Chapter

RETURN TO EQUILIBRIUM FOR SOME STOCHASTIC SCHRÖDINGER EQUATIONS

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ABSTRACT

The principle of "indirect continuous measurement" in "open quantum system theory" is usually described by non-usual types of stochastic differential equations. These equations are called "stochastic Schrödinger equations" and their solutions are called "quantum trajectories". Physically, they describe the random evolution of the state of a quantum system undergoing indirect quantum measurement (such models are widely used in quantum optics, quantum computing and quantum information theory). In this chapter, we consider a physically realistic discrete-time setup for two-level quantum systems and we present the theory of "discrete quantum trajectories". These discrete trajectories are Markov chains which can be expressed as solutions of "discrete-time" stochastic differential equations". In particular, these equations appear as time discretization of "stochastic Schrödinger equations". Going to the continuous-time limit, we justify the stochastic Schrödinger equations associated to the two-level systems. Within this approach, we obtain two different types of behaviors described either by jump-type or diffusive-type stochastic differential equations. Finally we investigate the large time behavior of the solutions and we prove return to equilibrium properties for the associated physical models.

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1. Introduction

Recent experiments of continuous measurement in quantum mechanics (Haroche's team in particular), or more precisely in quantum optics, have put into evidence the random evolution of the state of a quantum open system [25, 26]. In particular, one has experimentally observed "quantum jumps". These experiments allow to study the evolution of a quantum system interacting with some environment. They are based of the principle of indirect measurement on the environment, in order not to perturb the evolution of the small system [8, 12, 20, 22, 23, 33, 34].

The stochastic models attached to these phenomenons are described by stochastic differential equations, called "Stochastic Schrödinger Equations" or also "Belavkin Equations" [6–13, 15, 20, 22–24, 33, 34]. Their solutions are called "quantum trajectories", they describe the evolution of the state of the small open quantum system. The stochastic differential equations which are usually obtained in this context are of two different types. Either they are of "jump-type":

$$d\rho_t = L(\rho_t)dt + \left(\frac{\jmath(\rho_t)}{Tr[\jmath(\rho_t)]} - \rho_t\right)\left(d\tilde{N}_t - Tr[\jmath(\rho_t)]dt\right). \tag{1}$$

where $\tilde{N}_t$ is a stochastic counting process with stochatic intensity $\int_0^t Tr[\jmath(\rho_s)]ds$. The operator L corresponds to a Lindblad type operator and the operator $\jmath$ describes the evolution of the system during the quantum jumps. This equation describes experiments which are called "direct photon detection" (observation of the photon emission by an atom excited by a laser).

Or it can be an equation of diffusive type:

$$d\rho_t = L(\rho_t)dt + \Big(C\rho_t + \rho_t C^\star - \mathrm{Tr}[(C+C^\star)\rho_t]\rho_t\Big)dWt, \tag{2}$$

where W_t is a standard Brownian motion. In quantum optics, this equation describes experiments called "Heterodyne or Homodyne detection".

More complex models are described by jump-diffusion stochastic differential equations which are mixing of the two previous types [15, 32].

In the usual literature, obtaining and justifying rigorously these equations makes use of Quantum Filtering Theory [7, 10, 12, 19]. It is the quantum probability version of the usual filtering technics, it makes use of fine quantum stochastic calculus and heavy von Neumann algebra theory. Others approaches are based on classical probability and use of instrumental process and notion of a posteriori state [9, 11, 15, 29].

A maybe more intuitive and more physical approach for these equations is to start from a discrete-time procedure, that is, repeated quantum interactions with measurement of the environment ([2, 3, 16–18, 35]). Then one obtains the stochastic Schrödinger equations by passing to the limit to a continuous-time model [30, 31].

In this chapter, we come back and apply results obtained in [30] and [31], in which Belavkin equations are obtained with this approach. Here, we obtain the description of the stochastic Schrödinger equations for a two level system in contact with a spin chain. We adapt the result of [30] and [31] in order to describe the quantum trajectories in terms of wave function (in [30] and [31], the stochastic equations for the evolution of density

matrices have been derived from the approximation procedure). Next, for a special model, we show a property of return to equilibrium of the solution.

The chapter is structured as follows. Section 2 is devoted to the presentation of the model of quantum repeated interactions and quantum repeated measurements, that is the model of "discrete quantum trajectory" . In Section 3, we present the passage to the limit from discrete quantum trajectories to continuous quantum trajectories for two level systems. In parallel, we present the result of existence and uniqueness of the solutions[1] of equations (1) and (2). Next we concentrate on the property of return to equilibrium.

2. Discrete-Time Quantum Trajectories

In this section we describe the physical model and the mathematical setup of indirect repeated quantum measurements. We describe the evolution of the small system undergoing successive measurements through the "discrete quantum trajectories".

2.1. Repeated Quantum Interactions

The physical situation is the following. A quantum system, with state space $\mathcal{H}_S$ (often called *small system* for it is in general finite-dimensional and/or small compared to the environment) is undergoing repeated interactions with a chain of quantum systems $\otimes_{\mathbb{N}^*}\mathcal{H}$. This is to say that we consider an environment which is made up of a sequence of identical copies of a quantum system, each with state space $\mathcal{H}$. Each piece $\mathcal{H}$ of the environment is going to interact, one after the other, with the small system $\mathcal{H}_S$. This interaction lasts for a time duration τ and is driven by a total Hamiltonian H_{tot} on $\mathcal{H}_S \otimes \mathcal{H}$. Hence, each interaction is described by the unitary operator

$$U = e^{-i\tau H_{\text{tot}}}$$

on $\mathcal{H}_S \otimes \mathcal{H}$. In the Schrödinger picture, if ρ denotes any initial state on the tensor product $\mathcal{H}_S \otimes \mathcal{H}$ then the evolution of the state after this interaction is given by:

$$\rho \mapsto U\rho U^*.$$

After this interaction, the systems $\mathcal{H}_S$ and $\mathcal{H}$ stop interacting together, the system $\mathcal{H}_S$ comes to meet a second copy of $\mathcal{H}$ and they interact together in the same way as before (that is, with following the same unitary operator U). And so on... the small system interacts repeatedly with each of the independent copies of $\mathcal{H}$.

Let us develop the mathematical framework which allows describing these repeated quantum interactions. We follow the setup of the article [3], in which these models and their continuous limit were first introduced.

[1] This question is not straightforward since the coefficients, defining the equations, are not Lipschitz. Furthermore the equation (1) is ill defined: in expression (1), the driving process depends on the solution that it is supposed to drive. Hence the notion of solution is not immediate. In particular, the existence of (ρ_t) relies on the existence of $(\tilde{N}_t)$ and reciprocally.

The state space describing the whole game is

$$\Gamma = \mathcal{H}_S \otimes \bigotimes_{k\in\mathbb{N}^*} \mathcal{H}_k \,, \tag{3}$$

where each $\mathcal{H}_k$ is a copy of the Hilbert space $\mathcal{H}$. We have to be clear about what the above *countable* tensor product means:

$$T\Phi = \bigotimes_{k\in\mathbb{N}^*} \mathcal{H}_k \,.$$

Recall that a countable tensor product of Hilbert spaces can only be defined with respect to a choice of a particular unit vector u_k in each copy $\mathcal{H}_k$ (the so-called the *stabilizing sequence* of the countable tensor product). In our case, we assume that $\mathcal{H}_k$ is finite dimensional and we choose an orthonormal basis

$$\{X^i; i \in \mathcal{N} \cup \{0\}\}$$

where $\mathcal{N}$ is a set of the form $\{1,\dots N\}$, which is the same for each $\mathcal{H}_k$. A particular role is played by the vector X^0 which has to be considered as a reference vector for the system $\mathcal{H}$, as we choose the stabilizing sequence to be $u_k = X^0$ for all k.

Denote by X_k^i the basis vector X^i but leaving in the k-th copy $\mathcal{H}_k$ of $\mathcal{H}$. Then an Hilbertian orthonormal basis of $T\Phi$ is given by all the tensor products $\otimes_k v_k$ where all the vectors v_k are equal to X_k^0, except for a finite number of them which might be equal to some $X_k^{i_k}$, $i_k \in \mathcal{N}$. This stands for a definition of the countable tensor product $T\Phi = \otimes_{k\in\mathbb{N}^*}\mathcal{H}_k$.

The repeated quantum interaction setup is based on two elements: the time length τ and the Hamiltonian H_{tot} which describes each basic interaction. Consider the unitary operator $U = \exp(-i\tau H_{\text{tot}})$ acting on $\mathcal{H}_S \otimes \mathcal{H}$ and consider the unitary operator U_k on Γ which acts as U on $\mathcal{H}_S \otimes \mathcal{H}_k$ and which acts like the identity operator on the other copies $\mathcal{H}_{k'}$. This operator U_k describes the effect of the k-th interaction.

The unitary operator

$$V_k = U_k \dots U_1$$

describes the effect of the k first interactions. Indeed, if ρ is any initial state on Γ, then

$$V_k \rho V_k^*$$

is the state of the whole system (small system + environment) after k interactions.

Define the elementary operators a_j^i, $i,j \in \mathcal{N} \cap \{0\}$ on $\mathcal{H}$ by

$$a_j^i X^k = \delta_{i,k} X^j \,.$$

It is useful for further computations to notice that in Dirac notation $a_j^i = |X^j\rangle\langle X^i|$[2]. We denote by $a_j^i(n)$ their natural ampliation to $T\Phi$ acting on the n-th copy of $\mathcal{H}$ only.

[2]These are the usual "bra-ket" notations in quantum mechanics. The term $|X\rangle$ represent the vector X in the underlying Hilbert space. The term $\langle Y|$ represent the linear form which acts as $\langle Y|(|X\rangle) = \langle Y,X\rangle$ where $\langle,\rangle$ corresponds to the scalar product. This way, we have $|X^j\rangle\langle X^i|(|X\rangle) = \langle X^i,X\rangle|X^j\rangle$, for all vectors $|X\rangle$. In particular the operator $|X\rangle\langle X|$ corresponds to the orthogonal projector on the space generated by X.

Clearly, U can always be written as

$$U = \sum_{i,j \in \mathcal{K} \cup \{0\}} U_j^i \otimes a_j^i$$

for some operators U_j^i on $\mathcal{H}_0$ such that:

$$\sum_{k \in \mathcal{K} \cup \{0\}} (U_i^k)^* U_j^k = \sum_{k \in \mathcal{K} \cup \{0\}} U_j^k (U_i^k)^* = \delta_{i,j} I.$$

With this representation for U, it is clear that the operator U_n, representing the n-th interaction, is given by

$$U_n = \sum_{i,j \in \mathcal{K} \cup \{0\}} U_j^i \otimes a_j^i(n).$$

With these notations, the sequence (V_n) of unitary operators describing the n first repeated interactions can be represented as follows:

$$\begin{aligned} V_{n+1} &= U_{n+1} V_n \\ &= \sum_{i,j \in \mathcal{K} \cup \{0\}} U_j^i \otimes a_j^i(n+1) V_n . \end{aligned}$$

But, inductively, the operator V_n acts only on the n first sites of the chain $T\Phi$, whereas the operators $a_j^i(n+1)$ act on the $(n+1)$-th site only. Hence they commute. In the following, we shall drop the $\otimes$ symbols, identifying operators like $a_j^i(n+1)$ with $I_{\mathcal{H}_0} \otimes a_j^i(n+1)$, the operator U_j^i with $U_j^i \otimes I_{T\Phi}$, etc. This gives finally

$$V_{n+1} = \sum_{i,j \in \mathcal{K} \cup \{0\}} U_j^i V_n a_j^i(n+1). \tag{4}$$

On $T\Phi$, one vector plays a particular role, the vector

$$\Omega = \otimes_k X_k^0 .$$

For any bounded operator K on Γ, we define the operator $\mathbb{E}_0[K]$ on $\mathcal{H}_S$ as the unique operator on $\mathcal{H}_S$ such that, for all trace-class operator ρ on $\mathcal{H}_S$ we have

$$\mathrm{Tr}_{\mathcal{H}_S}(\rho \, \mathbb{E}_0[K]) = \mathrm{Tr}_\Gamma \left((\rho \otimes |\Omega\rangle\langle\Omega|) K \right) .$$

That is, $\mathbb{E}_0[K]$ is the partial trace of K with respect to the state $|\Omega\rangle\langle\Omega|$ on $T\Phi$.

We then have the following fundamental action of the repeated interactions, when restricted to the small system.

Theorem 1 (cf [3])**.** *The effect of the repeated interaction dynamics when restricted to $\mathcal{H}_S$ is given as follows. For all observable X on $\mathcal{H}_S$, for all $n \in \mathbb{N}$, we have*

$$\mathbb{E}_0[V_n^*(X \otimes I) V_n] = L^n(X),$$

where L is a completely positive map on $\mathcal{H}_S$ whose Krauss decomposition is

$$L(X) = \sum_{i \in \mathcal{K}} (U_i^0)^* X U_i^0 .$$

Any (discrete) semigroup (L^n) of completely positive maps can be obtained this way.

Note that the completely positive map L defined above acts on observables. It also induces a completely positive "dual map" L^* acting on states as follows:

$$L^*(\rho) = \sum_{i\in\mathcal{N}} U_i^0 \rho (U_i^0)^* \tag{5}$$

and which satisfies

$$\mathrm{Tr}\,(\rho L(X)) = \mathrm{Tr}\,(L^*(\rho)X)$$

for all state ρ and all bounded operator X on $\mathcal{H}_S$. Recall the usual notion of partial trace defined as follows.

Definition-Theorem 1. *Given any state α on a tensor product $\mathcal{H}\otimes\mathcal{K}$, then there exists a unique state η on $\mathcal{H}$ which is characterized by the property:*

$$\mathrm{Tr}\,[\eta X] = \mathrm{Tr}\,[\alpha(X\otimes I)],$$

for all $X\in\mathcal{B}(\mathcal{H})$. The state η is denoted by $\mathrm{Tr}_{\mathcal{K}}(\alpha)$ and is called the partial trace of η with respect to $\mathcal{K}$.

With these notations we have the following result.

Theorem 2. *For every state ρ on $\mathcal{H}_S$ and all $n\in\mathcal{N}$ we have*

$$\mathrm{Tr}_{T\Phi}(V_n(\rho\otimes|\Omega\rangle\langle\Omega|)V_n^*) = (L^*)^n(\rho).$$

Proof: We have, for all X bounded operator on $\mathcal{H}_S$,

$$\begin{aligned}
\mathrm{Tr}\,((L^*)^n(\rho)X) &= \mathrm{Tr}\,(\rho L^n(X)) \\
&= \mathrm{Tr}\,(\rho\,\mathbb{E}_0[V_n^*(X\otimes I)V_n]) \\
&= \mathrm{Tr}\,((\rho\otimes|\Omega\rangle\langle\Omega|)V_n^*(X\otimes I)V_n) \\
&= \mathrm{Tr}\,(V_n(\rho\otimes|\Omega\rangle\langle\Omega|)V_n^*(X\otimes I)) \\
&= \mathrm{Tr}\,(\mathrm{Tr}_{T\Phi}(V_n(\rho\otimes|\Omega\rangle\langle\Omega|)V_n^*)\,X)\,.
\end{aligned}$$

This proves the announced result. □

2.2. Repeated Quantum Measurements

We now somehow consider a more complicated procedure. After each interaction is finished, the piece $\mathcal{H}_k$ of environment which has just finished to interact with $\mathcal{H}_S$ is undergoing a quantum measurement of one of its observables. The random result of this quantum measurement will give some information on the state of the whole system and in particular on the state of $\mathcal{H}_S$. The so-called quantum trajectory is the random process we obtain this way, by looking at the knowledge we have of the state of $\mathcal{H}_S$ after each measurement.

Let A be any observable on $\mathcal{H}$, with spectral decomposition

$$A = \sum_{j=1}^{p} \lambda_j P_j\,,$$

the λ_j's being the eigenvalues, the P_j's being the eigenprojectors. We consider the natural ampliations of A which defines an observable on Γ by making A acting on the k-th site $\mathcal{H}_k$ only:

$$\begin{aligned} A^k &= \bigotimes_{j=0}^{k-1} I \otimes A \otimes \bigotimes_{j\geq k+1} I \\ &= \bigotimes_{j=0}^{k-1} I \otimes \left(\sum_{j=1}^{p} \lambda_j P_j \right) \otimes \bigotimes_{j\geq k+1} I \\ &= \sum_{j=1}^{p} \lambda_j P_j^k , \end{aligned}$$

with obvious notations.

As a consequence, if ρ is the state of Γ then a quantum measurement of the observable A^k gives the values λ_j with probability:

$$P[\text{to observe } \lambda_j] = Tr[\rho P_j^k], \;\; j \in \{1, \ldots, p\} .$$

If we have observed the eigenvalue λ_j for the observable A^k, the new sate of the system is

$$\rho_j = \frac{P_j^k \rho P_j^k}{Tr[\rho P_j^k]} .$$

This principle is the so-called "von Neumann projection postulate". Now, if we perform another measurement of the observable A^k we obtain $P[\text{to observe } \lambda_j] = 1$. As a consequence, a naive repeated measurement operation gives no information on the evolution of the system. The repeated measurement procedure has to be combined with the repeated interaction procedure in order to give non-trivial informations on the behavior of the system.

The quantum repeated measurement principle is the combination of the measurement principle and the repeated quantum interactions. Physically, this means that each copy $\mathcal{H}_k$ of $\mathcal{H}$ interacts with $\mathcal{H}_S$ and we perform a measurement of A^k on $\mathcal{H}_k$ after it has interacted with $\mathcal{H}_S$. After each measurement we have a new (random) state of the whole system, given by the projection postulate. This is the so-called *discrete quantum trajectory*.

More precisely, the initial state on Γ is chosen to be of the form

$$\mu = \rho \otimes \bigotimes_{j\geq 1} \eta_j ,$$

where ρ is any state on $\mathcal{H}_0$ and each $\eta_i = \eta$ is a reference state on $\mathcal{H}$. We denote by μ_k the state representing the new state after the k first interactions, that is,

$$\mu_k = V_k \mu V_k^* .$$

Let us now define the probabilistic framework in order to describe the effect of the successive measurements. We put $\Omega = \{1, \ldots, p\}$ and on $\Omega^{\mathbb{N}}$ we define the cylinders of size k:

$$\Lambda_{i_1,\ldots,i_k} = \{\omega \in \Omega^{\mathbb{N}} / \omega_1 = i_1, \ldots, \omega_k = i_k\} .$$

We endow $\Omega^{\mathbb{N}}$ with the σ-algebra $\mathcal{F}$ generated by all these sets, this is the *cylinder* σ-*algebra*. Note that for all j, the unitary operator U_j commutes with all the projectors P_i^k such that $k < j$. Hence, the state of the system after k interactions and k measurements which have given the respective values $\lambda_{i_1}, \dots, \lambda_{i_k}$ is (up to normalization by the trace)

$$\begin{aligned} P_{i_k}^k U_k \dots P_{i_1}^1 U_1 \, \mu \, (U_1)^* P_{i_1}^1 \dots (U_k)^* P_{i_k}^k = \\ &= P_{i_k}^k \dots P_{i_1}^1 U_k \dots U_1 \, \mu \, (U_1)^* \dots (U_k)^* P_{i_1}^1 \dots P_{i_k}^k \\ &= P_{i_k}^k \dots P_{i_1}^1 \, \mu_k \, P_{i_1}^1 \dots P_{i_k}^k , \end{aligned}$$

where we have used that U_k commutes with any $P_{k'}$ such that $k' \neq k$.

We denote by $\tilde{\mu}(i_1, \dots, i_k)$ the quantity

$$P_{i_k}^k \dots P_{i_1}^1 \, \mu_k \, P_{i_1}^1 \dots P_{i_k}^k .$$

By the Kolmogorov Consistency Theorem we can define a probability measure P on $(\Omega^{\mathbf{N}}, \mathcal{F})$ only by specifying

$$P[\Lambda_{i_1, \dots, i_k}] = Tr[\tilde{\mu}(i_1, \dots, i_k)] .$$

We also define a random sequence of states on Γ by

$$\begin{array}{rccc} \tilde{\rho}^k(.) : & \Omega^{\mathbf{N}} & \longrightarrow & \mathcal{B}(\Gamma) \\ & \omega & \longmapsto & \tilde{\rho}_k(\omega_1 \dots \omega_k) = \dfrac{\tilde{\mu}(\omega_1 \dots \omega_k)}{Tr[\tilde{\mu}(\omega_1 \dots \omega_k)]} . \end{array}$$

This random sequence of states is our discrete quantum trajectory and the operator $\tilde{\rho}^k(i_1, \dots, i_k)$ represents the state of the system, after having observed the results $\lambda_{i_1}, \dots, \lambda_{i_k}$ for the k first measurements. This fact is made precise in the following proposition.

Proposition 1. *Let $(\tilde{\rho}_k)$ be the above random sequence of states we have, for all $\omega \in \Omega^{\mathbb{N}}$*

$$\tilde{\rho}_{k+1}(\omega) = \frac{P_{\omega_{k+1}}^{k+1} U_{k+1} \, \tilde{\rho}_k(\omega) \, (U_{k+1})^* P_{\omega_{k+1}}^{k+1}}{\mathrm{Tr}\left[\tilde{\rho}_k(\omega) \, (U_{k+1})^* P_{\omega_{k+1}}^{k+1} U_{k+1}\right]} .$$

This proposition is obvious but summarizes the quantum repeated measurement principle. The sequence $\tilde{\rho}_k$ is the quantum trajectory, showing up the effect of the successive measurements on Γ. The following theorem is an easy consequence of the previous proposition.

Theorem 3. *The sequence $(\tilde{\rho}^n)_n$ is a Markov chain, valued in the set of states of Γ. It is described as follows:*

$$P\left[\tilde{\rho}^{n+1} = \mu \,|\, \tilde{\rho}^n = \theta_n, \dots, \tilde{\rho}^0 = \theta_0\right] = P\left[\tilde{\rho}^{n+1} = \mu \,|\, \tilde{\rho}^n = \theta_n\right] .$$

If $\tilde{\rho}^n = \theta_n$ then $\tilde{\rho}^{n+1}$ takes one of the values:

$$\frac{P_i^{n+1} U_{n+1} \, \theta_n \, (U_{n+1})^* P_i^{n+1}}{\mathrm{Tr}\left[U_{n+1} \, \theta_n \, (U_{n+1})^* P_i^{n+1}\right]}, \quad i = 1, \dots, p ,$$

with probability $\mathrm{Tr}\left[U_{n+1} \, \theta_n \, (U_{n+1})^* P_i^{n+1}\right]$.

The most interesting behavior of the Markov chain of states above is obtained when one restricts it to the small system $\mathcal{H}_S$. This way we obtain a quantum trajectory on the states of $\mathcal{H}_S$ by considering the sequence of random states on $\mathcal{H}_S$:

$$\rho_n(\omega) = \mathrm{Tr}_{T\Phi}(\tilde{\rho}_n(\omega)). \tag{6}$$

This defines a sequence of state on $\mathcal{H}_S$ which contains the "partial" information given by the measurement and we have the following theorem which completely describes the behavior of this random sequence.

Theorem 4. *The random sequence defined by formula (6) is a Markov chain with values in the set of states on $\mathcal{H}_S$. If $\rho_n = \chi_n$ then ρ_{n+1} takes one of the values:*

$$\frac{\mathrm{Tr}_{\mathcal{H}}\left[(I\otimes P_i)U(\chi_n\otimes\eta)U^*(I\otimes P_i)\right]}{Tr[U(\chi_n\otimes\eta)U^*(I\otimes P_i)]},\ i=1,\dots,p,$$

with probability $\mathrm{Tr}\left[U(\chi_n\otimes\eta)U^*(I\otimes P_i)\right]$.

The expectation of ρ_n satisfies

$$\mathbb{E}[\rho_n] = (L^*)^n(\rho_0),$$

where L^* is the completely positive map described in Theorem 2.

Proof: Assume, by induction, that ρ_n is given. This means that $\mathrm{Tr}_{T\Phi}(\tilde{\rho}_n) = \rho_n$. The next step of the quantum measurement gives (by Theorem 3)

$$\tilde{\rho}_{n+1} = \frac{P_i^{n+1}U_{n+1}\tilde{\rho}_n(U_{n+1})^*P_i^{n+1}}{\mathrm{Tr}\left[U_{n+1}\theta_n(U_{n+1})^*P_i^{n+1}\right]},$$

for some i. Hence, we have to compute

$$\mathrm{Tr}_{T\Phi}(P_i^{n+1}U_{n+1}\tilde{\rho}_n(U_{n+1})^*P_i^{n+1}).$$

Decomposing, with obvious notations, the space $T\Phi$ into $\mathcal{H}_S\otimes\Gamma_{[0,n]}\otimes\mathcal{H}_{n+1}\otimes\Gamma_{[n+2,+\infty[}$, one notes that, by induction, the state $\tilde{\rho}_n$ is of the form

$$\theta_n\otimes\eta\otimes\bigotimes_{k\geq n+2}\eta$$

where θ_n is a state on $\mathcal{H}_S\otimes\Gamma_{[0,n]}$, satisfying

$$\mathrm{Tr}_{\Gamma_{[0,n]}}(\theta_n) = \rho_n.$$

Hence, for all X, bounded operator on $\mathcal{H}_S$, we have

$$\begin{aligned}
\mathrm{Tr}\left(\mathrm{Tr}_{T\Phi}(P_i^{n+1}U_{n+1}\tilde{\rho}_n(U_{n+1})^*P_i^{n+1})X\right) &= \\
&= \mathrm{Tr}\left((P_i^{n+1}U_{n+1}\tilde{\rho}_n(U_{n+1})^*P_i^{n+1})(X\otimes I)\right) \\
&= \mathrm{Tr}\left(U_{n+1}\tilde{\rho}_n(U_{n+1})^*(X\otimes I_{[0,n]}\otimes P_i^{n+1}\otimes I_{[n+2,+\infty[})\right) \\
&= \mathrm{Tr}\left(U_{n+1}\left(\theta_n\otimes\eta\otimes\bigotimes_{k\geq n+2}\eta\right)(U_{n+1})^*(X\otimes I_{[0,n]}\otimes P_i^{n+1}\otimes I_{[n+2,+\infty[})\right) \\
&= \mathrm{Tr}\left((\theta_n\otimes\eta)(U_{n+1})^*(X\otimes I_{[0,n]}\otimes P_i^{n+1})U_{n+1}\right).
\end{aligned} \tag{7}$$

But U_{n+1} acts only on $\mathcal{H}_S \otimes \mathcal{H}_{n+1}$, hence the operator $(U_{n+1})^*(X \otimes I_{[0,n]} \otimes P_i^{n+1})U_{n+1}$ is an operator on $\mathcal{H}_S \otimes \mathcal{H}_{n+1} \otimes \Gamma_{[0,n]}$ (note the interchange of space, for simplicity of the notations) which is of the form

$$((U_{n+1})^*(X \otimes P_i^{n+1})U_{n+1}) \otimes I_{[0,n]}\,.$$

Hence, the quantity (7) is equal to

$$\mathrm{Tr}\left(\mathrm{Tr}_{\Gamma_{[0,n]}}(\theta_n \otimes \eta)\,(U_{n+1})^*(X \otimes P_i^{n+1})U_{n+1}\right).$$

But $\mathrm{Tr}_{\Gamma_[0,n]}(\theta_n \otimes \eta)$ is equal to $\mathrm{Tr}_{\Gamma_[0,n]}(\theta_n) = \rho_n \otimes \eta$. This gives finally

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{Tr}_{T\Phi}(P_i^{n+1}U_{n+1}\tilde{\rho}_n\,(U_{n+1})^*P_i^{n+1})\,X\right) &= \\ &= \mathrm{Tr}\left((P_i^{n+1}U_{n+1}\,(\rho_n \otimes \eta)\,(U_{n+1})^*P_i^{n+1})\,X\right).\end{aligned}$$

But in this expression, the index $n+1$ plays no more role and the expression above may as well be written

$$\mathrm{Tr}\left((P_iU\,(\rho_n \otimes \eta)\,(U)^*P_i)\,X\right)$$

on $\mathcal{H}_S \otimes \mathcal{H}$. This proves the first part of the theorem.

Let us check, the one concerning the expectation of ρ_n. Note that the expectation of ρ_1 is equal to

$$\begin{aligned}\mathbb{E}[\rho_1] &= \sum_{i=1}^{p} P(\{i\})\,\frac{\mathrm{Tr}_{\mathcal{H}}(P_iU(\rho_0 \otimes \eta)U^*P_i)}{P(\{i\})} \\ &= \sum_{i=1}^{p} \mathrm{Tr}_{\mathcal{H}}(U(\rho_0 \otimes \eta)U^*P_iP_i) \qquad \text{for } P_i \text{ acts on } \mathcal{H} \text{ only} \\ &= \mathrm{Tr}_{\mathcal{H}}(U(\rho_0 \otimes \eta)U^* \sum_{i=1}^{p} P_i) \\ &= \mathrm{Tr}_{\mathcal{H}}(U(\rho_0 \otimes \eta)U^*) \\ &= L^*(\rho_0)\,.\end{aligned}$$

By induction, we conclude easily. □

Thanks to the above description we can express a discrete-time evolution equation for the quantum trajectories. Let us put

$$\mathcal{L}_i(\rho) = \mathbf{E}_0\left[(I \otimes P_i)\,U(\rho \otimes \eta)U^*\,(I \otimes P_i)\right],$$

$i = 1, \ldots, p$. We then have for all $\omega \in \Sigma^{\mathbb{N}}$ and all $k > 0$:

$$\rho_{k+1}(\omega) = \sum_{i=0}^{p} \frac{\mathcal{L}_i(\rho_k)(\omega)}{Tr[\mathcal{L}_i(\rho_k)(\omega)]}\mathbf{1}_i^{k+1}(\omega) \tag{8}$$

where $\mathbf{1}_i^k(\omega) = \mathbf{1}_i(\omega_k)$.

2.3. The two-level atom model

In this section we specialise to the case where $\mathcal{H}_0 = \mathbb{C}^2$, this is the so-called *two-level atom model*. In most of the physical applications that we have in mind, the interacting system is also of the form $\mathcal{H} = \mathbb{C}^2$. We denote by X_0, X_1 an orthonormal basis where the reference state η is diagonal:

$$\eta = \begin{pmatrix} \eta_0 & 0 \\ 0 & \eta_1 \end{pmatrix}.$$

Let Ω, X be any orthonormal basis of $\mathcal{H}_0$. For describing the interactions between $\mathcal{H}_0$ and $\mathcal{H}$ we choose $\Omega\otimes X_0, X\otimes X_0, \Omega\otimes X_1, X\otimes X_1$ as an orthonormal basis of $\mathcal{H}_0\otimes\mathcal{H}$. In such a basis, the unitary operator U, describing the elementary interaction, can be written as a 2×2 matrix with coefficients being operators on $\mathcal{H}_0$. That is, we can write U as:

$$U = \begin{pmatrix} U_0^0 & U_0^1 \\ U_1^0 & U_1^1 \end{pmatrix}.$$

Let A be an observable of $\mathcal{H}$ on which we want to perform a measurement. It can be written as $A = \lambda_0 P_0 + \lambda_1 P_1$ where λ_i are its eigenvalues and P_i the corresponding eigenprojectors. Let $(P^i_{k,l})_{k,l=0,1}$ be the matrix elements of the projector P^i in the basis X_0, X_1. Put

$$\mathcal{L}_i(\rho) = \sum_{k,l=0,1} P^i_{k,l}\left(\eta_0 U_0^k \rho\,(U_0^l)^* + \eta_1 U_1^k \rho\,(U_1^l)^*\right).$$

Then, if ρ_k denotes the state of the system $\mathcal{H}_S$ after the k-th measurement, the state ρ_{k+1} takes one of the two possibles values

$$\frac{\mathcal{L}_i(\rho_k)}{\mathrm{Tr}[\mathcal{L}_i(\rho_k)]}.$$

We denote $p_{k+1} = Tr[\mathcal{L}_0(\rho_k)]$ or $q_{k+1} = Tr[\mathcal{L}_1(\rho_k)]$ the corresponding transitions probabilities.

In the rest of the chapter, we concentrate on a special case of environment, where $\eta = |X_0\rangle\langle X_0|$. This situation corresponds to a model of heat bath at zero temperature, see [5] for more explanations and for positive temperature models (let us just stress that this choice is crucial and that positive temperature gives rise to completely different continuous-time behaviours). In this situation, the discrete quantum trajectory can be described in terms of pure states. More precisely, if the initial state of $\mathcal{H}_0$ is pure, the random sequence (ρ_k) remains pure. This way, we can describe the evolution of $\mathcal{H}_0$ with a random sequence of vectors (wave functions).

Proposition 2. *Let $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$ and $\eta = |X_0\rangle\langle X_0|$. Let (ρ_k) be the discrete quantum trajectories corresponding to the indirect measurement of an obervable A.*

If ρ_0 is a pure state, that is $\rho_0 = |\psi_0\rangle\langle\psi_0|$, and if the measurement is non-trivial (A is not a multiple of the identity), then the state of the small system ρ_n is always a pure state. In other terms, there exists a random sequence of wave functions $(|\psi_n\rangle)$ such that $\||\psi_n\|| = 1$ and such that $\rho_n = |\psi_n\rangle\langle\psi_n|$, for all $n\in\mathbb{N}$.

The sequence $(|\psi_n\rangle)$ is also called a discrete quantum trajectory.

Proof: Since we work in 2-dimension and since A is not a multiple of identity, we have $A = \lambda_0 P_0 + \lambda_1 P_1$ where P_i are one dimensionnal projector. Thus there exist two vectors α_i, $i = 0,1$ such that $P_i = |\alpha_i\rangle\langle\alpha_i|$. Now, let $\rho_0 = |\psi_0\rangle\langle\psi_0|$, after the first measurement if we have observed the eignevalue λ_i, the non normalized state describing the experiment is described by

$$\begin{aligned}
\tilde{\rho}_1(i) &= \mathbb{E}_0\big[I\otimes|\alpha_i\rangle\langle\alpha_i|\,U(|\psi_0\rangle\langle\psi_0|\otimes|X_0\rangle\langle X_0|)U^\star\, I\otimes|\alpha_i\rangle\langle\alpha_i|\big] \\
&= \sum_{k,l}\sum_{u,v}\mathbb{E}_0\left[I\otimes|\alpha_i\rangle\langle\alpha_i|\,\left(U_k^l(|\psi_0\rangle\langle\psi_0|(U_v^u)^\star\otimes a_k^l|X_0\rangle\langle X_0|a_u^v\right)I\otimes|\alpha_i\rangle\langle\alpha_i|\right] \\
&= \sum_{k,v}\mathbb{E}_0\left[I\otimes|\alpha_i\rangle\langle\alpha_i|\,\left(U_k^0(|\psi_0\rangle\langle\psi_0|(U_v^0)^\star\otimes a_k^v\right)I\otimes|\alpha_i\rangle\langle\alpha_i|\right] \\
&= \sum_{k,v}\mathbb{E}\left[\left|U_k^0\psi_0\right\rangle\left\langle U_v^0\psi_0\right|\otimes|\alpha_i\rangle\langle\alpha_i||X^k\rangle\langle X^v||\alpha_i\rangle\langle\alpha_i|\right] \\
&= \mathbb{E}\left[\left|\sum_k\langle\alpha_i,X^k\rangle U_k^0\psi_0\right\rangle\left\langle\sum_v\langle\alpha_i,X^v\rangle U_v^0\psi_0\right|\otimes|\alpha_i\rangle\langle\alpha_i|\right] \\
&= \left|\sum_k\langle\alpha_i,X^k\rangle U_k^0\psi_0\right\rangle\left\langle\sum_v\langle\alpha_i,X^v\rangle U_v^0\psi_0\right|
\end{aligned}$$

Now, by normalizing the vector $\sum_k\langle\alpha_i,X^k\rangle U_k^0\psi_0$, it is straightforward that we get a vector ψ_1 such that $\rho_1 = |\psi_1\rangle\langle\psi_1|$. Next, by induction we can construct a sequence ψ_n such that $\rho_n = |\psi_n\rangle\langle\psi_n|$ for all n. □

Remark: Such a property is at the basis of the use of "quantum trajectory theory" for numerical simulations of Lindblad master equations. Numerically, the description in terms of pure states reduces the number of parameters to control (in comparaison with density matrices) . We recover the "deterministic" dynamic by taking the expectation, that is, $\mathbb{E}[|\psi_n\rangle\langle\psi_n|] = \mathcal{L}^n(\rho_0)$. In the continuous time version, similar properties are called "unravelling[3]" of master equations and simulations use techniques called "Quantum Monte Carlo simulations".

Now, we can complete the description of our model with the help of discrete equation which describes the stochastic evolutions of discrete quantum trajectories. To this end, let us introduce some notations. Let P_0 be the projector on $\alpha_0 = (\mu,\nu)$, with $\|\alpha_0\| = 1$ and P_1 the projector on $\alpha_1 = (\bar{\nu},-\bar{\mu})$. Let define the following functions acting on vectors

$$\begin{aligned}
\mathcal{F}_0(|\psi\rangle) &= \left|\left[\mu U_0^0+\nu U_1^0)\right]\psi\right\rangle \\
\mathcal{F}_1(|\psi\rangle) &= \left|\left[\bar{\nu}U_0^0-\bar{\mu}U_1^0)\right]\psi\right\rangle\ .
\end{aligned}$$

Then, the dynamic of (ψ_n) can be described by the equation

$$|\psi_{k+1}(\omega)\rangle = \frac{\mathcal{F}_0(|\psi_k(\omega)\rangle)}{\|\mathcal{F}_0(|\psi_k(\omega)\rangle)\|}\mathbf{1}_0^{k+1}(\omega)+\frac{\mathcal{F}_1(|\psi_k(\omega)\rangle)}{\|\mathcal{F}_1(|\psi_k(\omega)\rangle)\|}\mathbf{1}_1^{k+1}(\omega), \tag{9}$$

for all $\omega\in\Sigma^{\mathbb{N}}$. This equation corresponds to equation (8) for a two level system in terms of wave functions, i.e. the sequence $(|\psi_k\rangle\langle\psi_k|)$ satisfies equation (8).

[3]Unravelling means the description of a wave function stochastic process (ψ_t) such that $\mathbb{E}[|\psi_t\rangle\langle\psi_t|] = e^{tL}(\rho_0)$

In the next section, we will describe the continuous time version of these equations. To this end, we aim at considering this discrete-time model but depending on a time-length parameter τ which we shall make tend to 0. That is, we want to pass from a discrete time interaction model to a continuous time one. This way, we shall obtain the classical Belavkin equations for quantum trajectories associated to continuous measurement. In the literature, these equations describe a model where a two-level atom is in contact with a photon-stream.

Let $\tau = \frac{1}{n}$ be the time of interaction between the small system and one element of the environment. Let us denote by $U(n)$ the unitary operator associated to each interaction, it now depends of the time of interaction. If we had no measurement process on the environment, we will be back to the problem of going from a discrete-time repeated quantum interaction model, to a continuous time one. This problem has been completely studied in [3]. In their article they show that, in order to get a limit evolution when τ goes to 0, we have to ask the operator $U(n)$ to satisfy certain renormalization conditions. They have shown that the coefficients $U_j^i(n)$ must follow well-defined time scaling in order to obtain a non-trivial limit. Namely they have shown that the operator $(V_{[nt]} = U([nt])\dots U_1)_{t>0}$ converges to an evolution $(V_t)_t$ which is a continuous operator process. This process naturally satisfies a quantum Langevin equation which represents the evolution equation of the small system + bath.

Our continuous measurement procedure does not differ much from their approach, except that we perform a measurement on the environment after each interaction. This is why we have to keep the same normalization for the coefficients $U_j^i(n)$ in order to get a limit. Following [3] we assume that the total Hamiltonian, describing one elementary interaction, is of the form

$$H_{tot} = H \otimes I + I \otimes \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_1 \end{pmatrix} + \sqrt{n}\left(C \otimes a_1^0 + C^* \otimes a_0^1\right).$$

That is, a typical dipole-type interaction Hamitonian with a renormalization in $\sqrt{n}$ of the field operator a_1^0 and a_0^1 in order to strengthen the force of the interaction while the interaction-time decreases.

With this Hamiltonian, it is easy to check that the coefficients of $U(n)$ are of the form

$$U_0^0(n) = I + \frac{1}{n}\left(-iH - i\gamma_0 I + \frac{1}{2}C^*C\right) + \circ(\frac{1}{n}) \tag{10}$$

$$U_1^0(n) = -i\frac{1}{\sqrt{n}}C + \circ(\frac{1}{n}) \tag{11}$$

$$U_0^1(n) = -i\frac{1}{\sqrt{n}}C^* + \circ(\frac{1}{n}) \tag{12}$$

$$U_1^1(n) = I + \frac{1}{n}\left(-iH - i\gamma_1 I + \frac{1}{2}CC^*\right) + \circ(\frac{1}{n}). \tag{13}$$

3. Continuous Trajectories

In this section, we implement the asymptotic expression of the coefficient $U_j^i(n)$ in the description of the quantum repeated measurements for the model of the two level atom. First

we recall the convergence of discrete models to continuous models of Belavkin equations. Second, we show return to equilibrium results in this context.

As in shown in [30] and [31], the continuous limit of the evolution equation is completely different, depending on wether the observable A is diagonal or not in the basis of η. The point is that the limit equation is of diffusive type when A is non-diagonal and of Poisson type in the diagonal case. Inside each case, the behaviors are very comparable and differ only by some coefficients. This is why, it is enough here to consider only two cases:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a_1^1,$$

as representing the diagonal case, or

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a_0^1 + a_1^0,$$

as representing the non-diagonal case. Here, we focus on the description of quantum trajectories in terms of pure states, while in [30, 31], the evolution for the density matrices is considered.

3.1. The Poisson case

We first start with the case $A = a_1^1$, for which we have $P_0 = a_0^0$. It is easy to see that we can choose $\mu = 1$, $\nu = 0$ for the description of the projectors P_i. Applying the hypothesis (10)-(13), we obtain the probabilities

$$\begin{aligned} p_{k+1} &= Tr[\rho_k P_0] = \|U_0^0|\psi_k\rangle\| = 1 - \frac{1}{n}\frac{1}{2}\mu_k(n) + \circ\left(\frac{1}{n}\right) \\ q_{k+1} &= Tr[\rho_k P_1] = \|U_1^0|\psi_k\rangle\| = \frac{1}{n}\frac{1}{2}\mu_k(n) + \circ\left(\frac{1}{n}\right), \end{aligned}$$

where $\mu_k(n) = \langle\psi_k, C^*C\psi_k\rangle$. By remarking that $\mathbf{1}_0^k = 1 - \mathbf{1}_1^k$, we have the following difference equation for (ψ_k):

$$|\psi_{k+1}\rangle - |\psi_k\rangle = \frac{1}{n}\left(-iH - \frac{1}{2}C^*C + \frac{1}{2}\mu_k + \circ(1)\right)|\psi_k\rangle + \\ + \left(\frac{C}{\sqrt{\mu_k}} - I + \circ(1)\right)|\psi_k\rangle\mathbf{1}_1^{k+1}. \quad (14)$$

In the continuous limit, we shall see that this difference equation converges to an equation of the form

$$d|\psi_t\rangle = \left(-iH - \frac{1}{2}(C^*C + \mu_{t-}I) + \sqrt{\mu_{t-}}C\right)|\psi_{t-}\rangle\,dt + \\ + \frac{(C - \sqrt{\mu_{t-}}I)}{\sqrt{\mu_{t-}}}|\psi_{t-}\rangle\,(d\tilde{N}_t - \mu_{t-}\,dt) \quad (15)$$

where $\mu_t = \langle \psi_t, C^*C\psi_t\rangle$ and $(\tilde{N}_t)$ is a counting process such that $t \to \tilde{N}_t - \int_0^t \mu_s ds$ is a martingale. This is to say that $(\tilde{N}_t)$ is a counting process with stochastic intensity equal to $\int_0^t \mu_s ds$. A first problem is that equation (15) is ill-defined. Indeed, the intensity of the counting process depends on the solution itself. We need to be more precise about what we mean by a "solution to equation (15)".

Definition 1. *Let $\Omega, \mathcal{F}, P)$ be a probability space. A* process-solution *of the jump-equation (15) is a process (ψ_t) and a counting process $\tilde{N}_t$, with intensity $\int_0^t \mu_s ds$ where $\mu_t = \langle \psi_t, C^*C\psi_t\rangle$, such that for all t we have*

$$|\psi_t\rangle = |\psi_0\rangle + \int_0^t \left(-iH - \frac{1}{2}(C^*C + \mu_{s-}I) + \sqrt{\mu_{s-}}C\right)|\psi_{s-}\rangle ds + \\ + \int_0^t \frac{(C - \sqrt{\mu_{s-}}I)}{\sqrt{\mu_{s-}}}|\psi_{s-}\rangle(d\tilde{N}_s - \mu_{s-}ds). \quad (16)$$

This notion of solution imposes the simultaneous existence of the process $|\psi_t\rangle$ and the counting process $\tilde{N}_t$. In order to construct such a counting process, we use a Poisson point process.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, on which is living a Poisson point process N on $\mathbb{R}^2$ such that the expectation of the number of points $N(\omega, B)$ lying inside a Borel set B is given by

$$\mathbf{E}[N(\cdot, B)] = \lambda(B)$$

where λ is the Lebesgue measure on $\mathbb{R}^2$.

This way, N defines a *random measure* $B \mapsto N(\omega, B)$ on $\mathbb{R}^2$, whose volume element is denoted by $N(\omega, dx \times ds)$. The following theorem shows how the random Poisson measure is used to construct the counting process.

Theorem 5 ([31]). *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space on which lives a Poisson point process N. The following equation*

$$|\psi_t\rangle = |\psi_0\rangle + \int_0^t \left(-iH - \frac{1}{2}(C^*C - \mu_{s-}I)\right)|\psi_{s-}\rangle ds + \\ + \int_0^t \int_{\mathbb{R}} \frac{(C - \sqrt{\mu_{s-}}I)}{\sqrt{\mu_{s-}}}|\psi_{s-}\rangle \mathbf{1}_{0\le x\le \mu_{s-}} N(dx, ds). \quad (17)$$

admits a unique solution (ψ_t) such that $\|\psi_t\| = 1$ almost surely. Then the process $(|\psi_t\rangle)$ together with the counting process

$$\tilde{N}_t = \int_0^t \int_{\mathbf{R}} \mathbf{1}_{0\le x\le \mu_{s-}} N(dx, ds) \quad (18)$$

constitute a process-solution for equation (15).

Even if this theorem is just an application of the results of [31], let us explain roughly how it is proved (this description will allow also to describe the return to equilibrium property in the jump case).

In equation (17) there are two parts: the ordinary differential part and the one driven by the Poisson process. Consider the collection of jumping times of the Poisson process. If there is no jump of the Poisson process N, we deal with an ordinary differential equation

$$|\psi_t\rangle = |\psi_0\rangle + \int_0^t \left(-iH - \frac{1}{2}(C^*C - \mu_{s-}I)\right)|\psi_{s-}\rangle\, ds\,.$$

This equation admits a unique solution, from which we deduce the curve $t \to \mu_t$. The first time T_1 when the Poisson process has a jump under this curve, the solution $|\psi_t\rangle$ jumps and takes the value

$$\frac{C|\psi_{T_1-}\rangle}{\sqrt{\mu_{T_1-}}}\,.$$

After this first jump, we have a new "initial" value for $|\psi_t\rangle$ and the process starts again in the same way: we solve the ordinary differential equation and the solution follows it, until it meets a jump of N which is bellow the curve, then it jumps. And so on.

Remark: The corresponding evolution for the density matrices can be obtained by computing the stochastic differential equation for $\rho_t = |\psi_t\rangle\langle\psi_t|$. By applying the stochastic calculus rules for random Poisson measure, we get the equation

$$\begin{aligned}\rho_t &= \rho_0 + \int_0^t \Big(L(\rho_{s-}) - C\rho_{s-}C^\star + \mathrm{Tr}[C\rho_{s-}C^\star]\rho_{s-}\Big)ds + &(19)\\ &\int_0^t\int_{\mathbb{R}} \left(\frac{C\rho_{s-}C^\star}{\mathrm{Tr}[C\rho_{s-}C^\star]} - \rho_{s-}\right)\mathbf{1}_{0<x<\mathrm{Tr}[C\rho_{s-}C^\star]}N(dx,ds), &(20)\end{aligned}$$

where L is the Lindblad operator defined by

$$L(\rho) = -i[H,\rho] - \frac{1}{2}\{C^\star C,\rho\} + C\rho C^\star.$$

Thus, by definig $\jmath(\rho) = C\rho C^\star$,we recover the equation (1) mentionned in Introduction.

Now, that equation (17) is well understood, we wish to pass to the continuous time limit on equation (15). The appropriate topology for the convergence theorem proved in [31] is the Skorohod topology. Let us recall it. For all $T > 0$ we denote by $\mathcal{D}([0,T])$ the space of all càdlàg matricial process on $[0,T]$ endowed with the Skorohod topology, that is, the topology of the weak convergence of càdlàg processes (the convergence in distribution).

The approximation result is based on the description of a quantum trajectory as the solution of a stochastic equation wich is a discretization of (17). In particular, from equation (14), we can write

$$\begin{aligned}|\psi_{[nt]}\rangle &= |\psi_0\rangle + \sum_{k=0}^{[nt]-1}(|\psi_{k+1}\rangle - |\psi_k\rangle))\\ &= |\psi_0\rangle + \sum_{k=0}^{[nt]-1}\frac{1}{n}\left(-iH - \frac{1}{2}C^*C + \frac{1}{2}\mu_k + \circ(1)\right)|\psi_k\rangle +\\ &\quad + \sum_{k=0}^{[nt]-1}\left(\frac{C}{\sqrt{\mu_k}} - I + \circ(1)\right)|\psi_k\rangle\mathbf{1}_1^{k+1}\,, &(21)\end{aligned}$$

for all $t \geq 0$. An adaptation of the result of [31] give us the following convergence.

Theorem 6 ([31]). *Let T be fixed. Let $(\Omega,\mathcal{F},P)$ be a probability space in which lives a Poisson point process N. Let $(|\psi_{[nt]}\rangle)_{0\le t\le T}$ be the discrete quantum trajectory defined by the equation (21). This discrete quantum trajectory converges in $\mathcal{D}([0,T])$ to the process $(|\tilde{\psi}_t\rangle)_{0\le t\le T}$ which is the unique solution of the stochastic differential equation*

$$\begin{aligned}|\tilde{\psi}_t\rangle &= |\psi_0\rangle+\int_0^t\left(-iH-\frac{1}{2}C^*C+\frac{1}{2}\mu_t I\right)|\tilde{\psi}_s\rangle ds\\ &\quad+\int_0^t\int_{\mathbb{R}}\left(\frac{C}{\sqrt{\mu_{s-}}}-I\right)|\tilde{\psi}_{s-}\rangle\mathbf{1}_{0<x<\mu_{s-}}N(dx,ds)\end{aligned}$$

where $\mu_t=\langle\tilde{\psi}_t\,,C^*C\tilde{\psi}_t\rangle$.

This result relies on the fact that Equation (21) can be interpreted as a discrete time stochastic differential equation which is a discretization of the jump equation.

3.2. The diffusive case

We now consider the case where $A=\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}-\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}$. We have $P_0=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}$ and $\mu=\nu=\frac{1}{\sqrt{2}}$. Hence, after computation we obtain:

$$p_{k+1} = Tr[\rho_k P_0]=\left\|\frac{1}{\sqrt{2}}(U_0^0+U_1^0)|\psi_k\rangle\right\|=\frac{1}{2}+\frac{\nu_k(n)}{\sqrt{n}}+\circ\left(\frac{1}{n}\right), \tag{22}$$

$$q_{k+1} = Tr[\rho_k P_1]=\left\|\frac{1}{\sqrt{2}}(U_1^0-U_0^0)|\psi_k\rangle\right\|=\frac{1}{2}-\frac{\nu_k(n)}{\sqrt{n}}+\circ\left(\frac{1}{n}\right), \tag{23}$$

where $\nu_k(n)=Re(\langle\psi_k,C\psi_k\rangle)$. here, we introduce the random variables (X_k) defined by

$$X_{k+1}=-\frac{\mathbf{1}_1^k-q_{k+1}}{\sqrt{p_{k+1}q_{k+1}}},$$

for all $k\ge 0$. in terms of (X_k), the evolution equation takes the form

$$\begin{aligned}|\psi_{k+1}\rangle-|\psi_k\rangle=\frac{1}{n}\left(-iH-\frac{1}{2}(C^*C-2\nu_k C+\nu_k^2 I)+\circ(1)\right)|\psi_k\rangle+\\+\left(C-\nu_k+\circ(1)\right)|\psi_k\rangle\frac{1}{\sqrt{n}}X_{k+1}.\end{aligned} \tag{24}$$

The continuous diffusive equation which is the natural candidate to be the limit of equation (24) is

$$d|\psi_t\rangle=\left(-iH-\frac{1}{2}\left(C^*C-2\nu_t C+\nu_t^2 I\right)\right)|\psi_t\rangle dt+(C-\nu_t I)|\psi_t\rangle dW_t, \tag{25}$$

where $\nu_t=Re(\langle\psi_t,C\psi_t\rangle)$ and $(W_t)_t$ is a one-dimensional Brownian motion.

In [30], it is shown that the convergence result is highly based on the existence and uniqueness of the solution for such equation (let us stress that the coefficients are not Lipschitz). In particular, by a truncation method the following Theorem is proven in [30].

Theorem 7 ([30]). *Let $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ be a probability space on which is defined a standard Brownian motion $(W_t)_t$. The following stochastic differential equation*

$$d|\psi_t\rangle = \left(-iH-\frac{1}{2}\left(C^*C-2\nu_t C+\nu_t^2 I\right)\right)|\psi_t\rangle dt+(C-\nu_t I)|\psi_t\rangle dW_t \tag{26}$$

admits a unique solution. Furthermore, almost surely, for all t we have $\|\psi_t\|=1$.

We can now consider the approximation procedure. In a similar way as the Poisson case, we can consider the difference equation

$$\begin{aligned}|\psi_{[nt]}\rangle &= |\psi_0\rangle+\sum_{k=0}^{[nt]-1}\left(|\psi_{k+1}\rangle-|\psi_k\rangle\right)\\ &= |\psi_0\rangle+\sum_{k=0}^{[nt]-1}\frac{1}{n}\left(-iH-\frac{1}{2}(C^*C-2\nu_k C+\nu_k^2 I)+\circ(1)\right)|\psi_k\rangle\\ &\quad+\sum_{k=0}^{[nt]-1}\left(C|\psi_k\rangle-\nu_k|\psi_k\rangle+\circ(1)\right)\frac{1}{\sqrt{n}}X_{k+1}\,.\end{aligned} \tag{27}$$

We have the following result.

Theorem 8. *Let T be fixed. Let $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ be a probability space on which is defined a standard Brownian motion $(W_t)_t$. Let $(|\psi_{[nt]}\rangle)_{0\leq t\leq T}$ be the discrete quantum trajectory defined by the equation (27). This discrete quantum trajectory converges in $\mathcal{D}([0,T])$ for all T to the process $(|\tilde{\psi}_t\rangle)_{0\leq t\leq T}$ which is the unique solution on Ω of the following stochastic differential equation:*

$$d|\psi_t\rangle = (C-\nu_t I)|\psi_t\rangle dW_t+\left(-iH-\frac{1}{2}\left(C^*C-2\nu_t C+\nu_t^2 I\right)\right)|\psi_t\rangle dt \tag{28}$$

where $\nu_t = Re(\langle\psi_t,C\psi_t\rangle)$.

In a sake of completeness we give some details of how proving such a convergence. In particular we show how to interpret the equation (27) as a discrete-time stochastic differential equation.

Proof: Define the processes

$$\psi_n(t)=|\psi_{[nt]}\rangle,\ V_n(t)=\frac{[nt]}{n},\ W_n(t)=\frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]-1}X_{k+1}.$$

The process $(\psi_n(t))$ can be considered as the solution of the following discrete-time stochastic differential equation

$$\begin{aligned}\psi_n(t) &= \int_0^t\left(-\frac{1}{2}C^*C\psi_n(s-)+Re(\psi_n(s-),C\psi_n(s-)\rangle)C\psi_n(s-)\right)dV_n(s)\\ &\quad+\int_0^t(C\psi_n(s-)-Re(\psi_n(s-),C\psi_n(s-)\rangle)\psi_n(s-)dW_n(s)+\varepsilon_n(t),\end{aligned} \tag{29}$$

where the terms $\varepsilon_n(t)$ corresponds of the $\circ$ terms in the equation in asymptotic form.

In order to show that $\psi_n(t)$ converges in the Skorohod space to the solution of

$$|\psi_t\rangle = \int_0^t \left(-\frac{1}{2}C^*C|\psi_s\rangle + \nu_t C|\psi_s\rangle\right) ds + \int_0^t (C|\psi_t\rangle - \nu_t|\psi_t\rangle) dW_s$$

we make use of the celebrated Kurtz-Protter theorem. Let us recall it.

Recall that $[X,X]$ is defined for a semi-martingale by the formula $[X,X]_t = X_t^2 - \int_0^t X_{s-} dX_s$. For a finite variation process V we put $T_t(V)$ to be the total variation of V on $[0,t]$.

Theorem 9 (Kurtz-Protter, [28])**.** *Suppose that W_n is a martingale and V_n is a finite variation process. Assume that for each $t \geq 0$:*

$$\sup_n \mathbf{E}[[W_n, W_n]_t] < \infty$$
$$\sup_n \mathbf{E}[T_t(V_n)] < \infty$$

and that $(W_n, V_n, \varepsilon_n)$ converges in distribution to $(W,V,0)$ where W is a standard brownian motion and $V(t) = t$ for all t. Let $X_n(t)$ be a process satisfying

$$X_n(t) = \rho_0 + \varepsilon_n(t) + \int_0^t L(X_n(s-) dV_n(s) + \int_0^t \Theta(X_n(s-)) dW_n(s)$$

Suppose that X satisfies:

$$X_t = X_0 + \int_0^t L(X_s) ds + \int_0^t \Theta(X_s) dW_s$$

and that the solution of this stochastic differential equation is unique. Then X_n converges in distribution to X.

In our case, the different hypothesis above are satisfied. Indeed, define a filtration for the process $(W_n(.))$:

$$\mathcal{F}_t^n = \sigma(X_i, i \leq [nt]).$$

The following is proved in [Pe1].

Proposition 3. *We have that $(W_n(.), \mathcal{F}_.^n)$ is a martingale. The process $(W_n(.))$ converges to a standard Brownian motion $W_.$ when n goes to infinity and $sup_n \mathbf{E}[[W_n, W_n]_t] < \infty$.*

Furthermore, we have the convergence in distribution for the process $(W_n, V_n, \varepsilon_n)$ to $(W,V,0)$ when n goes to infinity.

This proves the announced convergence. □

Remark: Using Ito rules on $|\psi_t\rangle\langle\psi_t|$, we get the equation for density matrices

$$d\rho_t = L(\rho_t)dt + \Big(C\rho_t + \rho_t C^\star - \mathrm{Tr}[(C+C^\star)\rho_t]\rho_t\Big) dW_t,$$

which corresponds to the equation (2) mentionned in the Introduction (the Linblad operator L has the same form as the Poisson case).

3.3. Return to Equilibrium

Now that the limit equations are established, we are interested into the long time behaviour of the solutions. We specify our investigations to the special case where $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a_0^1$ and $H = H_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Writing the processes (ψ_t) in terms of their coordinates; that is $(\psi_t := (x_t, y_t))$, the Belavkin equations take the form

$$\begin{cases} x_t &= x_0 + \int_0^t \left(-ix_s + Re(\bar{x_s}y_s)y_s - \frac{1}{2}Re(\bar{x_s}y_s)x_s \right) ds + \int_0^t (y_s - Re(\bar{x_s}y_s)x_s)\, dW_s \\ y_t &= y_0 + \int_0^t \left(-\frac{1}{2}y_s - \frac{1}{2}Re(\bar{x_s}y_s)^2 y_s \right) ds + \int_0^t \left(-Re(\bar{x_s}y_s)y_s \right) dW_s \end{cases} \tag{30}$$

in the diffusive case, and

$$\begin{cases} x_t &= x_0 + \int_0^t \left(-ix_s + \frac{1}{2}x_s|y_s|^2 \right) ds + \int_0^t \int_{0<x<|y_{s-}|^2} \left(-x_{s-} + 1 \right) N(dx, ds) \\ y_t &= y_0 + \int_0^t \left(-\frac{1}{2}y_s + \frac{1}{2}|y_s|^2 y_s \right) ds + \int_0^t \int_{0<x<|y_{s-}|^2} \left(-y_{s-} \right) N(dx, ds) \end{cases} \tag{31}$$

in the Poisson case.

In the Poisson case, note that the intensity is $\mu_t = |y_{t-}|^2$, so that one can restrict ourselves to the case where the jumps of the Poisson process are in between the lines $y = 1$ and $y = 0$ (we have namely $|y_{t-}|^2 \leq 1$, for all t). The function $t \to card(N(.,[0,1] \times [0,t])) = \mathcal{N}_t$ then defines a standard Poisson process with intensity 1. The Poisson random measure and the previous process generate on $[0,T]$ (for a fixed T) a sequence $\{(T_i, \xi_i), i \in \{1, \ldots, \mathcal{N}_t)\}\}$ where each T_i represents the jump time of $\mathcal{N}$. Moreover the random variables ξ_i are uniform random variables on $[0,1]$. Consequently we can write our quantum trajectory as follows

$$\begin{cases} x_t &= x_0 + \int_0^t \left(-ix_s + \frac{1}{2}x_s|y_s|^2 \right) ds + \sum_{i=1}^{\mathcal{N}_t} \left(-x_{T_i-} + 1 \right) \mathbf{1}_{0<\zeta_i<|y_{T_i-}|^2} \\ y_t &= y_0 + \int_0^t \left(-\frac{1}{2}y_s + \frac{1}{2}|y_s|^2 y_s \right) ds + \sum_{i=1}^{\mathcal{N}_t} \left(-y_{T_i-} \right) \mathbf{1}_{0<\zeta_i<|y_{T_i-}|^2} \end{cases} \tag{32}$$

Now, we shall investigate the large time behaviour of a solution of equation (30) or (31). To this end we need to notice the following lemma.

Lemma 1. *Let* $\begin{pmatrix} x_t \\ y_t \end{pmatrix}$ *be the either solution of equation (30) or (31) starting with an initial condition of the form* $\begin{pmatrix} x_0 \\ 0 \end{pmatrix}$. *Then, almost surely, we have* $y_t = 0$, *for all* t.

Proof: Starting from $y_0 = 0$, in each case, it is easy to verify that $y_t = 0$ is a particular solution for the corresponding stochastic differential equation describing the evolution of (y_t). As a consequence by uniqueness of solution, almost surely, $y_t = 0$ for all t. □

Remark: In both cases, if $y_t = 0$ for all t, it is easy to see that the evolution of (x_t) is given by the solution of $dx_t = -ix_t dt$.

Remark: In terms of states, this lemma expresses that if $\psi_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$, we have almost surely

$$|\psi_t\rangle\langle\psi_t| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |\Omega\rangle\langle\Omega|,$$

for all t. In other words, the state $|\Omega\rangle\langle\Omega|$ is an invariant (or stationnary state) for the stochastic dynamic of continuous measurement (let us stress that without measurement, i.e in the deterministic regime, it is easy to see that this state is already the invariant state).

Now we can make precise the result which states the return to equilibrium property. In particular we focus on the large time behaviour of the part y_t and we show that this process converges to zero when t goes to infinity.

Proposition 4. *Let* $|\psi_t\rangle = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ *be either the solution of the jump-equation or the solution of the diffusive equation, then we have*

$$|y_t|^2 \xrightarrow[t\to\infty]{a.s} 0\,. \tag{33}$$

Therefore, we have

$$y_t \xrightarrow[t\to\infty]{a.s} 0 \tag{34}$$

and the process of pure states $(|\psi_t\rangle\langle\psi_t|)$, *where* $\psi_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$, *for all* t, *satisfies*

$$|\psi_t\rangle\langle\psi_t| \xrightarrow[t\to\infty]{a.s} |\Omega\rangle\langle\Omega|\,. \tag{35}$$

Proof: Let us first treat the case of the jump-equation. We need to share into two cases, if there are jumps or if there are no jumps.

In the case where there is at least one jump. At the first jumping time T_1 we have

$$\begin{pmatrix} x_{T_1} \\ y_{T_1} \end{pmatrix} = \begin{pmatrix} x_{T_1-} \\ y_{T_1-} \end{pmatrix} + \begin{pmatrix} -x_{T_1-}+1 \\ -y_{T_1-} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Following the description of the solution of the jump equation, the solution after T_1 is given by the ordinary differential part with the new initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This initial condition satisfies $y_0 = 0$, then by Lemma 1, we get $y_t = 0$ for all $t \geq T_1$.

If there are no jumps, this corresponds to the event $A = \{\omega \in \Omega / N(\omega, \{(s,x) \in \mathbb{R}^2 / 0 < x < |y_s|^2\} = 0\}$. In this situation, the evolution of (y_t) is only given by the ordinary differential equation

$$y_t = y_0 + \frac{1}{2}\int_0^t (-y_s + |y_s|^2 y_s) ds\,.$$

We want to show that $|y_t|^2 \to 0$, when t goes to infinity. Derivating, we get

$$\frac{d}{dt}(|y_t|^2) = \frac{d}{dt}(y_t\bar{y_t}) = y_t\frac{d}{dt}(\bar{y_t}) + \bar{y_t}\frac{d}{dt}(y_t) = |y_t|^2(|y_t|^2-1)\,.$$

By Lemma 1, if $y_0 \neq 0$, we have $|y_t|^2 > 0$, for all t. In this case, dividing by y_t^2 we solve the equation and we get

$$|y_t|^2 = |y_s|^2 \times \exp\left(-(t-s) + \int_s^t (|y_u|^2)du\right),$$

for all $t > s$. In particular the function $t \to |y_t|^2$ is decreasing, then we get

$$|y_t|^2 \quad \leq \quad |y_s|^2 \exp\left(-2(t-s) + 2(t-s)|y_s|^2\right), \tag{36}$$

for all $t \leq s$. Since we have $|x_s|^2 + |y_s|^2 = 1$, for all s, we have $|y_s|^2 \leq 1$, for all s. With the estimation (36), in order to conclude, we need to remark that there exist s such that $|y_s|^2 < 1$. In the opposite case, we should have $|y_s| = 1$, for all s. Such situation appears actually with probability zero. Indeed, in this case the event A is $A = \{\omega \in \Omega / N(\omega, \{(s,x) \in \mathbb{R}^2 / 0 < x < 1 = 0, s > 0\}$ which is actually an event of probability zero (we have $P[\{\omega \in \Omega / N(\omega, \{(s,x) \in \mathbb{R}^2 / 0 < x < 1 = 0\}] = lim_n P[\{\omega \in \Omega / N(\omega, \{(s,x) \in \mathbb{R}^2 / 0 < x < 1 = 0, 0 < s < n\}]$ and $P[\{\omega \in \Omega / N(\omega, \{(s,x) \in \mathbb{R}^2 / 0 < x < 1 = 0, 0 < s < n\}] = \exp(-n)$). Thus, there exist s such that $|y_s|^2 < 1$. For this s, by taking the limit t goes to infinity in expression (36), we get $|y_t|^2 \to 0$.

With the above discussion, for the jump equation, it is easy to conclude that

$$y_t^2 \xrightarrow[t\to\infty]{a.s} 0\,.$$

Let us now treat the diffusive case. In order to prove the result we shall show first that $|y_t|^2$ converges almost surely to a random variable u_∞ when t goes to infinity. Second we show $u_\infty = 0$ almost surely. Using Ito rules, we get

$$\begin{aligned} d|y_t|^2 &= y_t d\bar{y_t} + \bar{y_t} dy_t + dy_t d\bar{y_t} \\ &= -|y_t|^2 dt - 2Re(\bar{x_t} y_t)|y_t|^2 dW_t \end{aligned}$$

As a consequence we have almost surely:

$$y_t^2 = y_s^2 + \int_s^t -|y_u|^2 du + \int_s^t -2Re(\bar{x_u} y_u)|y_u|^2 dW_u, \tag{37}$$

for all $t > s$. Let $(\mathcal{F}_t)$ be the filtration generated by the Brownian motion, that is $\mathcal{F}_t = \sigma\{W_u, u \leq t\}$. Since $\mathbb{E}\left[\int_s^t -2Re(\bar{x_u} y_u)|y_u|^2 dW_u | \mathcal{F}_s\right] = 0$, the above equation shows that

$$\mathbb{E}[|y_t|^2 | \mathcal{F}_s] \leq \mathbb{E}[|y_s|^2].$$

This way the process $(|y_t|^2)$ is a super martingale which is bounded (for all t, we have $0 \leq |y_t| \leq 1$). Therefore, this process converges almost surely to a non-negative random variable u_∞ when t goes to infinity. In order to show that this random variable is equal to

zero almost surely, we just have to show that $\mathbb{E}[u_\infty] = 0$. To this end, from Eq. 37 for $s = 0$, we get

$$\mathbf{E}[|y_t|^2] = y_0^2 + \int_0^t -\mathbf{E}[|y_s|^2]ds\,.$$

Solving the equation, we get

$$\mathbf{E}[|y_t|^2] = |y_0|^2 e^{-t}\,.$$

As a consequence, we get

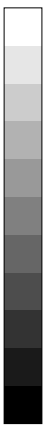
$$\mathbf{E}[y_t^2] \underset{t\to\infty}{\to} 0\,.$$

Now, using the Lebesque dominated convergence Theorem, we deduce that $\mathbb{E}[u_\infty] = 0$ and then $u_\infty = 0$ almost surely. The proposition is then proved. □

Remark: In the proof, we have supposed that the initial condition is deterministic. This result can be easily generalized by assuming that the initial condition is random and the same result holds.

Remark: In Probability Theory, for stochastic process, we usually consider invariant measure. Here, we are in a special case where the invariant measure is just the Dirac measure on the state $|\Omega\rangle\langle\Omega|$.





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