Lecture 8
FOCK SPACES

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Abstract This lecture is devoted to introducing a fundamental family of spaces in Quantum Mechanics: the Fock spaces. They are fundamental for they represent typical state spaces for gases of particles, thermal baths, etc. They are also an interesting playground as quantum probability spaces, for they produce striking examples of non-commutative laws. In this lecture we give some physical motivations for these spaces. We define the symmetric, antisymmetric and full Fock spaces, together with their fundamental structures (continuous tensor product, coherent vectors, Guichardet's representation) and their fundamental operators (creation, annihilation, second quantization and Weyl operators).

In order to read this lecture, one should be familiar with general Operator Theory, with basic Quantum Mechanics and with basic notions of Quantum Probability. If necessary, please read Lectures 1, 5 and 7.

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8.1 Motivations

8.1.1 Physical Motivations

The discussion below is rather informal and not really necessary for understanding the mathematical structure of the Fock spaces. The reader can possibly directly jump to Section 8.2.

Fock spaces have interpretations in physics, they have been built in order to represent the state space for a system containing an indefinite (and variable) number of identical particles (an electron gas, photons, etc.). They are also of great use in quantum field theory for they provide the setup for the so-called second quantization procedure.

In classical mechanics, a point system is characterized by its position coordinates $Q_i(t)$ and momentum coordinates $P_i(t)$, $i = 1, \dots, n$. In the Hamiltonian description of motion there exists a fundamental function $H(P, Q)$ (called the *Hamiltonian*) which represents the total energy of the system and satisfies the Hamilton equations:

$$\frac{\partial H}{\partial P_i} = \dot{Q}_i\,, \quad \frac{\partial H}{\partial Q_i} = -\dot{P}_i\,.$$

If $f(P, Q)$ is a functional of the trajectory, we then have the evolution equation

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial P_i}\frac{\partial P_i}{\partial t} + \frac{\partial f}{\partial Q_i}\frac{\partial Q_i}{\partial t}$$

or else

$$\frac{df}{dt} = \{f, H\}$$

where $\{\,\cdot\,,\,\cdot\,\}$ denotes the *Poisson bracket*:

$$\{g, h\} = \sum_i \frac{\partial g}{\partial P_i}\frac{\partial h}{\partial Q_i} - \frac{\partial g}{\partial Q_i}\frac{\partial h}{\partial P_i}\,.$$

In particular we have

$$\begin{cases} \{P_i, P_j\} = \{Q_i, Q_j\} = 0 \\ \{P_i, Q_j\} = \delta_{ij}\,. \end{cases} \tag{8.1}$$

It happens that this is not exactly the definitions of the P_i and Q_i which are important, but the relations (8.1). Indeed, a change of coordinates $P'(P, Q)$, $Q'(P, Q)$ will give rise to the same equations of motion if and only if P' and Q' satisfy (8.1).

In quantum mechanics the situation is essentially the same. We have a self-adjoint operator H (the Hamiltonian) which describes all the evolution of the state of the system via the Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt}\psi(t) = \mathsf{H}\,\psi(t)\,.$$

There are also self-adjoint operators Q_i, P_i which represent the position and the momentum of the system and which evolve as follows:

$$\begin{aligned}\mathsf{Q}_i(t) &= \mathrm{e}^{\mathrm{i}\hbar t\mathsf{H}} Q_i\, \mathrm{e}^{-\mathrm{i}\hbar t\mathsf{H}}\\ \mathsf{P}_i(t) &= \mathrm{e}^{\mathrm{i}\hbar t\mathsf{H}} \mathsf{P}_i\, \mathrm{e}^{-\mathrm{i}\hbar t\mathsf{H}}\,.\end{aligned}$$

Any observable A of the system satisfies the evolution equation

$$\frac{d}{dt}\mathsf{A}(t) = -\frac{\mathrm{i}}{\hbar}[\mathsf{A}(t), \mathsf{H}]\,.$$

The particular observables P_i, Q_i satisfy the relations

$$\begin{cases}[\mathsf{P}_i, \mathsf{P}_j] = [\mathsf{Q}_i, \mathsf{Q}_j] = 0\\ [\mathsf{Q}_i, \mathsf{P}_j] = \mathrm{i}\hbar\,\delta_{ij}\mathrm{I}\,.\end{cases} \tag{8.2}$$

Once again, it is not the choice of the representations of P_i and Q_i as concrete operators on a Hilbert space which is important, but the relations (8.2) which are fundamental. They are called the *Canonical Commutation Relations* or *C.C.R.*

In Quantum Field Theory one has to deal with an infinite number of degrees of freedom; the position and momentum operators are indexed by a continuous set ($\mathbb{R}^3$ for example). The relations (8.2) then become

$$\begin{cases}[\mathsf{P}(x), \mathsf{P}(y)] = [\mathsf{Q}(x), \mathsf{Q}(y)] = 0\\ [\mathsf{Q}(x), \mathsf{P}(y)] = \mathrm{i}\hbar\,\delta(x-y)\mathrm{I}\,.\end{cases} \tag{8.3}$$

If we define

$$a(x) = \frac{1}{\sqrt{2}}(\mathsf{Q}(x) + \mathrm{i}\mathsf{P}(x))$$

and

$$a^*(x) = \frac{1}{\sqrt{2}}(\mathsf{Q}(x) - \mathrm{i}\mathsf{P}(x))$$

then $a(x)$ and $a^*(x)$ are mutually adjoint and satisfy the following relations which are also called *Canonical Commutation Relations*:

$$\begin{cases}[a(x), a(y)] = [a^*(x), a^*(y)] = 0\\ [a(x), a^*(y)] = \hbar\,\delta(x-y)\mathrm{I}\,.\end{cases}$$

It happens that these equations are valid only for a particular family of particles: the *bosons* (photons, phonons, mesons, gravitons,...). There exists another family of particles: the *fermions* (electrons, muons, neutrinos, protons, neutrons, ...) for which the correct relations are the *Canonical Anticommutation Relations* or *C.A.R.*

$$\begin{cases} \{b(x), b(y)\} = \{b^*(x), b^*(y)\} = 0 \\ \{b(x), b^*(y)\} = \hbar\delta(x-y)\mathrm{I}\,, \end{cases}$$

where $\{\mathsf{A}, \mathsf{B}\} = \mathsf{AB} + \mathsf{BA}$ is the *anticommutator* of operators (it has nothing to do with the Poisson bracket but, as we never use the latter in the rest of the lecture, there shall not be any possible confusion).

8.1.2 Realization of the Commutation Relations

A natural problem, which has given rise to a huge literature, is to describe the possible concrete realisations of these relations. As an example, let us consider the simplest problem: find two self-adjoint operators P and Q such that

$$\mathsf{QP} - \mathsf{PQ} = \mathrm{i}\hbar\,\mathrm{I}\,. \tag{8.4}$$

In a certain sense there is only one solution to this problem. This solution is realized on $L^2(\mathbb{R};\mathbb{C})$ by the operators

$$\mathsf{Q} = x \quad \text{(multiplication by } x\text{)} \;\; \text{and} \quad \mathsf{P} = \mathrm{i}\hbar\frac{d}{dx}\,.$$

This is the so-called *Schrödinger representation of the C.C.R.* But in full generality the Problem (8.4) is not well-posed. Indeed, one can show that the solutions P and Q of the above problem cannot be bounded operators. One then needs to be able to define the operators PQ and QP on good common domains. It is actually possible to construct pathological counter-examples (cf [RS80]). The problem becomes well-posed if we rewrite it in terms of bounded operators.

If we put

$$\mathsf{W}_{x,y} = \mathrm{e}^{-\mathrm{i}(x\mathsf{P} - y\mathsf{Q})}$$

and $\mathsf{W}_z = \mathsf{W}_{x,y}$ when $z = x + iy \in \mathbb{C}$, it is then easy to see that the relation (8.4) is translated into the *Weyl commutation relations*

$$\mathsf{W}_z\mathsf{W}_{z'} = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,\langle z,z'\rangle/2}\,\mathsf{W}_{z+z'}\,. \tag{8.5}$$

When posed in these terms the problem has only one solution: the Schrödinger representation (this the so-called Stone-von Neumann Theorem, cf [BR97] if interested).

This solution, as well as the ones for more (even infinite) degrees of freedom, is realized through a particular family of spaces: the *symmetric Fock spaces.*

In the case of the anticommutation relations, one does not need to rewrite them, for $b(x)$ and $b^*(x)$ are always bounded operators (as we shall prove later). But the concrete realisation of the C.A.R. is always made through the *antisymmetric Fock spaces.*

The importance of Fock space comes from the fact they give a natural realization of the C.C.R. and C.A.R., whatever is the number of degrees of freedom involved. They are a natural tool for quantum field theory.

The basic physical idea hidden behind their definition is the following. If $\mathcal{H}$ is the Hilbert space describing the state space for one particle, then $\mathcal{H} \otimes \mathcal{H}$ describes the state space for two particles of the same type. The space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, the n-fold product, describes the state space for n such particles. Finally the space $\oplus_{n \in \mathbb{N}} \mathcal{H}^{\otimes n}$ describes a system where there can be any number of such particles which can disappear (be annihilated) or appear (be created). But depending on the type of particles (bosons or fermions), there are some symmetries which force to look at certain subspaces of $\oplus_n \mathcal{H}^{\otimes n}$.

We do not aim to describe all the physics behind Fock spaces (we are not able to), but we just wanted to motivate them. Let us now come back to mathematics.

8.2 Fock Spaces

We now enter into the concrete construction of the Fock spaces as Hilbert spaces.

8.2.1 Symmetric and Antisymmetric Tensor Products

Definition 8.1. Let $\mathcal{H}$ be a complex Hilbert space. For any integer $n \geq 1$ consider

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

the *n-fold tensor product* of $\mathcal{H}$. For $u_1, \ldots, u_n \in \mathcal{H}$ we define the *symmetric tensor product*

$$u_1 \circ \cdots \circ u_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \, ,$$

where $\mathcal{S}_n$ is the group of permutations of $\{1,2,\dots,n\}$, and the *antisymmetric tensor product*

$$u_1 \wedge \cdots \wedge u_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \varepsilon_\sigma \, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \,,$$

where ε_σ is the signature of the permutation σ.

The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the $u_1 \circ \cdots \circ u_n$ is denoted by $\mathcal{H}^{\circ n}$. It is called the n-fold *symmetric tensor product* of $\mathcal{H}$.

The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the $u_1 \wedge \cdots \wedge u_n$ is denoted by $\mathcal{H}^{\wedge n}$. It is called the n-fold *antisymmetric tensor product* of $\mathcal{H}$.

In the case $n = 0$ we put

$$\mathcal{H}^{\otimes 0} = \mathcal{H}^{\circ 0} = \mathcal{H}^{\wedge 0} = \mathbb{C} \,.$$

In physics the spaces $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$ are called the *n-particle spaces*.

Definition 8.2. The scalar product

$$\langle u_1 \wedge \cdots \wedge u_n \,,\, v_1 \wedge \cdots \wedge v_n \rangle = \frac{1}{(n!)^2} \sum_{\sigma, \tau \in S_n} \varepsilon_\sigma \, \varepsilon_\tau \, \langle u_{\sigma(1)}, v_{\tau(1)} \rangle \cdots \langle u_{\sigma(n)} \,,\, v_{\tau(n)} \rangle$$

can be easily seen to be equal to

$$\frac{1}{n!} \, \mathrm{Det}(\langle u_i, v_i \rangle)_{ij} \,.$$

In order to remove the $n!$ factor we put a scalar product on $\mathcal{H}^{\wedge n}$ which is different from the one induced by $\mathcal{H}^{\otimes n}$, namely:

$$\langle u_1 \wedge \cdots \wedge u_n \,,\, v_1 \wedge \cdots \wedge v_n \rangle_\wedge = \mathrm{Det}(\langle u_i \,,\, v_j \rangle)_{ij} \,. \tag{8.6}$$

This way, we have

$$\| u_1 \wedge \cdots \wedge u_n \|_\wedge^2 = n! \, \| u_1 \wedge \cdots \wedge u_n \|_\otimes^2 \,. \tag{8.7}$$

In the same way, on $\mathcal{H}^{\circ n}$ we put

$$\langle u_1 \circ \cdots \circ u_n \,,\, v_1 \circ \cdots \circ v_n \rangle_\circ = \mathrm{Per}(\langle u_i \,,\, v_j \rangle)_{ij} \,, \tag{8.8}$$

where Per denotes the *permanent* of the matrix (that is, the determinant without the minus signs). This way we get

$$\| u_1 \circ \cdots \circ u_n \|_\circ^2 = n! \, \| u_1 \circ \cdots \circ u_n \|_\otimes^2 \,.$$

8.2.2 Fock Spaces

Definition 8.3. We call *free* (or *full*) *Fock space* over $\mathcal{H}$ the space

$$\Gamma_f(\mathcal{H}) = \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\otimes n} .$$

We call *symmetric* (or *bosonic*) *Fock space* over $\mathcal{H}$ the space

$$\Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\circ n} .$$

We call *antisymmetric* (or *fermionic*) *Fock space* over $\mathcal{H}$ the space

$$\Gamma_a(\mathcal{H}) = \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\wedge n} .$$

The element $1 \in \mathbb{C}$ plays an important role when seen as an element of a Fock space. We denote it by Ω and call it the *vacuum vector*.

It is understood that in the definition of $\Gamma_f(\mathcal{H})$, $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ each of the spaces $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$ is equipped with its own scalar product $\langle \cdot, \cdot \rangle_\otimes$, $\langle \cdot, \cdot \rangle_\circ$ or $\langle \cdot, \cdot \rangle_\wedge$. In other words, the elements of $\Gamma_f(\mathcal{H})$ (*resp.* $\Gamma_s(\mathcal{H})$, $\Gamma_a(\mathcal{H})$) are those sequences $f = (f_n)$ such that $f_n \in \mathcal{H}^{\otimes n}$ (*resp.* $\mathcal{H}^{\circ n}$, $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} \|f_n\|_\varepsilon^2 < \infty$$

for $\varepsilon = \otimes$ (*resp.* $\circ$, $\wedge$).

If one wants to write everything in terms of the usual tensor norm, an element $f = (f_n)$ is in $\Gamma s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) if $f_n \in \mathcal{H}^{\circ n}$ (*resp.* $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} n! \|f_n\|_\otimes^2 < \infty .$$

The simplest case of a symmetric Fock space is obtained by taking $\mathcal{H} = \mathbb{C}$, this gives $\Gamma_s(\mathbb{C}) = \ell^2(\mathbb{N})$.

If $\mathcal{H}$ is of finite dimension n then $\mathcal{H}^{\wedge m} = 0$ for $m > n$ and thus $\Gamma_a(\mathcal{H})$ is of finite dimension 2^n. A symmetric Fock space $\Gamma_s(\mathcal{H})$ is never finite dimensional.

In physics, one usually considers bosonic or fermionic Fock spaces over $\mathcal{H} = L^2(\mathbb{R}^3)$. In Quantum Probability, the space $\Gamma_s(L^2(\mathbb{R}^+))$ is very important for quantum stochastic calculus.

8.2.3 Coherent Vectors

In this section we only consider symmetric Fock spaces $\Gamma_s(\mathcal{H})$.

Definition 8.4. Let $u \in \mathcal{H}$ be given. Note that $u \circ \cdots \circ u = u \otimes \cdots \otimes u$. The *coherent vector* (or *exponential vector*) associated to u is

$$\varepsilon(u) = \sum_{n \in \mathbb{N}} \frac{u^{\otimes n}}{n!} \tag{8.9}$$

so that

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u,v \rangle}$$

for the scalar product in $\Gamma_s(\mathcal{H})$. We denote by $\mathcal{E}$ the space of finite linear combinations of coherent vectors in $\Gamma_s(\mathcal{H})$.

Proposition 8.5.
1) The space $\mathcal{E}$ is dense in $\Gamma_s(\mathcal{H})$.
2) Every finite family of coherent vectors is linearly free.

Proof. We first prove the linear independence. Let $u_1, \ldots, u_n \in \mathcal{H}$ be fixed. Consider the sets

$$E_{i,j} = \left\{ u \in \mathcal{H}\, ;\ \langle u\,,\, u_i \rangle \neq \langle u\,,\, u_j \rangle \right\},$$

for $i \neq j$. They are dense open sets in $\mathcal{H}$. Hence, the set $\bigcap_{i,j} E_{i,j}$ is non empty. In particular, there exists a $v \in \mathcal{H}$ such that the quantities $\theta_j = \langle v\,,\, u_j \rangle$ are mutually distinct. The existence of scalars α_i such that $\sum_{i=1}^n \alpha_i\, \varepsilon(u_i) = 0$ would imply

$$0 = \Big\langle \varepsilon(zv)\,,\, \sum_{i=1}^n \alpha_i\, \varepsilon(u_i) \Big\rangle = \sum_{i=1}^n \alpha_i\, e^{z\theta_i}$$

for all $z \in \mathbb{C}$. The functions $z \mapsto e^{\theta_i z}$ are linearly independent and the α_i are thus all equal to 0. This proves that the family $\big\{\varepsilon(u_1), \ldots, \varepsilon(u_n)\big\}$ is free.

Let us now establish the density property. The identity

$$u_1 \circ \cdots \circ u_n = \frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n \left(\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n\right)^{\circ n}$$

shows that the set $\{u^{\circ n}\,;\ u \in \mathcal{H},\, n \in \mathbb{N}\}$ is total in $\Gamma_s(\mathcal{H})$. As a consequence, the identity

$$u^{\circ n} = \frac{d^n}{dt^n} \varepsilon(tu)\Big|_{t=0}$$

shows the density of the space $\mathcal{E}$. □

Corollary 8.6. *If $S \subset \mathcal{H}$ is a dense subset, then the space $\mathcal{E}(S)$ generated by the $\varepsilon(u)$, $u \in S$, is dense in $\Gamma_s(\mathcal{H})$.*

Proof. The equality

$$\|\varepsilon(u) - \varepsilon(v)\|^2 = e^{\|u\|^2} + e^{\|v\|^2} - 2\mathrm{Re}\left(e^{\langle u\,,\,v\rangle}\right)$$

shows that the mapping $u \mapsto \varepsilon(u)$ is continuous. We now conclude easily from Proposition 8.5. $\square$

There exist examples of subsets $S \subset \mathcal{H}$ which are not dense in $\mathcal{H}$ but for which $\mathcal{E}(S)$ is nevertheless dense in $\Gamma_s(\mathcal{H})$. A (non trivial) example, in the case $\mathcal{H} = L^2(\mathbb{R})$, is the set S of indicator functions of Borel sets (cf [PS98]). Whereas the set S' of indicator functions of intervals does not have this property ([AB02]). It is in general an open problem to characterize those $S \subset \mathcal{H}$ such that $\mathcal{E}(S)$ is dense in $\mathit{\Gamma}_s(\mathcal{H})$, even in the case $\mathcal{H} = \mathbb{C}$ (!)

One of the most important property of Fock spaces is their *exponential property.* For symmetric Fock spaces, this property is carried by the coherent vectors.

Theorem 8.7 (Exponential property). *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces. Then the mapping*

$$\mathsf{U}\varepsilon(u \oplus v) = \varepsilon(u) \otimes \varepsilon(v)$$

from $\Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2)$ to $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ extends to a unitary isomorphism.

Proof. We have

$$\begin{aligned}
\langle \varepsilon(u \oplus v)\,,\, \varepsilon(u' \oplus v')\rangle &= e^{\langle u\oplus v\,,\,u'\oplus v'\rangle} \\
&= e^{\langle u\,,\,u'\rangle + \langle v\,,\,v'\rangle} \\
&= e^{\langle u\,,\,u'\rangle} e^{\langle v\,,\,v'\rangle} \\
&= \langle \varepsilon(u)\,,\,\varepsilon(u')\rangle \, \langle \varepsilon(v)\,,\,\varepsilon(v')\rangle \\
&= \langle \varepsilon(u) \otimes \varepsilon(v)\,,\,\varepsilon(u') \otimes \varepsilon(v')\rangle\,.
\end{aligned}$$

This proves that the mapping U is isometric. As the space $\mathcal{E}(\mathcal{H}_i)$ is dense in $\Gamma_s(\mathcal{H}_i)$, $i = 1, 2$, and the set $\{\varepsilon(u) \otimes \varepsilon(v)\,;\ u \in \mathcal{H}_1,\, v \in \mathcal{H}_2\}$ is total in $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$, we get that U extends to a unitary operator. $\square$

This *exponential property* of Fock space justifies the fact Fock spaces are often considered as “exponentials of Hilbert spaces”.

There exists an interesting characterization of the space $\Gamma_s(\mathcal{H})$ which also goes in the direction of viewing $\Gamma_s(\mathcal{H})$ as “the exponential of $\mathcal{H}$”.

Theorem 8.8. *Let $\mathcal{H}$ be a separable Hilbert space. If $\mathcal{K}$ is a Hilbert space such that there exists a mapping*

$$\begin{aligned}
\lambda : \mathcal{H} &\longrightarrow \ \mathcal{K} \\
u &\longmapsto \lambda(u)
\end{aligned}$$

satisfying
i) $\langle \lambda(u)\,,\,\lambda(v)\rangle = e^{\langle u\,,\,v\rangle}$ *for all* $u, v \in \mathcal{H}$,
ii) $\{\lambda(u)\,;\; u \in \mathcal{H}\}$ *is total in* $\mathcal{K}$,
then there exists a unique unitary isomorphism

$$\begin{aligned} \mathsf{U}:\ \ \mathcal{K} \quad &\longrightarrow \Gamma_s(\mathcal{H}) \\ \lambda(u) &\longmapsto \ \varepsilon(u)\,. \end{aligned}$$

Proof. Clearly U, as defined above, is isometric and maps a dense subspace onto a dense subspace. Hence it extends to a unitary operator. □

Let us consider a simple example, that we shall follow throughout this lecture. This is the simplest example of a symmetric Fock space: the space $\Gamma_s(\mathbb{C})$. By definition this space is equal to $\oplus_{n\in\mathbb{N}}\mathbb{C}$ and thus can be naturally identified with $\ell^2(\mathbb{N})$. But it can be interpreted advantageously as $L^2(\mathbb{R})$ in the following way.

Let U be the mapping from $\Gamma_s(\mathbb{C})$ to $L^2(\mathbb{R}, \nu)$ which maps the coherent vectors $\varepsilon(z)$ of $\Gamma_s(\mathbb{C})$ to the functions

$$f_z(x) = \frac{1}{(2\pi)^{1/4}}\, e^{zx - z^2/2 - x^2/4}$$

of $L^2(\mathbb{R})$. It is easy to see that

$$\langle f_{z'}\,,\, f_z\rangle_{L^2(\mathbb{R})} = e^{\overline{z}' z}$$

and thus U extends to a unitary operator. We shall come back to this example later on and see that this unitary isomorphism gives a nice interpretation of the space $\Gamma_s(\mathbb{C})$.

8.2.4 Guichardet's Representation

We here make a little detour in order to describe the structure of the symmetric Fock space $\Gamma_s(\mathcal{H})$ when $\mathcal{H}$ is of the form $L^2(E, \mathcal{E}, m)$. We shall see that if $(E, \mathcal{E}, m)$ is a non atomic, σ-finite, separable measured space then $\Gamma_s(L^2(E, \mathcal{E}, m))$ can be written as $L^2(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$ for some explicit measured space $(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$.

Here is the main idea. If $\mathcal{H} = L^2(E, \mathcal{E}, m)$, then $\mathcal{H}^{\otimes n}$ is naturally interpreted as $L^2(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ and $\mathcal{H}^{\circ n}$ is interpreted as $L^2_{\mathrm{sym}}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ the space of symmetric, square integrable functions on E^n. If $f(x_1, \ldots, x_n)$ is a n-variable *symmetric* function on E then $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $\sigma \in S_n$. Thus, if the x_i are mutually distinct, we thus can see f as a function on the *set* $\{x_1, \dots, x_n\}$. But as m is non atomic, almost all the $(x_1, \dots, x_n) \in E^n$ satisfy $x_i \neq x_j$ for all $i \neq j$. An element of $\Gamma_s(\mathcal{H})$ is of the form $f = (f_n)$ where each f_n is a function on the n-element subsets of E. Thus f can be seen as a function on the finite subsets of E.

Definition 8.9. More rigorously, let $\mathcal{P}$ be the set of finite subsets of E. Then $\mathcal{P} = \cup_{n \in \mathbb{N}} \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n-element subsets of E. Let $f_n \in L^2_{\mathrm{sym}}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ and define f on $\mathcal{P}$ by

$$\begin{cases} f(\sigma) = 0 & \text{if } \ \sigma \in \mathcal{P} \ \text{ and } \ |\sigma| \neq n\,, \\ f(\{x_1, \dots, x_n\}) = f_n(x_1, \dots, x_n)\,, & \text{otherwise}\,. \end{cases}$$

Let $\mathcal{E}_{\mathcal{P}}$ be the smallest σ-field on $\mathcal{P}$ which makes all these functions measurable. Let $\Delta_n \subset E^n$ be the set of $(x_1, \dots, x_n)$ such that $x_i \neq x_j$ for all $i \neq j$. By the non-atomicity of m, we have $m(E^n \setminus \Delta^n) = 0$. For $F \in \mathcal{E}_{\mathcal{P}}$ we put

$$\mu(F) = \mathbb{1}_F(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Delta_n} \mathbb{1}_{F \cap \mathcal{P}_n}(x_1, \dots, x_n)\, dm(x_1) \cdots dm(x_n)\,.$$

For example, if $E = \mathbb{R}$ with the Lebesgue structure, then $\mathcal{P}_n$ can be identified with the increasing simplex $\Sigma_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n\,;\ x_1 < \cdots < x_n\}$. Thus $\mathcal{P}_n$ inherits the Lebesgue measure from $\mathbb{R}^n$.

Coming back to the general setup, the measure μ that we have defined is σ-finite and possesses only one atom: $\mu(\{\emptyset\}) = 1$. We call $(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ the *symmetric measure space* over $(E, \mathcal{E}, m)$.

For all $u \in L^2(E, \mathcal{E}, m)$ one defines by π_u the element of $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ which satisfies

$$\pi_u(\sigma) = \begin{cases} 1 & \text{if } \ \sigma = \emptyset \\ \prod_{s \in \sigma} u(s) & \text{otherwise} \end{cases}$$

for all $\sigma \in \mathcal{P}$.

Theorem 8.10. *The mapping $\pi_u \longmapsto \varepsilon(u)$ extends to a unitary isomorphism from $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ onto $\Gamma_s(L^2(E, \mathcal{E}, m))$.*

Proof. Clearly $\langle \pi_u\,,\, \pi_v \rangle = e^{\langle u\,,\, v \rangle}$ and the set of functions π_u is total in $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$. We conclude by Theorem 8.8. $\square$

8.3 Basic Operators

We now come back to general symmetric and antisymmetric Fock spaces $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$.

8.3.1 Creation and Annihilation Operators

Definition 8.11. For any $u \in \mathcal{H}$ we define the following operators:
– the *bosonic creation operator* $a^*(u)$ from $\mathcal{H}^{\circ n}$ to $\mathcal{H}^{\circ(n+1)}$ defined by

$$a^*(u)(u_1 \circ \cdots \circ u_n) = u \circ u_1 \circ \cdots \circ u_n\,;$$

– the *fermionic creation operator* $b^*(u)$ from $\mathcal{H}^{\wedge n}$ to $\mathcal{H}^{\wedge(n+1)}$ defined by

$$b^*(u)(u_1 \wedge \cdots \wedge u_n) = u \wedge u_1 \wedge \cdots \wedge u_n\,;$$

– the *bosonic annihilation operator* $a(u)$ from $\mathcal{H}^{\circ n}$ to $\mathcal{H}^{\circ(n-1)}$ defined by

$$a(u)(u_1 \circ \cdots \circ u_n) = \sum_{i=1}^{n} \langle u, u_i\rangle\, u_1 \circ \cdots \circ \widehat{u}_i \circ \cdots \circ u_n\,;$$

– the *fermionic annihilation operator* $b(u)$ from $\mathcal{H}^{\wedge n}$ to $\mathcal{H}^{\wedge(n-1)}$ defined by

$$b(u)(u_1 \wedge \cdots \wedge u_n) = \sum_{i=1}^{n} (-1)^i \langle u, u_i\rangle\, u_1 \wedge \cdots \wedge \widehat{u}_i \wedge \cdots \wedge u_n\,.$$

Note that $a^*(u)$ and $b^*(u)$ depend linearly on u, whereas $a(u)$ and $b(u)$ depend antilinearly on u. Actually, one often finds in the literature notations with "bras" and "kets": $a^*_{|u\rangle}$, $b^*_{|u\rangle}$, $a_{\langle u|}$, $b_{\langle u|}$.

Also note the following relations with respect to the vacuum vector:

$$\begin{aligned} a^*(u)\Omega &= b^*(u)\Omega = u \\ a(u)\Omega &= b(u)\Omega = 0\,. \end{aligned}$$

Definition 8.12. All the above operators extend to the space $\Gamma_s^f(\mathcal{H})$ (*resp.* $\Gamma_a^f(\mathcal{H})$) of finite sums of particle spaces, that is, the space of those $f = \sum_{n\in\mathbb{N}} f_n \in \Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) such that only a finite number of f_n do not vanish. In physics this space is often called the *finite particle space*. This subspace is dense in the corresponding Fock space. It is included in the domain of the operators $a^*(u)$, $b^*(u)$, $a(u)$ and $b(u)$ (defined as operators on $\Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$)), and it is stable under their action. On this subspace we have the following relations:

$$\begin{aligned}\langle a^*(u)f\,,\,f\rangle &= \langle f\,,\,a(u)g\rangle\\ [a(u)\,,\,a(v)] &= [a^*(u)\,,\,a^*(v)] = 0\\ [a(u)\,,\,a^*(v)] &= \langle u\,,\,v\rangle I\\ \langle b^*(u)f\,,\,f\rangle &= \langle f\,,\,b(u)g\rangle\\ \{b(u)\,,\,b(v)\} &= \{b^*(u)\,,\,b^*(v)\} = 0\\ \{b(u)\,,\,b^*(v)\} &= \langle u\,,\,v\rangle I\,.\end{aligned}$$

In other words, when restricted to $\Gamma^f_s(\mathcal{H})$ (*resp.* $\Gamma^f_a(\mathcal{H})$) the operators $a(u)$ and $a^*(u)$ (*resp.* $b(u)$ and $b^*(u)$) are mutually adjoint and they satisfy the *C.C.R.* (*resp. C.A.R.*).

Proposition 8.13. *For all $u \in \mathcal{H}$ we have*
1) $b(u)^2 = b^*(u)^2 = 0\,,$
2) $\|b(u)\| = \|b^*(u)\| = \|u\|\,.$

Proof. The anticommutation relation $\{b(u), b(u)\} = 0$ means $2b(u)b(u) = 0$ on $\Gamma^f_a(\mathcal{H})$, this gives 1). Furthermore we have, on $\Gamma^f_a(\mathcal{H})$

$$\begin{aligned}b^*(u)b(u)b^*(u)b(u) &= b^*(u)\{b(u), b^*(u)\}b(u)\\ &= \|u\|^2 b^*(u)b(u)\,.\end{aligned}$$

In particular, for all $f \in \Gamma^f_a(\mathcal{H})$ we have

$$\begin{aligned}\|b^*(u)b(u)f\|^2 &= \langle f\,,\,b^*(u)b(u)b^*(u)b(u)f\rangle = \|u\|^2\ \langle f\,,\,b^*(u)b(u)f\rangle\\ &\le \|u\|^2\ \|f\|\ \|b^*(u)b(u)f\|\ .\end{aligned}$$

In particular $b^*(u)b(u)$ is bounded and so is $b(u)$. The same identity above also implies that

$$\begin{aligned}\|b(u)\|^4 = \|b^*(u)b(u)b^*(u)b(u)\| &= \|u\|^2\|b^*(u)b(u)\|\\ &= \|u\|^2\|b(u)\|^2\,.\end{aligned}$$

As the operator $b(u)$ is null if and only if $u = 0$ we easily deduce that $\|b(u)\| = \|u\|$. □

The identity 1), for $b^*(u)$ expresses the so-called *Pauli exclusion principle*: "One cannot have two fermionic particles together in the same state".

The bosonic case is less simple for the operators $a^*(u)$ and $a(u)$ are never bounded. Indeed, we have $a(u)v^{\circ n} = n\langle u, v\rangle v^{\circ(n-1)}$, thus the coherent vectors are in the domain of $a(u)$ and

$$a(u)\varepsilon(v) = \langle u, v\rangle\varepsilon(v)\,. \tag{8.10}$$

In particular

$$\begin{aligned}\sup_{\|h\|=1} \|a(u)h\| &\geq \sup_{v\in\mathcal{H}} \|a(u)e^{-\|v\|^2/2}\varepsilon(v)\| \\ &= \sup_{v\in\mathcal{H}} |\langle u, v\rangle| = +\infty\,.\end{aligned}$$

Thus $a(u)$ is not bounded.

The action of $a^*(u)$ can also be made explicit. Indeed, we have

$$a^*(u)v^{\circ n} = u \circ v \circ \cdots \circ v = \frac{d}{d\varepsilon}(u+\varepsilon v)^{\circ n}_{\big|\varepsilon=0}\,.$$

Thus $\varepsilon(v)$ is in the domain of $a^*(u)$ and

$$a^*(u)\varepsilon(v) = \frac{d}{d\varepsilon}\varepsilon(u+\varepsilon v)_{\big|\varepsilon=0}\,. \tag{8.11}$$

The operators $a(u)$ and $a^*(u)$ are in particular closable for they have a densely defined adjoint. We extend them by closure, while keeping the same notations $a(u)$, $a^*(u)$ for their closure.

Proposition 8.14. *The operator $a^*(u)$ is the adjoint of $a(u)$.*

Proof. On $\Gamma_s^f(\mathcal{H})$ we have $\langle f\,,\, a(u)g\rangle = \langle a^*(u)f\,,\, g\rangle$. We extend this relation to $f \in \mathrm{Dom}\, a^*(u)$. The mapping $g \mapsto \langle f\,,\, a(u)g\rangle$ is thus continuous and $f \in \mathrm{Dom}\, a(u)^*$. We have proved that $a^*(u) \subset a(u)^*$.

Conversely, if $f \in \mathrm{Dom}\, a(u)^*$ and if $h = a(u)^* f$, we decompose f and h as $f = \sum_n f_n$ and $h = \sum_n h_n$. We have $\langle f\,,\, a(u)g\rangle = \langle h\,,\, g\rangle$ for all $g \in \Gamma_s^f(\mathcal{H})$. Thus, taking $g \in \mathcal{H}^{\circ n}$ we get $\langle f_{n-1}\,,\, a(u)g\rangle = \langle h_n\,,\, g\rangle$, that is, $\langle a^*(u)f_{n-1}\,,\, g\rangle = \langle h_n\,,\, g\rangle$. This shows that $h_n = a^*(u)f_{n-1}$. In particular $\sum_n \|a^*(u)f_n\|^2$ is finite, f belongs to $\mathrm{Dom}\, a^*(u)$ and $a^*(u)f = a(u)^* f$. $\square$

8.3.2 Examples

In physics, the space $\mathcal{H}$ is often $L^2(\mathbb{R}^3)$. An element h_n of $\mathcal{H}^{\circ n}$ is thus a symmetric function of n variables on $\mathbb{R}^3$. With our definitions we have

$$\big(a(f)h_n\big)(x_1, \ldots, x_{n-1}) = \int h_n(x_1, \ldots, x_{n-1}, x)\bar{f}(x)\, dx$$

and

$$\big(a^*(f)h_n\big)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} h_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})f(x_i)\,.$$

But in the physic literature one often uses creation and annihilation operators indexed by the points of $\mathbb{R}^3$, instead of the elements of $L^2(\mathbb{R}^3)$. One can find

$a(x)$ and $a^*(x)$ formally defined by

$$a(f) = \int \bar{f}(x) a(x)\, dx$$
$$a^*(f) = \int f(x) a^*(x)\, dx$$

with

$$\big(a(x)h_n\big)(x_1, \ldots, x_{n-1}) = h_n(x_1, \ldots, x_{n-1}, x)$$
$$\big(a^*(x)h_n\big)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} \delta(x - x_i) h_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})\,.$$

If we come back to our example $\Gamma_s(\mathbb{C}) \simeq L^2(\mathbb{R}, \nu)$, we have the creation and annihilation operators $a^*(z)$, $a(z)$, $z \in \mathbb{C}$. They are actually determined by two operators $a^* = a^*(1)$ and $a = a(1)$. They operate on coherent vectors by

$$a\varepsilon(z) = z\varepsilon(z), \quad a^*\varepsilon(z) = \frac{d}{dt}\varepsilon(z+t)_{|t=0}\,,$$

as can be easily checked. On $L^2(\mathbb{R}, \nu)$ this gives

$$af_z(x) = zf_z(x) = \left(\frac{x}{2} + \frac{d}{dx}\right) f_z(x)$$
$$a^* f_z(x) = \frac{d}{dt} f_{z+t}(x)_{|t=0} = (x - z) f_z(x) = \left(\frac{x}{2} - \frac{d}{dx}\right) f_z(x)\,.$$

The operators $\mathsf{Q} = a + a^*$ and $\mathsf{P} = i(a - a^*)$ are thus respectively represented by the operator x and $2i\frac{d}{dx}$ on $L^2(\mathbb{R})$, that is, the Schrödinger representation of the *C.C.R.* (with $\hbar = 2$!).

The operator $\mathsf{Q} = a + a^*$ is self-adjoint, so it is an observable. Let us compute its law in the *vacuum state* $|\Omega\rangle\langle\Omega|$. The n-th moment of this law is given by

$$\begin{aligned}\langle \Omega, \mathsf{Q}^n \Omega \rangle &= \langle f_0, x^n f_0 \rangle \\ &= \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2}\, dx\,.\end{aligned}$$

Thus the law of Q under the state $|\Omega\rangle\langle\Omega|$ is the centered gaussian law $\mathcal{N}(0, 1)$.

8.3.3 Second Quantization

Definition 8.15. If one is given an operator A from an Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$, it is possible to extend it naturally to an operator $\Gamma(\mathsf{A})$ from $\Gamma_s(\mathcal{H})$ to $\Gamma_s(K)$ (and in a similar way from $\Gamma_a(\mathcal{H})$ to $\Gamma_a(K)$) by putting

$$\Gamma(\mathsf{A})(u_1 \circ \cdots \circ u_n) = \mathsf{A}u_1 \circ \cdots \circ \mathsf{A}u_n \,. \tag{8.12}$$

One sees easily that

$$\mathit{\Gamma}(\mathsf{A})\,\varepsilon(u) = \varepsilon(\mathsf{A}u)\,. \tag{8.13}$$

This operator $\Gamma(\mathsf{A})$ is called the *second quantization* of A.

Definition 8.16. One must be careful that even if A is a bounded operator, $\Gamma(\mathsf{A})$ is not bounded in general. Indeed, if $\|\mathsf{A}\| > 1$ then $\Gamma(\mathsf{A})$ is not bounded. But one easily sees that

$$\Gamma(\mathsf{AB}) = \Gamma(\mathsf{A})\Gamma(\mathsf{B})$$

and

$$\Gamma(\mathsf{A}^*) = \Gamma(\mathsf{A})^*\,.$$

In particular if A is unitary, then so is $\Gamma(\mathsf{A})$. Even more, if $(\mathsf{U}_t)_{t\in\mathbb{R}}$ is a strongly continuous one parameter group of unitary operators then so is $(\Gamma(\mathsf{U}_t)_{t\in\mathbb{R}})$. In other words, by Stone's Theorem, if $\mathsf{U}_t = \mathrm{e}^{\mathrm{i}t\mathsf{H}}$ for some self-adjoint operator H, then $\Gamma(\mathsf{U}_t) = \mathrm{e}^{\mathrm{i}t\mathsf{H}'}$ for some self-adjoint operator H'. The operator H' is denoted by $\Lambda(\mathsf{H})$ or $d\Gamma(\mathsf{H})$ and is called the *differential second quantization of* H.

One can easily check that

$$\Lambda(\mathsf{H})\,u_1 \circ \cdots \circ u_n = \sum_{i=1}^n u_1 \circ \cdots \circ \mathsf{H}u_i \circ \cdots \circ u_n \tag{8.14}$$

and $\Lambda(\mathsf{H})\Omega = 0$.

In particular, if $\mathsf{H} = \mathrm{I}$ we have

$$\Lambda(\mathrm{I})\,u_1 \circ \cdots \circ u_n = n\,u_1 \circ \cdots \circ u_n\,.$$

This operator is called the *number operator*.

Proposition 8.17. *We have, for all* $u \in \mathcal{H}$

$$\Lambda(\mathsf{H})\,\varepsilon(u) = a^*(\mathsf{H}u)\,\varepsilon(u)\,.$$

Proof. We have $\Lambda(\mathsf{H})\,u^{\circ n} = n(\mathsf{H}u) \circ u \circ \cdots \circ u$, so that

$$\Lambda(\mathsf{H})\,\frac{u^{\circ n}}{n!} = (\mathsf{H}u) \circ \frac{u^{\circ(n-1)}}{(n-1)!} = a^*(\mathsf{H}u)\frac{u^{\circ(n-1)}}{(n-1)!}\,. \qquad \square$$

Proposition 8.18. *For all $u \in \mathcal{H}$, we have*

$$\Lambda(|u\rangle\langle u|) = a^*_{|u\rangle} a_{\langle u|}\,.$$

Proof. Indeed, we have

$$\begin{aligned}\Lambda(|u\rangle\langle u|)\,\varepsilon(v) &= a^*(\langle u\,,\, v\rangle\, u)\,\varepsilon(v)\\ &= \langle u\,,\, v\rangle a^*(u)\,\varepsilon(v)\\ &= a^*_{|u\rangle} a_{\langle u|}\,\varepsilon(v)\,.\end{aligned}$$

□

Coming back to our example $L^2(\mathbb{R},\nu)$, there is only one differential second quantization operator (up to a scalar factor):

$$\Lambda(\mathrm{I}) = \Lambda = a^* a\,.$$

With our identification, this operator is given by

$$\begin{aligned}\Lambda &= \left(\frac{x}{2} - \frac{d}{dx}\right)\left(\frac{x}{2} + \frac{d}{dx}\right)\\ &= \frac{x^2}{4} - \frac{d^2}{dx^2} - \frac{1}{2}\end{aligned}$$

that is

$$\Lambda + \frac{1}{2} = \frac{x^2}{4} - \frac{d^2}{dx^2}\,. \tag{8.15}$$

In quantum physics this exactly corresponds to the Hamiltonian of the one dimensional harmonic oscillator.

Note that Λ is self-adjoint and that its law in the vacuum state is just the Dirac mass at 0, for $\Lambda\Omega = 0$.

8.3.4 Weyl Operators

Definition 8.19. Let $\mathcal{H}$ be a Hilbert space. Let G be the Euclidian group of $\mathcal{H}$ that is,

$$G = \left\{(\mathsf{U}, u)\,;\ \mathsf{U} \in \mathcal{U}(\mathcal{H}),\, u \in \mathcal{H}\right\},$$

where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$. This group acts on $\mathcal{H}$ by

$$(\mathsf{U}, u)\, h = \mathsf{U}h + u\,.$$

The composition law in G is thus given by

$$(\mathsf{U}, u)(\mathsf{V}, v) = (\mathsf{U}\mathsf{V}, \mathsf{U}v + u)$$

and in particular the inverse is given by

$$(\mathsf{U}, u)^{-1} = (\mathsf{U}^*, -\mathsf{U}^* u)\,.$$

Definition 8.20. For every $\alpha = (\mathsf{U}, u) \in G$ one defines the *Weyl operator* W_α on $\Gamma_s(\mathcal{H})$ by

$$\mathsf{W}_\alpha\, \varepsilon(v) = e^{-\|u\|^2/2 - \langle u, \mathsf{U}v\rangle} \varepsilon(\mathsf{U}v + u)\,. \tag{8.16}$$

In particular

$$\mathsf{W}_\alpha \mathsf{W}_\beta = e^{-i\,\mathrm{Im}\,\langle u, \mathsf{U}v\rangle}\, \mathsf{W}_{\alpha\beta} \tag{8.17}$$

for all $\alpha = (\mathsf{U}, u)$, $\beta = (\mathsf{V}, v)$ in G. This relation is called the *Weyl commutation relation*.

Proposition 8.21. *The Weyl operators* W_α *are unitary.*

Proof. We have

$$\begin{aligned}\langle \mathsf{W}_\alpha\, \varepsilon(k)\,,\, \mathsf{W}_\alpha\, \varepsilon(\ell)\rangle &= e^{-\|u\|^2 - \langle \mathsf{U}k\,,\, u\rangle - \langle u\,,\, \mathsf{U}\ell\rangle} \langle \varepsilon(\mathsf{U}k + u)\,,\, \varepsilon(\mathsf{U}\ell + u)\rangle \\ &= e^{-\|u\|^2 - \langle \mathsf{U}k, u\rangle - \langle u, \mathsf{U}_\ell\rangle} e^{\langle \mathsf{U}k + u, \mathsf{U}\ell + u\rangle} \\ &= e^{\langle \mathsf{U}k\,,\, \mathsf{U}\ell\rangle} = e^{\langle k\,,\, \ell\rangle} = \langle \varepsilon(k)\,,\, \varepsilon(\ell)\rangle\,.\end{aligned}$$

Thus W_α extends to an isometry. But we furthermore have

$$\begin{aligned}\mathsf{W}_\alpha \mathsf{W}_{\alpha^{-1}} &= e^{-i\,\mathrm{Im}\,\langle u\,,\, -\mathsf{U}\mathsf{U}^* u\rangle} \mathsf{W}_{\alpha\alpha^{-1}} \\ &= e^{-i\,\mathrm{Im}\,(-\|u\|^2)} \mathsf{W}_{(\mathrm{I},0)} \\ &= \mathrm{I}\,.\end{aligned}$$

This proves that W_α is invertible and thus unitary. □

Definition 8.22. The relation (8.17) shows that the mapping:

$$\begin{aligned} G &\longrightarrow \mathcal{U}(\Gamma(\mathcal{H})) \\ \alpha &\longmapsto \quad \mathsf{W}_\alpha \end{aligned}$$

is a unitary projective representation of G. If we consider the group $\widetilde{G}$ made from the set $\{(\mathsf{U}, u, t),\ \mathsf{U} \in \mathcal{U}(\mathcal{H}),\ u \in \mathcal{H} \text{ and } t \in \mathbb{R}\}$ with composition law

$$(\mathsf{U}, u, t)(\mathsf{V}, v, s) = (\mathsf{U}\mathsf{V}, \mathsf{U}v + u, t + s + \mathrm{Im}\,\langle u\,,\, \mathsf{U}v\rangle)$$

we obtain the so-called *Heisenberg group* of $\mathcal{H}$ and the mapping $(\mathsf{U}, u, t) \longmapsto \mathrm{e}^{\mathrm{i}t}\, W_{(\mathsf{U},u)}$ is a unitary representation of $\widetilde{G}$.

Conversely, if $\mathsf{W}_{(\mathsf{U},u,t)}$ is a unitary representation of the Heisenberg group of $\mathcal{H}$ we then have

$$\mathsf{W}_{(\mathsf{U},u,t)} = \mathsf{W}_{(\mathsf{U},u,0)} \mathsf{W}_{(\mathrm{I},0,t)}$$

and

$$\mathsf{W}_{(\mathrm{I},0,t)} = \mathsf{W}_{(\mathrm{I},0,s)}\mathsf{W}_{(\mathrm{I},0,t+s)}\,.$$

This means that

$$\mathsf{W}_{(\mathsf{U},u,t)} = \mathsf{W}_{(\mathsf{U},u,0)}\mathrm{e}^{\mathrm{i}t\mathsf{H}}$$

for some self-adjoint operator H. Hence the operators $\mathsf{W}_{(\mathsf{U},u,0)}$ satisfy the Weyl commutation relations.

If we come back to our Weyl operators $\mathsf{W}_{(\mathsf{U},u)}$ one easily sees that

$$\mathsf{W}_{(\mathsf{U},u)} = \mathsf{W}_{(\mathrm{I},u)}\mathsf{W}_{(\mathsf{U},0)}\,.$$

By definition $\mathsf{W}_{(\mathsf{U},0)}\,\varepsilon(k) = \varepsilon(\mathsf{U}k)$ and thus

$$\mathsf{W}_{(\mathsf{U},0)} = \Lambda(\mathsf{U})\,.$$

Finally, writing W_u for $\mathsf{W}_{(\mathrm{I},u)}$, we get

$$\mathsf{W}_u\mathsf{W}_v = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,\langle u,v\rangle}\mathsf{W}_{u+v}\,. \tag{8.18}$$

These relations are also called *Weyl commutation relations*. As a consequence $(\mathsf{W}_{(\mathrm{I}+tu)})_{t\in\mathbb{R}}$ is a unitary group; it can be easily shown to be strongly continuous.

Proposition 8.23. *We have*

$$\mathsf{W}_{tu} = \mathrm{e}^{\mathrm{i}t\,\frac{1}{i}(a^*(u)-a(u))}\,.$$

Proof. By definition we have

$$\begin{aligned}
\frac{1}{i}\,\frac{d}{dt}\Big|_{t=0}\mathsf{W}_{(\mathrm{I},tu)}\,\varepsilon(k) &= \frac{1}{i}\,\frac{d}{dt}\Big|_{t=0} e^{-\frac{t^2}{2}\|u\|^2 - t\langle u\,,\,k\rangle}\,\varepsilon(k+tu)\\
&= -\frac{1}{i}\langle u\,,\,k\rangle\,\varepsilon(k) + \frac{1}{i}\,\frac{d}{dt}\Big|_{t=0}\,\varepsilon(k+tu)\\
&= \frac{1}{i}(-a(u)+a^*(u))\,\varepsilon(k)\,. \qquad \square
\end{aligned}$$

Coming back to our example on $L^2(\mathbb{R})$, the Weyl operators are defined by

$$\mathsf{W}_z\,\varepsilon(z') = \mathrm{e}^{-\frac{|z|^2}{2}-\bar{z}z'}\varepsilon(z+z')\,.$$

We shall use them to compute the laws of some basic observables.

Proposition 8.24. *The observable $\frac{1}{i}(za^*-\bar{z}a)$ obeys the law $\mathcal{N}(0,|z|^2)$ in the vacuum state. In particular, the observable $za^*+\bar{z}a$ obeys the law $\mathcal{N}(0,|z|^2)$ in the vacuum state.*

Proof. We have

$$\begin{aligned}\langle \Omega\,,\, e^{t(za^*-\bar z a)}\Omega\rangle &= \langle \Omega\,,\, \mathsf{W}_{tz}\Omega\rangle\\ &= \langle \varepsilon(0)\,,\, \mathsf{W}_{tz}\,\varepsilon(0)\rangle\\ &= \langle \varepsilon(0)\,,\, \varepsilon(tz)\rangle\, e^{-t^2|z|^2/2}\\ &= e^{-t^2|z|^2/2}\,. \end{aligned}$$

□

Proposition 8.25. *The observable $\Lambda+\alpha I$ obeys the law δ_α in the vacuum state.*

Proof. Indeed, we have

$$(\Lambda+\alpha \mathrm{I})\Omega=\alpha\Omega\,.$$

□

Lemma 8.26. *We have*

$$\mathsf{W}_{-z}\, \mathrm{e}^{\mathrm{i}t\Lambda}\, \mathsf{W}_z=\mathrm{e}^{\mathrm{i}t(\Lambda+za^*+\bar z a+|z|^2\mathrm{I})}\,.$$

Proof. It is sufficient to prove that $\mathsf{W}_{-z}\Lambda\mathsf{W}_z=\Lambda+za^*+\bar z a+|z|^2\mathrm{I}$. We have

$$\begin{aligned}\langle \varepsilon(z_1)\,,\, \mathsf{W}_{-z}\,\Lambda\mathsf{W}_z\,\varepsilon(z_2)\rangle &= \langle a\mathsf{W}_z\,\varepsilon(z_1)\,,\, a\mathsf{W}_z\,\varepsilon(z_2)\rangle\\ &= \mathrm{e}^{-|z|^2-\bar z z_2-\bar z_1 z}\langle (z_1+z)\varepsilon(z_1+z)\,,\,(z_2+z)\varepsilon(z_2+z)\rangle\\ &= (\bar z_1 z_2+\bar z_1 z+z_2\bar z+|z|^2)\mathrm{e}^{\bar z_1 z_2}\,.\end{aligned}$$

We immediatly recognize that this quantity is equal to

$$\left\langle \varepsilon(z_1)\,,\,(\Lambda+za^*+\bar z a+|z|^2\mathrm{I})\varepsilon(z_2)\right\rangle\,.$$

□

Proposition 8.27. *The observable $\Lambda+za^*+\bar z a+|z|^2\mathrm{I}$ obeys the Poisson law $\mathcal{P}(|z|^2)$ in the vacuum state.*

Proof. We have

$$\begin{aligned}\langle \Omega\,,\, \mathrm{e}^{\mathrm{i}t(\Lambda+za^*-\bar z a+|z|^2\mathrm{I})}\Omega\rangle &= \langle \mathsf{W}_z\,\Omega\,,\, \mathrm{e}^{\mathrm{i}t\Lambda}\mathsf{W}_z\,\Omega\rangle\\ &= \mathrm{e}^{-|z|^2}\langle \varepsilon(z)\,,\, \mathrm{e}^{\mathrm{i}t\Lambda}\,\varepsilon(z)\rangle\\ &= \mathrm{e}^{-|z|^2}\mathrm{e}^{|z|^2\mathrm{e}^{\mathrm{i}t}}\\ &= \mathrm{e}^{|z|^2(\mathrm{e}^{\mathrm{i}t}-1)}\end{aligned}$$

which is the characteristic function of the law $\mathcal{P}(|z|^2)$. □

This result is one of the most remarkable of quantum probability theory. We have two observables $za^*+\bar z a$ and $\Lambda+|z|^2\mathrm{I}$. In the vacuum state, the first one is gaussian $\mathcal{N}(0,|z|^2)$, and the second is deterministic, always equal to $|z|^2$. The sum of this gaussian observable and this deterministic one gives ... an observable which is Poisson! Of course these two observables do not commute and we had no reason to obtain the law of their convolution. But one must admit that the result here is really surprising!

Notes

There are plenty references introducing to the notion of Fock spaces, to their use in quantum physics, ... From a reference to another the definition are slightly different. In this chapter we mainly followed the presentation which is commonly shared in quantum probability theory, such as in the references [Par92], [Mey93], [Bia95]. The book of Meyer contains an original proof of the Stone-von Neumann theorem.

In the second volume of Bratelli and Robinson's books ([BR97]), the Fock spaces are put into perspective with applications in quantum statistical mechanics (and these two volumes are a reference in this domain). All the main theorems are there, including Stone-von Neumann, analogous characterizations of the C.A.R., etc.

The Guichardet interpretation of the Fock spaces is due to ... Guichardet of course. It was first developed in his book [Gui72], which has nothing to do with Quantum Probability. His approach on continuous tensor products of Hilbert spaces has become famous and very useful in Quantum Probability. In particular with the help of the article of H. Maassen on kernel operators on Fock space ([Maa85]).

References

[AB02] Stéphane Attal and Alain Bernard. Orthogonal spaces associated with exponentials of indicator functions on Fock space. *J. London Math. Soc. (2)*, 66(2):487–498, 2002.

[Bia95] Philippe Biane. Calcul stochastique non-commutatif. In *Lectures on probability theory (Saint-Flour, 1993)*, volume 1608 of *Lecture Notes in Math.*, pages 1–96. Springer, Berlin, 1995.

[BR97] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.

[Gui72] Alain Guichardet. *Symmetric Hilbert spaces and related topics.* Lecture Notes in Mathematics, Vol. 261. Springer-Verlag, Berlin, 1972. Infinitely divisible positive definite functions. Continuous products and tensor products. Gaussian and Poissonian stochastic processes.

[Maa85] Hans Maassen. Quantum Markov processes on Fock space described by integral kernels. In *Quantum probability and applications, II (Heidelberg, 1984)*, volume 1136 of *Lecture Notes in Math.*, pages 361–374. Springer, Berlin, 1985.

[Mey93] Paul-André Meyer. *Quantum probability for probabilists*, volume 1538 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1993.

[Par92] K. R. Parthasarathy. *An introduction to quantum stochastic calculus*, volume 85 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 1992.

[PS98] K. R. Parthasarathy and V. S. Sunder. Exponentials of indicator functions are total in the boson Fock space $\Gamma(L^2[0,1])$. In *Quantum probability communications*, QP-PQ, X, pages 281–284. World Sci. Publ., River Edge, NJ, 1998.

[RS80] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis.