Exercice 1

a) $\dfrac{n}{n^2+1} \geq 0$ et $\sim \dfrac{1}{n}$ donc série divergente.

b) $\dfrac{e^n+e^{-n}}{e^{2n}+e^{-2n}} \geq 0$ et $\sim e^{-n}$ car

$$\frac{e^n\left(e^n+e^{-n}\right)}{e^{2n}+e^{-2n}} = \frac{e^{2n}+1}{e^{2n}+e^{-2n}} = \frac{1+e^{-2n}}{1+e^{-4n}} \to 1$$

$\Rightarrow$ convergente.

c) $n^2-1 < n^2+1 \Rightarrow \dfrac{1}{\sqrt{n^2-1}} \leq \dfrac{1}{\sqrt{n^2+1}} \Rightarrow \geq 0$

$$= \frac{1}{n}\left(\frac{1}{\sqrt{1-\frac{1}{n^2}}} - \frac{1}{\sqrt{1+\frac{1}{n^2}}}\right) = \frac{1}{n}\left(1+\frac{1}{2n^2} - \left(1-\frac{1}{2n^2}\right) + o\left(\frac{1}{n^2}\right)\right)$$

$= \dfrac{1}{4}\dfrac{1}{n^3} + o\left(\dfrac{1}{n^3}\right) \Rightarrow$ convergente.

d) Signe de $e - \left(1+\frac{1}{n}\right)^n = e - e^{n\ln(1+1/n)}$

$1 \geq n\ln(1+1/n)$?

$\frac{1}{n} \geq \ln(1+1/n)$?

$\ln(1+x) \leq x$

$\left(\ln(1+x)-x\right)' = \dfrac{1}{1+x} - 1 \leq 0 \quad (x \geq 0)$

$0 \searrow \Rightarrow \leq 0$ oui

Série à termes ≥ 0.

$$e - e^{n\ln(1+1/n)} = e - e^{n\left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)}$$

$$= e - e^{1 - \frac{1}{2n} + o\left(\frac{1}{n}\right)}$$

$$= e\left(1 - \left(1 - \frac{1}{2n} + o\left(\frac{1}{n}\right)\right)\right)$$

$$= \frac{e}{2n} + o\left(\frac{1}{n}\right)$$

divergente.

Exercice 2

$$S_N = \sum_{k<\sqrt{N}} \frac{1}{k^2} + \sum_{\substack{n<N \\ n\neq k^2}} \frac{1}{n^2}$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k^2} + \sum_{n=0}^{\infty} \frac{1}{n^2} < \infty$$

Donc conv. (car à termes ≥ 0)

Exercice 3

a) $\sum_n \max(u_n, v_n) \leq \sum_n (u_n + v_n)$ conv.

b) $a^2+b^2-2ab \geq 0 \Rightarrow ab \leq \frac{1}{2}(a^2+b^2)$

$$\sqrt{ab} \leq \frac{1}{\sqrt{2}} \sqrt{a^2+b^2} \leq \frac{1}{\sqrt{2}}(a+b) \quad (a,b \geq 0)$$

$$\sum_n \sqrt{u_n v_n} \leq \frac{1}{\sqrt{2}} \sum_n (u_n + v_n).$$

c) $\frac{ab}{a+b} \leq k(a+b)$?

$ab \leq k(a+b)^2$?

$\frac{1}{k} ab \leq (a+b)^2$

$a^2+b^2+2ab - \cancel{\alpha}ab \geq 0$?

oui avec $\alpha = 4 \Rightarrow k = \frac{1}{4}$

$\frac{ab}{a+b} \leq \frac{1}{4}(a+b)$ $\qquad \sum_n \frac{u_n v_n}{u_n+v_n} \leq \frac{1}{4} \sum_n (u_n+v_n)$ conv.

$\boxed{\text{Exercice 4}}$

$$\frac{u_n}{v_n} = 1 + u_n$$

. Donc si $u_n \to 0$ alors $(u_n) \sim (v_n)$ et donc même nature.

. ~~Si $u_n \to \ell \neq 0$ ...~~

$u_n = \dfrac{v_n}{1-v_n}$ donc $u_n \to \ell \iff v_n \to \dfrac{\ell}{1+\ell}$

$u_n \to \dfrac{\alpha}{1-\alpha} \iff v_n \to \alpha$

. Donc div-grossière $\iff$ div-grossière

. $u_n \to +\infty \iff v_n \to 1$

. $u_n \to 1 \iff v_n \to +\infty$ idem.

. (u_n) pas de lim $\iff$ (v_n) pas de lim

à cause de ci-dessus.

Exercice 5

$u_n \geq 0$

$$e - \left(1+\frac{1}{n}\right)^n = \frac{e}{2n} + o\left(\frac{1}{n}\right)$$

$$n^{3/2} - [n^{3/2}] + n \in [n, n+1]$$

donc $\sim \frac{e}{2n^2}$ donc conv.

Exercice 6

$\exists n_0 \,/\, n \geq n_0 \Rightarrow a_n < 1$ (car $a_n \to 0$)

$$a_0 \cdots a_n = a_0 \cdots a_{n_0}\, a_{n_0+1} \cdots a_{n-1}\, a_n$$

$$u_n < (a_0 \cdots a_{n_0})\, a_n = k_0\, a_n \quad n \geq n_0.$$

Donc $\sum_n u_n$ converge,

Exercice 7

a) $\frac{(-1)^n}{n^2+1}$ Abel conv ~~...~~.

b) idem

c) $\ln\left(1+\frac{(-1)^n}{n+1}\right) = \frac{(-1)^n}{n+1} + O\left(\frac{1}{n^2}\right)$

$\Rightarrow$ conv.

d) $\cos\left(\pi\sqrt{n^2+n+1}\right) = \cos\left(n\pi\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}\right)$

$= \cos\left(n\pi\left(1+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n^2}\right)-\frac{1}{8}\left(\frac{1}{n}+\frac{1}{n^2}\right)^2+o\left(\frac{1}{n^2}\right)\right)\right)$

$= \cos\left(n\pi + \pi/2 + \frac{3}{8}\frac{\pi}{n} + o\left(\frac{1}{n}\right)\right)$

$= (-1)^n \sin\left(\frac{3}{8}\frac{\pi}{n} + o\left(\frac{1}{n}\right)\right)$

$= (-1)^n \frac{3}{8}\frac{\pi}{n} + o\left(\frac{1}{n^2}\right)$ $\Rightarrow$ conv.

$\boxed{\text{Exercice 8}}$

$$\frac{(-1)^n 8^n}{(2n)!}$$

1) convergente. (Abel. Alternée)

2) $R_n = \sum_{k=n+1}^{+\infty} (-1)^k u_k$

$$|R_n| \leq u_{n+1} = \frac{8^{n+1}}{(2n+2)!}$$

3) Par le calcul :

$S_0 = 1$

$S_1 = 1 - \frac{8}{2} = -3$

$S_2 = -3 + \frac{8}{3} = -\frac{1}{3}$

$S_3 \simeq -1$

$u_4 \simeq 0,1$

$\Rightarrow \underline{\underline{S < 0}}$

Exercice 9

$$\frac{1}{n(n+1)(n+2)} = \frac{\alpha}{n} + \frac{\beta}{n+1} + \frac{\gamma}{n+2}$$

$$\frac{1}{2} = \alpha$$

$$\frac{1}{(-1)(1)} = \beta = -1$$

$$\frac{1}{(-2)(-1)} = \gamma = \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{n} - 2\left(\sum_{k=1}^{\infty}\frac{1}{n} - 1\right) + \left(\sum_{k=1}^{\infty}\frac{1}{n} - 1 - \frac{1}{2}\right)\right)$$

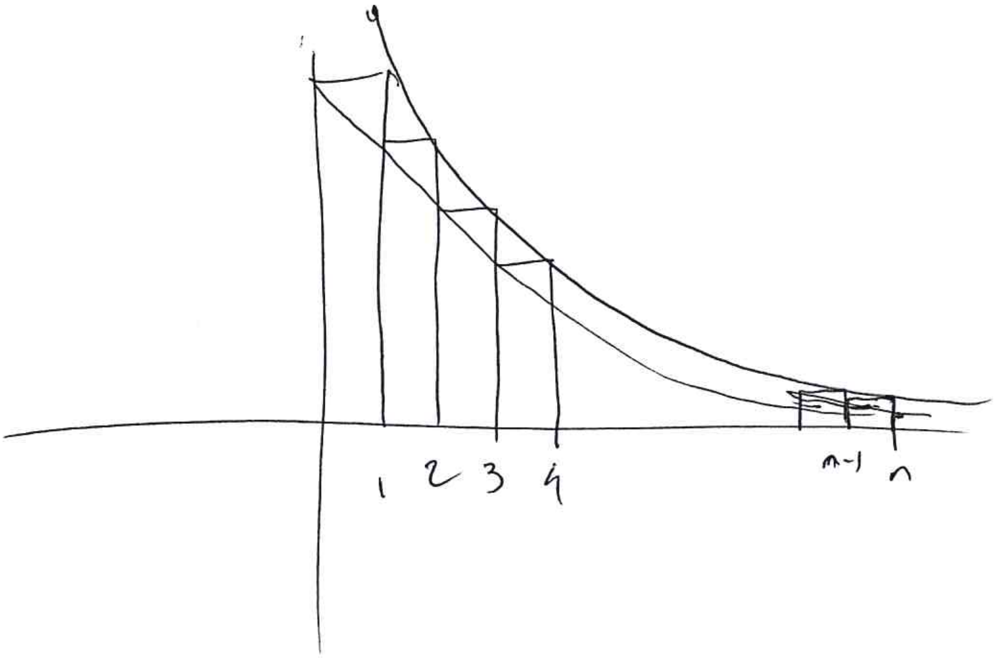
$$= \frac{1}{2}\left(2 - 1 - \frac{1}{2}\right) = \frac{1}{4}.$$

$$\cancel{\frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)}$$

$$\cancel{(n+1)(n+2) - 2n(n+1) + n(n+n) = n + 2n - 4n + n}$$

Exercice 10

1)

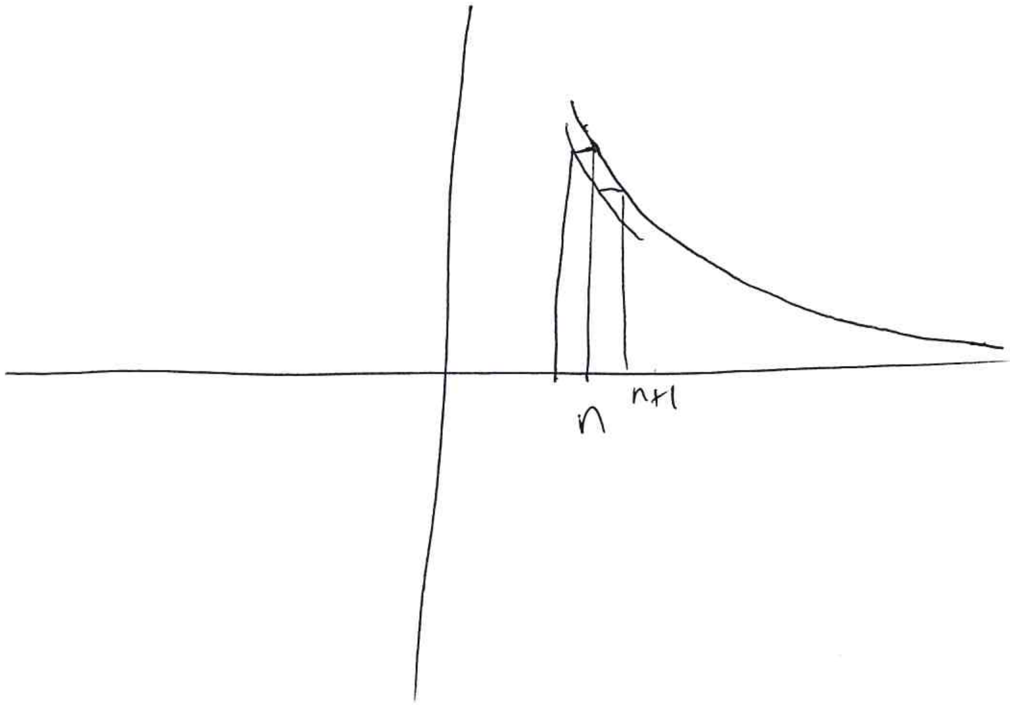


$$\int_0^n \frac{1}{(x+1)^\alpha}\,dx \leq \sum_{k=1}^{n} \frac{1}{k^\alpha} \leq 1 + \int_1^n \frac{1}{x^\alpha}\,dx$$

$$\left[\frac{(x+1)^{1-\alpha}}{1-\alpha}\right]_0^n \leq \qquad \leq 1 + \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^n$$

$$\frac{(n+1)^{1-\alpha}}{1-\alpha} \leq \qquad \leq \frac{1}{1-\alpha} n^{1-\alpha}.$$

$$\Rightarrow \sim \frac{n^{1-\alpha}}{1-\alpha}$$



$$\int_{n-1}^{\infty} \frac{1}{(x+1)^{\alpha}}\, dx \le R_n \le \int_{n-1}^{\infty} \frac{1}{x^{\alpha}}\, dx$$

$$\left[\frac{(x+1)^{1-\alpha}}{1-\alpha}\right]_{n-1}^{\infty} \qquad \qquad \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_{n-1}^{\infty}$$

$$\boxed{\frac{1}{\alpha-1}\, n^{1-\alpha}}$$

$\boxed{\text{Exercice 11}}$

a) $e^{-(n^\alpha)}$

Div. grossière $\quad e^{-n^\alpha} \to 0 \iff n^\alpha \to +\infty$

$\iff \alpha > 0$

Pour $\alpha \leq 0$: div. gross.

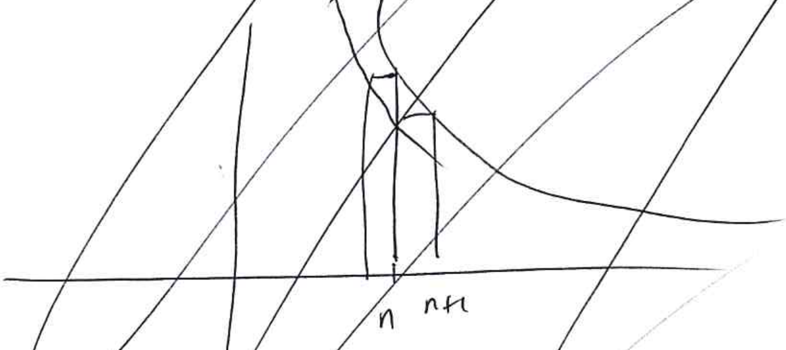
Pour $\alpha > 0 \quad n^2 e^{-(n^\alpha)} = e^{2\ln n - n^\alpha}$

$$2\ln n - n^\alpha \to -\infty$$

car $\dfrac{2\ln n - n^\alpha}{n^\alpha} = 2\dfrac{\ln n}{n^\alpha} - 1 \to -1$

$\Rightarrow n^2 e^{-(n^\alpha)} \to 0$, conv

b) $\displaystyle\int_{n-1}^{\infty} \frac{1}{(x+1)^\alpha} \leq R_n \leq \int_{n-1}^{\infty} \frac{1}{x^\alpha}\,dx$



$\Rightarrow \sim n^{1-\alpha}$.

TD 1-11

b) $\dfrac{\ln n}{n^\alpha}$

conv $\to 0 \iff \alpha > 0$

- $\alpha \leq 0$ div. gross

- $0 < \alpha \leq 1$: $\dfrac{\ln n}{n^\alpha} > \dfrac{1}{n}$ donc div.

- $\alpha > 1$ $\alpha = 1 + \varepsilon$

$$n^{1+\varepsilon/2} \frac{\ln n}{n^{1+\varepsilon}} = \frac{\ln n}{n^{\varepsilon/2}} \to 0$$

$$\Rightarrow \frac{\ln n}{n^{1+\varepsilon}} < n^{-(1+\varepsilon/2)} \Rightarrow \text{conv.}$$

c) $e^{-(\ln n)^\alpha}$

$\alpha \leq 0$: div. gross.

$\alpha > 0$: $n^2 e^{-(\ln n)^\alpha} = e^{2\ln n - (\ln n)^\alpha}$

Si $\alpha > 1$

$$2\ln n - (\ln n)^\alpha = \underbrace{\left(\frac{2\ln n}{(\ln n)^\alpha} - 1\right)}_{\to -1} (\ln n)^\alpha \to -\infty$$

conv.

Si $0 < \alpha \leq 1$:

$$(\ln n)^\alpha < \ln n$$

$$e^{-(\ln n)^\alpha} \geq e^{-\ln n} = n$$

$\Rightarrow$ div. geom.

d) $\dfrac{(-1)^n}{n^\alpha + (-1)^n} = \dfrac{1}{(-1)^n n^\alpha + 1}$

$$= \frac{(-1)^n}{n^\alpha}\left(\frac{1}{1 + \frac{(-1)^n}{n^\alpha}}\right)$$

$$= \frac{(-1)^n}{n^\alpha}\left(1 - \frac{(-1)^n}{n^\alpha} + o\left(\frac{1}{n^\alpha}\right)\right)$$

$$= \underbrace{\frac{(-1)^n}{n^\alpha}}_{\text{série conv}} - \underbrace{\frac{1}{n^{2\alpha}}}_{\text{conv. si } 2\alpha > 1} + o\left(\frac{1}{n^{2\alpha}}\right)$$

$\alpha > 1/2$

$\boxed{\text{Ex 12}}$

1) Comparaison série/intégrale

$$\int_1^n \frac{1}{x+\sqrt{x}}\,dx$$

$$y = \sqrt{x}$$

$$\int_1^{\sqrt{n}} \frac{1}{y^2+y}\,2y\,dy$$

$$\frac{2y+1}{y^2+y} - \frac{1}{y^2+y}$$

$$\Rightarrow \ln(y^2+y) - \int\left(\frac{1}{y} - \frac{1}{y+1}\right)$$

$$\ln(y^2+y) - \ln y + \ln(y+1)$$

$$\ln(n+\sqrt{n}) - \ln\sqrt{n} + \ln(\sqrt{n}+1)$$

$$\sim \underline{\underline{\ln n}}$$

$$S_n - \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{\ln(n)+\gamma+o(1)} = \sum_{k=1}^{n} \frac{1}{k+\sqrt{k}} - \frac{1}{k}$$

$$= \sum_{k=1}^{n} \frac{k-k-\sqrt{k}}{(k+\sqrt{k})k} = \underbrace{\sum_{k=1}^{n} \frac{-\sqrt{k}}{(k+\sqrt{k})k}}_{\alpha+o(1)}$$

$\boxed{B}$

$\boxed{\text{Ex } 13}$

* Conv. ss pb

* $$\sum_{n=0}^{\infty} \frac{n+1}{3^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{3^k} \frac{1}{3^{n-k}}$$

$$= \left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right)\left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right) = \frac{9}{4}.$$