# Feuille 3

## Ex 1

$$f_n(x) = \left(\frac{x}{n}\right)^{nx}$$

1) $f_n(x) = \left(\frac{x}{n}\right)^{nx} \leq \left(\frac{1}{2}\right)^{nx}$ pour n assez grand $\Rightarrow \longrightarrow 0$.

2) $e^{nx\ln(x/n)} \to e^0 = 1$

3) Pas de conv. unif.

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## Ex 2

$f_n(x) = \min\left(n, \frac{1}{\sqrt{1-x}}\right)$ sur $[0,1[$.

1) $f_n(x) \longrightarrow \frac{1}{\sqrt{1-x}}$

2) $\lim\limits_{x \to 1^-} f_n(x) = n$

3) Pas unif.

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## Ex 3

1) $u_n = \int_0^{\pi/4} (\tan x)^n \, dx$.

$|\tan x| < 1$ sur $[0, \pi/4[$

$\Rightarrow (\tan x)^n \to 0$ et $\leq 1$ intégrable

CV Dom $\Rightarrow u_n \to \int_0^{\pi/4} 0 \, dx = 0$

2) $\int_0^{\infty} \frac{1}{x^n + e^x}\, dx$

$$\frac{1}{x^n+e^x} \to \begin{cases} 1/e^x, & x<1 \\ \frac{1}{1+e^x}, & x=1 \\ 0, & x>1 \end{cases}$$

$\frac{1}{x^n+e^x} \leq \frac{1}{e^x}$ int. sur $[0,+\infty[$

$\Rightarrow \lim u_n = \int_0^1 1/e^x\, dx = \left[-e^{-x}\right]_0^1 = 1-e^{-1}$.

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$\boxed{\text{Ex 4}}$

1) $\underbrace{\text{Arctan}(nx)}_{\to \pi/2}\, \underbrace{e^{-(x^n)}}_{}$ $\leq \pi/2 \left( 1\!\!1_{[0,1]} + e^{-x}\, 1\!\!1_{[1,+\infty[}\right)$ intégrable.

$$\begin{matrix} 0 & x>1 \\ e^{-1} & x=1 \\ 1 & x<1 \end{matrix}$$

$\Rightarrow \lim = \int_0^1 \pi/2\, dx = \pi/2$

2) $\frac{1}{(1+x^2)(1+x^n)^{1/n}}$

$(1+x^n)^{1/n} = e^{\frac{1}{n}\ln(1+x^n)}$

$$\begin{cases} x<1: \quad e^{\frac{1}{n}\left(x^n + o(x^n)\right)} \to e^0 = 1. \\ x=1 \quad 2^{1/n} \to 0 \\ x>1 \quad e^{1/n\left(\ln(x^n) + \ln\left(\frac{1}{x^n}+1\right)\right)} = x\, e^{\frac{1}{n}\left(\frac{1}{x^n} + o\left(\frac{1}{x^n}\right)\right)} \to x. \end{cases}$$

$f_n(x) \leq \frac{1}{1+x^2}$ intégrable

$$\to \int_0^1 \frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{x(1+x^2)}\,dx$$

$$= \text{Arctan}(1) + \int_1^\infty \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$$

$$= \pi/4 + \left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^\infty$$

$$= \pi/4 + \left[\ln \frac{x}{\sqrt{1+x^2}}\right]_1^\infty = \frac{\pi}{4} + \left(\ln(1) - \ln\left(\frac{1}{\sqrt{2}}\right)\right)$$

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$\boxed{\text{Ex 5}}$

$$\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$$

$$\left(1+\frac{x}{n}\right)^n \mathbb{1}_{[0,n]}(x)\, e^{-2x} \xrightarrow[n\to\infty]{} e^{x} e^{-2x} = e^{-x}.$$

Majoration: $e^{n\ln(1+x/n)} \leq e^{n\,x/n} = e^{x}$ $\left(\times e^{-2x} \Rightarrow e^{-x} \text{ int}\right).$

$$\to \int_0^\infty e^{-x}\,dx = 1.$$

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$\boxed{\text{Ex 6}}$

$n\int_0^1 \frac{f(nt)}{1+t}\,dt = \int_0^n \frac{f(u)}{1+u/n}\,du \to \int_0^\infty f(u)\,du$ car cvs $f(u)$ et $\leq f(u)$ int..

$u = nt$

$$\boxed{\text{Ex } 7}$$

$$n\int_1^{\infty} e^{-x^n}\,dx = n\int_1^{\infty} e^{-u}\,\frac{1}{n}\,u^{1/n-1}\,du. \longrightarrow \int_1^{\infty} \frac{e^{-u}}{u}\,du \quad \text{avec la majo : } \frac{e^{-u}}{u^{1-1/n}} \le \frac{e^{-u}}{\sqrt{u}}$$

$u = x^n \qquad x = u^{1/n}$

$\qquad\qquad dx = \frac{1}{n}\,u^{1/n-1}\,du$