# SERIES OF ITERATED QUANTUM STOCHASTIC INTEGRALS[1]

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## Abstract

We consider series of iterated non-commutative stochastic integrals of scalar operators on the boson Fock space. We give a sufficient condition for these series to converge and to define a reasonable operator. An application of this criterion gives a condition for the convergence of some formal series of generalized integrator processes such as considered in [CEH].

## 1 Introduction

On the multiple boson Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+;\mathbb{C}^N))$ the quantum stochastic calculus ([HP1], [Me1], [Par]) gives the definition of stochastic integrals of the form $\int_0^t H^i_j(s)\,dA^i_j(s)$ where $A^i_0$, $A^0_i$, $A^i_j$, $i,j \in \{1,\ldots,N\}$ are the creation, annihilation and conservation processes respectively; the $H^i_j$ beeing adapted processes of operators on $\Phi$. These quantum stochastic integrals can be seen as non-commutative extensions of the usual stochastic integrals with respect to the Brownian motion (for example). In the classical stochastic calculus one considers chaotic expansion of random variables, that is, series of iterated stochastic integrals of scalar processes. It is also useful in many problems of quantum stochastic calculus to consider series of iterated non-commutative stochastic integrals that is, operators of the form

$$T_t = \lambda I + \sum_{n=1}^{\infty} \sum_{\varepsilon\in E^n} \int_{0<t_1<\ldots<t_n<t} h^{\varepsilon}_{t_1\ldots t_n}\, dA^{\varepsilon_1}_{t_1}\ldots dA^{\varepsilon_n}_{t_n} \tag{1}$$

where $I$ is the identity operator on $\Phi$ and $\lambda$ is a complex number; where $E = \{0,1,\ldots,N\}^2 \setminus \{(0,0)\}$ and for each $\eta = (\eta^1,\eta^2) \in E$ the symbol $A^\eta_t$ denotes the operator $A^{\eta^1}_{\eta^2}(t)$; where $\{h^{\varepsilon}_{t_1\ldots t_n}; n \in \mathbb{N}^*, 0 < t_1 < \ldots < t_n, \varepsilon \in E^n\}$ are scalar

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operators; and finally $\int_{0<t_1<\ldots<t_n<t} h^{\varepsilon}_{t_1\ldots t_n}\, dA^{\varepsilon_1}_{t_1}\ldots dA^{\varepsilon_n}_{t_n}$ denotes the iterated non-commutative stochastic integral

$$\int_0^t \left[\int_0^{t_n}\left[\cdots\int_0^{t_2} h^{\varepsilon}_{t_1\ldots t_n}\, dA^{\varepsilon_1}_{t_1}\right] dA^{\varepsilon_2}_{t_2}\right]\ldots dA^{\varepsilon_{n-1}}_{t_{n-1}}\Big] dA^{\varepsilon_n}_{t_n}.$$

We propose here to give a sufficient condition on the family $\{h^{\varepsilon}_{t_1\ldots t_n}\}$ for the expression (1) to define a reasonable operator on $\Phi$, that is for each iterated stochastic integral to be well-defined and for the series to converge weakly.

Let us first present a useful "short notation": the *Guichardet space* notation. Let $\mathcal{I}$ be the set $\{1,\ldots,N\}$. Let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$, then $\mathcal{P}=\cup_{n\in\mathbb{N}}\mathcal{P}_n$ where $\mathcal{P}_0=\{\emptyset\}$ and $\mathcal{P}_n$ is the set of $n$-element subsets of $\mathbb{R}^+$, for $n\geq 1$. Each set $\mathcal{P}_n$ can be identified with the increasing simplex $\{(t_1,\ldots,t_n)\in\mathbb{R}^n; 0<t_1<\ldots<t_n\}$, so it is equipped with the restriction of the Lebesgue measure on $\mathbb{R}^n$. Thus the set $\mathcal{P}$ can be equipped with a measure space structure whose element of volume is simply denoted $d\sigma$, $\sigma\in\mathcal{P}$ (in the following, elements of $\mathcal{P}$ will always be denoted by small greek letters $\alpha,\beta,\gamma,\sigma,\tau,\ldots$). It can be easily seen that the Fock space $\Phi=\Gamma(L^2(\mathbb{R}^+;\mathbb{C}^N))$ is isomorphic to the space $L^2(\mathcal{P}^{\mathcal{I}})$ (see [Me1], p. 103-104), called the *Guichardet space*. Thus a vector $f$ of $\Phi$ is determined by the family of complex numbers $f(\sigma)$, $\sigma=(\sigma_i)_{i\in\mathcal{I}}\in\mathcal{P}^{\mathcal{I}}$ which satisfies $\int_{\mathcal{P}^{\mathcal{I}}}|f(\sigma)|^2\, d\sigma<\infty$ where $d\sigma$ denotes $\prod_{i\in\mathcal{I}} d\sigma_i$. The family $\{f(\sigma),\sigma\in\mathcal{P}^{\mathcal{I}}\}$ is called the *chaotic expansion* of $f$.

In the following the symbol $+$ used for elements of $\mathcal{P}$ denotes the union of *disjoints* elements of $\mathcal{P}$.

Recall a fundamental property of integrals over Guichardet space, known as the $\oint$-Lemma.

**$\oint$-Lemma** (see [L-P])– *Let $\varphi$ be a positive (resp. integrable) measurable function on $\mathcal{P}^n$, then the function*

$$\alpha\mapsto\sum_{\alpha_1+\ldots+\alpha_n=\alpha}\varphi(\alpha_1,\ldots,\alpha_n)$$

*is a positive (resp. integrable) measurable function on $\mathcal{P}$ and one has*

$$\int_{\mathcal{P}}\cdots\int_{\mathcal{P}}\varphi(\alpha_1,\ldots,\alpha_n)\, d\alpha_1\ldots d\alpha_n=\int_{\mathcal{P}}\sum_{\alpha_1+\ldots+\alpha_n=\alpha}\varphi(\alpha_1,\ldots,\alpha_n)\, d\alpha. \qquad \blacksquare$$

Let $L^2_{lb}(\mathbb{R}^+;\mathbb{C}^N)$ be the space of locally bounded elements of $L^2(\mathbb{R}^+;\mathbb{C}^N)$. For $f$ in $L^2_{lb}(\mathbb{R}^+;\mathbb{C}^N)$, one puts $f_{t]}=f\mathbb{1}_{[0,t]}$, $f_{[t}=f\mathbb{1}_{[t,+\infty[}$ and denotes by $\varepsilon(f)$ the associated *coherent vector*, that is the element of $\Phi$ whose chaotic expansion is given by $[\varepsilon(f)](\sigma)=\prod_{i\in\mathcal{I}}\prod_{s\in\sigma_i} f_i(s)$ (where as usual the empty product is equal to 1), where the $f_i$'s are the coordinates of $f$. The space of finite linear combinaisons of coherent vectors is denoted $\mathcal{E}_{lb}$. It is a dense subspace of $\Phi$.

Let $\Phi_{t]}$ be the space $\Gamma(L^2([0,t];\mathbb{C}^N))$ and $\Phi_{[t}$ the space $\Gamma(L^2([t,\infty[;\mathbb{C}^N))$, then one has the *continuous tensor product structure* $\Phi\simeq\Phi_{t]}\otimes\Phi_{[t}$, in which we have $\varepsilon(f) = \varepsilon(f_{t]}) \otimes \varepsilon(f_{[t})$ (cf [Me2]). Actually, in the rest of the article, the spaces $\Phi$ and $\Phi_{t]} \otimes \Phi_{[t}$ are not distiguished. The tensor product symbol is even omited in the rest of the article ($\varepsilon(f) = \varepsilon(f_{t]})\varepsilon(f_{[t})$).

Recall that an *adapted process* of operators (in the sense of [HP1]) is a family $(H_t)_{t\geq 0}$ of operators from $\Phi$ to $\Phi$, defined on $\mathcal{E}_{lb}$ and such that:

i) the mapping $t \mapsto H_t\,\varepsilon(f)$ is strongly measurable for all $f$;

ii) $H_t\,\varepsilon(f_{t]}) \in \Phi_{t]}$ for all $t$, all $f$;

iii) $H_t\,\varepsilon(f) = [H_t\varepsilon(f_{t]})]\varepsilon(f_{[t})$ for all $t$, all $f$.

Recall that if an adapted process of operators $(H_t)_{t\geq 0}$ is such that for all $f \in L^2_{lb}(\mathbb{R}^+; \mathbb{C}^N)$

$$\int_0^t \|H_s\,\varepsilon(f)\|^2\,ds < \infty \quad \text{for all } t \geq 0, \tag{2}$$

then for every $\eta = (\eta^1, \eta^2) \in E$ the process $\left(\int_0^t H_s\,dA^\eta_s\right)_{s\geq 0}$ is well-defined as an adapted process of operators on $\mathcal{E}_{lb}$ given by

$$<\varepsilon(g)\,,\,\int_0^t H_s\,dA^\eta_s\,\varepsilon(f)> = \int_0^t \overline{g}_{\eta^1}(s) f_{\eta^2}(s) <\varepsilon(g)\,,\,H_s\,\varepsilon(f)> ds \tag{3}$$

where $g_0(s) = f_0(s) \equiv 1$.

## 2 Definition of the iterated integrals

**Lemma 1** – *Let $(H_t)_{t\geq 0}$ be an adapted process of operators satisfying* (2). *Then, for every $T > 0$, every $f \in L^2_{lb}(\mathbb{R}^+; \mathbb{C}^N)$ there exist two constants $C, C' \geq 0$ such that for all $0 \leq t < T$, all $\eta \in E$ one has*

$$\left\|\int_0^t H_s\,dA^\eta_s\,\varepsilon(f)\right\|^2 \leq C e^{C't} \int_0^t \|H_s\,\varepsilon(f)\|^2\,ds.$$

**Proof**

One can find many proofs of this kind of estimate. This particular one is taken from [At2] p. 93. One can find an analogous one in [Par] p. 188. ■

**Lemma 2** – *Let $t \in \mathbb{R}^+$. If $h$ is a function on $\mathcal{P}_n$ such that*

$$\int_{0<t_1<\ldots<t_n<t} |h^\varepsilon_{t_1\ldots t_n}|^2\,dt_1\ldots dt_n < \infty$$

*then the iterated non-commutative stochastic integrals*

$$T_t = \int_{0<t_1<\ldots<t_n<t} h(t_1,\ldots,t_n)\,dA^{\varepsilon_1}_{t_1}\ldots dA^{\varepsilon_n}_{t_n}$$

*are well-defined on $\mathcal{E}_{lb}$ for every $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \in E^n$.*

*If $h$ satisfies*

$$\int_{0<t_1<\ldots<t_n<\infty} e^{Ct_n}|h^\varepsilon_{t_1\ldots t_n}|^2\,dt_1\ldots dt_n < \infty$$

*for all $C \in \mathbb{R}^+$ then the iterated integral*

$$T = \int_{0<t_1<\ldots<t_n<\infty} h(t_1, \ldots, t_n)\, dA_{t_1}^{\varepsilon_1} \ldots dA_{t_n}^{\varepsilon_n}$$

*is well-defined on $\mathcal{E}_{lb}$ for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E^n$.*

**Proof**

From (2) it is sufficient to have for every $f \in L^2_{lb}(\mathbb{R}^+; \mathbb{C}^N)$

$$\int_0^t \Big|\Big| \Big(\int_{0<t_1<\ldots<t_{n-1}<t_n} h(t_1, \ldots, t_{n-1}, t_n)\, dA_{t_1}^{\varepsilon_1} \ldots dA_{t_{n-1}}^{\varepsilon_{n-1}}\Big)\varepsilon(f)\Big|\Big|^2 \, dt_n < \infty.$$

By successive applications of Lemma 1, one gets that this quantity is indeed dominated by

$$C^{n-1} e^{(n-1)C't} \int_{0<t_1<\ldots<t_n<t} ||h(t_1, \ldots, t_n)\varepsilon(f)||^2 \, dt_1 \ldots dt_n$$

$$= C^{n-1} e^{(n-1)C't} ||\varepsilon(f)||^2 \int_{0<t_1<\ldots<t_n<t} |h(t_1, \ldots, t_n)|^2 \, dt_1 \ldots dt_n.$$

Which is finite if $h$ is locally square integrable.

The case $t = +\infty$ is easy to get in the same way. ■

## 3 Correspondance between non-commutative chaotic expansions and Maassen-Meyer kernels

We now consider operators of the form (1) and call their representation as series of iterated non-commutative stochastic integrals, the *non-commutative chaotic expansion* of the operator. Of course, for the moment, we have not given a sense to the series; so let us consider operators of the form (1) such that the $\sum_{\varepsilon\in E^n} \int_{0<t_1<\ldots<t_n<t} h^\varepsilon_{t_1\ldots t_n}\, dA_{t_1}^{\varepsilon_1} \ldots dA_{t_n}^{\varepsilon_n}$, $n \in \mathbb{N}$, do not vanish only for a finite number of $n$. We are going to show that the operators of the form (1) have a *Maassen-Meyer kernel.* The reader does not need to know the theory of Maassen-Meyer kernels, we just use them as a useful language for our computations. All what is needed in this note is going to be defined. Note that what we here call Maassen-Meyer kernels are in fact Dermoune's extension ([Der]) to multiple Fock space of Maassen-Meyer kernels ([Maa], [Me2]).

Recall that $\mathcal{I} = \{1, \ldots N\}$. Let $\mathcal{M} = \mathcal{I}^2$. We consider in the following elements of $\mathcal{P}^\mathcal{I}$ and of $\mathcal{P}^\mathcal{M}$. An element $\alpha \in \mathcal{P}^\mathcal{I}$ is then a "vector" $(\alpha_i)_{i\in\mathcal{I}}$, an element $\beta \in \mathcal{P}^\mathcal{M}$ is written as a "matrix" $(\beta_i^j)_{i,j\in\mathcal{I}}$. We also underline ($\underline{\beta}$) the elements of $\mathcal{P}^\mathcal{M}$ in order to distinguish them from the elements of $\mathcal{P}^\mathcal{I}$. Thus when one integrates with respect to $\underline{\beta} \in \mathcal{P}^\mathcal{M}$, the symbol $d\underline{\beta}$ actually denotes $\prod_{i\in\mathcal{I}} \prod_{j\in\mathcal{I}} d\beta_j^i$.

An operator $T$ from $\Phi$ to $\Phi$ is said to have a *Maassen-Meyer kernel* on a domain $\mathcal{D}$ if there exists a measurable mapping, also denoted $T$, from $\mathcal{P}^\mathcal{I} \times \mathcal{P}^\mathcal{M} \times \mathcal{P}^\mathcal{I}$

to $\mathbb{C}$ such that $\mathcal{D}$ is included in $\mathrm{Dom}\, T$ and for every $f \in \mathcal{D}$ the chaotic expansion of $Tf$ is given by

$$[Tf](\alpha) = \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma_i^j + \tau_i = \alpha_i} T(\rho, \underline{\sigma}, \mu) f\big((\mu_i + \Sigma_j \sigma_j^i + \tau_i)_i\big)\, d\mu.$$

Let us make precise some more notations. For $(\rho, \underline{\sigma}, \tau) \in \mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}}$ we denote by $\mathrm{ord}(\rho, \underline{\sigma}, \tau)$ the set $\cup_{i \in \mathcal{I}} (\rho_i \cup \cup_j \sigma_i^j \cup \tau_i)$ but *ordered* in the increasing direction. We denote by $\varepsilon(\rho, \underline{\sigma}, \tau)$ the element of $E^n$, where $n = \#\mathrm{ord}(\rho, \underline{\sigma}, \tau)$, such that

$$\varepsilon(\rho, \underline{\sigma}, \tau)_k = \begin{cases} (i, 0) & \text{if the } k\text{-th smallest element of } \mathrm{ord}(\rho, \underline{\sigma}, \tau) \text{ is in } \rho_i \\ (j, i) & \text{if the } k\text{-th smallest element of } \mathrm{ord}(\rho, \underline{\sigma}, \tau) \text{ is in } \sigma_i^j \\ (0, i) & \text{if the } k\text{-th smallest element of } \mathrm{ord}(\rho, \underline{\sigma}, \tau) \text{ is in } \tau_i. \end{cases}$$

Conversely, for every $\varepsilon \in E^n$, every $(i, j) \in E$, let $\varepsilon(i, j) = \{k \in \{1, \dots n\};\, \varepsilon_k = (i, j)\}$. For given $\varepsilon \in E^n$ and $\sigma = \{t_1, \dots, t_n\} \in \mathcal{P}_n$, let $\sigma^+$ be the element of $\mathcal{P}^{\mathcal{I}}$ defined by $\sigma_i^+ = \{t_k\,;\, k \in \varepsilon(i, 0)\}$; let $\underline{\sigma}$ be the element of $\mathcal{P}^{\mathcal{M}}$ defined by $(\underline{\sigma})_j^i = \{t_k\,;\, k \in \varepsilon(i, j)\}$; let $\sigma^-$ be the element of $\mathcal{P}^{\mathcal{I}}$ defined by $\sigma_i^- = \{t_k\,;\, k \in \varepsilon(0, i)\}$.

**Proposition 3** – *Let $t \in \mathbb{R}^+ \cup \{+\infty\}$. Let $T_t$ be an operator on $\Phi$ which admits a non-commutative chaotic expansion*

$$T_t = \lambda I + \sum_{n=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0 < t_1 < \dots < t_n < t} h^{\varepsilon}_{t_1 \dots t_n}\, dA^{\varepsilon_1}_{t_1} \dots dA^{\varepsilon_n}_{t_n}$$

*which is well-defined on $\mathcal{E}_{lb}$ in so far as the corresponding conditions of Lemma 2 are satisfied and that the sum over $n$ is finite. Then $T_t$ admits a Maassen-Meyer kernel described by*

$$\begin{cases} T_t(\emptyset, \underline{\emptyset}, \emptyset) & = \lambda \\ T_t(\rho, \underline{\sigma}, \tau) & = h^{\varepsilon(\rho, \underline{\sigma}, \tau)}_{\mathrm{ord}(\rho, \underline{\sigma}, \tau)} \mathbb{1}_{[0,t]}(\rho, \underline{\sigma}, \tau) \end{cases}$$

*where $\mathbb{1}_{[0,t]}(\rho, \underline{\sigma}, \tau)$ is the indicator function of "$\rho_i, \sigma_i^j, \tau_i \subset [0, t]$, for all $i, j \in \mathcal{I}$".*

*Conversely, let $T$ be an operator on $\Phi$ having a Maassen-Meyer kernel representation given by the kernel $(\rho, \underline{\sigma}, \tau) \mapsto T(\rho, \underline{\sigma}, \tau)$. Then $T$ admits a non-commutative chaotic expansion*

$$T = \lambda I + \sum_{n=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0 < t_1 < \dots < t_n < \infty} h^{\varepsilon}_{t_1 \dots t_n}\, dA^{\varepsilon_1}_{t_1} \dots dA^{\varepsilon_n}_{t_n}$$

*where, for $\sigma = \{t_1, \dots t_n\}$, $h^{\varepsilon}_{\sigma} = T(\sigma^+, \underline{\sigma}, \sigma^-)$.*

**Proof**

By (3) one has

$$\begin{aligned} < \varepsilon(g)\,,\, T_t\, \varepsilon(f) > = \lambda + \sum_{n=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0 < t_1 < \dots < t_n < t} & \overline{g}_{\varepsilon_1^1}(t_1) \dots \overline{g}_{\varepsilon_n^1}(t_n) \\ & \times f_{\varepsilon_1^2}(t_1) \dots f_{\varepsilon_n^2}(t_n)\, h^{\varepsilon}_{t_1 \dots t_n}\, dt_1 \dots dt_n < \varepsilon(g)\,,\, \varepsilon(f) >. \end{aligned}$$

For every $\varepsilon \in E^n$, every $(i,j) \in E$, recall that $\varepsilon(i,j) = \{k \in \{1,\dots n\}; \varepsilon_k = (i,j)\}$. Then

$$< \varepsilon(g)\,,\, T_t\,\varepsilon(f) > = \lambda + \sum_{n=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0<t_1<\dots<t_n<t} \prod_{i\in\mathcal{I}} \prod_{k\in\varepsilon(i,0)} \overline{g}_i(t_k)$$
$$\times \prod_{i\in\mathcal{I}} \prod_{j\in\mathcal{I}} \prod_{k\in\varepsilon(i,j)} f_j(t_k)\,\overline{g}_i(t_k) \prod_{i\in\mathcal{I}} \prod_{k\in\varepsilon(0,i)} f_i(t_k)$$
$$\times\, h^{\varepsilon}_{t_1\dots t_n}\, dt_1 \dots dt_n < \varepsilon(g)\,,\, \varepsilon(f) >.$$

But, for a given $\alpha = \{0 < t_1 < \dots < t_n < t\} \in \mathcal{P}_n$, every $\varepsilon \in E^n$ defines a partition, $\alpha = \sum_i \alpha_i$ and $\rho_i + \sum_j \sigma_i^j + \tau_i = \alpha_i$, of $\alpha$ simply by taking $\rho_i = \{t_k; k \in \varepsilon(i,0)\}$, $\sigma_j^i = \{t_k; k \in \varepsilon(i,j)\}$, $\tau_i = \{t_k; k \in \varepsilon(0,i)\}$. Conversely, let $(\rho, \underline{\sigma}, \tau) \in \mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}}$, then $\varepsilon(\rho,\underline{\sigma},\tau)$ is an element of $E^n$ which corresponds to the partition $\rho_i + \sum_j \sigma_i^j + \tau_i = \alpha_i$ of $\mathrm{ord}(\rho,\underline{\sigma},\tau)$. So

$$< \varepsilon(g)\,,\, T_t\,\varepsilon(f) > = \lambda + \sum_{n=1}^{\infty} \int_{\alpha\subset\mathcal{P}_n} \mathbb{1}_{[0,t]}(\alpha) \sum_{\Sigma_i \alpha_i = \alpha} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma_i^j + \tau_i = \alpha_i} \prod_{i\in\mathcal{I}} \prod_{s\in\rho_i} \overline{g}_i(s)$$
$$\times \prod_{i\in\mathcal{I}} \prod_{j\in\mathcal{I}} \prod_{s\in\sigma_i^j} f_j(s)\,\overline{g}_i(s) \prod_{i\in\mathcal{I}} \prod_{s\in\tau_i} f_i(s)$$
$$\times\, h_\alpha^{\varepsilon(\rho,\underline{\sigma},\tau)}\, d\alpha < \varepsilon(g)\,,\, \varepsilon(f) >$$
$$= \int_{\alpha\in\mathcal{P}} \mathbb{1}_{[0,t]}(\alpha) \sum_{\Sigma_i \alpha_i = \alpha} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma_i^j + \tau_i = \alpha_i} \overline{[\varepsilon(g)]}\big((\rho_i + \Sigma_j \sigma_i^j)_i\big)$$
$$\times [\varepsilon(f)]\big((\Sigma_j \sigma_j^i + \tau_i)_i\big) h_\alpha^{\varepsilon(\rho,\underline{\sigma},\tau)}\, d\alpha < \varepsilon(g)\,,\, \varepsilon(f) >$$

by putting $h_\emptyset^{\varepsilon(\emptyset,\underline{\emptyset},\emptyset)} = \lambda$. By the $\pounds$-Lemma, this gives

$$< \varepsilon(g)\,,\, T_t\,\varepsilon(f) > = \int_{\rho\in\mathcal{P}^{\mathcal{I}}} \int_{\underline{\sigma}\in\mathcal{P}^{\mathcal{M}}} \int_{\tau\in\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\rho,\underline{\sigma},\tau)\, \overline{[\varepsilon(g)]}\big((\rho_i + \Sigma_j \sigma_i^j)_i\big)$$
$$\times [\varepsilon(f)]\big((\Sigma_j \sigma_j^i + \tau_i)_i\big)\; h^{\varepsilon(\rho,\underline{\sigma},\tau)}_{\mathrm{ord}(\rho,\underline{\sigma},\tau)}\; d\rho\, d\underline{\sigma}\, d\tau < \varepsilon(g)\,,\, \varepsilon(f) >.$$

But $< \varepsilon(g)\,,\, \varepsilon(f) > = \int_{\mathcal{P}^{\mathcal{I}}} \overline{[\varepsilon(g)]}(\mu)[\varepsilon(f)](\mu)\, d\mu$, therefore

$$< \varepsilon(g)\,,\, T_t\,\varepsilon(f) > = \int_{\rho\in\mathcal{P}^{\mathcal{I}}} \int_{\underline{\sigma}\in\mathcal{P}^{\mathcal{M}}} \int_{\tau\in\mathcal{P}^{\mathcal{I}}} \int_{\mu\in\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\rho,\underline{\sigma},\tau)$$
$$\times \overline{[\varepsilon(g)]}\big((\rho_i + \Sigma_j \sigma_i^j + \mu_i)_i\big)[\varepsilon(f)]\big((\mu_i + \Sigma_j \sigma_j^i + \tau_i)_i\big)$$
$$\times\, h^{\varepsilon(\rho,\underline{\sigma},\tau)}_{\mathrm{ord}(\rho,\underline{\sigma},\tau)}\, d\rho\, d\underline{\sigma}\, d\tau\, d\mu.$$

Now, in order to get coherent notations, we exchange $\mu$ and $\tau$ in the previous identity. This gives

$$< \varepsilon(g)\,,\, T_t\,\varepsilon(f) > = \int_{\rho\in\mathcal{P}^{\mathcal{I}}} \int_{\underline{\sigma}\in\mathcal{P}^{\mathcal{M}}} \int_{\tau\in\mathcal{P}^{\mathcal{I}}} \int_{\mu\in\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\rho,\underline{\sigma},\mu)$$
$$\times \overline{[\varepsilon(g)]}\big((\rho_i + \Sigma_j \sigma_i^j + \tau_i)_i\big)\, h^{\varepsilon(\rho,\underline{\sigma},\mu)}_{\mathrm{ord}(\rho,\underline{\sigma},\mu)}$$

$$\times [\varepsilon(f)]\big((\mu_i + \Sigma_j \sigma^i_j + \tau_i)_i\big)\, d\mu\, d\rho\, d\underline{\sigma}\, d\tau.$$

Using the $\oint$-Lemma once again, we get

$$<\varepsilon(g)\,,\, T_t\, \varepsilon(f)> = \int_{\alpha \in \mathcal{P}^{\mathcal{I}}} \overline{[\varepsilon(g)]}(\alpha) \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma^j_i + \tau_i = \alpha_i} \mathbb{1}_{[0,t]}(\rho, \underline{\sigma}, \mu)$$

$$\times\, h^{\varepsilon(\rho,\underline{\sigma},\mu)}_{\mathrm{ord}(\rho,\underline{\sigma},\mu)}\, [\varepsilon(f)]\big((\mu_i + \Sigma_j \sigma^i_j + \tau_i)_i\big)\, d\mu\, d\sigma.$$

Let $\widetilde{T}_t$ be the Maassen-Meyer kernel given by the statement of the Proposition. We have proved that for every $f, g \in L^2_{lb}(\mathbb{R}^+)$ we have

$$<\varepsilon(g)\,,\, T_t\, \varepsilon(f)> = <\varepsilon(g)\,,\, \widetilde{T}_t\, \varepsilon(f)>.$$

So one concludes.

The converse is now easy. ■

## 4 A criterion for the convergence of Maassen-Meyer kernels

In the previous section we have identified any finite series of iterated non-commutative stochastic integrals with a Maassen-Meyer kernel. Now, in the Fock space with multiplicity one (that is, $N = 1$) Belavkin and Lindsay [B-L] have given a criterion for a mapping $T : \mathcal{P}^3 \to \mathbb{C}$ to define a "reasonable operator" on $\Phi$ whose Maassen-Meyer kernel is $T$. We give an extension of their result to the case of any finite multiplicity.

Let $a \in (0, +\infty)$. Define

$$\Phi(a) = \{f \in L^0(\mathcal{P}^{\mathcal{I}});\ \int_{\mathcal{P}^{\mathcal{I}}} a^{\Sigma_i \# \sigma_i}\, |f(\sigma)|^2\, d\sigma < \infty\}.$$

Equipped with the scalar product $<g\,,\, f>_{(a)} = \int_{\mathcal{P}^{\mathcal{I}}} a^{\Sigma_i \# \sigma_i}\, \overline{g}(\sigma) f(\sigma)\, d\sigma$ the space $\Phi(a)$ is a Hilbert space, whose norm is denoted $||\cdot||_{(a)}$; it is a dense subspace of $\Phi$ for $a \geq 1$.

Let $T$ be a measurable mapping from $\mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$. One identifies the mapping $T$ with the operator $T$ from $L^0(\mathcal{P}^{\mathcal{I}})$ into itself defined by

$$[Tf](\alpha) = \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma^j_i + \tau_i = \alpha_i} T(\rho, \underline{\sigma}, \mu) f\big((\mu_i + \Sigma_j \sigma^i_j + \tau_i)_i\big)\, d\mu. \tag{4}$$

Let us consider the quantity

$$\int_{\mathcal{P}^{\mathcal{I}}} |\overline{g}(\alpha)\, [Tf](\alpha)|\, d\alpha.$$

It is dominated by

$$\int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma^j_i + \tau_i = \alpha_i} \big| T(\rho, \underline{\sigma}, \mu) f\big((\mu_i + \Sigma_j \sigma^i_j + \tau_i)_i\big)$$

$$\times g\big((\rho_i + \Sigma_j \sigma^j_i + \tau_i)_i\big)\big|\, d\mu\, d\alpha$$

$$= \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} \big| T(\rho, \underline{\sigma}, \mu) f\big((\mu_i + \Sigma_j \sigma^i_j + \tau_i)_i\big)$$

$$\times g\big((\rho_i + \Sigma_j \sigma^j_i + \tau_i)_i\big)\big|\, d\mu\, d\rho\, d\tau\, d\underline{\sigma}.$$

Let us change the notations and put $\beta_i^i = \sigma_i^i + \tau_i$ and $\beta_j^i = \sigma_j^i$ for $i \neq j$. The previous expression then becomes

$$\int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} \sum_{\forall i,\, \gamma_i \subset \beta_i^i} \left| T(\rho, \underline{\beta \dot{\gamma}}, \mu) f\big((\mu_i + \Sigma_j \beta_j^i)_i\big) g\big((\rho_i + \Sigma_j \beta_i^j)_i\big) \right| d\mu\, d\rho\, d\underline{\beta}$$

where $\underline{\beta \dot{\gamma}}$ denotes the element $\underline{\delta}$ of $\mathcal{P}^{\mathcal{M}}$ such that $\delta_i^i = \gamma_i$ and $\delta_j^i = \beta_j^i$ for $i \neq j$.

Define the measurable mapping

$$\begin{aligned} T' \ &: \mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}} \ \longrightarrow \ \mathbb{C} \\ &(\rho, \underline{\beta}, \tau) \mapsto \sum_{\forall i,\, \gamma_i \subset \beta_i^i} T(\rho, \underline{\beta \dot{\gamma}}, \tau). \end{aligned}$$

From now on, for $\alpha \in \mathcal{P}^{\mathcal{I}}$ we denote by $|\alpha|$ the quantity $\sum_i \#\alpha_i$, and for $\underline{\beta} \in \mathcal{P}^{\mathcal{M}}$ we denote by $|\underline{\beta}|$ the quantity $\sum_{i,j} \#\beta_j^i$.

For $a, b, c \in (0, +\infty)$, let

$$T'_{a,b,c}(\alpha, \underline{\beta}, \gamma) = \frac{T'(\alpha, \underline{\beta}, \gamma)}{\sqrt{a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|}}}$$

$$||T'||_{a,b,c} = \Big( \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \sup_{\underline{\beta} \in \mathcal{P}^{\mathcal{M}}} |T'_{a,b,c}(\alpha, \underline{\beta}, \gamma)|^2 \, d\alpha\, d\gamma \Big)^{1/2}.$$

The following estimate is inspired by [B-L].

**Lemma 4** – *Let $p, a, q, c \in (0, +\infty)$ with $p > a$, $q > c$. Let $f, g \in L^0(\mathcal{P}^{\mathcal{I}})$ and $T$ be a measurable map from $\mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$. Then one has*

$$\int_{\mathcal{P}^{\mathcal{I}}} |\overline{g}(\sigma)\, [Tf](\sigma)| \, d\sigma \leq ||g||_{(p)}\, ||f||_{(q)}\, ||T'||_{a,b,c}$$

*where $b = \frac{\sqrt{(p-a)(q-c)}}{N}$.*

**Proof**

Let $b = \frac{\sqrt{(p-a)(q-c)}}{N}$. One has

$$\begin{aligned} &\int_{\mathcal{P}^{\mathcal{I}}} |\overline{g}(\sigma)\, [Tf](\sigma)| \, d\sigma \\ &\qquad \leq \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} \left| T'(\rho, \underline{\beta}, \mu) f\big((\mu_i + \Sigma_j \beta_j^i)_i\big) g\big((\rho_i + \Sigma_j \beta_i^j)_i\big) \right| d\mu\, d\rho\, d\underline{\beta} \\ &\qquad \leq \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} \sqrt{a^{|\rho|} \Big(\frac{p-a}{N}\Big)^{|\underline{\beta}|} \Big(\frac{q-c}{N}\Big)^{|\underline{\beta}|} c^{|\mu|}} \\ &\qquad\quad \times \left| T'_{a,b,c}(\rho, \underline{\beta}, \mu) f\big((\mu_i + \Sigma_j \beta_j^i)_i\big) g\big((\rho_i + \Sigma_j \beta_i^j)_i\big) \right| d\mu\, d\rho\, d\underline{\beta} \\ &\qquad \leq \Big( \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} a^{|\rho|} \Big(\frac{p-a}{N}\Big)^{|\underline{\beta}|} |g\big((\rho_i + \Sigma_j \beta_i^j)_i\big)|^2 \, d\rho\, d\underline{\beta} \Big)^{\frac{1}{2}} \\ &\qquad\quad \times \int_{\mathcal{P}^{\mathcal{I}}} \sqrt{c^{|\mu|}} \Big( \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{M}}} \Big(\frac{q-c}{N}\Big)^{|\underline{\beta}|} \left| T'_{a,b,c}(\rho, \underline{\beta}, \mu) \right|^2 \end{aligned}$$

$$\times \left|f\left((\mu_i+\Sigma_j\beta_j^i)_i\right)\right|^2 d\rho\, d\underline{\beta}\Big)^{\frac12} d\mu$$

$$\leq \Big(\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}}\sum_{\forall i,\,\Sigma_j\beta_i^j=\lambda_i} a^{|\rho|}\Big(\frac{p-a}{N}\Big)^{|\lambda|}|g\left((\rho_i+\lambda_i)_i\right)|^2\, d\rho\, d\lambda\Big)^{\frac12}$$

$$\times \int_{\mathcal{P}^{\mathcal{I}}}\sqrt{c^{|\mu|}}\Big(\int_{\mathcal{P}^{\mathcal{M}}}\Big(\frac{q-c}{N}\Big)^{|\underline{\beta}|}\left|f\left((\mu_i+\Sigma_j\beta_j^i)_i\right)\right|^2 d\underline{\beta}$$

$$\times \int_{\mathcal{P}^{\mathcal{I}}}\sup_{\underline{\beta}}\left|T'_{a,b,c}(\rho,\underline{\beta},\mu)\right|^2 d\rho\Big)^{\frac12} d\mu$$

$$\leq \Big(\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} N^{|\lambda|}a^{|\rho|}\Big(\frac{p-a}{N}\Big)^{|\lambda|}|g\left((\rho_i+\lambda_i)_i\right)|^2\, d\rho\, d\lambda\Big)^{\frac12}$$

$$\times \Big(\int_{\mathcal{P}^{\mathcal{I}}} c^{|\mu|}\int_{\mathcal{P}^{\mathcal{M}}}\Big(\frac{q-c}{N}\Big)^{|\underline{\beta}|}\left|f\left((\mu_i+\Sigma_j\beta_j^i)_i\right)\right|^2 d\underline{\beta}\, d\mu\Big)^{\frac12}$$

$$\times \Big(\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}}\sup_{\underline{\beta}}\left|T'_{a,b,c}(\rho,\underline{\beta},\mu)\right|^2 d\rho\, d\mu\Big)^{\frac12}$$

$$\leq ||g||_{(p)}||f||_{(q)}||T'||_{a,b,c}. \qquad \blacksquare$$

**Proposition 5** – *Let $T$ be a measurable map from $\mathcal{P}^{\mathcal{I}}\times\mathcal{P}^{\mathcal{M}}\times\mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$ satisfying $||T'||_{a,b,c}<\infty$ for some $a,b,c\in(0,+\infty)$. Then, for every $e\in(0,+\infty)$, $T$ defines a bounded operator from $\Phi(c+b\sqrt{N}/e)$ to $\Phi((a+b\sqrt{N}e)^{-1})$ with norm at most $||T'||_{a,b,c}$.*

## Proof

By Lemma 4 we have proved that $|<g\,,\,\widetilde{T}f>|\leq ||g||_{(p)}\,||f||_{(q)}\,||T'||_{a,b,c}$ for any $p,q$ such that $p>a$, $q>c$ and $b=\frac{\sqrt{(p-a)(q-c)}}{N}$. Take $e\in(0,+\infty)$, put $b'=\sqrt{N}be$ and $b''=\sqrt{N}b/e$, put $p=a+b'$ and $q=c+b''$, we then have $b=\frac{\sqrt{(p-a)(q-c)}}{N}$. Take $f,g\in L^0(\mathcal{P}^{\mathcal{I}})$ and define $\widetilde{g}$ in $L^0(\mathcal{P}^{\mathcal{I}})$ by $\widetilde{g}(\sigma)=1/(a+b')^{|\sigma|}g(\sigma)$. Then applying Lemma 4 one gets

$$|<\widetilde{g}\,,\,Tf>|$$

$$=\Big|\int_{\mathcal{P}^{\mathcal{I}}}\frac{1}{(a+b')^{|\sigma|}}\overline{g}(\sigma)[Tf](\sigma)\,d\sigma\Big|$$

$$\leq \Big(\int_{\mathcal{P}^{\mathcal{I}}}\Big(\frac{p}{(a+b')^2}\Big)^{|\sigma|}|g(\sigma)|^2\,d\sigma\Big)^{1/2}\Big(\int_{\mathcal{P}^{\mathcal{I}}} q^{|\sigma|}\,|f(\sigma)|^2\,d\sigma\Big)^{1/2}||T'||_{a,b,c}$$

$$=\Big(\int_{\mathcal{P}^{\mathcal{I}}}\Big(\frac{1}{a+b'}\Big)^{|\sigma|}|g(\sigma)|^2\,d\sigma\Big)^{1/2}\Big(\int_{\mathcal{P}^{\mathcal{I}}}(c+b'')^{|\sigma|}\,|f(\sigma)|^2\,d\sigma\Big)^{1/2}||T'||_{a,b,c}.$$

So one concludes easily. $\blacksquare$

Now notice that if $T$ is a measurable mapping from $\mathcal{P}^{\mathcal{I}}\times\mathcal{P}^{\mathcal{M}}\times\mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$, then the operator $T$ on the Fock space whose Maassen-Meyer kernel is given by the mapping $T$ satisfies identity (4) wherever it is meaningful. So by Proposition 5, one easily gets the following results.

**Proposition 6** – *Let $T$ be a measurable mapping from $\mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{M}} \times \mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$ such that $||T'||_{a,b,c} < \infty$ for some $a, b, c \in (0, +\infty)$.*

*i) If $a < 1$ then the associated Maassen-Meyer kernel is well defined as an operator from $\Phi$ to $\Phi$, with a dense domain containing $\Phi(c + N^2b^2/(1-a))$. This operator is bounded from $\Phi(c + N^2b^2/(1-a))$ to $\Phi$ with norm at most $||T'||_{a,b,c}$.*

*ii) If $a < 1$, $c < 1$ and $b = \frac{\sqrt{(1-a)(1-c)}}{N}$ then the Maassen-Meyer kernel $T$ is a bounded operator on $\Phi$, with same bound for the norm.* ■

Let us give a simple example to illustrate these estimates.

If $H$ is a Hilbert-Schmidt operator on $\Phi$, it is then a Hilbert-Schmidt operator on $L^2(\mathcal{P}^{\mathcal{I}})$, thus there exists a mapping $\varphi$ from $\mathcal{P}^{\mathcal{I}} \times \mathcal{P}^{\mathcal{I}}$ to $\mathbb{C}$ such that

$$\int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} |\varphi(\alpha, \gamma)|^2 \, d\alpha \, d\gamma < \infty$$

and satisfying

$$[Hf](\sigma) = \int_{\mathcal{P}^{\mathcal{I}}} \varphi(\sigma, \mu) f(\mu) \, d\mu \tag{5}$$

for all $f \in \Phi$. Now consider the Maassen-Meyer kernel $T$ defined by

$$T(\alpha, \underline{\beta}, \gamma) = \begin{cases} (-1)^{\Sigma_i \# \beta_i^i} \varphi(\alpha, \gamma) & \text{if } \beta_j^i = \emptyset \text{ for all } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Applying (4) we get

$$\begin{aligned}
[Tf](\alpha) &= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma_i^j + \tau_i = \alpha_i} T(\rho, \underline{\sigma}, \mu) f\big((\mu_i + \Sigma_j \sigma_j^i + \tau_i)_i\big) \, d\mu \\
&= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \Sigma_j \sigma_i^j + \tau_i = \alpha_i} \mathbb{1}_{\forall i \neq j,\, \sigma_j^i = \emptyset} (-1)^{\Sigma_i \# \sigma_i^i} \varphi(\rho, \mu) \\
&\qquad\qquad \times f\big((\mu_i + \Sigma_j \sigma_j^i + \tau_i)_i\big) \, d\mu \\
&= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \rho_i + \sigma_i^i + \tau_i = \alpha_i} (-1)^{\Sigma_i \# \sigma_i^i} \varphi(\rho, \mu) f\big((\mu_i + \sigma_i^i + \tau_i)_i\big) \, d\mu \\
&= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \delta_i \subset \alpha_i} \varphi(\sigma \setminus \delta, \mu) f\big((\mu_i + \delta_i)_i\big) \sum_{\forall i,\, \sigma_i^i \subset \delta_i} (-1)^{\Sigma_i \# \sigma_i^i} \, d\mu \\
&= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \sum_{\forall i,\, \delta_i \subset \alpha_i} \varphi(\sigma \setminus \delta, \mu) f\big((\mu_i + \delta_i)_i\big) \mathbb{1}_{\delta = \emptyset} \, d\mu \\
&\quad \text{(by Mœbius inversion formula)} \\
&= \int_{\mu \in \mathcal{P}^{\mathcal{I}}} \varphi(\sigma, \mu) f(\mu) \, d\mu \\
&= [Hf](\sigma).
\end{aligned}$$

Thus the Hilbert-Schmidt operator $H$ admits $T(\alpha, \underline{\beta}, \gamma)$ as a Maassen-Meyer kernel. Let us apply our criterion to this kernel. We get

$$\begin{aligned}
&\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \sup_{\underline{\beta}} \frac{\left|\sum_{\forall i,\, C_i \subset \beta^i_i} T(\alpha, \underline{\beta : C}, \gamma)\right|^2}{\sqrt{a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|}}}\, d\alpha\, d\gamma \\
&\quad = \int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \sup_{\underline{\beta}} \frac{\sum_{\forall i,\, C_i \subset \beta^i_i} (-1)^{\Sigma_i \# C_i} \mathbb{1}_{\forall i \neq j,\, \beta^i_j = \emptyset} \left|\varphi(\alpha, \gamma)\right|^2}{\sqrt{a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|}}}\, d\alpha\, d\gamma \\
&\quad = \int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \frac{\mathbb{1}_{\forall i, j,\, \beta^i_j = \emptyset} \left|\varphi(\alpha, \gamma)\right|^2}{\sqrt{a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|}}}\, d\alpha\, d\gamma \\
&\quad = \int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \frac{\left|\varphi(\alpha, \gamma)\right|^2}{\sqrt{a^{|\alpha|} c^{|\gamma|}}}\, d\alpha\, d\gamma.
\end{aligned}$$

This quantity is finite for $a = 1$, $c = 1$ and $b = 0$. Hence we recover that this kernel defines a bounded operator (Proposition 6).

## 5 Convergence of series of iterated non-commutative stochastic integrals

We are now able to give our final result, which gives a condition for a series of iterated non-commutative stochastic integrals of the form (1) to define a densily defined operator on $\Phi$.

**Theorem 7** – *Let $t \in \mathbb{R}^+ \cup \{+\infty\}$. For all $n \in \mathbb{N}$, all $\varepsilon \in E^n$, let $h^\varepsilon$ be a function on $\mathcal{P}_n$ satisfying the condition of Lemma 2. Suppose that the functions $h^\varepsilon$ satisfy*

$$||T'||^2_{a,b,c} = \int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\alpha, \gamma) \sup_{\underline{\beta} \in \mathcal{P}^{\mathcal{M}}} \frac{\left|\sum_{\forall i,\, C_i \subset \beta^i_i} h^{\varepsilon(\alpha, \underline{\beta : C}, \gamma)}_{\mathrm{ord}(\alpha, \underline{\beta : C}, \gamma)}\right|^2 \mathbb{1}_{[0,t]}(\beta)}{(a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|})}\, d\alpha\, d\gamma < \infty$$

*for some $a \in (0,1)$, $b, c \in (0, +\infty)$. Then the operator*

$$T_t = \lambda I + \sum_{i=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0 < t_1 < \ldots < t_n < t} h^\varepsilon_{t_1 \ldots t_n}\, dA^{\varepsilon_1}_{t_1} \ldots dA^{\varepsilon_n}_{t_n}$$

*is well-defined as an operator on $\Phi$ with (dense) domain $\Phi(c + N^2 b^2/(1-a))$, bounded from $\Phi(c + N^2 b^2/(1-a))$ to $\Phi$ with norm at most $||T'||_{a,b,c}$. Furthermore, if $c < 1$ and $b = \frac{\sqrt{(1-c)(1-a)}}{N}$ the operator $T_t$ is then a bounded operator on $\Phi$, with the same bound for the norm.*

**Proof**

Combine Lemma 2, Proposition 3 and Proposition 5. ■

We also have a uniqueness theorem for the representation of operators as non-commutative chaotic expansions.

**Theorem 8** – *Let $T$ be an operator on $\Phi$ having a representation of the form*

$$T = \lambda I + \sum_{i=1}^{\infty} \sum_{\varepsilon \in E^n} \int_{0<t_1<\ldots<t_n<t} h^{\varepsilon}_{t_1,\ldots,t_n} \, dA^{\varepsilon_1}_{t_1} \ldots dA^{\varepsilon_n}_{t_n}$$

*for some $t \in \mathbb{R}^+ \cup \{+\infty\}$. Then $T$ vanishes if and only if for all $n \in \mathbb{N}$, all $\varepsilon \in E^n$, almost all $(t_1, \ldots, t_n) \in \mathcal{P}_n$ one has $h^{\varepsilon}_{t_1\ldots t_n} = 0$.*

**Proof**

We have proved in Proposition 3 that there is a one-to-one correspondance between non-commutative chaotic expansions and Maassen-Meyer kernels. In [At3], Theorem IV.6, the uniqueness of Maassen-Meyer kernels representation is proved, for any multiplicity of the Fock space. So one concludes. ■

## 6 The case of iterated non-commutative stochastic integrals containing the time integrator

In all the results considered previously we have never considered the case where the iterated non-commutative stochastic integrals are containing the time process as an integrator. Indeed, we have only considered the integrators $dA^i_j$ for $(i, j) \in \{0, 1, \ldots, N\}^2 \setminus \{(0, 0)\}$, avoiding the term $dA^0_0(t)$ which in fact corresponds to the time integrator $dt$. There are two reasons for that. The case of series of iterated non-commutative stochastic integrals without time integrator really corresponds to the notion of *non-commutative chaotic expansion*; that is, the representation of a given operator in terms of a series of iterated integrals of scalar operators with respect to the "quantum noises" : the creation, annihilation and exchange processes. Another way to understand these series as non-commutative chaotic expansions is to see that, in some cases like the case of Hilbert-Schmidt operators on the Fock space (cf [At1]), these series can be obtained by iterating the integral representation of the operators; exactely like one can prove the chaotic representation property of the Brownian motion by iterating the predictable representation property. The second reason is that in the case where there is no time integrators, we have uniqueness of the series (Theorem 8); this uniqueness is lost when allowing the time integrator. But it is convenient in some problems (such as the application of the next section) to consider series with the time integrator. That is why we present here the results corresponding to this case. Let us first make precise some new notations.

Let $F = \{0, 1, \ldots, N\}^2$. For $\varepsilon = (\varepsilon^1, \varepsilon^2) \in F$ the operator $A^{\varepsilon}_t$ denotes, as previously, $A^{\varepsilon^1}_{\varepsilon^2}(t)$ for $\varepsilon \neq (0, 0)$ and $A^0_0(t) = tI$ if $\varepsilon = (0, 0)$. We now want to consider operators of the form

$$T_t = \lambda I + \sum_{n=1}^{\infty} \sum_{\varepsilon \in F^n} \int_{0<t_1<\ldots<t_n<t} h^{\varepsilon}_{t_1\ldots t_n} \, dA^{\varepsilon_1}_{t_1} \ldots dA^{\varepsilon_n}_{t_n}. \tag{6}$$

In the following we consider elements of $\mathcal{P}^{\mathcal{I}}$, $\mathcal{P}^{\mathcal{M}}$ and $\mathcal{P}$, so in order to distinguish them we will denote as previously elements of $\mathcal{P}^{\mathcal{I}}$ with small greek letters $(\rho)$, elements of $\mathcal{P}^{\mathcal{M}}$ with underlined small greek letter ($\underline{\sigma}$) and elements of $\mathcal{P}$ with upperlined small greek letters ($\overline{\delta}$). A quadruple $(\rho,\underline{\sigma},\tau,\overline{\delta})$ in $\mathcal{P}^{\mathcal{I}}\otimes\mathcal{P}^{\mathcal{M}}\otimes\mathcal{P}^{\mathcal{I}}\otimes\mathcal{P}$ can be seen as a $(N+1)\otimes(N+1)$ matrix $(\alpha^i_j)_{i,j\in\{0,1,\ldots,N\}}$ of elements of $\mathcal{P}$ by putting $\alpha^0_0=\delta$, $\alpha^i_0=\rho_i$, $\alpha^i_j=\sigma^i_j$ and $\alpha^0_i=\tau_i$, for $i,j=1,\ldots,N$. So, by $\mathrm{ord}(\rho,\underline{\sigma},\tau,\overline{\delta})$ we mean the ordered set $\cup_{i,j=0,\ldots,N}\,\alpha^i_j$; by $\varepsilon(\rho,\underline{\sigma},\tau,\overline{\delta})$ we mean the element of $\mathcal{P}^n$ (where $n=\#\mathrm{ord}(\rho,\underline{\sigma},\tau,\overline{\delta})$) such that $\varepsilon(\rho,\underline{\sigma},\tau,\overline{\delta})_k=(i,j)$ if the $k$-th smallest element of $\mathrm{ord}(\rho,\underline{\sigma},\tau,\overline{\delta})$ is in $\alpha^i_j$.

By using the same kind of proof as in Proposition 3, one can prove that an operator $T_t$ given by a series of the form (6) admits a Maassen-Meyer kernel $T_t$ which given by

$$T_t(\rho,\underline{\sigma},\tau)=\int_{\mathcal{P}} h^{\varepsilon(\rho,\underline{\sigma},\tau,\overline{\delta})}_{\mathrm{ord}(\rho,\underline{\sigma},\tau,\overline{\delta})}\,d\overline{\delta}\,.$$

So we obtain the following easy extension of Theorem 7.

**Theorem 9** – *Let $t\in\mathbb{R}^+\cup\{+\infty\}$. For all $n\in\mathbb{N}$, all $\varepsilon\in\mathcal{E}^n$ let $h^\varepsilon$ be a function on $\mathcal{P}_n$ satisfying the condition of Lemma 2. Suppose that the function $h^\varepsilon$ is such that the quantity*

$$||T'||^2_{a,b,c}\stackrel{\mathrm{def}}{=}$$

$$=\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}}\mathbb{1}_{[0,t]}(\alpha,\gamma)\sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}}\frac{\left|\sum_{\forall i,\,C_i\subset\beta^i_i}\int_{\mathcal{P}} h^{\varepsilon(\alpha,\underline{\beta\cdot C},\gamma,\overline{\delta})}_{\mathrm{ord}(\alpha,\underline{\beta\cdot C},\gamma,\overline{\delta})}\mathbb{1}_{[0,t]}(\overline{\delta})\,d\overline{\delta}\right|^2\mathbb{1}_{[0,t]}(\beta)}{(a^{|\alpha|}b^{|\underline{\beta}|}c^{|\gamma|})}\,d\alpha\,d\gamma$$

*is finite for some $a\in(0,1)$, $b,c\in(0,+\infty)$. Then the operator*

$$T_t=\lambda I+\sum_{i=1}^{\infty}\sum_{\varepsilon\in F^n}\int_{0<t_1<\ldots<t_n<t}h^{\varepsilon}_{t_1,\ldots,t_n}\,dA^{\varepsilon_1}_{t_1}\ldots dA^{\varepsilon_n}_{t_n}$$

*is well-defined as an operator on $\Phi$ with (dense) domain $\Phi(c+N^2b^2/(1-a))$, bounded from $\Phi(c+N^2b^2/(1-a))$ to $\Phi$ with norm at most $||T'||_{a,b,c}$. Furthermore, if $c<1$ and $b=\frac{\sqrt{(1-a)(1-c)}}{N}$ then the operator $T_t$ is a bounded operator on $\Phi$, with the same bound for the norm.* ■

## 7 An application to series of generalized integrators

In [CEH], Cohen, Eyre and Hudson are considering some generalized integrator processes. Let $\mathcal{K}$ be the space $\mathbb{C}^N$. For a $(N+1)\times(N+1)$ matrix $H$, that is an element of the space $\mathcal{J}$ of linear transformation of the space $\mathbb{C}\oplus\mathcal{K}$, they consider the process $(\Lambda_t(H))_{t\geq 0}$, defined on the exponential domain $\mathcal{E}$ by the identity

$$<\varepsilon(g)\,,\,\Lambda_t(H)\varepsilon(f)>=\int_0^t<\widetilde{g}(s)\,,\,H\widetilde{f}(s)>ds<\varepsilon(g)\,,\,\varepsilon(f)>,$$

where for $u\in\mathcal{K}$, $\widetilde{u}\stackrel{\mathrm{def}}{=}(1,u)\in\mathbb{C}\oplus\mathcal{K}$. This means that, with our notations, $\Lambda_t(H)$ is actually equal to $\sum_{i,j}H(i,j)\,A^i_j(t)$, where $H=(H(i,j))_{i,j}$. Now take $H^n\in\mathcal{J}^{\otimes n}$

of the form $H^n = H_1^n \otimes \ldots \otimes H_n^n$. Consider processes of the form

$$\begin{aligned} I_t(H^n) &= \int_{0<t_1<\ldots<t_n<t} d\Lambda_{t_1}(H_1^n)\ \ldots\ d\Lambda_{t_n}(H_n^n) \\ &= \sum_{\substack{l_1,\ldots,l_n,k_1,\ldots,k_n \\ \in\{0,1,\ldots N\}}} \int_{0<t_1<\ldots<t_n<t} H_1^n(k_1,l_1)\ldots H_n^n(k_n,l_n) \\ &\qquad\qquad \times dA_{l_1}^{k_1}(t_1)\ \ldots\ dA_{l_n}^{k_n}(t_n) \\ &= \sum_{\varepsilon\in F^n} \int_{0<t_1<\ldots<t_n<t} h^{\varepsilon}_{t_1\ldots t_n}\, dA^{\varepsilon^1}_{t_1}\ \ldots\ dA^{\varepsilon^n}_{t_n} \end{aligned}$$

where $h^{\varepsilon}_{t_1,\ldots,t_n} = H_1^n(\varepsilon_1^1,\varepsilon_1^2)\ldots H_n^n(\varepsilon_n^1,\varepsilon_n^2)$.

By extending it linearly the mapping $I_t$ can be defined on the tensor space over $\mathcal{J}$: $J = \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J}^{\otimes 2} \oplus \ldots$.

In [CEH] such operators have been *formally* considered, and were proved to form an "Ito algebra" of which the product was described. This extended the formula given in [HP2] where only the purely conservation generalized processes case was considered. Indeed, the matrices $H$ of $\mathcal{J}$ are composed of four elements: a complex number, a vector in $\mathcal{K}$, a linear form on $\mathcal{K}$ and a linear transformation of $\mathcal{K}$, each of them corresponding to the time, creation, annihilation and multidimensional conservation component of $\Lambda_t(H)$. So, when only the fourth component does not vanish, we are in the purely conservation case.

Our purpose here is, for elements $\mathcal{H}$ of the tensor algebra $J$, to give a meaning to $I_t(\mathcal{H})$ as a well-defined operator on the Fock space, a series of iterated non-commutative stochastic integrals.

**Proposition 10** – *Let $\mathcal{H}$ be an element of the tensor algebra $J$ over $\mathcal{J}$ of the form $\mathcal{H} = \oplus_n H^n$, with $H^n = \otimes_{i\le n} H_i^n$ and each $H_i^n$ beeing a $(N+1)\times(N+1)$ matrix. If $K = \sup\{|H_i^n(k,l)|\,;\, n\in\mathbb{N}\,;\, i\le n\,;\, k,l = 0,1,\ldots,N\}$ is finite then, for all $t\in\mathbb{R}^+$, the operator*

$$I_t(\mathcal{H}) = \sum_{n=0}^{\infty} \int_{0<t_1<\ldots<t_n<t} d\Lambda_{t_1}(H_1^n)\ \ldots\ d\Lambda_{t_n}(H_n^n)$$

*is well-defined as a bounded operator from $\Phi(r)$ to $\Phi$, with any $r > N^2(K+1)^2$.*

**Proof**

We have to apply Theorem 9 to the familly of scalar operators $h^{\varepsilon}_{t_1,\ldots,t_n} = H_1^n(\varepsilon_1^1,\varepsilon_2^1)\ldots H_n^n(\varepsilon_1^n,\varepsilon_2^n)$. The corresponding conditions of Lemma 2 are obviously satisfied.

Let $K = \sup\{H_i^n(k,l); n\in\mathbb{N}; i\le n; k,l = 0,1,\ldots,N\} < \infty$. We then have $|h^{\varepsilon}_{t_1,\ldots,t_n}| \le K^n$. Thus we get

$$\int_{\mathcal{P}^{\mathcal{I}}}\int_{\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\alpha,\gamma) \sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}} \frac{\left|\sum_{\forall i,\, C_i\subset\beta_i^i} \int_{\mathcal{P}} h^{\varepsilon(\alpha,\underline{\beta\cdot C},\gamma,\overline{\delta})}_{\mathrm{ord}(\alpha,\underline{\beta\cdot C},\gamma,\overline{\delta})}\, \mathbb{1}_{[0,t]}(\overline{\delta})\, d\overline{\delta}\right|^2 \mathbb{1}_{[0,t]}(\underline{\beta})}{(a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|})}\, d\alpha\, d\gamma$$

$$\leq \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}} \frac{\left(\sum_{\forall i,\, C_i\subset\beta_i^i} \int_{\mathcal{P}} K^{|\alpha|}\, K^{|\underline{\beta}.C|}\, K^{|\gamma|}\, K^{|\overline{\delta}|}\, \mathbb{1}_{[0,t]}(\overline{\delta})\; d\overline{\delta}\right)^2}{(a^{|\alpha|} b^{|\underline{\beta}|} c^{|\gamma|})}$$

$$\times \mathbb{1}_{[0,t]}(\alpha,\gamma)\, d\alpha\, d\gamma\,.$$

$$= \int_{\mathcal{P}^{\mathcal{I}}} \int_{\mathcal{P}^{\mathcal{I}}} \mathbb{1}_{[0,t]}(\alpha,\gamma) \Big( \sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}} \frac{\left(\sum_{\forall i,\, C_i\subset\beta_i^i} K^{|\underline{\beta}.C|}\right)^2}{b^{|\underline{\beta}|}} \Big) \frac{(K^{|\alpha|})^2}{a^{|\alpha|}} \frac{(K^{|\gamma|})^2}{c^{|\gamma|}}$$

$$\times \Big(\sum_n \frac{t^n}{n!} K^n\Big)^2\, d\alpha\, d\gamma$$

$$\leq \Big(\int_{\mathcal{P}^{\mathcal{I}}} \Big(\frac{K^2}{a}\Big)^{|\alpha|} \mathbb{1}_{[0,t]}(\alpha)\, d\alpha\Big) \Big(\int_{\mathcal{P}^{\mathcal{I}}} \Big(\frac{K^2}{c}\Big)^{|\gamma|} \mathbb{1}_{[0,t]}(\gamma)\, d\gamma\Big) \exp\left(2tK\right)$$

$$\times \sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}} \Big(\frac{(K+1)^2}{b}\Big)^{|\underline{\beta}|}$$

$$= \exp\big(t(\frac{NK^2}{a} + \frac{NK^2}{c} + 2K)\big) \sup_{\underline{\beta}\in\mathcal{P}^{\mathcal{M}}} \Big(\frac{(K+1)^2}{b}\Big)^{|\underline{\beta}|}\,.$$

So taking any $a, c \in (0, +\infty)$, any $b > (K+1)^2$ one gets that the quantity $||T'||^2_{a,b,c}$ is finite. Taking $a$ and $c$ as close of 0 as possible, applying theorem 9, gives the result. ■

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## References


[At1] ATTAL S. : "Non-commutative chaotic expansion of Hilbert-Schmidt operators on Fock space", *Comm. Math. Phys.* **175**, (1996), p. 43-62.

[At2] ATTAL S. : "*Semimartingales non commutatives et applications aux endomorphismes browniens*", PhD thesis, Univ. L. Pasteur, Strasbourg (1993).

[At3] ATTAL S. : "Problèmes d'unicité dans les représentations d'opérateurs sur l'espace de Fock", *Séminaire de probabilités XXVI*, Springer Verlag L.N.M. 1526 (1992), p. 619-632.

[B-L] BELAVKIN V.P. & LINDSAY J.M. : "The kernel of a Fock space operator II", *Quantum Prob. & Rel. Topics* IX, World Scientific p. 87-94.

[CEH] COHEN P.B., EYRE T.W.M. & HUDSON R.L. : "Higher order Ito product formula, and generators of evolutions and flows", *Int. Journ. Theor. Phys.*, 34 (1995), p. 1481-1486.

[Der] DERMOUNE A. : "Formule de composition pour une classe d'opérateurs", *Séminaire de probabilités XXIV*, Springer Verlag L.N.M. 1426 (1990), p 397-401.

[HP1] HUDSON R.L. & PARTHASARATHY K.R. : "Quantum Itô's formula and stochastic evolutions", *Comm. Math. Phys.* 93 (1984), p 301-323.

[HP2] HUDSON R.L. & PARTHASARATHY K.R. : "The Casimir chaos map for $U(N)$", *Tatra Mountains Mathematicals Proceedings* 3 (1993), p 81-88.

[L-P] LINDSAY J.M. & PARTHASARATHY K.R. : "Cohomology of power sets with applications in quantum probability", *Comm. in Math. Phys.* 124 (1989), p. 337-364.

[Maa] MAASSEN H. : "Quantum Markov processes on Fock space described by integral kernels", *Quantum Prob. & Appl.* II, Springer Verlag L.N.M. 1136 1985, p 361-374.

[Me1] MEYER P.A. : "*Quantum probability for probabilists*", second edition, Springer Verlag L.N.M. 1538, (1995).

[Me2] MEYER P.A. : "Eléments de probabilités quantiques", *Séminaire de Probabilités* XX, Springer Verlag L.N.M. 1204 (1986), p 186-312.

[Par] PARTHASARATHY K.R. : "*An introduction to quantum stochastic calculus*", Monographs in Mathematics, Birkhäuser, 1992.