a_j^i, \qquad (11)$$

where the $U_j^i(h)$'s are operators on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$. This way, we shall find the asymptotic expression of $U_j^i(h)$ in order to apply the convergence results of [2].

In order to obtain the asymptotic expression of $U_j^i(h)$, let us study H_{tot}^A and H_{tot}^B in details. Using a decomposition similar to (11), the operators H_{tot}^A and H_{tot}^B can be seen as matrices whose the coefficients are operators on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$. In particular, they can be written as follows

$$H_{\text{tot}}^A = \begin{pmatrix} H^A \otimes I + \lambda_0 I \otimes I & \frac{1}{\sqrt{h}} V_1^* \otimes I & \frac{1}{\sqrt{h}} V_2^* \otimes I & \cdots & \frac{1}{\sqrt{h}} V_N^* \otimes I \\ \frac{1}{\sqrt{h}} V_1 \otimes I & H^A \otimes I + \lambda_1 I \otimes I & 0 & \cdots & 0 \\ \frac{1}{\sqrt{h}} V_2 \otimes I & 0 & H^A \otimes I + \lambda_2 I \otimes I & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\sqrt{h}} V_N \otimes I & 0 & 0 & \cdots & H^A \otimes I + \lambda_N I \otimes I \end{pmatrix}$$

and

$$H_{\text{tot}}^B = \begin{pmatrix} I \otimes H^B + \lambda_0 I \otimes I & \frac{1}{\sqrt{h}} I \otimes W_1^* & \frac{1}{\sqrt{h}} I \otimes W_2^* & \cdots & \frac{1}{\sqrt{h}} I \otimes W_N^* \\ \frac{1}{\sqrt{h}} I \otimes W_1 & I \otimes H^B + \lambda_1 I \otimes I & 0 & \cdots & 0 \\ \frac{1}{\sqrt{h}} I \otimes W_2 & 0 & I \otimes H^B + \lambda_2 I \otimes I & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\sqrt{h}} I \otimes W_N & 0 & 0 & \cdots & I \otimes H^B + \lambda_N I \otimes I \end{pmatrix}.$$

Hence we obtain a Taylor expansion of U^A as

$$U^A = \begin{pmatrix} D_0^A & -i\sqrt{h} V_1^* \otimes I & -i\sqrt{h} V_2^* \otimes I & \cdots & -i\sqrt{h} V_N^* \otimes I \\ -i\sqrt{h} V_1 \otimes I & D_1^A & -\frac{1}{2} h V_1 V_2^* \otimes I & \cdots & -\frac{1}{2} h V_1 V_N^* \otimes I \\ \vdots & -\frac{1}{2} h V_2 V_1^* \otimes I & D_2^A & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -i\sqrt{h} V_N \otimes I & -\frac{1}{2} h V_N V_1^* \otimes I & -\frac{1}{2} h V_N V_2^* \otimes I & \cdots & D_N^A \end{pmatrix} + O(h^{3/2})$$

and of U^B as

$$U^B = \begin{pmatrix} D_0^B & -i\sqrt{h} I \otimes W_1^* & -i\sqrt{h} I \otimes W_2^* & \cdots & -i\sqrt{h} I \otimes W_N^* \\ -i\sqrt{h} I \otimes W_1 & D_1^B & -\frac{1}{2} h I \otimes W_1 W_2^* & \cdots & -\frac{1}{2} h I \otimes W_1 W_N^* \\ \vdots & -\frac{1}{2} h I \otimes W_2 W_1^* & D_2^B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -i\sqrt{h} I \otimes W_N & -\frac{1}{2} h I \otimes W_N W_1^* & -\frac{1}{2} h I \otimes W_N W_2^* & \cdots & D_N^B \end{pmatrix} + O(h^{3/2}),$$

where the diagonal coefficients are, for all $j = 1, \ldots, N$,

$$D_0^A = I \otimes I - ihH^A \otimes I - ih\lambda_0 I \otimes I - \frac{1}{2} h \sum_j V_j^* V_j \otimes I,$$

$$D_j^A = I \otimes I - ihH^A \otimes I - ih\lambda_j I \otimes I - \frac{1}{2} h V_j V_j^* \otimes I,$$

$$D_0^B = I \otimes I - ihI \otimes H^B - ih\lambda_0 I \otimes I - \frac{1}{2} h \sum_j I \otimes W_j^* W_j,$$

$$D_j^B = I \otimes I - ihI \otimes H^B - ih\lambda_j I \otimes I - \frac{1}{2} h I \otimes W_j W_j^*.$$

This way, computing $U^B U^A$ in asymptotic form, the coefficients $U^i_j(h)$ of the matrix U for $i,j=0,\dots,N$ are, up to terms in $h^{3/2}$ or higher

$$U^0_0 = I\otimes I - ih(H_A\otimes I + I\otimes H_B + 2\lambda_0 I\otimes I) \\ \qquad - \frac{1}{2}h\sum_j \left(V_j^*V_j\otimes I + I\otimes W_j^*W_j + 2V_j\otimes W_j^*\right)\,, \tag{12}$$

$$U^j_0 = -i\sqrt{h}(V_j^*\otimes I + I\otimes W_j^*)\,, \tag{13}$$

$$U^0_j = -i\sqrt{h}(V_j\otimes I + I\otimes W_j)\,, \tag{14}$$

$$U^j_j = I\otimes I - ih(H_A\otimes I + I\otimes H_B + 2\lambda_j I\otimes I) \\ \qquad - \frac{1}{2}h(V_jV_j^*\otimes I + I\otimes W_jW_j^* + 2V_j^*\otimes W_j)\,, \tag{15}$$

$$U^k_j = -\frac{1}{2}h(V_jV_k^*\otimes I + I\otimes W_jW_k^* + 2V_k^*\otimes W_j)\,. \tag{16}$$

One can easily check that

$$\lim_{h\to 0}\frac{U^i_j(h) - \delta_{ij}\, I\otimes I}{h^{\epsilon_{i,j}}} = L^i_j\,,$$

where $\epsilon_{0,0}=1$, $\epsilon_{0,j}=\epsilon_{j,0}=1/2$ and $\epsilon_{i,j}=0$ and where

$$L^0_0 = -i(H^A\otimes I + I\otimes H^B + 2\lambda_0 I\otimes I) - \frac{1}{2}\sum_j V_j^*V_j\otimes I + I\otimes W_j^*W_j + 2V_j\otimes W_j^*\,,$$
$$L^j_0 = -i(V_j^*\otimes I + I\otimes W_j^*)\,,$$
$$L^0_j = -i(V_j\otimes I + I\otimes W_j)\,,$$
$$L^i_j = 0.$$

These are exactly the conditions of [2] and the result follows. □

Now that we have derived Eq. (10), we are in the position to identify the interaction Hamiltonian which has been "created" by the environment. To this end, we compare the limit equation (10) with the one one could have obtained with a usual repeated quantum interaction scheme.

Theorem 3.2. *The quantum stochastic differential equation* (10) *represents an evolution, on $\mathcal{H}_S^A\otimes\mathcal{H}_S^B$ coupled to a Fock space Φ, which can be obtained from the continuous-time limit of a usual repeated interaction scheme with the following Hamiltonian on $\mathcal{H}_S^A\otimes\mathcal{H}_S^B\otimes\mathcal{H}$*

$$\widetilde{H}_{\text{tot}} = H_0^{A,B}\otimes I + 2\,I\otimes I\otimes H^R + \frac{1}{\sqrt{h}}\sum_j S_j\otimes a^0_j + S_j^*\otimes a^j_0\,, \tag{17}$$

where $S_j = V_j\otimes I + I\otimes W_j$ and where the free Hamiltonian of $\mathcal{H}_S^A\otimes\mathcal{H}_S^B$ is given by

$$H_0^{A,B} = H^A\otimes I + I\otimes H^B + \frac{i}{2}\sum_j V_j^*\otimes W_j - V_j\otimes W_j^*\,. \tag{18}$$

In particular the term

$$\frac{i}{2}\sum_j \left(V_j^* \otimes W_j - V_j \otimes W_j^*\right)$$

represents an effective interaction Hamiltonian term created by the environment between $\mathcal{H}_S^A$ *and* $\mathcal{H}_S^B$*.*

Proof. With the expression of the Hamiltonian (17), using again the results of [2], the continuous-time limit (h goes to zero) gives rise to the QSDE

$$d\widetilde{U}_t = L_0^0 \widetilde{U}_t\, dt + \sum_j L_0^j \widetilde{U}_t\, da_0^j(t) + L_j^0 \widetilde{U}_t\, da_j^0(t)\,, \tag{19}$$

where

$$L_0^0 = -i(H_0^{A,B} + 2\lambda_0 I \otimes I) - \frac{1}{2}\sum_j S_j^* S_j,$$

$$L_j^0 = -iS_j \qquad \text{and} \qquad L_0^j = -iS_j^*\,.$$

In Eq. (10), the coefficient L_0^0 is

$$L_0^0 = -i(H^A \otimes I + I \otimes H^B + 2\lambda_0 I \otimes I) - \frac{1}{2}\sum_j V_j^* V_j \otimes I + I \otimes W_j^* W_j + 2V_j \otimes W_j^*$$

which can also be written as

$$\begin{aligned} L_0^0 = &- i\left(H^A \otimes I + I \otimes H^B + 2\lambda_0 I \otimes I + \frac{i}{2}\sum_j V_j^* \otimes W_j - V_j \otimes W_j^*\right) \\ &- \frac{1}{2}\sum_j (V_j \otimes I + I \otimes W_j)^*(V_j \otimes I + I \otimes W_j)\,. \end{aligned}$$

We now see that Eq.(10) is exactly a QSDE of the same form as Eq.(19) but with

$$H_0^{A,B} = H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_j V_j^* \otimes W_j - V_j \otimes W_j^* \quad \text{and} \quad S_j = V_j \otimes I + I \otimes W_j\,.$$

The result follows. □

Remark 1. One can wonder if we can recover the above result and the description of the created interaction Hamiltonian only by knowing the separate evolutions (that is when only $\mathcal{H}_S^A$ or $\mathcal{H}_S^B$ is involved). Using again the results of [2] one can describe the separate evolution and get

$$d\widetilde{U}_t^A = [-i(H^A + \lambda_0 I) - \frac{1}{2}\sum_j V_j^* V_j]\widetilde{U}_t^A\, dt - i\sum_j V_j \widetilde{U}_t^A\, da_j^0(t) + V_j^* \widetilde{U}_t^A\, da_0^j(t)\,, \tag{20}$$

which is the limit of $V^A_{[t/h]} = U^A_{[t/h]} \dots U^A_1$. The corresponding evolution for $\mathcal{H}^B_S$ coupled to the Fock space Φ is given by

$$d\widetilde{U}^B_t = [-i(H^B + \lambda_0 I) - \frac{1}{2}\sum_j W^*_j W_j]\widetilde{U}^B_t \, dt - i\sum_j W_j \widetilde{U}^B_t \, da^0_j(t) + W^*_j \widetilde{U}^B_t \, da^j_0(t) \,, \quad (21)$$

which is the limit of $V^B_{[t/h]} = U^B_{[t/h]} \dots U^B_1$. At that stage, with only (20) and (21) in hands it is not clear how to derive Eq.(10). In particular, it is not obvious how to describe the fact that the quantum field, at time t, acts first with $\mathcal{H}^A_S$ and then with $\mathcal{H}^B_S$.

Remark 2. Note that the interaction Hamiltonian is not symmetric in $\mathcal{H}^A_S$ and $\mathcal{H}^B_S$ due to the fact that each auxiliary system $\mathcal{H}$ acts with $\mathcal{H}^A_S$ before $\mathcal{H}^B_S$. Somehow the evolution keeps the memory of the order of the interaction.

We shall now illustrate our results by studying the creation of entanglement in some simple physical model.

4 Evolution of Entanglement for Spontaneous Emission

The physical model considered in this section is the spontaneous emission of photons. More precisely, the systems $\mathcal{H}^A_S$, $\mathcal{H}^B_S$ and $\mathcal{H}$ are 2-level systems, hence both represented by the state space $\mathbb{C}^2$. The free dynamics H^A, H^B and H^R are given by the Pauli matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The operators V_1 and W_1 are $V_1 = W_1 = a^1_0$. Applying Theorem 3.1, the limit evolution is

$$dU_t = \left[-i(\sigma_z \otimes I + I \otimes \sigma_z + 2I \otimes I) - \frac{1}{2}S^* S + \frac{1}{2}(a^0_1 \otimes a^1_0 - a^1_0 \otimes a^0_1)\right] U_t \, dt \\ - iSU_t \, da^0_1(t) - iS^* U_t \, da^1_0(t) \,, \quad (22)$$

where $S = a^1_0 \otimes I + I \otimes a^1_0$.

In order to study the entanglement of a system evolving according to Eq.(22), we compute its Lindblad generator. Indeed, from the solution $(U_t)_{t\in\mathbb{R}^+}$ of Eq.(22), we consider associate the semigroup of completely positive maps $(T_t)_{t\in\mathbb{R}^+}$ defined by

$$T_t(\rho) = Tr_{\mathcal{H}}(U_t(\rho \otimes |\Omega\rangle\langle\Omega|)U^*_t) \,,$$

for all state ρ of $\mathcal{H}^A_S \otimes \mathcal{H}^B_S$ and all $t \in \mathbb{R}^+$, where Ω represents the ground (or vacuum) state of the associated Fock space Φ. The infinitesimal generator of (T_t) is then given by

$$L(\rho) = -\, i\left[\sigma_z \otimes I + I \otimes \sigma_z + \frac{i}{2}(a^0_1 \otimes a^1_0 - a^1_0 \otimes a^0_1), \rho\right] + \frac{1}{2}\Big(2S\rho S^* - S^* S\rho - \rho S^* S\Big) .$$

Note that this generator can also be simply recovered from the limit of the completely positive discrete-time semigroup associated to the completely positive operator $l(h)$ defined by

$$
\begin{aligned}
l(h)(\rho) &= \mathrm{Tr}_{\mathcal{H}}(U(\rho \otimes |e_0\rangle\langle e_0|)U) && (23)\\
&= \sum_i U_i^0(h)\,\rho\, U_i^0(h)^* && (24)\\
&= \rho + hL(\rho) + \circ(h)\,. && (25)
\end{aligned}
$$

Now we are in the position to study the entanglement between the system $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$. In particular, we shall study the so-called *entanglement of formation* (see [14] for an introduction). It is worth noticing that an explicit formula does not hold in general; though, in [14] an explicit formula has been derived for particular initial states. These initial states are called "X states" for their matrix representations look like an X. A particular feature of such states is that their particular form is preserved under the dynamics and the entanglement of formation can be computed explicitly in terms of the concurrence of Wooters [15].

In order to make concrete the X representation, we consider the following basis of $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$:

$$\mathcal{B} = (|e_0 \otimes e_0\rangle, |e_0 \otimes e_1\rangle, |e_1 \otimes e_0\rangle, |e_1 \otimes e_1\rangle)\,.$$

A general X state in this basis is then

$$\rho = \begin{pmatrix} a & 0 & 0 & y \\ 0 & b & x & 0 \\ 0 & \overline{x} & c & 0 \\ \overline{y} & 0 & 0 & d \end{pmatrix}$$

with the conditions that a, b, c, d are non-negative reals such that $a+b+c+d = 1$, $|y|^2 \leq ad$ and $|x|^2 \leq bc$. As proved in [14], the concurrence of Wooters of such a state is

$$C(\rho) = 2\max(0, |y| - \sqrt{bc}, |x| - \sqrt{ad}) \tag{26}$$

and its entanglement of formation is given by the general formula, shown by Wooters [15],

$$E(\rho) = h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right)\,, \tag{27}$$

where $h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$.

One can now compute the action of L on a X state and after computation we get

$$L(\rho) = \begin{pmatrix} \overline{x}+x+b+c & 0 & 0 & y(-1-4i) \\ 0 & d-b-\overline{x}-x & d-c-x & 0 \\ 0 & d-c-\overline{x} & d-c & 0 \\ \overline{y}(4i-1) & 0 & 0 & -2d \end{pmatrix}.$$

Using the development of e^{tL} in series, it is obvious to see that the X representation is preserved during the evolution. Unfortunately, in general, the expression of $L^n(\rho)$ for all n is not computable and we cannot obtain the expression of $e^{tL}(\rho)$ for all ρ. However, we are able to compute the expression of $e^{tL}(\rho)$ for those states defining the basis $\mathcal{B}$.

• A straightforward computation shows that $|e_0 \otimes e_0\rangle\langle e_0 \otimes e_0|$ is an invariant state of the dynamics (one can check that $L(|e_0 \otimes e_0\rangle\langle e_0 \otimes e_0|) = 0$) and there is no entanglement of formation. Of course, for such an initial state the dynamics of spontaneous emission generates no interaction at all with the environment!

• Consider now another initial state $\rho^{01} = |e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|$ corresponding to the case $a = 0$, $b = 1$, $c = 0$, $d = 0$, $x = 0$ and $y = 0$. This state represents the system $\mathcal{H}_S^A$ in its ground state and $\mathcal{H}_S^B$ in its excited state. One can easily check that we get for all $n \geq 1$

$$L^n(\rho^{01}) = \begin{pmatrix} (-1)^{n+1} & 0 & 0 & 0 \\ 0 & (-1)^n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives directly that

$$\begin{aligned} \rho_t^{01} &= e^{tL}(|e_0 \otimes e_1\rangle\langle e_0 \otimes e_1|) = \begin{pmatrix} 1-e^{-t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (28) \\ &= |e_0 \otimes (1-e^{-t})e_0 + e^{-t}e_1\rangle\langle e_0 \otimes (1-e^{-t})e_0 + e^{-t}e_1| \qquad (29) \end{aligned}$$

The entanglement of formation is obviously zero. Which was to be expected also, as the initial state of $\mathcal{H}_S^A$ is $|e_0\rangle$ is invariant under the repeated interactions and generates no interaction with the environment; hence the environment is here interacting with $\mathcal{H}_S^B$ only.

In the two next cases we shall see effective creation of entanglement.

• Consider the initial state $\rho^{10} = |e_1 \otimes e_0\rangle\langle e_1 \otimes e_0|$ corresponding to the case $a = 0$, $b = 0$, $c = 1$, $d = 0$, $x = 0$ and $y = 0$. For all $n \geq 1$, we get

$$L^n(\rho^{10}) = \begin{pmatrix} (-1)^{n+1}(n^2-n+1) & 0 & 0 & 0 \\ 0 & (-1)^n(n-1)n & (-1)^n n & 0 \\ 0 & (-1)^n n & (-1)^n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This way, we have for all time t,

$$\rho_t^{10} = e^{tL}(|e_1 \otimes e_0\rangle\langle e_1 \otimes e_0|) = \begin{pmatrix} 1-(1+t^2)e^{-t} & 0 & 0 & 0 \\ 0 & t^2e^{-t} & -te^{-t} & 0 \\ 0 & -te^{-t} & e^{-t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (30)$$

In this case the entanglement of formation is then

$$E(\rho_t^{10}) = h\left(\frac{1+\sqrt{1-4t^2e^{-2t}}}{2}\right). \tag{31}$$

In particular, this quantity is positive for all $t > 0$ (see figure). One can check that the maximum is reached at time 1 when the state is

$$\begin{pmatrix} 1-2\,e^{-1} & 0 & 0 & 0 \\ 0 & e^{-1} & -e^{-1} & 0 \\ 0 & -e^{-1} & e^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case we see that there is spontaneous creation of entanglement which increases until time 1 and next decreases exponentially fast to zero, see Fig. 1.

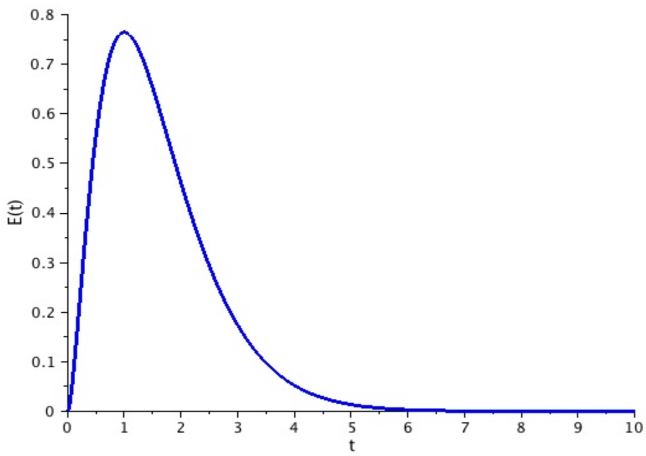

Figure 1: *Time evolution of Wooters' concurrence, initial state* $e_1 \otimes e_0$

• The last case concerns $\rho^{11} = |e_1 \otimes e_1\rangle\langle e_1 \otimes e_1|$. In particular, this corresponds to the case $a = 0$, $b = 0$, $c = 0$, $d = 1$, $x = 0$ and $y = 0$. After computations we get

$$L^n(\rho^{11}) = (-1)^n \begin{pmatrix} 5\times 2^n - 6 - n(n+3) & 0 & 0 & 0 \\ 0 & -5(2^n-1)+n(n+3) & -2^{n+1}+n+2 & 0 \\ 0 & -2^{n+1}+n+2 & -2^n+1 & 0 \\ 0 & 0 & 0 & 2^n \end{pmatrix}.$$

This gives for all time t,

$$\rho_t^{11} = \begin{pmatrix} 1-(t^2-4t+6)e^{-t}+5e^{-2t} & 0 & 0 & 0 \\ 0 & (t^2-4t+5)e^{-t}-5e^{-2t} & (2-t)e^{-t}-2te^{-2t} & 0 \\ 0 & (2-t)e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix}. \tag{32}$$

The concurrence of Wooters is then

$$C(\rho_t^{11}) = 2\max[0, |(2-t)e^{-t} - 2e^{-2t}| - \sqrt{(1-(t^2-4t+6)e^{-t}+5e^{-2t})e^{-2t}}] \,.$$

and the entanglement of formation is

$$E(\rho_t^{11}) = h\left(\frac{1+\sqrt{1-C(\rho_t^{11})^2}}{2}\right) \,. \tag{33}$$

The behavior is mostly the same as in the previous case (see Fig 2), with the important difference that the entanglement, initially starting at 0, takes a strictly positive time to leave the value 0.

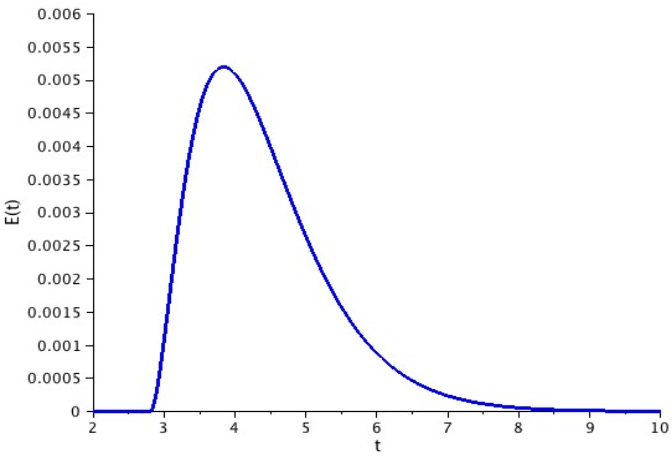

Figure 2: *Time evolution of Wooters' concurrence, initial state* $e_1 \otimes e_1$

5 Thermal Environment

In this section, we investigate the bipartite model in presence of a thermal environment. To this end, we consider that the reference state of each copy of $\mathcal{H}$ is the Gibbs state

$$\omega_\beta = \frac{1}{Z}\, e^{-\beta H^R} \,,$$

where β is positive and Z is a normalizing constant (as usual β is the inverse of the temperature). In the orthonormal basis $\{e_0, \ldots, e_N\}$ of eigenvectors of the Hamiltonian H^R, the state ω_β is diagonal and is expressed as

$$\omega_\beta = \sum_j \beta_j \, |e_j\rangle\langle e_j| \,, \tag{34}$$

where $\beta_j = e^{-\beta\lambda_j}/Z$, with $\sum_j \beta_j = 1$.

Let us stress that the limit evolution described in [2] is crucially related to the fact that the state of $\mathcal{H}$ is a pure state. With a general state of the form ω_β, in order to compute the limit evolution in terms of a unitary evolution on a Fock space, one has to consider the so-called of G.N.S. representation of the dynamics. This techniques has been successfully developed in [1] in order to derive the quantum Langevin equation associated to the action of a quantum heat bath.

5.1 Limit Lindblad Generator

Here we shall not describe such results but we focus only on the Lindblad generator. As in the previous section this generator can be obtained from the continuous-limit of the discrete one. To this end we define the discrete generator $l_\beta(h)$ including temperature by

$$l_\beta(h)(\rho) = \mathrm{Tr}_{\mathcal{H}}(U(\rho \otimes \omega_\beta)U) = \sum_k \beta_k \mathrm{Tr}_{\mathcal{H}}(U(\rho \otimes |e_k\rangle\langle e_k|)U) = \sum_{j,k} \beta_k\, U_j^k\, \rho\, U_j^{k*}\,. \tag{35}$$

Proposition 5.1. *In terms of h, the asymptotic expression of $l_\beta(h)$ is given by*

$$l_\beta(h)(\rho) = \rho + hL_\beta(\rho) + \circ(h),$$

where

$$\begin{aligned} L_\beta(\rho) =& -i\left[H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_{j=1}^N (\beta_j - \beta_0)(V_j \otimes W_j^* - V_j^* \otimes W_j), \rho\right] \\ & - \frac{1}{2}\sum_{j=1}^N \beta_j \big(S_j S_j^* \rho + \rho S_j S_j^* - 2S_j^* \rho S_j\big) - \frac{1}{2}\sum_{j=1}^N \beta_0 \big(S_j^* S_j \rho + \rho S_j^* S_j - 2S_j \rho S_j^*\big)\,, \end{aligned} \tag{36}$$

where $S_j = V_j \otimes I + I \otimes W_j$.

Furthermore, the interaction Hamiltonian between $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ created by repeated interactions with the environment is

$$\frac{i}{2}\sum_{j=1}^N (\beta_j - \beta_0)(V_j \otimes W_j^* - V_j^* \otimes W_j)\,.$$

Proof. Plugging the asymptotic expressions (12) – (16) into (35) and putting

$S_j = V_j \otimes I + I \otimes W_j$ for all $j \geq 1$, we get, up to terms in $h^{3/2}$ or higher,

$$
\begin{aligned}
l_\beta(\rho) = \rho + h\Big(- i \left[H^A \otimes I + I \otimes H^B, \rho\right] & - \frac{1}{2}\sum_{j=1}^N \beta_j (V_j V_j^* \otimes I + I \otimes W_j W_j^* + 2V_j^* \otimes W_j)\rho \\
& - \frac{1}{2}\sum_{j=1}^N \beta_j \rho (V_j V_j^* \otimes I + I \otimes W_j W_j^* + 2V_j^* \otimes W_j)^* \\
& - \frac{1}{2}\sum_{j=1}^N \beta_0 (V_j^* V_j \otimes I + I \otimes W_j^* W_j + 2V_j \otimes W_j^*)\rho \\
& - \frac{1}{2}\sum_{j=1}^N \beta_0 \rho (V_j^* V_j \otimes I + I \otimes W_j^* W_j + 2V_j \otimes W_j^*)^* \\
& + \sum_{j=1}^N \beta_0 S_j \rho S_j^* + \beta_j S_j^* \rho S_j \Big),
\end{aligned}
$$

which can be written in the usual form

$$
\begin{aligned}
l_\beta(\rho) =& \rho + h\Big(- i \left[H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_{j=1}^N (\beta_j - \beta_0)(V_j \otimes W_j^* - V_j^* \otimes W_j), \rho \right] \\
& - \frac{1}{2}\sum_{j=1}^N \beta_j \big(S_j S_j^* \rho + \rho S_j S_j^* - 2 S_j^* \rho S_j\big) - \frac{1}{2}\sum_{j=1}^N \beta_0 \big(S_j^* S_j \rho + \beta_0 \rho S_j^* S_j - 2 S_j \rho S_j^*\big)\Big) .
\end{aligned}
$$

This way, the interaction Hamiltonian naturally appears in the Hamiltonian part. □

5.2 Return to Equilibrium in a Physical Example, Thermalization

On a particular example we shall study the asymptotic behavior of the dynamics described above.

Recall that $\mathcal{H}_S^A$, $\mathcal{H}_S^B$ and $\mathcal{H}$ are $\mathbb{C}^{N+1}$. We assume that the free evolutions satisfy $H^A = H^B = H^R$. The total Hamiltonian operators are

$$H_{tot}^A = H^A \otimes I \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^N a_j^0 \otimes I \otimes a_0^j + a_0^j \otimes I \otimes a_j^0 \, , \qquad (37)$$

$$H_{tot}^B = I \otimes H^B \otimes I + I \otimes I \otimes H^R + \frac{1}{\sqrt{h}} \sum_{j=1}^N I \otimes a_j^0 \otimes a_0^j + I \otimes a_0^j \otimes a_j^0 . \qquad (38)$$

This is a generalization of the spontaneous emission (see [1]).

Applying Proposition 5.1 we get the expression of the Lindblad generator

$$L_\beta(\rho) = -\, i \left[H^A \otimes I + I \otimes H^B + \frac{i}{2} \sum_{j=1}^N (\beta_j - \beta_0)(a_0^j \otimes a_j^0 - a_j^0 \otimes a_0^j), \rho \right]$$
$$- \frac{1}{2} \sum_{j=1}^N \beta_j \big(S_j S_j^* \rho + \rho S_j S_j^* - 2 S_j^* \rho S_j \big) - \frac{1}{2} \sum_{j=1}^N \beta_0 \big(S_j^* S_j \rho + \rho S_j^* S_j - 2 S_j \rho S_j^* \big)\,,$$

where $S_j = a_0^j \otimes I + I \otimes a_0^j$.

Now, we are in the position to consider the problem of return to equilibrium. More precisely, we shall show that there exists a unique state ρ_∞ such that

$$\lim_{t \to +\infty} \mathrm{Tr}(e^{tL_\beta}(\rho) X) = \mathrm{Tr}(\rho_\infty X)\,,$$

for all initial state ρ and all observable X on $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$. The state ρ_∞ is an invariant state.

In the case of finite dimensional Hilbert spaces, a general result, proved by Frigerio and Verri [9] and extended by Fagnola and Rebolledo [8] gives a sufficient condition, in the case where the system has a faithful invariant state ρ_∞. The criterion is the following. Let L, defined by

$$L(\rho) = -i[H, \rho] + \sum_j -\frac{1}{2} \{ C_j^\star C_j, \rho \} + C_j \rho C_j^\star\,,$$

be the Lindblad generator of a quantum dynamical system. The property of return to equilibrium is satisfied if

$$\big\{ H, L_j, L_j^*; j = 1, \ldots, N \big\}' = \quad \big\{ L_j, L_j^*; j = 1, \ldots, N \big\}'\,, \tag{39}$$

where the notation $\{\}'$ refers to the commutant of the ensemble.

In our context we shall prove the following return to equilibrium result.

Theorem 5.2. *On $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$, the dynamical system whose Lindblad generator is given by*

$$L_\beta(\rho) = -\, i \left[H^A \otimes I + I \otimes H^B + \frac{i}{2} \sum_{j=1}^N (\beta_j - \beta_0)(a_0^j \otimes a_j^0 - a_j^0 \otimes a_0^j), \rho \right]$$
$$- \frac{1}{2} \sum_{j=1}^N \beta_j \big(S_j S_j^* \rho + \rho S_j S_j^* - 2 S_j^* \rho S_j \big) - \frac{1}{2} \sum_{j=1}^N \beta_0 \big(S_j^* S_j \rho + \rho S_j^* S_j - 2 S_j \rho S_j^* \big)\,,$$

where $S_j = a_0^j \otimes I + I \otimes a_0^j$, has the property of return to equilibrium.

Moreover, the limit invariant state is

$$\rho_\beta = \frac{e^{-\beta(H^A \otimes I + I \otimes H^B)}}{Z}\,,$$

where Z is a normalizing constant.

Proof. First, one can check that ρ_β is a faithful invariant state since

$$L_\beta(\rho_\beta) = 0\,.$$

The rest of proof is then based on the result of Fagnola and Rebolledo by showing that the commutants

$$\Big\{H^A \otimes I + I \otimes H^B + \frac{i}{2}\sum_{j=1}^{N}(\beta_j - \beta_0)(a_0^j \otimes a_j^0 - a_j^0 \otimes a_0^j), S_k, S_k^*; k = 1, \dots, N\Big\}'$$

and

$$\{S_k, S_k^*; k = 1, \dots, N\}'$$

are simply trivial.

Recall that in this physical system the operators S_k are $a_0^k \otimes I + I \otimes a_0^k$ for all $k \geq 1$. Let us prove now that $\{S_k, S_k^*; k = 1, \dots, N\}'$ is trivial. Consider an element K of this commutant. This element K can be written with respect to the canonical basis $(a_j^i)_{i,j=0,\dots,N}$ as

$$K = \sum_{i,j=0}^{N} K_j^i \otimes a_j^i\,,$$

where the K_j^i's are operators on $\mathbb{C}^{N+1}$. Since the operators K and S_k commute for all $k \geq 1$, we get equality between

$$\begin{aligned} KS_k = \left(\sum_{i,j=0}^{N} K_j^i \otimes a_j^i\right)\left(a_0^k \otimes I + I \otimes a_0^k\right) &= \sum_{i,j=0}^{N} K_j^i a_0^k \otimes a_j^i + \sum_{i,j=0}^{N} K_j^i \otimes a_j^i a_0^k \\ &= \sum_{i,j=0}^{N} K_j^i a_0^k \otimes a_j^i + \sum_{j=0}^{N} K_j^0 \otimes a_j^k \end{aligned}$$

and

$$\begin{aligned} S_k K = \left(a_0^k \otimes I + I \otimes a_0^k\right)\left(\sum_{i,j=0}^{N} K_j^i \otimes a_j^i\right) &= \sum_{i,j=0}^{N} a_0^k K_j^i \otimes a_j^i + \sum_{i,j=0}^{N} K_j^i \otimes a_0^k a_j^i \\ &= \sum_{i,j=0}^{N} a_0^k K_j^i \otimes a_j^i + \sum_{i=0}^{N} K_k^i \otimes a_0^i\,. \end{aligned}$$

From the commutation of K and S_k^*, we also have equality between

$$\begin{aligned} KS_k^* = \left(\sum_{i,j=0}^{N} K_j^i \otimes a_j^i\right)\left(a_k^0 \otimes I + I \otimes a_k^0\right) &= \sum_{i,j=0}^{N} K_j^i a_k^0 \otimes a_j^i + \sum_{i,j=0}^{N} K_j^i \otimes a_j^i a_k^0 \\ &= \sum_{i,j=0}^{N} K_j^i a_k^0 \otimes a_j^i + \sum_{j=0}^{N} K_j^k \otimes a_j^0 \end{aligned}$$

and

$$S_k^* K = \left(a_k^0 \otimes I + I \otimes a_k^0\right) \left(\sum_{i,j=0}^{N} K_j^i \otimes a_j^i \right) = \sum_{i,j=0}^{N} a_k^0 K_j^i \otimes a_j^i + \sum_{i,j=0}^{N} K_j^i \otimes a_k^0 a_j^i$$
$$= \sum_{i,j=0}^{N} a_k^0 K_j^i \otimes a_j^i + \sum_{i=0}^{N} K_0^i \otimes a_k^i \,.$$

From these equalities and since the operators $(a_j^i)_{i,j=0,\ldots,N}$ form a basis, the following system of equations is obtained for $k = 1, \ldots, N$,

$$K_0^0 a_0^k = a_0^k K_0^0 + K_k^0$$
$$a_k^0 K_0^0 = K_0^0 a_k^0 + K_0^k \,,$$

for $j, k, l = 1, \ldots, N$ with $k \neq j$

$$K_j^0 a_0^l = a_0^l K_j^0$$
$$a_k^0 K_j^0 = K_j^0 a_k^0 + K_j^k$$
$$K_j^0 a_j^0 + K_j^j = a_j^0 K_j^0 + K_0^0 \,,$$

and

$$K_0^j a_l^0 = a_l^0 K_0^j$$
$$K_0^j a_0^k = a_0^k K_0^j + K_k^j$$
$$K_0^j a_0^j + K_0^0 = a_0^j K_0^j + K_j^j \,,$$

and for $i, j, k, l = 1, \ldots, N$ with $k \neq i$ and $l \neq j$

$$K_j^i a_0^k = a_0^k K_j^i$$
$$K_j^i a_l^0 = a_l^0 K_j^i$$
$$K_j^i a_0^i + K_j^0 = a_0^i K_j^i$$
$$K_j^i a_j^0 = a_j^0 K_j^i + K_0^i \,.$$

We now concentrate on all these equations in order to prove that the K_j^i's are all equal to 0. Note that the commutation of a matrix $M = (m_{ij})_{i,j=0,\ldots,N}$ with a_0^k for $k = 1, \ldots, N$ implies that for all $p \geq 1$ and $q = 0, \ldots, N$ with $q \neq k$

$$m_{00} = m_{kk} \;, \quad m_{p,0} = 0 \quad \text{and} \quad m_{k,q} = 0 \,.$$

The commutation of M with a_k^0 gives that for all $p \geq 1$ and $q = 0, \ldots, N$ with $q \neq k$

$$m_{00} = m_{kk} \;, \quad m_{0,p} = 0 \quad \text{and} \quad m_{q,k} = 0 \,.$$

Thus since $K_j^0 a_0^l = a_0^l K_j^0$ for all $j, l = 1, \dots, N$, the matrices K_j^0 are of the form

$$\begin{pmatrix} m_{00} & m_{01} & \dots & m_{0N} \\ 0 & m_{00} & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & m_{00} \end{pmatrix}.$$

In the same way, from $K_0^j a_l^0 = a_l^0 K_0^j$ for all $j, l = 1, \dots, N$, we deduce that the matrices K_0^j are of the form

$$\begin{pmatrix} m_{00} & 0 & \dots & 0 \\ m_{10} & m_{00} & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ m_{N0} & \dots & 0 & m_{00} \end{pmatrix}.$$

Consider now the equations associated to K_j^i for $i, j \neq 0$. Since we get $K_j^i a_0^k = a_0^k K_j^i$ and $K_j^i a_l^0 = a_l^0 K_j^i$ for $k, l = 1, \dots, N$ with $k \neq i$ and $l \neq j$, the matrix K_j^i is a diagonal matrix whose coefficients are all equal to m_{00} except the column j and the row i with for the moment only zero coefficients on the first row and the first column.

In the following, the coefficients of a matrix K_j^i are denoted by $(m_{kl}^{ij})_{k,l=0,\dots,N}$. We work then on the equation $K_j^i a_0^i + K_j^0 = a_0^i K_j^i$. This equality gives that the diagonal coefficients of K_j^0 are 0 and for $l = 1, \dots, N$ with $l \neq i$,

$$m_{00}^{ij} = m_{ii}^{ij} + m_{0i}^{0j} \quad \text{and} \quad m_{0l}^{0j} = -m_{il}^{ij}\,.$$

Then, from $K_j^i a_j^0 = a_j^0 K_j^i + K_0^i$ we deduce that the diagonal coefficients of K_0^i are 0 and, for $l = 1, \dots, N$ with $l \neq j$,

$$m_{00}^{ij} = m_{jj}^{ij} + m_{0j}^{i0} \quad \text{and} \quad m_{l0}^{i0} = -m_{lj}^{ij}\,.$$

From the equalities $K_0^j a_0^k = a_0^k K_0^j + K_k^j$ and $a_k^0 K_j^0 = K_j^0 a_k^0 + K_j^k$ with $k \neq j$, we finally obtain that all the matrices K_0^j, K_j^0 and K_k^j are vanishing for $j \neq k$. For $j = k$, the equalities $K_0^j a_0^j + K_0^0 = a_0^j K_0^j + K_j^j$ allow us to conclude that the only non zero operators are the K_j^j's for $j = 0, \dots, N$ and all equal to $m_{00}^{00} I$.

Hence, we have proved that the commutant $\{S_k, S_k^*; k = 1, \dots, N\}'$ is reduced to the operators of the form $\lambda I \otimes I$ with λ in $\mathbb{C}$. Then the commutant

$$\left\{ H^A \otimes I + I \otimes H^B + \frac{i}{2} \sum_{j=1}^{N} (\beta_j - \beta_0)(a_0^j \otimes a_j^0 - a_j^0 \otimes a_0^j), S_k, S_k^*; k = 1, \dots, N \right\}'$$

is by definition a subset of $\{S_k, S_k^*; k = 1, \dots, N\}'$. Therefore it is trivial too. This proves that the system has the property of return to equilibrium, applying [8]. □

Since this state ρ_β is the invariant state of $\mathcal{H}_S^A \otimes \mathcal{H}_S^B$, one deduces the thermalization of $\mathcal{H}_S^A$ and $\mathcal{H}_S^B$ by the environment.

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