ORTHOGONAL SPACES ASSOCIATED TO EXponentialS OF indicator FUNCTIONS ON Fock SPACE

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Abstract. — Parthasarathy and Sunder have proved in [P-S] that the set of coherent vectors associated to the indicator function of Borel sets is total in the boson Fock space \( \Gamma_1 L^2(\mathbb{R}^+,\mathbb{C}) \). In this article we study the space generated by coherent vectors associated to the union of \( n \) intervals. We give a complete characterization of their orthogonal space in terms of their chaos expansion. By the way, we recover Parthasarathy-Sunder’s result in a very simple way. In the cases of the Brownian motion or Poisson process interpretation of the Fock space, our result characterizes those random variables which are orthogonal to the exponential of any sum of \( n \) increments of the Brownian motion or Poisson process.

I – The Fock space and main results

Let \( \mathcal{P} \) be the set of finite subsets of \( \mathbb{R}^+ \). Then \( \mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n \) where \( \mathcal{P}_n \) is the set of \( n \)-element subsets of \( \mathbb{R}^+ \) (with \( \mathcal{P}_0 = \{ \emptyset \} \)). The set \( \mathcal{P}_n \) can be clearly identified to the simplex \( \Sigma_n = \{ 0 < t_1 < \cdots < t_n \in \mathbb{R} \} \) and thus inherits the Lebesgue measure structure of \( \mathbb{R}^n \). By putting the Dirac mass \( \delta_\emptyset \) on \( \mathcal{P}_0 \), we have finally equipped \( \mathcal{P} \) with a measured space structure. Elements of \( \mathcal{P} \) are denoted by small greek letters \( \sigma, \alpha, \beta, \cdots \). Any element \( f \) of \( L^2(\mathcal{P}; \mathbb{C}) \) is thus of the form

\[
 f = \sum_{n \in \mathbb{N}} f_n
\]

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with \( f_0 \in \mathbb{C} \) and \( f_n \in L^2(\mathbb{S}_n; \mathbb{C}), \ n \geq 0 \) and

\[
\|f\|^2_{L^2(P)} = |f_0|^2 + \sum_{n=1}^{\infty} \|f_n\|^2_{L^2(\mathbb{S}_n)} = |f_0|^2 + \sum_{n=1}^{\infty} \int_{0\leq t_1 < \cdots < t_n} |f_n(t_1, \ldots, t_n)|^2 \, dt_1 \cdots dt_n.
\]

The above expression is simply denoted, with obvious notations,

\[
\|f\|^2_{L^2(P)} = \int_P |f(\sigma)|^2 \, d\sigma.
\]

The space \( L^2(P) \) is denoted \( \Phi \) and called the Fock space (it is the usual symmetric, or boson, Fock space over \( L^2(\mathbb{R}^+) \)).

The following lemma is very helpful (cf. [L-P]).

**Lemma.** — If \( f \) is a positive (resp. integrable) measurable function on \( P \times P \), then the function

\[
g(\sigma) = \sum_{\alpha \subseteq \sigma} f(\alpha, \sigma \setminus \alpha)
\]

is positive (resp. integrable) measurable on \( P \) and we have

\[
\int_P \int_P f(\alpha, \beta) \, d\alpha d\beta = \int_P g(\sigma) \, d\sigma.
\]

Particular elements of \( \Phi \) will be of interest for us: the coherent (or exponential) vectors: for any \( u \in L^2(\mathbb{R}^+) \) define \( \varepsilon(u) \) to be the element of \( \Phi \) defined by

\[
[\varepsilon(u)](\sigma) = \prod_{s \in \sigma} u(s).
\]

We then have \( \|\varepsilon(u)\|^2_\Phi = e^{\|u\|^2_{L^2(\mathbb{R}^+)}} \).

The coherent vectors are linearly independent and total in \( \Phi \). If \( \mathcal{M} \) is a dense subset of \( L^2(\mathbb{R}^+) \), it is also easy to check that \( \mathcal{E}(\mathcal{M}) = \{\varepsilon(u) : u \in \mathcal{M}\} \) is total in \( \Phi \). The open problem is to characterise all those subsets \( \mathcal{M} \subset L^2(\mathbb{R}^+) \) such that \( \mathcal{E}(\mathcal{M}) \) is total in \( \Phi \) (this problem is far from being obvious even in the case where \( L^2(\mathbb{R}^+) \) is replaced by \( \mathbb{C} \)). In [P-S], Parthasarathy and Sunder have proved that if one take \( \mathcal{M} = \{1_B : B \text{ is a bounded Borel subset of } \mathbb{R}^+\} \) then \( \mathcal{E}(\mathcal{M}) \) is total in \( \Phi \). Note that by the continuity of the mapping \( u \mapsto \varepsilon(u) \), it suffices to take \( \mathcal{M} = \{1_B : B \text{ is the union of disjoint bounded intervals of } \mathbb{R}^+\} \) to get the same conclusion.

In this article we consider, for all \( n \in \mathbb{N} \), the space \( E_n \) generated by the \( \{\varepsilon(u) : u \) is the indicator of the union of \( n \) bounded intervals of \( \mathbb{R}^+\} \). We first recover Parthasarathy
-Sunder result in a much simpler way. We characterise $E_n^\perp$, the orthogonal space of $E_n$ in $\Phi$, completely, showing that the typical element $f$ of $E_n^\perp$ is as follows:

$$f = \sum_{q \in \mathbb{N}} f_q, \quad \text{where } f_q \in L^2(\mathcal{P}_q), \quad q \in \mathbb{N}$$

with

- $f_0, f_1, \ldots, f_n$ null
- $f_{2n+1}, f_{2n+2}, \ldots$ can be chosen arbitrarily (modulo a small integrability condition)
- $f_{n+1}, f_{n+2}, \ldots, f_{2n}$ are uniquely determined by the choice of $f_{2n+1}, f_{2n+2}, \ldots$

II – Probabilistic interpretations

It is interesting to note that our results have nice probabilistic interpretations in terms of the Brownian motion and Poisson process.

Let $(\Omega, \mathcal{F}, P)$ be the Wiener space and $(w_t)_{t \geq 0}$ be the canonical Brownian motion on $\Omega$. It is well-known that every random variable $f \in L^2(\Omega, \mathcal{F}, P)$ admits a unique chaotic expansion

$$f = \mathbb{E}[f] + \sum_{q=1}^{\infty} \int_{0 < t_1 < \cdots < t_q} f_q(t_1, \ldots, t_q) \, d\omega_{t_1} \cdots d\omega_{t_q}$$

with $f_q \in L^2(\Sigma_q)$ for all $q \in \mathbb{N}^*$. We also have

$$\|f\|_{L^2(\Omega)}^2 = |\mathbb{E}[f]|^2 + \sum_{q=1}^{\infty} \int_{0 < t_1 < \cdots < t_q} |f(t_1, \ldots, t_q)|^2 \, dt_1 \cdots dt_q.$$  

Thus the space $L^2(\Omega, \mathcal{F}, P)$ canonically identifies to our Fock space $\Phi$ (simply by identifying the coefficients $(f_q)_{q \in \mathbb{N}^*}$; the norm is then the same in both spaces).

If $u$ belongs to $L^2(\mathbb{R}^+)$, the element of $L^2(\Omega, \mathcal{F}, P)$ which corresponds to $\varepsilon(u)$ is simply the Dooleans’ exponential

$$\varepsilon(u) = \exp \left( \int_0^\infty u(s) \, d\omega_s - \frac{1}{2} \int_0^\infty u(s)^2 \, ds \right).$$

If $u = \mathbb{1}_{[s_1, s_2]} \cup \ldots \cup [s_n, t_n]$ is the indicator of the union of $n$ bounded intervals then

$$\varepsilon(u) = \exp(w_{s_1} - w_{s_2} + \cdots + w_{s_n} - w_{t_n}) \exp \left( -\frac{1}{2}(s_1 - s_2 + \cdots + t_n - s_n) \right).$$

Thus our space $E_n$ is exactly the space of random variables generated by exponentials of the sum of $n$ increments of the Brownian motion.
What we have said for the Brownian motion, also holds for the compensated Poisson process $X_t = N_t - t$, $t \in \mathbb{R}^+$, on its canonical space. In this context, we have

$$
\epsilon(u) = e^{\int_0^\infty u(s) \, dX_s} \prod_s (1 + u(s) \Delta X_s) e^{-u(s) \Delta X_s} \\
= e^{\int_0^\infty u(s) \, ds - \sum_s u(s) \Delta N_s} \prod_s (1 + u(s) \Delta N_s) \\
= e^{-\int_0^\infty u(s) \, ds} \prod_s (1 + u(s) \Delta N_s).
$$

In the case where $u = 1_B$ this gives, where $\lambda$ denotes the Lebesgue measure,

$$
\epsilon(u) = e^{-\lambda(\emptyset)} \prod_{s \in B} (1 + \Delta N_s) \\
= e^{-\lambda(B)} \prod_{s \in B} 2 \\
= e^{-\lambda(B) 2^{\#(\{s \in B : \delta \neq 0\})}}
$$

thus

$$
\epsilon(1_{[t_1,t_2] \cup \cdots \cup [t_n,t_n]}) = e^{-(n_1 + \cdots + n_n) 2^{N_{t_1} - N_{t_1} + \cdots + N_{t_n} - N_{t_n}}}
$$

Thus, in this case, our space $E_n$ is the space generated by the exponentials of sums of $n$ increments of the Poisson process (times ln 2).

**III – A simple proof of Parthasarathy -Sunder’s result**

We come back to the general setting of the Fock space $\Phi$. For any $f \in \Phi$ we put for all $t \in \mathbb{R}^+$, all $\sigma \in \mathcal{P}$,

$$
[D_t f](\sigma) = f(\sigma \cup \{t\}) 1_{\sigma \subset [0,t]}.
$$

It is easy to check by the $\mathcal{F}$-lemma (cf. [Att]) that

$$
\int_0^\infty \int_{\mathcal{P}} \|D_t f(\sigma)\|^2 \, d\sigma \, dt = \|f\|^2_\Phi - \|f(\emptyset)\|^2 < \infty.
$$

Thus $D_t f$ is a well-defined element of $\Phi$, for almost all $t$. Furthermore, for all $g \in \Phi$ we have (same reference as above)

$$
\langle f, g \rangle = \int_{\emptyset} g(\emptyset) + \int_0^\infty \langle D_t f, D_t g \rangle \, dt.
$$

Finally note that

$$
D_t \epsilon(u) = u(t) \epsilon(u 1_{[0,t]}).
$$
Let $n \in \mathbb{N}$ be fixed. Define $E_n$ to be the subspace of finite linear combinations of $\varepsilon(u)$, with

$$
u = \mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_n]}$$

for $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n \in \mathbb{R}^+$. Let $E_n^\perp$ be the orthogonal space of $E_n$ in $\Phi$. As $\varepsilon(0) = \delta_0$, we clearly have $E_0 = L^2(\mathcal{P}_0) = \mathbb{C}\delta_0$ and $E_0^\perp = \{ f \in \Phi : f(\emptyset) = 0 \}$.

Note that $E_n \subset E_m$ and $E_m^\perp \subset E_n^\perp$ for all $n \leq m$.

**Proposition 1.** — If $f$ belongs to $E_n^\perp$ for some $n \geq 1$, then $D_t f$ belongs to $E_{n-1}^\perp$ for almost all $t$.

**Proof.** — If $f$ belongs to $E_n^\perp$ then $f$ belongs to $E_0^\perp$ and thus $f(\emptyset) = 0$. Now, we have, for all $s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$

$$0 = \langle f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_n]}) \rangle = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{i}, t_{i+1}]}) \rangle \, dt .$$

Deriving with respect to $t_n$ at $t_n = t$ gives the following: for all $s_1 \leq t_1 \leq \cdots \leq s_n$, for almost all $t \geq s_n$

$$\langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_n]}) \rangle = 0 .$$

The above expression is continuous in $s_1, t_1, \ldots, s_n$, thus we have: for almost all $t$, for all $s_1 \leq t_1 \leq \cdots \leq s_n \leq t$

$$\langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n}, t_n]}) \rangle = 0 .$$

Taking $s_n = t$ we get

$$\langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_{n-1}]} \cup [t_n, t_n]) \rangle = 0 .$$

Now, if $s_1 \leq t_1 \leq \cdots \leq s_{n-1} \leq t_{n-1}$ and $t \in [s_i, t_i]$ we have

$$\langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_{n-1}]} \cup [t_n, t_n]) \rangle = \langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_n, t_n]}) \rangle = \lim_{t_i \to t} \langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_i, t_i]}) \rangle = 0 .$$

Finally, if $t \in [t_i, s_{i+1}]$, we have

$$\langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_{n-1}, t_{n-1}]} \cup [t_n, t_n]) \rangle = \langle D_t f, \varepsilon(\mathbf{1}_{[t_1, t_2] \cup \cdots \cup [t_i, t_i]}) \rangle = 0 .$$

We thus have an easy proof of Parthasarathy -Sunder's result.

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Theorem 1. — The space $\bigcup_n E_n$ is dense in $\Phi$.

Proof. — If $f$ belongs to $\bigcap_n E_n^\perp$, $f(\emptyset) = 0$ and $D_t f$ belongs to $\bigcap_n E_n^\perp$ for almost all $t$. Thus $[D_t f](\emptyset) = 0$ but $[D_t f](\{t\})$ is equal to $f(\{t\})$. Thus the first chaos of $f$ vanishes.

Furthermore, $D_s D_t f$ belongs to $\bigcap_n E_n^\perp$ for almost all $(s, t)$ and thus $f(\{s, t\}) = [D_s D_t f](\emptyset) = 0$ for almost all $(s, t)$. And so on, by induction we get $f(\sigma) = 0$ for a.a. $\sigma \in \mathcal{P}$.

IV – A characterization of $E_n^\perp$

IV.1. Introduction.

For any $f = \sum_q f_q$ belonging to $\Phi$ we write $f_q = f_1 + \cdots + f_q$, $f_q^\perp = f_q + f_{q+1} + \cdots$, etc. The space $\Phi^{(q)}$ is the space $L^2(\mathcal{P}_q)$, the space $\Phi^{\perp}$ is $L^2(\bigcup_{i=0}^q \mathcal{P}_i)$ and $\Phi^{\perp q}$ is $L^2(\bigcup_{i=q}^\infty \mathcal{P}_i)$.

The characterization we are going to prove implies that the space $\{h \in \Phi^{2n+1};$ there exists $f \in E_n^\perp$ with $f_{2n+1} = h\}$ is dense in $\Phi^{2n+1}$ and that, given any such $h$ there exists a unique $f \in E_n^\perp$ such that $f_{2n+1} = h$. Furthermore, we will make explicit the so announced bijection between this dense subspace and $E_n^\perp$.

IV.2. The enlarged Fock space.

In order to state our result we need to introduce a family $(A_q)_{q \in \mathbb{N}}$ of projectors and this will be easier on an enlargement of the Fock space. Let $\tilde{\Phi}$ be defined by

$$\tilde{\Phi} = \{ \text{measurable functions } f \text{ on } \mathcal{P} \text{ such that, for all } N \geq 1, \text{all } T \geq 0, \int_\mathcal{P} N^{d_T} |f(\sigma)| \mathbf{1}_{\sigma \in [0, T]} \, d\sigma < \infty \} .$$

The following remarks and notations will be of constant use in the sequel:

Remark 1. — Let $q \in \mathbb{N}$. For each $f \in \tilde{\Phi}$, the vectors $f_q$, $f_q^\perp$ and $f_q^\perp q$ are in $\tilde{\Phi}$ (obvious).

Notation. — For each $q \in \mathbb{N}$, let $\tilde{\Phi}^{(q)}$, $\tilde{\Phi}^{(q)} q$, $\tilde{\Phi}^{\perp q}$ be defined in the obvious way, in the same way as the corresponding definitions for $\Phi$.  

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Remark 2. — The space $\Phi$ is a subspace of $\tilde{\Phi}$ (left to the reader).

Remark 3. — For any $f \in \tilde{\Phi}$ and any $t \in \mathbb{R}^+$, we put

$$[\nabla_t f](\sigma) = f(\sigma \cup \{t\}).$$

Then $\nabla_t f$ belongs to $\tilde{\Phi}$ for almost all $t$. Indeed, take $N \geq 1$ and $T \geq 0$, for any $S \leq T$
we have

$$\int_0^S \int_{\mathcal{P}} \|\nabla_t f\|(\sigma)|\mathbf{1}_{\sigma \subset [0,T]} N^{\sigma_0}d\sigma dt \leq \int_0^S \int_{\mathcal{P}} |f(\sigma \cup \{t\})|\mathbf{1}_{\sigma \cup \{t\} \subset [0,T]} N^{\sigma_0 \cup \{t\}} d\sigma dt$$

$$= \int_{\mathcal{P}} (\#\beta)|f(\beta)|\mathbf{1}_{\beta \subset [0,T]} N^{\#\beta} d\beta$$

(by the $\mathcal{F}$-lemma)

which is finite for $\#\beta N^{\#\beta} \leq (N + 1)^{\#\beta}$.

Remark 4. — If $f$ belongs to $\tilde{\Phi}$, if $B$ is a bounded Borel set in $\mathbb{R}^+$, then $f \epsilon(\mathbf{1}_B)$ is integrable on $\mathcal{P}$. Indeed, take $T \in \mathbb{R}^+$ such that $\beta \subset [0, T]$, then $|f(\sigma)\epsilon(\mathbf{1}_B)(\sigma)| = |f(\sigma)|\mathbf{1}_{\sigma \subset B} \leq |f(\sigma)|\mathbf{1}_{\sigma \subset [0,T]}$ which is integrable.

Notation. — If $f$ belongs to $\tilde{\Phi}$ and if $\beta$ is a bounded Borel set in $\mathbb{R}^+$, we write $\langle f, \epsilon(\mathbf{1}_B) \rangle$ for

$$\int f(\alpha)\mathbf{1}_{\alpha \subset B} d\alpha.$$

IV.3. The characterising projectors.

For $\sigma \in \mathcal{P}_q$, we write $\sigma_1, \sigma_2, \ldots, \sigma_q$ the elements of $\sigma$, arranged in the increasing order.

If $\#\sigma = 2p$, we write $[\sigma]$ for union of $n$ intervals

$$[\sigma_1, \sigma_2] \cup \cdots \cup [\sigma_{2p-1}, \sigma_{2p}];$$

if $\#\sigma = 2p + 1$, then $[\sigma]$ denotes the union of intervals

$$[\sigma_1, \sigma_2] \cup \cdots \cup [\sigma_{2p-1}, \sigma_{2p}].$$

In any case, we write $\epsilon_{[\sigma]}$ for $\epsilon(\mathbf{1}_{[\sigma]})$. Note that if $\#\sigma$ is odd the $\epsilon_{[\sigma]}$ does not depend on max $\sigma$.

For any $\sigma \in \mathcal{P}$, we write $\nabla_\sigma$ for $\nabla_{\sigma_1} \cdots \nabla_{\sigma_q}$, and $\nabla_\emptyset = I$. Note that for all $s, t, \nabla_s \nabla_t = \nabla_t \nabla_s$.

Lemma 3. — Let $f \in \tilde{\Phi}$, then $Af$ defined on $\mathcal{P}$ by

$$[Af](\sigma) = \langle \nabla_\sigma f, \epsilon_{[\sigma]} \rangle$$

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belongs to $\tilde{\Phi}$.

Proof. — Let $N \geq 1$ and $T \geq 0$ be fixed. Write
\[ \varphi(\alpha, \sigma) = f(\alpha \cup \sigma)1_{\alpha \subset [0, T]}N^{\alpha}\sigma \]
defined on $\mathcal{P} \times \mathcal{P}$. Then $\varphi$ is measurable and satisfies
\[ \int_{\mathcal{P} \times \mathcal{P}} |\varphi(\alpha, \sigma)| \, d\alpha \, d\sigma \leq \int_{\mathcal{P} \times \mathcal{P}} |f(\alpha \cup \sigma)1_{\alpha \subset [0, T]}N^{\alpha(\alpha \cup \sigma)}| \, d\alpha \, d\sigma \]
\[ = \int_{\mathcal{P}} 2^{|\varepsilon|}1_{[0, T]}N^{\varepsilon} \, d\beta < \infty. \]
Thus, by Fubini’s theorem, we have that the mapping
\[ \sigma \mapsto \int_{\mathcal{P}} \varphi(\alpha, \sigma) \, d\alpha = \langle \nabla_{\sigma f}, \varepsilon_{[\sigma]} \rangle 1_{\sigma \subset [0, T]}N^{\sigma} \]
is measurable and integrable on $\mathcal{P}$. ■

For any $q \in \mathbb{N}$ we define $A_q$ as the operator from $\tilde{\Phi}$ to $\tilde{\Phi}$ such that
\[ [A_q f](\sigma) = [A f](\sigma)1_{\sigma^{q}=q}. \]

Proposition 4. — Let $q \in \mathbb{N}$. The operator $A_q$ is a projector from $\tilde{\Phi}$ onto $\tilde{\Phi}[q]$. Moreover $A_q A_r = 0$ if $r < q$. If $q = 0$ then $A_q f = f_0$. If $q = 1$ then $A_q f = f_1$. If $q = 2p+1$ then
\[ [A_{2p+1} f](\{ \sigma_1, \ldots, \sigma_{2p+1} \}) = [A_{2p} D_{\sigma_{2p+1}} f](\{ \sigma_1, \ldots, \sigma_{2p} \}), \]
for a.a. $\sigma = \{ \sigma_1, \ldots, \sigma_{2p+1} \}$.

Proof. — All these results are simple verifications from the definitions. ■

Proposition 5. — Let $p \in \mathbb{N}$ and $f \in \tilde{\Phi}$, then the mapping
\[ (\sigma_1, \sigma_2, \ldots, \sigma_{2p}) \mapsto \langle f, \varepsilon_{[\sigma_1, \sigma_2, \ldots, \sigma_{2p-1}, \sigma_{2p}]} \rangle \]
adsmits a.e. on $\Sigma_{2p}$ a $\frac{\partial^{2p}}{\partial \sigma_1 \cdots \partial \sigma_{2p}}$ derivative and one has
\[ [A_{2p} f](\sigma) = (-1)^p \frac{\partial^{2p}}{\partial \sigma_1 \cdots \partial \sigma_{2p}}(f, \varepsilon_{[\sigma]}1_{\sigma^{2p-2p}}) \text{ a.e. on } \mathcal{P}, \]
and
\[ [A_{2p+1} f](\sigma) = (-1)^p \frac{\partial^{2p}}{\partial \sigma_1 \cdots \partial \sigma_{2p}}(\nabla_{\sigma_{2p+1}} f, \varepsilon_{[\sigma]}1_{\sigma^{2p+1}}) \text{ a.e. on } \mathcal{P}. \]

The above proposition is an easy consequence of the following lemma, which will be of constant use in the sequel.
**Lemma 6.** Let \( f \in \tilde{\Phi} \), let \( B_1, B_2 \) be two bounded Borel sets of \( \mathbb{R}^+ \). Let \([a, b]\) be an interval of \( \mathbb{R}^+ \) with
\[
\max B_1 \leq a < b \leq \min B_2.
\]
Then
\[
i) \text{ the mapping } (s, t) \mapsto \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle \text{ is continuous on } [a, b] \times [a, b];
\]
\[
ii) \text{ the mapping } (s, t) \mapsto \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle \text{ is derivable in } s \text{ and } t \in ]a, b[ \text{ with derivatives given by}
\]
\[
\frac{\partial}{\partial t} \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle = \langle \nabla_x f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle \text{ a.e.}
\]
\[
\frac{\partial}{\partial s} \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle = \langle \nabla_x f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle \text{ a.e.}
\]

Proof.

i) comes from Lebesgue’s theorem: when \((s_n, t_n)\) tends to \((s, t)\), then \( f(\sigma) 1_{\sigma \in B_1]a,b[} \cup 1_{B_2} \) tends to \( f(\sigma) 1_{\sigma \in [a,b]} \) a.e. on \( \mathcal{P} \) and \( |f(\sigma)| 1_{\sigma \in B_1]a,b[} \cup 1_{B_2} \) is dominated by \( |f(\sigma)| 1_{\sigma \in [0,T]} \) (where \( T \) satisfies \( T \geq \max B_2 \)) which is integrable.

ii) Let us prove the formula for \( t \):
we have
\[
\int_s^t \langle \nabla_x f, \epsilon(1_{\sigma \in B_1]a,b[} \cup 1_{B_2}) \rangle \, dx = \int_s^t \int_\mathcal{P} f(\sigma) 1_{\sigma \in B_1]a,b[} \cup 1_{B_2} \, d\sigma \, dx
\]
\[
= \int_\mathcal{P} f(\beta) 1_{\beta \in B_1]a,b[} \cup 1_{B_2} 1_{\beta \in [a,T]} \, d\beta
\]
\[
= \int_\mathcal{P} f(\beta) 1_{\beta \in B_1]a,b[} \cup 1_{B_2} \, d\beta - \int_\mathcal{P} f(\beta) 1_{\beta \in B_1]a,b[} \cup 1_{B_2} \, d\beta
\]
\[
= \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle - \langle f, \epsilon(1_{B_1]a,b[} \cup 1_{B_2}) \rangle
\]
which gives the desired formula.

For \( n \in \mathbb{N} \), let us define
\[
B_n = I - (I - A_0)(I - A_1) \cdots (I - A_{2n}).
\]

**Proposition 7.** The operator \( B_n \) is a projector from \( \tilde{\Phi} \) onto \( \tilde{\Phi}^\odot [n] \).

Proof. The fact that \( B_n \) is a projector comes immediately from \( \lambda^*_{A_q} = A_q \) and \( A_r A_r = 0 \) for \( r < q \). As each \( A_q \), for \( q \leq n \), has its range included in \( \tilde{\Phi}^\odot [q] \), the same holds for \( B_n \). To show that \( B_n \) is onto, just notice that if \( f \in \tilde{\Phi}^\odot [q] \) for some \( q \leq 2n \), then \( (I - A_0)(I - A_1) \cdots (I - A_{2n}) f = 0 \) and thus \( B_n f = f \).
Let $B_n$ be the set of $f \in \Phi$ such that $B_n f$ belongs to $\Phi$.

**Proposition 8.** — The space $B_n$ is dense in $\Phi$.

**Proof.** — Let $\Phi_{0,0}$ be \( \{ f \in \Phi ; \exists k \geq 1 \text{ and } \tau \geq 0 \text{ with } f(\sigma) = f(\sigma) I_{\sigma \in [k,\tau]} \} \). Then $\Phi_{0,0}$ is dense in $\Phi$ and is contained in $B_n$ (for $A_q f \in \Phi_{0,0}$ for all $q \in \mathbb{N}$ and all $f \in \Phi_{0,0}$).

**Remark.** — The above proof also implies that $B_n \cap \Phi_{2^{n+1}}$ is dense in $\Phi_{2^{n+1}}$.

**IV.4. Characterization of $E_n^\perp$.**

We finally come to our characterization.

**Theorem 8.** — Let $n \in \mathbb{N}$. Let $f \in \Phi$. The following assertions are equivalent.

i) $f \in E_n^\perp$.

ii) $A_q f = 0$ for all $q \leq 2n$.

iii) $B_n f = 0$.

iv) There exists $h \in \Phi$ such that

\[ f = (I - A_0)(I - A_1) \cdots (I - A_{2^n}) h. \]

v) $f_{2^{n+1}} = B_n f_{2^{n+1}}$.

**Proof of Theorem 8.**

ii) $\Rightarrow$ iii) is obvious since $A_q f = 0$ for each $q \leq 2n$ implies $(I - A_0)(I - A_1) \cdots (I - A_{2^n}) f = f$.

iii) $\Rightarrow$ iv) comes from the fact that $B_n$ is a projector.

iv) $\Rightarrow$ ii) comes from $A_q A_r = 0$ if $r < q$ and $A_q^2 = A_q$.

v) $\Leftrightarrow$ iv) since iv) is equivalent, by definition of $B_n$, to “$\exists h \in B_n$; $f = (I - B_n) h$”, and since $B_n$ projects $B_n$ onto $\Phi_{2^n}$.

We are just left to prove i) $\Rightarrow$ ii).

The case $n = 0$ is obvious. So by induction our result is equivalent to proving

\[ f \in E_n^\perp \iff f \in E_{n-1}^\perp, \quad A_{2n} f = 0, \quad A_{2n-1} f = 0. \]
Knowing that $f \in E_n^+ \iff f \in E_{n-1}^+$, $A_2f = 0$, $A_{2n-1}f = 0$ for each $p < n$.

$\rightarrow$: Take $f \in E_n^+$. Of course $f$ belongs to $E_{n-1}^+$, so $A_{2n-2}D_t f = 0$. This exactly means $[A_{2n-1}f](\sigma) = 0$, a.e. on $\mathcal{P}$, i.e., $A_{2n-1}f = 0$. We are left to prove $A_{2n}f = 0$,

\[
\langle \nabla \sigma f, \epsilon_n[^0] \rangle = 0 \text{ for almost all } \sigma \text{ such that } \#\sigma = 2n. \]

This comes immediately from $\langle f, \epsilon_n[^0] \rangle = 0$ (since $f \in E_n^+$) and $\langle \nabla \sigma f, \epsilon_n[^0] \rangle = (-1)^n \frac{\partial^n}{\epsilon_1 \cdots \epsilon_{2n}} \langle f, \epsilon_n[^0] \rangle$ a.e. on $\mathcal{P}_{2n}$.

$\leftarrow$: Take $f \in E_{n-1}^+$, with $A_{2n}f = 0$ and $A_{2n-1}f = 0$. We want to prove $\langle f, \epsilon_n[^0] \rangle = 0$ for every $\sigma$ such that $\#\sigma = 2n$. By a continuity argument it is enough to do it a.e. This will come step by step as follows.

First step: From $A_{2n}f = 0$, i.e., $\langle \nabla \sigma f, \epsilon_n[^0] \rangle = 0$ a.e. we deduce, by Lemma 6, that $\langle \nabla \sigma_1 \cdots \sigma_{2n-1} f, \epsilon_n[^0] \rangle$ does not depend on $\sigma_{2n}$ and is equal (making $\sigma_{2n}$ tend to $\sigma_{2n-1}$) to $\langle \nabla \sigma_1 \cdots \sigma_{2n-1} f, \epsilon_n[^0\cdots 2n-1 \sigma_{2n}] \rangle$, where $\sigma^{2n-1} = \sigma \cup \{\sigma_{2n-1}, \sigma_{2n}\}$. But this last equality is $\langle A_{2n-1} f \rangle(\sigma_1, \ldots, \sigma_{2n-1})$ which is null by hypothesis. We have proved that

$\langle \nabla \sigma_1 \cdots \sigma_{2n-1} f, \epsilon_n[^0] \rangle = 0$ a.e. on $\mathcal{P}_{2n}$.

General step: Suppose we have $\langle \nabla \sigma_1 \cdots \sigma_{2n-1} f, \epsilon_n[^0] \rangle = 0$ a.e. on $\mathcal{P}_{2n}$, with $j \leq 2n - 1$. Then, as above, $\langle \nabla \sigma_1 \cdots \sigma_{2n-j} f, \epsilon_n[^0] \rangle$ does not depend on $\sigma_{2n-j}$ and is equal (making $\sigma_{2n-j}$ tend to $\sigma_{2n-j+1}$) to $\langle \nabla \sigma_1 \cdots \sigma_{2n-j+1} f, \epsilon_n[^0\cdots 2n-1 \sigma_{2n-j+1}] \rangle$ (where $\sigma^{2n-1} = \sigma \cup \{\sigma_{2n-j+1}\}$). But this quantity is $\langle \nabla \sigma_1 \cdots \sigma_{2n-j} f, \epsilon_n[^0] \rangle$ a.e. and $\langle f, \epsilon_n[^0] \rangle = 0$ since $\#\sigma_{2n-j+1} = 2n - 2$ and $f \in E_{n-1}^+$ by hypothesis. So we have

$\langle \nabla \sigma_1 \cdots \sigma_{2n-j} f, \epsilon_n[^0] \rangle = 0$ a.e. on $\mathcal{P}_{2n}$.

Last step: With $j = 1$, i.e., $\langle \nabla \sigma_1 f, \epsilon_n[^0] \rangle = 0$ a.e. on $\mathcal{P}_{2n}$, we arrive to $\langle f, \epsilon_n[^0] \rangle = 0$ a.e. on $\mathcal{P}_{2n}$ and so $f \in E_n^+$.

The following consequence of Theorem 8 together with Proposition 1 gives more details on the construction of the elements of $E_n^+$.

**Proposition 9.** Let $n \in \mathbb{N}$. The orthogonal projection of $E_n^+$ on $\Phi^{[2n+1]}$ is equal to $B_n \cap \Phi^{[2n+1]}$. For each $h$ belonging to this dense subspace of $\Phi^{[2n+1]}$, there exists a unique $f \in E_n^+$ such that $f_{[2n+1]} = h$. The coefficients $f_0, f_1, \ldots, f_{2n}$ of $f$ are given by

$\quad f_q = -(B_n)_{q,h}$, $0 \leq q \leq 2n$

where $(B_n)_{q,h}$ is the $q$-th coefficient of $B_n$, given by

$(B_n)_{q,h} = A_q \times \prod_{q+1 \leq j \leq 2n} (I - A_j)$, $0 \leq q \leq 2n$.

Moreover, the $n + 1$ first coefficients $f_0, f_1, \ldots, f_n$ of $f$ all vanish.
Proof. — The equivalence between $i)$ and $ii)$ in Theorem 8 shows that the space \( \{ h \in \Phi^{2n+1}; \exists f \in E_n^+ \text{ such that } f_{2n+1} = h \} \) is equal to \( B_n \cap \Phi^{2n+1} \) (which is dense in \( \Phi^{2n+1} \), see remark at the end of IV.3), and that any \( f \in E_n^+ \) is determined by its projection \( h = f_{2n+1} \) with \( f_{2n} = -B_n h \).

To compute the coefficients \( (B_n)_{q} \), \( 0 \leq q < 2n \), write down, for \( g \in \tilde{\Phi} \):
\[
(B_n g)_q = \left[ I - (I - A_0)(I - A_1) \cdots (I - A_{2n})g \right]_q = g_q - [(I - A_0) \cdots (I - A_{2n})g]_q = A_q(I - A_{q+1}) \cdots (I - A_{2n})g
\]
(the result for \( q = 2n \) is evident).

The fact that the \( n + 1 \) first coefficients \( f_0, f_1, \ldots, f_n \) of any \( f \in E_n^+ \) must vanish comes from repeated applications of Proposition 1. ■

Remark. — The fact that each element of \( E_n^+ \) is determined by its projection \( f_{2n+1} \) is equivalent, by linearity, to \( E_n^+ \cap \Phi_{2n} = \{0\} \). This last equality is an immediate consequence of the following lemma, which we think could be of independent interest.

**Lemma 10.** — Let \( u_1, \ldots, u_n \) be the indicators of \( n \) disjoint bounded intervals of \( \mathbb{R}^+ \). Then one has, for a.a. \( \sigma \in \mathbb{P} \)
\[
\sum_{0 \leq \sigma \leq n} \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} (-1)^p \varepsilon(u_{j_1} + \cdots + u_{j_p})(\sigma)
= \sum_{k \geq n} \sum_{k_1, \ldots, k_n \geq 1} \sum_{j_1, j_2, \ldots, j_n = 1} \left[ u_1^{k_1} \circ u_2^{k_2} \circ \cdots \circ u_n^{k_n} \right](\sigma)
\]
where \( \left[ u_1^{k_1} \circ u_2^{k_2} \circ \cdots \circ u_n^{k_n} \right](\sigma) = 1 \) if the \( k_1 \) first elements of \( \sigma \) lie in \( u_1 \), the \( k_2 \) following ones lie in \( u_2, \ldots, \) the \( k_n \) last elements of \( \sigma \) lie in \( u_n \), and \( \left[ u_1^{k_1} \circ u_2^{k_2} \circ \cdots \circ u_n^{k_n} \right](\sigma) = 0 \) otherwise.

Proof. — Writing down the left hand side for a fixed \( \sigma \), the formula above is just an expression of the usual inclusion-exclusion principle. ■

Proposition 9 shows that for each \( 0 \leq q \leq n \), the coefficient \( (B_n)_{q} \) is equal to the natural projector \( l_q \) of \( \tilde{\Phi} \) onto \( \tilde{\Phi}^q \). We have not been able to derive a direct proof of this fact. That is, to prove that
\[
A_q(I - A_{q+1}) \cdots (I - A_{2n}) = l_q \text{ for } q \leq n.
\]
Another challenge is to develop \( B_n \) in order to give more explicit formulas for \( f_{n+1}, \ldots, f_{2n} \). The coefficient \( (B_n)_{2n} = A_{2n} \) is evident. We give in Propositions 11 and 14 below expressions for \( (B_n)_{2n-1} \) and \( (B_n)_{2n-2} \), just to put in evidence a certain underlying complexity.
IV.5. Some additional computations.

**Proposition 11.** — Let \( n \geq 1 \). One has, for each \( h \in \tilde{\Phi} \), for a.a. \( \sigma \in \mathcal{P}_{2n-1} \),

\[
[A_{2n-2}(I - A_{2n})h](\sigma) = \sum_{1 \leq i \leq 2n-1} (-1)^{i+1} [A_{2n-2} \nabla_{\sigma_i} h](\sigma^i) .
\]

The proof of Proposition 11 is based on two lemmas.

**Lemma 12.** — Let \( g \in \tilde{\Phi} \) be such that \( A_{2n}g = 0 \). Then for almost all \( \sigma \in \mathcal{P}_{2n-1} \), the quantity \([A_{2n-2} \nabla_{\sigma^i} f](\sigma^i)\) is independent of \( i \in \{1, \ldots, 2n-1\} \).

**Proof of Lemma 12.** — Let \( i \in \{1, \ldots, 2n-2\} \). By hypothesis we have, for a.a. \( \sigma \in \mathcal{P}_{2n-1} \) and a.a. \( t \in [\sigma_i, \sigma_{i+1}] \), that \( \langle \nabla_{\sigma_i \cup I} g, \xi_{[\sigma_i,\sigma_{i+1}]} \rangle = 0 \). By Lemma 6 this implies \( \langle \nabla_{\sigma^i} g, \xi_{[\sigma^i]} \rangle \) is independent of \( t \). Making \( t \) tend to \( \sigma_i \) or \( \sigma_{i+1} \) gives \( \langle \nabla_{\sigma^i} g, \xi_{[\sigma^i]} \rangle = \langle \nabla_{\sigma^i} g, \xi_{[\sigma^i]} \rangle \).

**Lemma 13.** — Let \( \sigma \in \mathcal{P}_{2n-1} \), then

\[
\sum_{1 \leq i \leq 2n-1} (-1)^{i+1} 1_{[\sigma^i]} = 0 .
\]

**Proof of Lemma 13.** — Evaluating the left hand side on any \( x \in \mathbb{R}^+ \), gives the same number of sign + and sign − in the sum. ■

**Proof of Proposition 11.** — Define \( C_{n,2n-1} \) on \( \tilde{\Phi} \) by

\[
(C_{n,2n-1}h)(\sigma) = \sum_{1 \leq i \leq 2n-1} (-1)^{i+1} (A_{2n-2} \nabla_{\sigma_i} h)(\sigma^i) 1_{[\sigma_{2n-1}]} .
\]

for each \( h \in \tilde{\Phi} \) and a.a. \( \sigma \in \mathcal{P} \). By Lemma 12, if \( g \in \tilde{\Phi} \) is such that \( A_{2n}g = 0 \) then \( C_{n,2n-1}g = A_{2n-1}g \). Thus for each \( h \in \tilde{\Phi} \), \( C_{n,2n-1}(I - A_{2n})h = A_{2n-1}(I - A_{2n})h \). But by Lemma 8 we have \( C_{n,2n-1}A_{2n} = 0 \). Finally \( C_{n,2n-1} = A_{2n-1}(I - A_{2n}) \).

**Proposition 14.** — Let \( n \geq 1 \). One has for each \( h \in \tilde{\Phi} \) and a.a. \( \sigma \in \mathcal{P}_{2n-2} \)

\[
[A_{2n-2}(I - A_{2n})h](\sigma) = \sum_{1 \leq i,j \leq 2n-2} (-1)^{i+j+1} [A_{2n-4} \nabla_{\sigma_i \sigma_j} h](\sigma^{i,j}) - (n - 2) [A_{2n-2} h](\sigma) .
\]

We again need two lemmas.

**Lemma 15.** — Let \( g \in \tilde{\Phi} \) be such that \( A_{2n}g = 0 \) and \( A_{2n-1}g = 0 \). Then for a.a. \( \sigma \in \mathcal{P}_{2n-2} \), the quantity \([A_{2n-4} \nabla_{\sigma_i \sigma_j} h](\sigma^{i,j})\) is independent of \( i, j \) with \( 1 \leq i < j \leq 2n-2 \).
Proof of Lemma 15. — Apply Lemma 12 to $[\nabla_{\sigma^j} f](\sigma^i)$ and to $[\nabla_{\sigma^j} f](\sigma^i)$.  

Lemma 16. — Let $\sigma \in \mathcal{P}_{2n-2}$. Then

$$
\sum_{1 \leq i < j \leq 2n-2} \varepsilon_{\sigma^j}(\alpha) = (n - 2)\varepsilon_{\sigma}(\alpha) \text{ if } \#\alpha = 1 \text{ or } \#\alpha = 2.
$$

Proof of Lemma 16. — For each $k \in \{1, \ldots, 2n-1\}$ define

$$J_k = \{(i, j); \ 1 \leq i < j \leq 2n-2 \text{ and } [\sigma_k, \sigma_{k+1}] \subset [\sigma^{i,j}]\}.$$

Note that:

- if $k$ is odd then $(i, j) \in J_k$ if and only if $i$ and $j$ are on the same side of $k + \frac{1}{2}$;
- if $k$ is even then $(i, j) \in J_k$ if and only if $i$ and $j$ are not on the same side of $k + \frac{1}{2}$.
- for $p \leq q$ we have

$$
\sum_{2p \leq i < j \leq 2q} (-1)^{i+j+1} = \sum_{2p+1 \leq i < j \leq 2q} (-1)^{i+j+1} = \sum_{2p+1 \leq i < j \leq 2q+1} (-1)^{i+j+1} = \sum_{2p+2 \leq i < j \leq 2q+1} (-1)^{i+j+1} = q - p.
$$

To get Lemma 16 for $\#\alpha = 1$ we have to prove that for $1 \leq k \leq 2n-3$ the quantity

$$
\sum_{(i,j) \in J_k} (i+j+1)
$$
equals $n-2$ if $k$ is odd and equals 0 if $k$ is even. This comes immediately from the remarks above:

- if $k = 2p+1$ then

$$
\sum_{(i,j) \in J_k} (i+j+1) = \sum_{1 \leq i < j \leq 2p+1} (i+j+1) + \sum_{2p+2 \leq i < j \leq 2n-2} (i+j+1) = p + (n-1) - (p+1) = n-2.
$$

- if $k = 2p$ then

$$
\sum_{(i,j) \in J_k} (i+j+1) = \sum_{1 \leq i < j \leq 2p} (i+j+1) = 0.
$$

Finally, to get Lemma 16 for $\#\alpha = 2$ we have to prove that if $h$ and $k$ are between $1$ and $2n-1$, we have

$$
\sum_{(i,j) \in J_k \cap J_{\sigma^j}} (i+j+1) = n-2 \text{ if } h \text{ and } k \text{ are odd, } 0 \text{ otherwise.}
$$

That comes again in the same way as above.  

Proof of Proposition 14. — Define $C_{n,2n-2}$ on $\tilde{\Phi}$ by

$$
[C_{n,2n-2}]h(\alpha) = \left[ \sum_{1 \leq i < j \leq 2n-2} (-1)^{i+j+1} (A_{2n-4} \nabla_{\sigma^j} h)(\sigma^i) \right] (n-2)(A_{2n-2} h)(\alpha) \mathbb{I}_{\#\sigma = 2n-2}
$$
for each $h \in \hat{\Phi}$ and a.a. $\sigma \in \mathcal{P}$.

By Lemma 15, if $g \in \hat{\Phi}$ is such that $A_{2n}g = 0$ and $A_{2n-1}g = 0$ then $C_{n,2n-2}g = A_{2n-2}g (\text{use } \sum_{1 \leq i < j \leq 2n-1} (-1)^{i+j+1} = n - 1)$. So for each $h \in \hat{\Phi}$, we have

$$C_{n,2n-2}(I - A_{2n-1})(I - A_{2n})h = A_{2n-2}(I - A_{2n-1})(I - A_{2n})h.$$ 

But by Lemma 16, we have $C_{n,2n-2}A_{2n} = 0$ and $C_{2n,2n-2}A_{2n-1} = 0$. Thus $C_{n,2n-2} = A_{2n-2}(I - A_{2n-2})(I - A_{2n})$.

\textbf{References}


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