STRONG MARKOV PROCESSES AND THE DIRICHLET PROBLEM IN C*-ALGEBRAS

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Abstract

From Bhat and Parthasarathy (Proc. Indian Acad. Sciences, 104 (1994)) it is known that a one-parameter semigroup of completely positive maps on a C^*-algebra can be dilated to a family of *-homomorphisms. This provides a non-commutative analogue of Kolmogorov's construction of a canonical Markov process starting from an initial distribution and a one-parameter semigroup of transition probability functions.

Here we formulate the notions of stop time and strong Markov property for quantum Markov processes. Sufficient conditions on the process as well as its generator are obtained for the strong Markov property to hold. We define a quantum analogue of the exit time from a domain and show that the expectation value of an observable at the exit time satisfies a harmonicity property. This leads to a formulation of the Dirichlet problem on a non-commutative algebra and its solution.

Perturbations of Markov semigroups by multiplicative functionals are investigated and a quantum analogue of Feymann-Kac's formula is established. Finally, we study an example of an additive functional of the process and obtain, by a random time change, an analogue of the classical perturbation of the Laplacian by multiplication by a strictly positive function.

1. Introduction

Let us recall some basic results from the classical theory of Markov processes (see for example [Dyn]).

Let $(X_t)_{t\geq 0}$ be a homogeneous Markov process on the measurable state space $(E,\mathcal{E})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration, $(T_t)_{t\geq 0}$ its associated semigroup and

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$\mathcal{L}$ its infinitesimal generator. Denote by $I\!E_x$ the expectation value conditioned by the event $(X_0 = x)$. The Markov property of $(X_t)_{t\geq 0}$ can be expressed as

$$I\!E_x[f(X_{s+t})/\mathcal{F}_s] = T_t f(X_s) \tag{1.1}$$

for all bounded measurable functions f on $(E, \mathcal{E})$.

If furthermore $(X_t)_{t\geq 0}$ is a strong Markov process we have

$$I\!E_x[f(X_{\tau+t})/\mathcal{F}_\tau] = T_t f(X_\tau) \tag{1.2}$$

for every $(\mathcal{F}_t)_{t\geq 0}$-stop time τ.

The property that $(X_t)_{t\geq 0}$ solves the martingale problem associated to the generator $\mathcal{L}$ can be written as

$$I\!E_x[f(X_\tau)] = f(x) + I\!E_x[\int_0^\tau \mathcal{L}f(X_s)\, ds]. \tag{1.3}$$

It is also well-known that in "good cases" the strong Markov process $(X_t)_{t\geq 0}$ solves the Dirichlet problem associated to $\mathcal{L}$. Indeed, let D be a "good domain" in E, let $\tau(D)$ be the first exit time of $(X_t)_{t\geq 0}$ from D (that is $\tau(D) = \inf\{t; X_t \notin D\}$). Then the harmonicity property for two such domains $\widetilde{D} \subset D$ can be expressed as

$$I\!E_x[I\!E_{X_{\tau(\widetilde{D})}}[f(X_{\tau(D)})]] = I\!E_x[f(X_{\tau(D)})], \tag{1.4}$$

which together with (1.3), implies that the Dirichlet problem

$$\begin{cases} \mathcal{L}g(x) &= 0, & x \in D \\ g(x) &= f(x), & x \notin D \end{cases} \tag{1.5}$$

can be solved by taking $g(x) = I\!E_x[f(X_{\tau(D)})]$.

But all these classical properties can be expressed in terms of some operator relations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the canonical space of the Markov process $(X_t)_{t\geq 0}$ given by Kolmogorov's existence theorem, with some initial distribution. Let H be the Hilbert space $L^2(\Omega, \mathcal{F}, P)$. Let $\mathcal{A}$ be the *-algebra of all bounded measurable functions on $(E, \mathcal{E})$. Then the process $(X_t)_{t\geq 0}$ determines a family of *-homomorphisms $j_t : \mathcal{A} \to \mathcal{B}(H)$ by

$$[j_t(f)h](x) = f(x(t))I\!E[h(x)/\mathcal{F}_t].$$

Here x is the whole path and $x(t)$ is its value at time t. Let $H_t = L^2(\Omega, \mathcal{F}_t, P)$ and F_t be the orthogonal projection from H onto H_t. The Markov property (1.1) becomes

$$\begin{cases} j_t(1\!\!1) &= F_t \\ F_s\, j_{t+s}(f) F_s &= j_s(T_t f). \end{cases} \tag{1.1'}$$

For a stop time τ, if F_τ denotes the projection determined by the conditional expectation given $\mathcal{F}_\tau$ then the strong Markov property becomes

$$F_\tau\, j_{\tau+t}(f) F_\tau = j_\tau(T_t f). \tag{1.2'}$$

The martingale property (1.3) becomes :

$$F_0\, j_\tau(f) F_0 = f + F_0 \int_0^\infty 1\!\!1_{\tau\geq s}\, j_s(\mathcal{L}f)\, ds\ F_0. \tag{1.3'}$$

Denoting by $\Gamma_D(f)$ the expression $F_0\, j_{\tau(D)}(f) F_0$, the relations (1.4) and (1.5) respectively become

$$\Gamma_{\tilde{D}}(\Gamma_D(f)) = \Gamma_D(f), \quad \text{for } \tilde{D} \subset D \tag{1.4'}$$

and

$$\begin{cases} \mathbb{1}_D\, \mathcal{L}(g) &= 0 \\ (\mathbb{1} - \mathbb{1}_D) g &= (\mathbb{1} - \mathbb{1}_D) f \end{cases} \tag{1.5'}$$

with $g = \Gamma_D(f)$.

Under this form the properties $(1.1') - (1.5')$ can be meaningful for a general C^*-algebra $\mathcal{A}$ which is not necessarily commutative. Furthermore, a stop time τ, when identified with the family $(\mathbb{1}_{\tau \leq t})_{t \geq 0}$, is a particular spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the space of orthogonal projections on H.

When $\mathcal{A}$ is a general unital C^*-algebra of bounded operators on a Hilbert space H_0 and $(T_t)_{t \geq 0}$ is a semigroup of contractive, unital and completely positive maps from $\mathcal{A}$ into itself, it is proved in [B-P] that there exists a Hilbert space H, an injection $H_0 \hookrightarrow H$ and a family of *-homomorphisms $j_t : \mathcal{A} \to \mathcal{B}(H)$ satisfying $(1.1')$. In that way, one gets a non-commutative extension of the notion of a Markov process.

In this article we consider the same non-commutative context. When one is able to give a meaning to the expression j_τ for all quantum stop times τ (that is, spectral measures of projections on H satisfying some adaptedness condition), we get that the Markov process $(j_t)_{t \geq 0}$ is a strong Markov process in the sense that it satisfies (1.2').

We give two kind of sufficient conditions for the process to be strong Markov. The first one is a Hilbertian quasimartingale-like condition, inspired from Enchev's characterization ([Enc]). As quantum stochastic integral processes on Fock space ([H-P]), when applied to vectors, always give rise to such quasimartingales ([A-M]), this sufficient condition is valid for example in the case of any minimal Evans-Hudson flows on Fock space ([E-H]). The second sufficient condition is actually stronger than the first one but has the advantage to be expressed as a regularity assumption on the semigroup and the generator of the quantum Markov process. It applies for example in the case where the generator $\mathcal{L}$ is bounded.

We then prove that the martingale property (1.3') is obtained for any quantum strong Markov process.

We define the notion of "exit time from a domain" for a quantum strong Markov process. What we call a domain here is actually a central projection in $\mathcal{A}$, and the exit time we obtain so is a quantum stop time. We then get the harmonicity property (1.4'), which together with (1.3') allows to show that the general non-commutative Dirichlet problem (1.5') can be solved by taking the "expectation" of the Markov process "killed when it hits the boundary of the domain".

We then pursue the analogy with the classical theory by investigating several examples of perturbations of the semigroup by multiplicative functionals. In one particular example we obtain a quantum generalization of Feynman-Kac's formula;

that is, a perturbation of the generator by addition of the multiplication operator by a negative central element of $\mathcal{A}$. Finally we study one example of perturbation of the semigroup by additive functionals and obtain, with the help of the quantum Feynman-Kac's formula, a multiplicative perturbation of the generator by a change of the time scale.

2. The basic construction

The results presented here are proved in [B-P] in a more general context.

Let $\mathcal{A}$ be a unital C^*-algebra of bounded operators on a complex Hilbert space H_0. Let $(T_t)_{t\geq 0}$ be a semigroup of contractive, unital and completely positive maps from $\mathcal{A}$ into itself. Recall that complete positivity means

$$\sum_{i,j} X_i^* T_t(Y_i^* Y_j) X_j \geq 0$$

for all $X_i \in \mathcal{B}(H_0)$, $Y_i \in \mathcal{A}$, $i = 1, ..., n$. Let

$$\mathcal{D} = \{(\mathbf{r}, \mathbf{Y}, u); u \in H_0, \mathbf{r} = (r_1, \cdots, r_n) \in (I\!R^+)^n,\ r_1 > \cdots > r_n, \\ \mathbf{Y} = (Y_1, ..., Y_n) \in \mathcal{A}^n,\ n \geq 1\}.$$

If $(\mathbf{r}, \mathbf{Y}, u) \in \mathcal{D}$ and $\mathbf{s} = \{s_1 > \cdots > s_m \geq 0\}$ is such that $\{r_1, ..., r_n\} \subset \{s_1, ..., s_m\}$, we define $(\mathbf{s}, \tilde{\mathbf{Y}}, u) \in \mathcal{D}$ by

$$\tilde{Y}_j = \begin{cases} Y_i & \text{if } s_j = r_i \text{ for some } i \\ I & \text{otherwise} \end{cases}$$

where I denotes the identity operator on H_0.

The semigroup $(T_t)_{t\geq 0}$ defines a map $L_T : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ by

$$L_T((\mathbf{r}, \mathbf{X}, u), (\mathbf{r}, \mathbf{Y}, v)) \\ = < u\,,\, T_{r_n}(X_n^* T_{r_{n-1}-r_n}(X_{n-1}^* \cdots X_2^* T_{r_1-r_2}(X_1^* Y_1) Y_2 \cdots Y_{n-1}) Y_n) v >$$

and $L_T((\mathbf{r}, \mathbf{X}, u), (\mathbf{s}, \mathbf{Y}, v)) = L_T((\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u), (\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{Y}}, v))$ where $\mathbf{r} \cup \mathbf{s}$ is obtained by arranging $\{r_1, ..., r_n\} \cup \{s_1, ..., s_n\}$ in the decreasing order.

The complete positivity of $(T_t)_{t\geq 0}$ implies easily (see [B-P]) that L_T is a positive definite kernel on $\mathcal{D} \times \mathcal{D}$. With the help of G.N.S. principle (see [Par], Proposition 15.4, for example) the following result is proved in [B-P].

Theorem 2.1 – *There exists a Hilbert space H, an increasing family $(F_t)_{t\geq 0}$ of projection operators on H, a family of *-homomorphisms $j_t : \mathcal{A} \to \mathcal{B}(H), t \geq 0$ and a unitary isomorphism V from H_0 onto the range of F_0 satisfying the following properties :*

(i) $j_t(I) = F_t$, *for all* $t \geq 0$;

(ii) $F_s j_t(X) F_s = j_s(T_{t-s}(X))$ *for any* $s \leq t$, $X \in \mathcal{A}$

(iii) $j_0(X)V = VX$ *for all* $X \in \mathcal{A}$;

(iv) *the set* $\{j_{t_1}(X_1)...j_{t_n}(X_n)Vu; t_1 > t_2 > \cdots > t_n \geq 0, X_i \in \mathcal{A}, i \in \{1, ..., n\}, n \geq 1, u \in H_0\}$ *is total in* H;

(v) *for any* $u, v \in H, \mathbf{r} = \{r_1 > \cdots > r_n \geq 0\}$, $\mathbf{s} = \{s_1 > \cdots > s_m \geq 0\}, X_i, Y_j \in \mathcal{A}, i \in \{1, ..., n\}, j \in \{1, ..., m\}$ *one has*

$$< j_{r_1}(X_1)...j_{r_n}(X_n)Vu \,,\, j_{s_1}(Y_1)...j_{s_m}(Y_m)Vv > = L_T((\mathbf{r}, \mathbf{X}, u), (\mathbf{s}, \mathbf{Y}, v)). \quad \blacksquare$$

As explained in the introduction such a quadruple $\{H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0}, V\}$ may be considered as the *non-commutative Markov process* with contraction semigroup $(T_t)_{t\geq 0}$. Following [B-P], a quadruple $\{H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0} V\}$ satisfying conditions (i), (ii) and (iii) of Theorem 2.1 is called a *conservative Markov process*. The process is said to be *minimal* if, in addition, it satisfies condition (iv). Two such minimal Markov processes $\{H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0}, V\}$ and $\{H', (F'_t)_{t\geq 0}, (j'_t)_{t\geq 0}, V'\}$ are said to be *equivalent* if there exists a unitary isomorphism $W : H \to H'$ such that $WF_tW^* = F'_t$, $Wj_t(X)W^* = j'_t(X)$ and $WV = V'$, for all $t \geq 0, X \in \mathcal{A}$. In [B-P] it is proved that the minimal Markov process given by Theorem 2.1 is unique upto equivalence.

Let $\mathcal{Z}$ be the center of the C^*-algebra $\mathcal{A}$. It is possible that T_t may not map $\mathcal{Z}$ into itself, thus the family $\{j_t(Z);\ Z \in \mathcal{Z}, t \geq 0\}$ may not be a commutative family. But, noticing that $j_t(X) = j_t(XI) = j_t(X)j_t(I) = j_t(X)F_t$, the operator $j_t(X)$ is determined by its restriction to the range of F_t, for all $X \in \mathcal{A}$. Hereafter the range of F_t is denoted H_t, for all $t \geq 0$. In [B-P] it is proved that if Z belongs to $\mathcal{Z}$, the operator $j_t(Z)$ can be lifted into an operator $k_t(Z) \in \mathcal{B}(H)$ such that $j_t(Z) = k_t(Z)F_t$ and such that the family $\{k_t(Z); Z \in \mathcal{Z},\ t \geq 0\}$ is commutative. In other words, the minimal Markov process $(j_t)_{t\geq 0}$ when restricted to the center of $\mathcal{A}$ can be obtained as a conditional expectation of a purely commutative process. This is expressed by the following result.

Theorem 2.2 – *There exists a unique *-unital homomorphism* $k_t : \mathcal{Z} \to \mathcal{B}(H)$ *such that*

(i) $k_t(Z)j_{t_1}(X_1)\cdots j_{t_n}(X_n)Vu$
$= j_{t_1}(X_1)\cdots j_{t_{i-1}}(X_{i-1})j_t(Z)j_{t_i}(X_i)\cdots j_{t_n}(X_n)Vu$ *where* $t_{i-1} > t \geq t_i$*;*

(ii) *the family* $\{k_t(Z); Z \in \mathcal{Z}, t \geq 0\}$ *is commutative;*

(iii) $j_t(Z) = k_t(Z)F_t$ *for all* $t \geq 0,\ z \in \mathcal{Z}$. ■

From now on we make the following assumptions on $(T_t)_{t\geq 0}$:

(i) $(T_t)_{t\geq 0}$ is strongly continuous that is, $\lim_{s\to t} \|T_s(X) - T_t(X)\| = 0$ for all $X \in \mathcal{A}$, all $t \geq 0$;

(ii) $T_0(X) = X$ for all $X \in \mathcal{A}$.

Furthermore, recall an important property: as j_t and k_t are homomorphisms from a C^*-algebra to a Banach algebra we have, from a well-known theorem of functional analysis (see [Dix] for example), that

$$\|j_t(X)\| \leq \|X\| \quad \text{and} \quad \|k_t(Z)\| \leq \|Z\|$$

for all $t \in \mathbb{R}^+, X \in \mathcal{A}, Z \in \mathcal{Z}$.

3. Preliminaries

Let $\mathcal{N} = \{(\mathbf{r}, \mathbf{X}); \mathbf{r} = (r_1, ..., r_n) \in (\mathbb{R}^+)^n, \mathbf{X} = (X_1, ..., X_n) \in \mathcal{A}^n, n \geq 1\}$. For all $(\mathbf{r}, \mathbf{X}) \in \mathcal{N}$ let $j(\mathbf{r}, \mathbf{X}) = j_{r_1}(X_1) \cdots j_{r_n}(X_n)$. For all $(\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}$ let $\lambda(\mathbf{r}, \mathbf{X}, u) = j(\mathbf{r}, \mathbf{X}) V u$.

The following proposition is a consequence of [B-P] Proposition 2.6 and some remarks in [Bh1]; these results can be easily recovered by several applications of Theorem 2.1 (i) and (ii).

Proposition 3.1 –

(a) *For all* $(\mathbf{r}, \mathbf{X}) \in \mathcal{N}$ *the expression* $j(\mathbf{r}, \mathbf{X})$ *can be reduced to an expression of the form* $j(\mathbf{s}, \mathbf{Y})$ *where* $s_1 > \cdots > s_k < s_{k+1} < \cdots < s_n$ *for some* $k \in \{1, ..., m\}$.

(b) *There exists a map* $\mathcal{E} : \mathcal{N} \to \mathcal{A}$, *independent of the Markov process and satisfying*

$$F_0\, j(\mathbf{r}, \mathbf{X}) F_0 = j_0(\varepsilon(\mathbf{r}, \mathbf{X})). \tag{3.1}$$

Furthermore the map ε *satisfies the following properties :*

$$F_0\, j(\mathbf{r}, \mathbf{X})^* F_0 = j_0(\mathcal{E}(\mathbf{r}^*, \mathbf{X}^*)) \tag{3.2}$$

$$\mathcal{E}(\mathbf{r} + s, X) = T_s(\mathcal{E}(\mathbf{r}, \mathbf{X})) \tag{3.3}$$

$$F_s\, j(\mathbf{r} + s, \mathbf{X}) F_s = j_s(\mathcal{E}(\mathbf{r}, \mathbf{X})) \tag{3.4}$$

where $\mathbf{r}^* = (r_n, r_{n-1}, ..., r_1)$, $\mathbf{X}^* = (X_n^*, X_{n-1}^*, ..., X_1^*)$ *and* $\mathbf{r} + s = (r_1 + s, r_2 + s, ..., r_n + s)$. ■

Lemma 3.2 – *For all* $(\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}$, *all* $\mathbf{s} = \{s_1 > \cdots > s_m \geq 0\}$, *one has* $\lambda(\mathbf{r}, \mathbf{X}, u) = \lambda(\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u)$.

Proof

One has

$$\begin{aligned} &\|\lambda(\mathbf{r}, \mathbf{X}, u) - \lambda(\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u)\|^2 \\ &\quad = \|\lambda(\mathbf{r}, \mathbf{X}, u)\|^2 + \|\lambda(\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u)\|^2 - 2\, \Re < \lambda(\mathbf{r}, \mathbf{X}, u)\,,\ \lambda(\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u) >. \end{aligned}$$

By Theorem 2.1 (v) we have

$$< \lambda(\mathbf{r}, \mathbf{X}, u)\,,\ \lambda(\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u) > = L_T((\mathbf{r}, \mathbf{X}, u), (\mathbf{r} \cup \mathbf{s}, \tilde{\mathbf{X}}, u))$$

which is equal to $L_T((\mathbf{r}, \mathbf{X}, u), (\mathbf{r}, \mathbf{X}, u))$ by definition. ■

For all $X \in \mathcal{A}$, let R_X denote the mapping

$$\begin{aligned} R_X \quad : \mathcal{A} \quad &\to \mathcal{A} \\ Y \quad &\mapsto YX. \end{aligned}$$

Lemma 3.3 – *For all* $t \in \mathbb{R}^+$, *all* $(\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}$ *one has*

$$F_t\, \lambda(\mathbf{r}, \mathbf{X}, u) = \begin{cases} \lambda(\mathbf{r}, \mathbf{X}, u) & \text{if } t \geq r_1 \\ j_t(T_{r_{i-1}-t} R_{X_{i-1}} T_{r_{i-2}-r_{i-1}} R_{X_{i-2}} \cdots R_{X_2} T_{r_1 - r_2}(X_1)) & \\ \quad \times j_{r_i}(X_i) \cdots j_{r_n}(X_n) V u & \text{if } r_{i-1} > t \geq r_i. \end{cases}$$

Proof

The case $t \geq r_1$ is obvious from Theorem 2.1 (i) and Lemma 3.2. Suppose $r_{i-1} > t \geq r_1$. Notice that if $r_1 > r_2 > t$ one has $F_t\, j_{r_1}(X_1) j_{r_2}(X_2) = F_t\, F_{r_2}\, j_{r_1}(X_1) F_{r_2}\, j_{r_2}(X_2) = F_t\, j_{r_2}(T_{r_1-r_2}(X_1)) j_{r_2}(X_2) = F_t\, j_{r_2}(R_{X_2} T_{r_1-r_2}(X_1))$. Thus the result is obtained by repeating this procedure to the expression

$$F_t\, \lambda(\mathbf{r}, \mathbf{X}, u) = F_t j_{r_1}(X_1) \cdots j_{r_{i-1}}(X_{i-1}) j_{r_i}(X_i) \cdots j_{r_n}(X_n) V u. \qquad \blacksquare$$

Lemma 3.4 – *For all $s \leq t \leq u$, $X \in \mathcal{A}, Z \in \mathcal{Z}$, one has the following properties :*

(i) $k_s(Z) F_t = F_t\, k_s(Z)$

(ii) $k_s(Z) j_t(X) = j_t(X) k_s(Z)$

(iii) $k_u(Z) F_t = j_u(Z) F_t$

(iv) $F_t\, k_u(Z) = F_t\, j_u(Z)$

(v) $k_u(Z) j_t(X) = j_u(Z) j_t(X)$

(vi) $j_t(X) k_u(Z) = j_t(X) j_u(Z)$.

Proof

From Lemma 3.3 (resp. Theorem 2.2 (i)) one sees that when F_t (respectively $k_t(Z)$) is applied to $\lambda(\mathbf{r}, \mathbf{X}, u)$ all the $j_{r_i}(X_i)$ in $\lambda(\mathbf{r}, \mathbf{X}, u)$ with $r_i \leq t$ (resp. $r_i \neq t$) remain unchanged. Thus, for all $s \leq t$ one has $k_s(Z) F_t\, \lambda(\mathbf{r}, \mathbf{X}, u) = F_t\, k_s(Z) \lambda(\mathbf{r}, \mathbf{X}, u)$. One gets (i) by totality of the vectors $\lambda(\mathbf{r}, \mathbf{X}, u)$ in H. Property (ii) is clear from Theorem 2.2 (i) and the previous result. If $u \geq t$ then $F_t = F_u\, F_t = F_t\, F_u$, thus $k_u(Z) F_t = k_u(Z) F_u\, F_t = j_u(Z) F_t$. Furthermore $F_t\, k_u(Z) = F_t\, F_u\, k_u(Z) = F_t\, k_u(Z) F_u = F_t\, j_u(Z)$. This proves (iii) and (iv). Properties (v) and (vi) are easily deduced. $\blacksquare$

Let $\mathcal{B}_0$ be the *-algebra generated by $\{j_t(X), k_s(Z); s, t \in I\!R^+, X \in \mathcal{A}, Z \in \mathcal{Z}\}$. Let $\widetilde{\mathcal{N}} = \{(\mathbf{r}, \mathbf{Z}) \in \mathcal{N}; Z_i \in \mathcal{Z} \text{ for all } i\}$. Let $\mathcal{M} = \{(\mathbf{s}, \mathbf{r}, \mathbf{X}, \mathbf{Z});\ \mathbf{s} = (s_1, ..., s_n) \in (I\!R^+)^n, \mathbf{r} = (r_1, ..., r_p) \subseteq \mathbf{s}, \mathbf{X} = (X_1, ..., X_{n-p}) \in \mathcal{A}^{n-p}, \mathbf{Z} = (Z_1, ..., Z_p) \in \mathcal{Z}^p,\ n \geq 1,\ p \leq n\}$. For $(\mathbf{r}, \mathbf{Z}) \in \widetilde{\mathcal{N}}$, $k(\mathbf{r}, \mathbf{Z})$ denotes

$$k_{r_1}(Z_1) \cdots k_{r_n}(Z_n).$$

For $(\mathbf{s}, \mathbf{r}, \mathbf{X}, \mathbf{Z}) \in \mathcal{M}$, $h(\mathbf{s}, \mathbf{r}, \mathbf{X}, \mathbf{Z})$ denotes the element $h_{s_1}(Y_1) \cdots h_{s_n}(Y_n) \in \mathcal{B}_0$ such that

$$h_{s_i}(Y_i) = \begin{cases} k_{s_i}(Z_i) & \text{if } s_i \in \mathbf{r} \\ j_{s_i}(X_i) & \text{otherwise.} \end{cases}$$

Lemma 3.5 – *Let $(\mathbf{s}, \mathbf{r}, \mathbf{X}, \mathbf{Z}) \in \mathcal{M}$. Then there exists $(\mathbf{t}, \mathbf{Z}') \in \widetilde{\mathcal{N}}$ and $(\mathbf{u}, \mathbf{X}') \in \mathcal{N}$ such that $\mathbf{t} \cup \mathbf{u} = \mathbf{s}$ and*

$$h(\mathbf{s}, \mathbf{r}, \mathbf{Z}, \mathbf{X}) = k(\mathbf{t}, \mathbf{Z}') j(\mathbf{u}, \mathbf{X}').$$

Proof

From Lemma 3.4 (ii) and (v) one sees that in $h_{s_1}(Y_1)\cdots h_{s_n}(Y_n)$ every term of the form $k_{s_i}(Y_i)$ either commutes with a $j_{s_k}(Y_k)$ on its left (if $s_k \geq s_i$) or is transformed into a $j_{s_i}(Y_i)$. Thus one can always write $h(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})$ as

$$k_{t_1}(Z'_1)\cdots k_{t_n}(Z'_n)j_{u_1}(X'_1)\cdots j_{u_k}(X'_k).$$ ■

Lemma 3.6 – *There exists a mapping $\tilde{\mathcal{E}} : \mathcal{M} \to \mathcal{A}$, independent of the Markov process, satisfying*

$$F_0\, h(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})F_0 = j_0(\tilde{\mathcal{E}}(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})). \tag{3.5}$$

Furthermore, the mapping $\tilde{\mathcal{E}}$ satisfies the following properties :

$$\tilde{\mathcal{E}}(\mathbf{s}+t,\mathbf{r}+t,\mathbf{X},\mathbf{Z}) = T_t(\tilde{\mathcal{E}}(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})); \tag{3.6}$$

$$F_t\, h(\mathbf{s}+t,\mathbf{r}+t,\mathbf{X},\mathbf{Z})F_t = j_t(\tilde{\mathcal{E}}(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})). \tag{3.7}$$

Proof

By Lemma 3.5, the element $h(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})$ can be written $k(\mathbf{t},\mathbf{Z}')j(\mathbf{u},\mathbf{X}')$. As the $k_{t_i}(Z'_i)$ are pairwise commuting one can assume that $t_1 < \cdots < t_n$. Thus by Lemma 3.4 (iv) and (vi) we have $F_0\, h(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z})F_0 = F_0\, j(\mathbf{t},\mathbf{Z}')j(\mathbf{u},\mathbf{X}')F_0 = F_0\, j((\mathbf{t},\mathbf{u}),(\mathbf{Z}',\mathbf{X}'))F_0 = j_0(\mathcal{E}((\mathbf{t},\mathbf{u}),(\mathbf{Z}',\mathbf{X}')))$. Thus the mapping

$$\tilde{\mathcal{E}}(\mathbf{s},\mathbf{r},\mathbf{X},\mathbf{Z}) = \mathcal{E}((\mathbf{t},\mathbf{u}),(\mathbf{Z}',\mathbf{X}'))$$

satisfies (3.5).

Furthermore, if one considers $\mathbf{t}$ and $\mathbf{u}$ as functions of $\mathbf{s}$ and $\mathbf{r}$, it is clear, since $\mathbf{t}$ and $\mathbf{u}$ form simply a partition of $\mathbf{s}$, that $\mathbf{t}(\mathbf{s}+h,\mathbf{r}+h) = \mathbf{t}(\mathbf{s},\mathbf{r})+h$ and $\mathbf{u}(\mathbf{s}+h,\mathbf{r}+h) = \mathbf{u}(\mathbf{s},\mathbf{r})+h$. Properties (3.6) and (3.7) are then easily deduced from (3.3) and (3.4). ■

4. The time-shift

Let $\mathcal{B}$ be the C^*-algebra generated by $\mathcal{B}_0$. The aim of this section is to construct a time-shift that is a semigroup $(\theta_t)_{t\geq 0}$ of *-endomorphisms of a *-algebra $\mathcal{P}$ containing $\mathcal{B}$, satisfying $\theta_t(j_s(X)) = j_{s+t}(X)$ and $\theta_t(k_s(Z)) = k_{s+t}(Z)$.

The proof of the following proposition is an extension of the proof of [Bh1] where the time shift is constructed on the C^*-algebra generated by the $j_s(X)$ only.

Proposition 4.1 – *There exists a unique semigroup $(\theta_t)_{t\geq 0}$ of *-endomorphisms of $\mathcal{B}$ such that $\theta_t(j_s(X)) = j_{s+t}(X)$ and $\theta_t(k_s(Z)) = k_{s+t}(Z)$ for all $s,t \in \mathbb{R}^+, X \in \mathcal{A}, Z \in \mathcal{Z}$.*

Proof

Define $\theta_t(j_s(X))$ to be $j_{s+t}(X)$ and $\theta_t(k_s(Z))$ to be $k_{s+t}(Z)$ for all $s,t \in \mathbb{R}^+$, $X \in \mathcal{A}$, $Z \in \mathcal{Z}$. The proposition will be true if we prove that θ_t extends to a well-defined contractive semigroup of *-endomorphisms of $\mathcal{B}_0$ as then we can extend θ_t to $\mathcal{B}$ by taking norm-limits.

Let $A = \sum_n h(\mathbf{s}_n, \mathbf{r}_n, \mathbf{X}_n, \mathbf{Z}_n)$ be an element of $\mathcal{B}_0$, for some $(\mathbf{s}_n, \mathbf{r}_n, \mathbf{X}_n, \mathbf{Z}_n)$ belonging to $\mathcal{M}$, $n = 1, ..., N$. For all $(\mathbf{v}_k, \mathbf{Y}_k, w_k) \in \mathcal{D}$, all $k \in \{1, ..., K\}$ we have

$$\sum_{k,\ell} < A\, j(\mathbf{v}_k, \mathbf{Y}_k) V w_k \,,\, A\, j(\mathbf{v}_\ell, \mathbf{Y}_\ell) V w_\ell >$$

$$\leq \|A\|^2 \sum_{k,\ell} < j(\mathbf{v}_k, \mathbf{Y}_k) V w_k \,,\, j(\mathbf{v}_\ell, \mathbf{Y}_\ell) V w_\ell >. \tag{4.1}$$

But the left hand side of (4.1) is equal to

$$\sum_{k,\ell,p,q} < h(\mathbf{s}_p, \mathbf{r}_p, \mathbf{X}_p, \mathbf{Z}_p) j(\mathbf{v}_k, \mathbf{Y}_k) V w_k \,,\, h(\mathbf{s}_q, \mathbf{r}_q, \mathbf{X}_q, \mathbf{Z}_q) j(\mathbf{v}_\ell, \mathbf{Y}_\ell) V w_\ell >$$

$$= \sum_{k,\ell,p,q} < k(\mathbf{t}_p, \mathbf{Z}'_p) j(\mathbf{u}_p, \mathbf{X}'_p) j(\mathbf{v}_k, \mathbf{Y}_k) V w_k \,,\, k(\mathbf{t}_q, \mathbf{Z}'_q) j(\mathbf{u}_q, \mathbf{X}'_q) j(\mathbf{v}_\ell, \mathbf{Y}_\ell) V w_\ell >$$

(by Lemma 3.5)

$$= \sum_{k,\ell,p,q} < V w_k \,,\, j(\mathbf{v}^*_k, \mathbf{Y}^*_k) j(\mathbf{u}^*_p, \mathbf{X}'^*_p) k(\mathbf{t}^*_p, \mathbf{Z}'^*_p) k(\mathbf{t}_q, \mathbf{Z}'_q)$$

$$j(\mathbf{u}_q, \mathbf{X}'_q) j(\mathbf{v}_\ell, \mathbf{Y}_\ell) V w_\ell >. \tag{4.2}$$

Let $\mathbf{a}_{k,\ell,p,q} = \mathbf{v}^*_k \cup \mathbf{u}^*_p \cup \mathbf{t}^*_p \cup \mathbf{t}_q \cup \mathbf{u}_\ell \cup \mathbf{v}_\ell$, let $\mathbf{b}_{k,\ell,p,q} = \mathbf{t}^*_p \cup \mathbf{t}_q$, let $\mathbf{J}_{k,\ell,p,q} = (\mathbf{Y}^*_k, \mathbf{X}'^*_p, \mathbf{X}'_q, \mathbf{Y}_\ell)$ and let $\mathbf{K}_{k,\ell,p,q} = (\mathbf{Z}'^*_p, \mathbf{Z}'_q)$. Then, as the range of V is included in the range of F_0, by Lemma 3.6 one has that (4.2) is equal to

$$\sum_{k,\ell,p,q} < V w_k \,,\, j_0(\tilde{\mathcal{E}}(\mathbf{a}_{k,\ell,p,q}, \mathbf{b}_{k,\ell,p,q}, \mathbf{J}_{k,\ell,p,q}, \mathbf{K}_{k,\ell,p,q})) V w_k >. \tag{4.3}$$

The right hand side of (4.1) is equal to

$$\|A\|^2 \sum_{k,\ell} < V w_k \,,\, j_0(\mathcal{E}((\mathbf{v}^*_k, \mathbf{v}_\ell), (\mathbf{Y}^*_k, \mathbf{Y}_\ell))) V w_\ell >.$$

As $j_0(X) = V X V^*$ for all $X \in \mathcal{A}$ we finally get

$$\sum_{k,\ell,p,q} < w_k \,,\, \tilde{\mathcal{E}}(\mathbf{a}_{k,\ell,p,q}, \mathbf{b}_{k,\ell,p,q}, \mathbf{J}_{k,\ell,p,q}, \mathbf{K}_{k,\ell,p,q}) w_\ell >$$

$$\leq \|A\|^2 \sum_{k,\ell} < w_k \,,\, \mathcal{E}((\mathbf{v}^*_k, \mathbf{v}_\ell), (\mathbf{Y}^*_k, \mathbf{Y}_\ell)) w_\ell >.$$

That is,

$$((\sum_{p,q} \tilde{\mathcal{E}}(\mathbf{a}_{k,\ell,p,q}, \mathbf{b}_{k,\ell,p,q}, \mathbf{J}_{k,\ell,p,q}, \mathbf{K}_{k,\ell,p,q}))) \leq \|A\|^2 ((\mathcal{E}((\mathbf{v}^*_k, \mathbf{v}_\ell), (\mathbf{Y}^*_k, \mathbf{Y}_\ell))) \tag{4.4}$$

as operators on $\oplus_k H_0$.

Now let $B_t = \theta_t(A) = \sum_n h(\mathbf{s}_n + t, \mathbf{r}_n + t, \mathbf{X}_n, \mathbf{Z}_n)$. Let $\eta = \sum_k j(\mathbf{v}_k + t, \mathbf{Y}_k) \xi_k$ for arbitrary ξ_k in H_t, for $k = 1, ..., K$. Then we have

$$< B_t \eta \,,\, B_t \eta > = \sum_{k,\ell,p,q} < \xi_k \,,\, j(\mathbf{v}^*_k + t, \mathbf{Y}^*_k) j(\mathbf{u}^*_p + t, \mathbf{X}'^*_p) k(\mathbf{t}^*_p + t, \mathbf{Z}'^*_p)$$

$$k(\mathbf{t}_q + t, \mathbf{Z}'_q) j(\mathbf{u}_q + t, \mathbf{X}'_q) j(\mathbf{v}_\ell + t, \mathbf{Y}_\ell) \xi_\ell >$$

$$= \sum_{k,\ell,p,q} < \xi_k \, , \, h(\mathbf{a}_{k,\ell,p,q} + t, \mathbf{b}_{k,\ell,p,q} + t, \mathbf{J}_{k,\ell,p,q}, \mathbf{K}_{k,\ell,p,q}) \xi_\ell >$$
$$= \sum_{k,\ell,p,q} < \xi_k \, , \, j_t(\tilde{\mathcal{E}}(\mathbf{a}_{k,\ell,p,q}, \mathbf{b}_{k,\ell,p,q}, \mathbf{J}_{k,\ell,p,q}, \mathbf{K}_{k,\ell,p,q}) \xi_\ell > \text{ by (3.7).}$$

From (4.4) and the complete positivity of the endomorphism j_t we finally obtain

$$< B_t \, \eta \, , \, B_t \, \eta > = \|A\|^2 \sum_{k,\ell} < \xi_k \, , \, j_t(\mathcal{E}(\mathbf{v}_k^*, \mathbf{v}_\ell), (\mathbf{Y}_k^*, \mathbf{Y}_\ell))) \xi_\ell >$$
$$= \|A\|^2 < \eta \, , \, \eta >.$$

As vectors of the form of η are dense in H we have $\|\theta_t(A)\| \leq \|A\|$. Hence $(\theta_t)_{t \geq 0}$ is a semigroup of contractive *-endomorphisms of $\mathcal{B}_0$. It extends to a semigroup of contractive *-endomorphisms of $\mathcal{B}$. ■

We now introduce a non-commutative analogue of the classical "intrinsic topology" associated to a Markov process.

For every $t \in \mathbb{R}^+, \psi \in H$, define on $\mathcal{B}_0$ the seminorm $\|Y\|_{t,\psi} = \|\theta_t(Y)\psi\|$. The family fo seminorms $\{\| \cdot \|_{t,\psi}, t \in \mathbb{R}^+, \psi \in H\}$ is separating. Let τ be the topology induced by this family of seminorms. Notice that a sequence $(Y_n)_n$ in $\mathcal{B}_0$ is convergent for the topology τ if and only if the sequence $(\theta_t(Y_n))_n$ is strongly convergent for all $t \geq 0$. Let $\mathcal{P}$ be the closure of B_0 under the topology τ.

Lemma 4.2 – *The space $\mathcal{P}$ is a *-algebra containing $\mathcal{B}$.*

Proof

If $(Y_n)_n$ is a norm-convergent sequence in $\mathcal{B}_0$ then so is $(\theta_t(Y_n))_n$, for θ_t is contractive; hence $(\theta_t(Y_n))_n$ is strongly convergent. As $\mathcal{B}$ is the norm-closure of $\mathcal{B}_0$ we get $\mathcal{B} \subset \mathcal{P}$.

The space $\mathcal{P}$ is clearly stable under the adjoint mapping $Y \to Y^*$. Now, recall that if $(A_n)_n$ (resp. $(B_n)_n$) is a sequence of bounded operators converging strongly to A (resp. B), then by the Uniform Boundedness Principle, $(A_n B_n)_n$ converges strongly to AB. This remark, together with the fact that θ_t is a homomorphism, imply that if A and B belong to $\mathcal{P}$ then so does AB. ■

Theorem 4.3 – *There exists a unique semigroup $(\theta_t)_{t \geq 0}$ of contractive *-endomorphisms of $\mathcal{P}$ such that $\theta_t(j_s(X)) = j_{s+t}(X)$ and $\theta_t(k_s(Z)) = k_{s+t}(Z)$ for all $s, t \in \mathbb{R}^+$, $X \in \mathcal{A}$, $Z \in \mathcal{Z}$.*

Proof

From Lemma 4.2, $\mathcal{P}$ is the τ-closure of $\mathcal{B}$. If $(Y_n)_n$ is a sequence in $\mathcal{B}$ converging to $Y \in \mathcal{P}$ then $(Y_n)_n$ converges strongly to Y and $(\theta_t(Y_n))_n$ is a strongly convergent sequence, for all $t \geq 0$. The operator $\lim_{n \to \infty} \theta_t(Y_n)$ is an element of $\mathcal{P}$ for if $(X_n)_n$ is a τ-convergent sequence then so is $(\theta_s(X_n))_n$ for all s. Furthermore, this limit depends only on Y for if $(Y'_n)_n$ is another sequence converging to Y then $(Y_n - Y'_n)_n$ τ-converges to 0 and consequently $(\theta_t(Y_n - Y'_n))_n$ converges strongly to $\theta_t(0) = 0$. Define $\theta_t(Y)$ to be $\lim_{n\to\infty} \theta_t(Y_n)$. Then, restricted to $\mathcal{B}$, the mapping θ_t is the one given by Proposition 4.1. Consequently, if $X, Y \in \mathcal{P}$ and $(X_n)_n$ (resp. $(Y_n)_n$) is any sequence in $\mathcal{B}$ τ-converging to X (resp. Y) then $\theta_t(XY) = \mathrm{s} - \lim_{n\to\infty} \theta_t(X_nY_n) = \mathrm{s} - \lim_{n\to\infty} \theta_t(X_n)\theta_t(Y_n) = \theta_t(X)\theta_t(Y)$. ■

5. Non-commutative stop times

In classical probability theory a stop time τ on the canonical space of a Markov process $(x_t)_{t\geq 0}$ is characterized by the family of projections $\mathbb{1}_{\tau\leq t}$ on $L^2(\Omega)$. Furthermore, the event $(\tau \leq t)$ is measurable for $\mathcal{F}_t$ that is, it is "independent" of $f(x_u)$ for $u \geq t$ (more precisely it is conditionally independent of $f(x_u)$ given x_t). Translating these properties in our context we take the following definition.

A *stop time* (or a *Markov time*, to follow Dynkin's terminology) τ on H is a spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the set of orthogonal projection operators on H and such that

$$\tau([0,t])j_u(X) = j_u(X)\tau([0,t]) \quad \text{for all } u \geq t,\ X \in \mathcal{A}. \tag{5.1}$$

The commutation relation (5.1) expresses that $\tau([0,t])$ does not interfere with the future "trajectories" of the Markov process. In other words "knowing that the event of stopping at time τ has occured before time t does not affect the Markov process after time t". In the language of physicists this is called the *non-demolition property*. Note that in the context of commutative Markov processes the condition (5.1) exactly expresses the fact that $(\tau \leq t)$ is a $\mathcal{F}_t$-measurable event.

In the following we adopt a probabilistic-like notation : for every Borel set $E \subset \mathbb{R}^+ \cup \{+\infty\}$ we write $\mathbb{1}_{\tau\in E}$ instead of $\tau(E)$, in the same way $\tau([0,t])$ is denoted $\mathbb{1}_{\tau\leq t,}$ $\tau(\{t\})$ is denoted $\mathbb{1}_{\tau=t}$ etc...

A stop time τ is *finite* if $\mathbb{1}_{\tau=+\infty} = 0$.

A stop time τ is *bounded* by $T \in \mathbb{R}^+$ if $\mathbb{1}_{\tau\leq T} = I$.

A stop time τ is *discrete* if there exists a finite set $E = \{0 \leq t_1 < \cdots < t_n \leq +\infty\}$ such that $\mathbb{1}_{\tau\in E} = I$.

A point $t \in \mathbb{R}^+$ is a *continuity point* for τ if $\mathbb{1}_{\tau=t} = 0$. Notice that any stop time τ has an at most countable set of points which are not continuity points for τ.

A sequence $(\tau_n)_n$ of stop times *converges* to a stop time τ if for every continuity point t of τ, the operators $\mathbb{1}_{\tau_n\leq t}$ converge strongly to $\mathbb{1}_{\tau\leq t}$.

For any two stop times τ and τ' one says that $\tau \leq \tau'$ if $\mathbb{1}_{\tau\leq t} \geq \mathbb{1}_{\tau'\leq t}$ for all t (in the sense of the positivity of the operator $\mathbb{1}_{\tau\leq t} - \mathbb{1}_{\tau'\leq t}$).

Let τ be any stop time. By a sequence of *refining τ-partitions* of $\mathbb{R}^+$ we mean a sequence $(E_n)_n$ of finite sets $E_n = \{0 \leq t_1^n < t_2^n < \cdots < t_{i_n}^n < +\infty\}$ such that

(i) all the t_j^n are continuity points for τ

(ii) $E_n \subseteq E_{n+1}$ for all n

(iii) the diameter, sup $\{t_{i+1}^n - t_i^n, i = 1, ..., i_n\}$, of E_n tends to 0 when n tends to $+\infty$.

(iv) $t_{i_n}^n$ tends to $+\infty$ when n tends to $+\infty$.

The following result is inspired by [P-S], Proposition 3.3 and [Me1].

Proposition 5.1 – *For every stop time τ there exists a sequence $(\tau_n)_n$ of discrete stop times such that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau$ and $(\tau_n)_n$ converges to τ.*

Proof

Let $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define a spectral measure τ_E by

$$\tau_E(\{t_i\}) = \begin{cases} \mathbb{1}_{\tau < t_1} & \text{if } i = 1 \\ \mathbb{1}_{\tau \in [t_{i-1}, t_i[}, & i > 1 \end{cases}$$
$$\tau_E(\{+\infty\}) = \mathbb{1}_{\tau \geq t_n}.$$

The spectral measure τ_E clearly defines a stop time on H and we have $\tau_E \geq \tau$. Taking a sequence $(E_n)_n$ of refining τ-partitions of $\mathbb{R}^+$ gives the required sequence $(\tau_n)_n = (\tau_{E_n})_n$. Details are left to the reader. ■

When $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ is a partition of $\mathbb{R}^+$ the stop time τ_E is then supported in $\{t_1, ..., t_n\} \cup \{+\infty\}$. In that case we put $t_0 = 0$ and $t_{n+1} = +\infty$ so that we get $\mathbb{1}_{\tau_E = t_i} = \mathbb{1}_{\tau \in [t_{i-1}, t_i]}, i = 1, ..., n+1$.

The rest of this section is devoted to constructing the value F_τ of the family of projections $(F_t)_{t \geq 0}$ at any stop time τ .

Lemma 5.2 – *One has* $F_\infty \stackrel{\text{def}}{=} \lim_{t \to +\infty} F_t = I$.

Proof

The family $(F_t)_{t \geq 0}$ is an increasing family of projections so it admits a strong limit which is the projection on the Hilbert space spanned by $\cup_t H_t$. But, by Lemma 3.3, we have that $\lambda(\mathbf{r}, \mathbf{X}, u)$ belongs to H_t for all $(\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}$ such that $r_1 \leq t$. Thus, every $\lambda(\mathbf{r}, \mathbf{X}, u)$ belongs to $\cup_t H_t$. We conclude, by totality of the $\lambda(\mathbf{r}, \mathbf{X}, u)$ in H, that $\cup_t H_t$ is equal to H. ■

For all $t \geq 0$ we have $F_t = j_t(I)$, thus by (5.1) we have

$$\mathbb{1}_{\tau \leq s} F_t = F_t \, \mathbb{1}_{\tau \leq s} \text{ for all } s \leq t. \tag{5.2}$$

Note that (5.2) is obviously valid for $t = +\infty$.

The following result is proved in [Bh2], Theorem 4.10 and can be deduced by computing the explicit formula giving the mapping $\mathcal{E}$ in Proposition 3.1.

Proposition 5.3 – *If $(T_t)_{t \geq 0}$ is strongly continuous then the maps $t \to F_t \psi$ and $t \to j_t(X)\psi$ are continuous for every $X \in \mathcal{A}$, $\psi \in H$.* ■

Since we made the assumption that $(T_t)_{t\geq 0}$ is strongly continuous we have the continuity of $t \to F_t\,\psi$ for all $\psi \in H$.

For a discrete stop time τ we define

$$F_\tau = \sum_i \mathbb{1}_{\tau=t_i} F_{t_i} \ (= \sum_i F_{t_i}\ \mathbb{1}_{\tau=t_i}).$$

The operator F_τ is clearly a projection operator, let H_τ denote its range. In order to define F_τ for any stop time τ we are going to approximate τ by a sequence of discrete stop times $(\tau_n)_n$ as described in Proposition 5.1 and prove the convergence of $(F_{\tau_n})_n$. That is, we want to prove the convergence of $\sum_i \mathbb{1}_{\tau\in[t_{i-1},t_i]} F_{t_i} (= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}$ by changing the indices) when the diameter δ of the partition $\{t_i, i=1,...,n\}$ tends to 0.

Proposition 5.4 – *Let τ be any stop time. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Let $\tau_n = \tau_{E_n}$, $n \in \mathbb{N}$. Then the projections F_{τ_n} converge strongly to a projection F_τ which is the projection on the space $H_\tau = \cap_n H_{\tau_n}$.*

Proof

Let $(E_n)_n$ be a refining sequence of τ-partitions of $\mathbb{R}^+$. Let $\tau_n = \tau_{E_n}, n \in \mathbb{N}$ as described in the proof of Proposition 5.1. Let $n < m \in \mathbb{N}$. As $E_n \subset E_m$ we can suppose that E_n is of the form $\{0 \leq t_1 < \cdots < t_n\}$ and E_m is of the form $\{0 \leq \cdots \leq t_i = t_i^0 < t_i^1 < \cdots < t_i^{n_i} = t_{i+1} < \cdots\}$. Let $\psi \in H$. One has

$$\begin{aligned}
&< \psi\,,\,(F_{\tau_n} - F_{\tau_m})\psi > \\
&\quad = < \psi\,,\,\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi > - < \psi\,,\,\sum_i \sum_j \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} F_{t_i^{j+1}}\psi > \\
&\quad = \sum_i \sum_j < \psi\,,\,\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(F_{t_{i+1}}\psi - F_{t_i^{j+1}}\psi) >.
\end{aligned}$$

But as all the t_i^j are continuity points for τ we have $\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} = \mathbb{1}_{\tau\in]t_i^j,t_i^{j+1}]} = \mathbb{1}_{\tau\leq t_i^{j+1}} - \mathbb{1}_{\tau\leq t_i^j}$, thus by (5.2) the operator $\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}$ commutes with $F_{t_{i+1}}$ and $F_{t_i^{j+1}}$ (we do not discuss the case where $t_{i+1} = t_i^{j+1} = +\infty$ as it is obvious). Consequently, we have

$$\begin{aligned}
&< \psi\,,\,(F_{\tau_n} - F_{\tau_m})\psi > \\
&\quad = \sum_i \sum_j [< F_{t_{i+1}}\psi\,,\,\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} F_{t_{i+1}}\psi > - < F_{t_i^{j+1}}\psi\,,\,\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} F_{t_i^{j+1}}\psi >] \\
&\quad = \sum_i \sum_j [\|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} F_{t_{i+1}}\psi\|^2 - \|F_{t_i^{j+1}} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} F_{t_{i+1}}\psi\|^2] \\
&\quad \geq 0
\end{aligned}$$

The sequence $(F_{\tau_n})_n$ is thus a decreasing sequence of projections. Hence it converges strongly to a projection F_τ which is the projection on $H_\tau = \cap_n\ H_{\tau_n}$. ■

By denoting the limit F_τ and its range H_τ we have anticipated a result to be proved now. Indeed, the limit F_τ may depend on the sequence $(\tau_n)_n$ and not only on τ. We are going to prove a result which is interesting in itself as it makes the connection with the usual properties of classical stop times, and as a corollary we will get that F_τ depends only on τ.

Lemma 5.5 – *For every stop time τ on H, every $\psi \in H$, every $t \in \mathbb{R}^+$ one has :*
(i) $\mathbb{1}_{\tau \leq t}\, F_\tau \psi \in H_t$
(ii) $\mathbb{1}_{\tau < t}\, F_\tau \psi \in H_t$.

Proof

(i) One has

$$\begin{aligned}\mathbb{1}_{\tau\leq t}\, F_{\tau_n}\psi &= \mathbb{1}_{\tau\leq t} \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi \\ &= \sum_{i\leq i_0-1} \mathbb{1}_{\tau\leq t}\mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi + \mathbb{1}_{\tau\in[t_{i_0},t]} F_{t_{i_0+1}}\psi\end{aligned}$$

where $t_{i_0} \leq t < t_{i_0+1}$

$$\begin{aligned}&= F_t \sum_{i\leq i_0-1} \mathbb{1}_{\tau\leq t}\mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi + \mathbb{1}_{\tau\in[t_{i_0},t]} F_{t_{i_0+1}}\psi \\ &\quad + F_t \sum_{i\geq i_0+1} \mathbb{1}_{\tau\leq t}\ \mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi\end{aligned}$$

as the last term actually vanishes

$$\begin{aligned}&= F_t\, \mathbb{1}_{\tau\leq t} \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} F_{t_{i+1}}\psi + \mathbb{1}_{\tau\in[t_{i_0},t]} F_{t_{i_0+1}}\psi \\ &\quad - F_t\, \mathbb{1}_{\tau\leq t}\ \mathbb{1}_{\tau\in[t_{i_0},t_{i_0+1}]} F_{t_{i_0+1}}\psi \\ &= F_t\, \mathbb{1}_{\tau\leq t}\, F_{\tau_n}\psi + \mathbb{1}_{\tau\in[t_{i_0},t]}(F_{t_{i_0+1}}\psi - F_t\psi).\end{aligned}$$

The left hand side of this identity converges to $\mathbb{1}_{\tau\leq t}F_\tau\psi$, the first term of the right hand side converges to $F_t\mathbb{1}_{\tau\leq t}F_\tau\psi$ and the second term converges to 0 by continuity of $t \to F_t\,\psi$. This proves (i).

The proof of (ii) is the same, one just has to check that the operator $\mathbb{1}_{\tau<t}$ commutes with F_u for all $u \geq t$, even for those t which are not continuity points for τ. Let $u \geq t$, $\psi \in H$. One has

$$\begin{aligned}&\|\mathbb{1}_{\tau<t}F_u\psi - F_u\mathbb{1}_{\tau<t}\psi\|^2 \\ &\leq 2\|\mathbb{1}_{\tau<t}F_u\psi - \mathbb{1}_{\tau\leq t-\frac{1}{n}}F_u\psi\|^2 + 2\|F_u\mathbb{1}_{\tau\leq t-\frac{1}{n}}\psi - F_u\mathbb{1}_{\tau<t}\psi\|^2 \\ &\leq 2\|\mathbb{1}_{\tau\in]t-\frac{1}{n},t[}F_u\psi\|^2 + 2\|\mathbb{1}_{\tau\in]t-\frac{1}{n},t[}\psi\|^2.\end{aligned}$$

This quantity converges to 0 when n tends to $+\infty$. ∎

Theorem 5.6 – *For every stop time τ one has*

$$\begin{aligned}H_\tau &= \{\psi \in H;\ \mathbb{1}_{\tau\leq t}\psi \in H_t \text{ for all } t\} \\ &= \{\psi \in H;\ \mathbb{1}_{\tau<t}\,\psi \in H_t \text{ for all } t\}.\end{aligned}$$

Proof

Let $E_{\leq}$ be the set $\{\psi \in H; \mathbb{1}_{\tau\leq t}\psi \in H_t \text{ for all } t\}$ and let $E_{<}$ be the set $\{\psi \in H; \mathbb{1}_{\tau<t}\psi \in H_t \text{ for all } t\}$. If $\psi \in H_\tau$ then $\psi = F_\tau\psi$ and by Lemma 5.5 $\mathbb{1}_{\tau\leq t}\,\psi$ and $\mathbb{1}_{\tau<t}\,\psi$ are elements of H_t for all t. Thus $H_\tau \subseteq E_{\leq}$ and $H_\tau \subseteq E_{<}$.

Let $\psi \in E_{<}$. One has

$$
\begin{aligned}
F_{\tau_n}\psi &= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}F_{t_{i+1}}\psi = \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}[}F_{t_{i+1}}\psi \\
&= \sum_i (\mathbb{1}_{\tau<t_{i+1}} - \mathbb{1}_{\tau<t_i})F_{t_{i+1}}\psi \\
&= \sum_i F_{t_{i+1}}(\mathbb{1}_{\tau<t_{i+1}} - \mathbb{1}_{\tau<t_i})\psi
\end{aligned}
$$

But, as ψ belongs to $E_{<}$, $(\mathbb{1}_{\tau<t_{i+1}} - \mathbb{1}_{\tau<t_i})\psi$ is an element of $H_{t_{i+1}}$. Thus

$$
F_{\tau_n}\psi = \sum_i (\mathbb{1}_{\tau<t_{i+1}} - \mathbb{1}_{\tau<t_i})\psi = \psi.
$$

Passing to the limit we get $F_\tau\psi = \psi$, thus ψ belongs to H_τ and we have proved that $E_{<} \subseteq H_\tau$.

The proof of $E_{\leq} \subseteq H_\tau$ is the same as $E_{<} \subseteq H_\tau$ by noticing that $\mathbb{1}_{\tau\in[t_i,t_{i+1}[} = \mathbb{1}_{\tau\in]t_i,t_{i+1}]} = \mathbb{1}_{\tau\leq t_{i+1}} - \mathbb{1}_{\tau\leq t_i}$. ■

Corollary 5.7 – *H_τ and thus F_τ depend only on τ.* ■

Two stop times τ_1 and τ_2 are said to be *commuting* if $\mathbb{1}_{\tau_1\in E}$ and $\mathbb{1}_{\tau_2\in F}$ commute for any two Borel sets $E, F \subset \mathbb{R}^+ \cup \{+\infty\}$.

Proposition 5.8 – *For any two commuting stop times τ_1 and τ_2, there exists a stop time $\tau_1 \wedge \tau_2$ satisfying*

$$
\mathbb{1}_{\tau_1\wedge\tau_2\leq t} = \mathbb{1}_{\tau_1\leq t} + \mathbb{1}_{\tau_2\leq t} - \mathbb{1}_{\tau_1\leq t}\,\mathbb{1}_{\tau_2\leq t}. \tag{5.3}
$$

Furthermore one has $\tau_1 \wedge \tau_2 \leq \tau_i$, $i = 1, 2$ and if τ is any stop time such that $\tau \leq \tau_i$, $i = 1, 2$ then one has $\tau \leq \tau_1 \wedge \tau_2$.

Proof

Identity (5.3) can also be written as : $\mathbb{1}_{\tau_1\wedge\tau_2>t} = \mathbb{1}_{\tau_1>t}\mathbb{1}_{\tau_2>t}$. Thus the family $(\mathbb{1}_{\tau_1\wedge\tau_2\leq t})_{t\geq 0}$ is clearly a right continuous nondecreasing projection valued function of t on $\mathbb{R}^+ \cup \{+\infty\}$ and hence can be extended uniquely to a spectral measure $\tau_1 \wedge \tau_2$. The fact that $\mathbb{1}_{\tau_1\wedge\tau_2\leq t}$ commutes with all the $j_u(X), u \geq t, X \in \mathcal{A}$ is clear. Furthermore we have $\mathbb{1}_{\tau_1\wedge\tau_2>t} \geq \mathbb{1}_{\tau_i>t}$, $i = 1, 2, t \in \mathbb{R}^+$ thus $\tau_1 \wedge \tau_2 \leq \tau_i$, $i = 1, 2$. If τ is another stop time satisfying $\tau \leq \tau_i$, $i = 1, 2$ we then have $\mathbb{1}_{\tau>t} \geq \mathbb{1}_{\tau_i>t}, i = 1, 2$, thus $\mathbb{1}_{\tau>t} \geq \mathbb{1}_{\tau_1>t}\mathbb{1}_{\tau_2>t} = \mathbb{1}_{\tau_1\wedge\tau_2>t}$. This proves $\tau \leq \tau_1 \wedge \tau_2$.
■

We now prove an analogue, in our non-commutative context, of Doob's conditioning theorem.

Theorem 5.9 – *For any two commuting stop time τ_1 and τ_2 one has*

$$F_{\tau_1} F_{\tau_2} = F_{\tau_2} F_{\tau_1} = F_{\tau_1 \wedge \tau_2}$$

Proof

Let $E = \{t_i, i = 1, .., n\}$ be a partition of $\mathbb{R}^+$. Let $\tilde{\tau}_1$ (resp. $\tilde{\tau}_2$) be the discrete stop time built from τ_1 (resp. τ_2) and E as in Proposition 5.1. We have

$$\begin{aligned}
F_{\tilde{\tau}_1} F_{\tilde{\tau}_2} &= \sum_i \sum_j \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}]} F_{t_{i+1}} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}]} F_{t_{j+1}} \\
&= \sum_i \sum_j \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}]} F_{t_{i+1}} \; F_{t_{j+1}} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}]} \\
&= \sum_i \sum_j \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}]} F_{t_{i+1} \wedge t_{j+1}} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}]} \\
&= \sum_i \sum_{j \leq i} \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}]} F_{t_{j+1}} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}]} \\
&\quad + \sum_i \sum_{j \geq i+1} \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}]} F_{t_{i+1}} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}]} \\
&= \sum_j \mathbb{1}_{\tau_1 \geq t_j} \mathbb{1}_{\tau_2 \in [t_j, t_{j+1}[} F_{t_{j+1}} + \sum_i \mathbb{1}_{\tau_1 \in [t_i, t_{i+1}[} \mathbb{1}_{\tau_2 \geq t_{i+1}} F_{t_{i+1}} \\
&= \sum_i [\mathbb{1}_{\tau_1 \geq t_i} \mathbb{1}_{\tau_2 \geq t_i} - \mathbb{1}_{\tau_1 \geq t_i} \mathbb{1}_{\tau_2 \geq t_{i+1}} + \mathbb{1}_{\tau_1 \geq t_i} \mathbb{1}_{\tau_2 \geq t_{i+1}} \\
&\quad - \mathbb{1}_{\tau_1 \geq t_{i+1}} \mathbb{1}_{\tau_2 \geq t_{i+1}}] F_{t_{i+1}} \\
&= \sum_i [\mathbb{1}_{\tau_1 \wedge \tau_2 \geq t_i} - \mathbb{1}_{\tau_1 \wedge \tau_2 \geq t_{i+1}}] F_{t_{i+1}} \\
&= \sum_i \mathbb{1}_{\tau_1 \wedge \tau_2 \in [t_i, t_{i+1}[} F_{t_{i+1}} \\
&= F_{\widetilde{\tau_1 \wedge \tau_2}}
\end{aligned}$$

where $\widetilde{\tau_1 \wedge \tau_2}$ is the discrete approximation of $\tau_1 \wedge \tau_2$ based on the partition E. Thus, passing to the limit, we get the result. ■

6. Strong Markov processes

The process $(j_t)_{t \geq 0}$ of *-homomorphisms on $\mathcal{A}$ represents our Markov process. In order to define strong Markov processes we need to define the value j_τ of $(j_t)_{t \geq 0}$ at a stop time τ. When τ is discrete, j_τ can be obviously defined by

$$j_\tau(X) = \sum_i \mathbb{1}_{\tau = t_i} j_{t_i}(X) = \sum_i j_{t_i}(X) \mathbb{1}_{\tau = t_i} = \sum_i \mathbb{1}_{\tau = t_i} j_{t_i}(X) \mathbb{1}_{\tau = t_i}.$$

For a general stop time τ we wish to pass to the limit through a sequence of discrete stop time $(\tau_n)_n$ converging to τ.

We say that the Markov process $(j_t)_{t\geq 0}$ is *τ-regular* on $(\mathcal{A}_0, \widetilde{H})$, for a quantum stop time τ, a dense subspace $\mathcal{A}_0$ of $\mathcal{A}$ and a dense subspace $\widetilde{H}$ of H if for every sequence $(\tau_n)_n$ of discrete stop times converging to τ, every $X \in \mathcal{A}_0$, every $\Psi \in \widetilde{H}$ the sequence $(j_{\tau_n}(X)\Psi)_n$ converges in H to a limit $j_\tau(X)\Psi$ which is independent of the choice of the sequence $(\tau_n)_n$.

We say that the Markov process $(j_t)_{t\geq 0}$ is a (quantum) *strong Markov process* if there exists a dense subspace $\mathcal{A}_0$ of $\mathcal{A}$ and a dense subspace $\widetilde{H}$ of H such that $(j_t)_{t\geq 0}$ is τ-regular on $(\mathcal{A}_0, \widetilde{H})$ for every finite stop time τ.

Proposition 6.1 – *Let $(j_t)_{t\geq 0}$ be a strong Markov process. The mapping $\psi \mapsto j_\tau(X)\psi$ defines a bounded linear mapping which extends to a bounded linear operator $j_\tau(X)$ on H with norm dominated by $\|X\|$. The mapping $X \mapsto j_\tau(X)$ extends to a (contractive) *-homomorphism on $\mathcal{A}$.*

Proof

Let $\psi \in \tilde{H}, X \in \mathcal{A}_0$. Let τ be a discrete stop time. Then

$$\begin{aligned}\|j_\tau(X)\psi\|^2 &= \|\sum_i \mathbb{1}_{\tau=t_i}\, j_{t_i}(X)\psi\|^2 = \sum_i \|\mathbb{1}_{\tau=t_i}\, j_{t_i}(X)\psi\|^2 \\ &\leq \|X\|^2 \sum_i \|\mathbb{1}_{\tau=t_i}\psi\|^2 = \|X\|^2\|\psi\|^2.\end{aligned}$$

If τ is now any finite stop time and $(\tau_n)_n$ is a sequence of discrete stop times converging to τ then

$$\|j_\tau(X)\psi\| \leq \|j_\tau(X)\psi - j_{\tau_n}(X)\psi\| + \|j_{\tau_n}(X)\psi\| \leq \frac{1}{n} + \|X\|\;\|\psi\|$$

for n large enough. Thus $\|j_\tau(X)\psi\| \leq \|X\|\;\|\psi\|$. As $\tilde{H}$ is dense in H we get that $j_\tau(X)$ extends to a bounded linear operator on H with norm dominated by $\|X\|$.

The mapping $X \mapsto j_\tau(X)$ is then a continuous linear mapping on $\mathcal{A}_0$ which is dense on $\mathcal{A}$. Thus it extends to $\mathcal{A}$. Let us check the morphism property. If τ is discrete we have

$$\begin{aligned}j_\tau(X)j_\tau(Y) &= \sum_i \sum_j \mathbb{1}_{\tau=t_i} j_{t_i}(X)\mathbb{1}_{\tau=t_j}\, j_{t_j}(Y) \\ &= \sum_i \sum_j j_{t_i}(X)\mathbb{1}_{\tau=t_i}\mathbb{1}_{\tau=t_j}\, j_{t_j}(Y) \\ &= \sum_i \mathbb{1}_{\tau=t_i}\, j_{t_i}(X)j_{t_i}(Y) \\ &= \sum_i \mathbb{1}_{\tau=t_i}\, j_{t_i}(XY) \\ &= j_\tau(XY).\end{aligned}$$

Now let τ be any bounded stop time and $(\tau_n)_n$ be a sequence of discrete stop times converging to τ. Let $\psi \in \tilde{H}$, $X, Y \in \mathcal{A}_0$. One has

$$\begin{aligned}
&\|[j_\tau(X)j_\tau(Y) - j_\tau(XY)]\psi\| \\
&\quad \leq \|[j_\tau(XY) - j_{\tau_n}(XY)]\psi\| + \|[j_{\tau_n}(X)j_{\tau_n}(Y) - j_{\tau_n}(X)j_\tau(Y)]\psi\| + \\
&\quad\ + \|[j_{\tau_n}(X)j_\tau(Y) - j_\tau(X)j_\tau(Y)]\psi\| \\
&\quad \leq \|[j_\tau(XY) - j_{\tau_n}(XY)]\psi\| + \|X\|\ \|[j_{\tau_n}(Y) - j_\tau(Y)]\psi\| + \\
&\quad\ + \|[j_{\tau_n}(X) - j_\tau(X)]j_\tau(Y)\psi\|.
\end{aligned}$$

This expression converges to 0 when n tends to $+\infty$. Thus for $X, Y \in \mathcal{A}_0$ one has $j_\tau(XY) = j_\tau(X)j_\tau(Y)$ on $\tilde{H}$. As all these operators are bounded this equality holds on the whole H. Furthermore, as $\|j_\tau(X)\| \leq \|X\|$ for all $X \in \mathcal{A}$, the morphism property extends to $\mathcal{A}$. ■

For any stop time τ and any $t \in \mathbb{R}^+$ define the stop time $\tau + t$ by

$$\mathbb{1}_{\tau+t\leq s} = \begin{cases} 0 & \text{if} \quad s < t \\ \mathbb{1}_{\tau \leq s-t} & \text{if} \quad s \geq t. \end{cases}$$

If τ is finite then so is $\tau + t$.

Now the naming "strong Markov process" is justified by the following result.

Theorem 6.2 (Strong Markov property) – *Let $(j_t)_{t\geq 0}$ be a strong Markov process. For any finite stop time τ, any $X \in \mathcal{A}$, any $t \in \mathbb{R}^+$, one has*

$$F_\tau j_{\tau+t}(X) F_\tau = j_\tau(T_t(X)).$$

Proof

Let τ be a discrete stop time, then

$$\begin{aligned}
F_\tau j_{\tau+t}(X)F_\tau &= \sum_{i,j,k} \mathbb{1}_{\tau=t_i}\, F_{t_i} \mathbb{1}_{\tau=t_j}\, j_{t_j+t}(X) \mathbb{1}_{\tau=t_k}\, F_{t_k} \\
&= \sum_{i,j,k} F_{t_i} \mathbb{1}_{\tau=t_i}\, \mathbb{1}_{\tau=t_j}\, j_{t_j+t}(X) \mathbb{1}_{\tau=t_k}\, F_{t_k} \\
&= \sum_{i,k} F_{t_i} \mathbb{1}_{\tau=t_i}\, j_{t_i+t}(X) \mathbb{1}_{\tau=t_k}\, F_{t_k} \\
&= \sum_{i} \mathbb{1}_{\tau=t_i}\, F_{t_i}\, j_{t_i+t}(X) F_{t_i} \\
&= \sum_{i} \mathbb{1}_{\tau=t_i}\, j_{t_i}(T_t(X)) \\
&= j_\tau(T_t(X)).
\end{aligned}$$

When τ is any finite stop time one approximates τ by discrete stop times. Hence there exists a sequence $(\tau_n)_n$ of discrete stop times such that F_{τ_n} converges to

F_τ, $j_{\tau_n}(X)$ converges to $j_\tau(X)$ and $j_{\tau_n+t}(X)$ converges to $j_{\tau+t}(X)$. As all these operators are uniformly bounded one concludes easily. ■

In the case of a strong Markov process $(j_t)_{t\geq 0}$ we are now going to define the value at time τ of the lifted Markov process $(k_t(Z))_{t\geq 0}, Z \in \mathcal{Z}$.

Lemma 6.3 – *For every stop time τ, every $u \geq t$ and every $Z \in \mathcal{Z}$ the operators $\mathbb{1}_{\tau\leq t}$ and $k_u(Z)$ are commuting.*

Proof

For any $(\mathbf{r}, \mathbf{X}, v) \in \mathcal{D}$ let us compute $\mathbb{1}_{\tau\leq t}\, k_u(Z)\lambda(\mathbf{r}, \mathbf{X}, v)$. One has

$$\begin{aligned}
&\mathbb{1}_{\tau\leq t}\, k_u(Z)\lambda(\mathbf{r}, \mathbf{X}, v)\\
&\quad= \mathbb{1}_{\tau\leq t} k_u(Z) j_{r_1}(X_1)\cdots j_{r_n}(X_n)Vv\\
&\quad= j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1})\mathbb{1}_{\tau\leq t} k_u(Z) j_{r_k}(X_k)\cdots j_{r_n}(X)Vv
\end{aligned}$$

(where $r_k < u \leq r_{k-1}$; the cases $u > r_1$ or $u < r_n$ do not contain any supplementary difficulty)

$$\begin{aligned}
&\quad= j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1})\mathbb{1}_{\tau\leq t} j_u(Z) j_{r_k}(X_k)\cdots j_{r_n}(X_n)Vv\\
&\quad= j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1}) j_u(Z)\mathbb{1}_{\tau\leq t}\, j_{r_k}(X_k)\cdots j_{r_n}(X_n)Vv\\
&\quad= j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1}) k_u(Z) F_u\mathbb{1}_{\tau\leq t}\, j_{r_k}(X_k)\cdots j_{r_n}(X_n)Vv\\
&\quad= k_u(Z) j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1}) F_u\mathbb{1}_{\tau\leq t}\, j_{r_k}(X_k)\cdots j_{r_n}(X_n)Vv\\
&\quad= k_u(Z)\mathbb{1}_{\tau\leq t}\, j_{r_1}(X_1)\cdots j_{r_{k-1}}(X_{k-1}) F_u j_{r_k}(X_k)\cdots j_{r_n}(X_n)Vv\\
&\quad= k_u(Z)\mathbb{1}_{\tau\leq t}\,\lambda(\mathbf{r}, \mathbf{X}, u).
\end{aligned}$$

We conclude by boundedness of $k_u(Z)$ and $\mathbb{1}_{\tau\leq t}$, and by totality of the $\lambda(\mathbf{r}, \mathbf{X}, u)$ in H. ■

For a discrete stop time τ we define $k_\tau(Z)$ by

$$k_\tau(Z) = \sum_i \mathbb{1}_{\tau=t_i}\, k_{t_i}(Z) = \sum_i k_{t_i}(Z)\mathbb{1}_{\tau=t_i}$$

Proposition 6.4 – *Let τ be a discrete stop time. Let $Z \in \mathcal{Z}$. Let $(\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}$ with $\mathbf{r} = \{r_1 > \ldots > r_n\}$. Put $r_0 = +\infty$ and $r_{n+1} = 0$. Then the operator $k_\tau(Z)$ is bounded with norm dominated by $\|Z\|$ and it satisfies*

$$\begin{aligned}
k_\tau(Z)\lambda(\mathbf{r}, \mathbf{X}, u) = \sum_{i=1}^{n+1} \mathbb{1}_{\tau\in[r_i, r_{i-1}[} j_{r_1}(X_1)\cdots j_{r_{i-1}}(X_{i-1}) j_\tau(Z) j_{r_i}(X_i)&\\
\cdots j_{r_n}(X_n)Vu.&
\end{aligned}\tag{6.1}$$

*Furthermore, the mapping $Z \mapsto k_\tau(Z)$ defines a *-unital homomorphism on $\mathcal{Z}$.*

Proof

For all $\psi \in H$ one has

$$\|k_\tau(Z)\|^2 = \|\sum_i \mathbb{1}_{\tau=t_i} k_{t_i}(Z)\psi\|^2 = \sum_i \|\mathbb{1}_{\tau=t_i} k_{t_i}(Z)\psi\|^2 = \sum_i \|k_{t_i}(Z)\mathbb{1}_{\tau=t_i}\psi\|^2$$

$$\leq \|Z\|^2 \sum_i \|\mathbb{1}_{\tau=t_i}\psi\|^2 = \|Z\|^2\|\psi\|^2.$$

This proves the boundedness and the norm estimate for $k_\tau(Z)$. The mapping $Z \mapsto k_\tau(Z)$ is clearly linear and *-preserving. The multiplication-preservation property is obtained in the same way as for j_τ in Proposition 6.1.

Furthermore, we have

$$\begin{aligned}
&k_\tau(Z) j_{r_1}(X_1)\cdots j_{r_n}(X_n)Vu \\
&\quad= \sum_i \mathbb{1}_{\tau=t_i} k_{t_i}(Z) j_{r_1}(X_1)\cdots j_{r_n}(X_n)Vu \\
&\quad= \sum_{j=1}^{n+1} j_{r_1}(X_1)\cdots j_{r_{j-1}}(X_{j-1}) \Big(\sum_{i;t_i\in[r_j,r_{j-1}[} \mathbb{1}_{\tau=t_i} k_{t_i}(Z)\Big) \\
&\qquad\qquad \times j_{r_j}(X_j)\cdots j_{r_n}(X_n)Vu \\
&\quad= \sum_{j=1}^{n+1} j_{r_1}(X_1)\cdots j_{r_{j-1}}(X_{j-1}) \Big(\sum_{i;t_i\in[r_j,r_{j-1}[} \mathbb{1}_{\tau=t_i} j_{t_i}(Z)\Big) \\
&\qquad\qquad \times j_{r_j}(X_j)\cdots j_{r_n}(X_n)Vu \\
&\quad= \sum_{j=1}^{n+1} j_{r_1}(X_1)\cdots j_{r_{j-1}}(X_{j-1}) \mathbb{1}_{\tau\in[r_j,r_{j-1}[}\, j_\tau(Z) j_{r_j}(X_j)\cdots j_{r_n}(X_n)Vu \\
&\quad= \sum_{j=1}^{n+1} \mathbb{1}_{\tau\in[r_j,r_{j-1}[}\, j_{r_1}(X_1)\cdots j_{r_{j-1}}(X_{j-1}) j_\tau(Z) j_{r_j}(X_j)\cdots j_{r_n}(X_n)Vu.
\end{aligned}$$

This proves identity (6.1). ■

Theorem 6.5 – *Let τ be any finite stop time. Let $(E_n)_n$ be a sequence of refining τ-partitions. Let $\tau_n = \tau_{E_n}$, $n \in \mathbb{N}$. Then the sequence $(k_{\tau_n}(Z))_n$ converges to an operator $k_\tau(Z)$, for all $Z \in \mathcal{Z}$, which satisfies*

$$\begin{aligned}
k_\tau(Z)\lambda(\mathbf{r},\mathbf{X},u) = \sum_{j=1}^{n+1} \mathbb{1}_{\tau\in[r_j,r_{j-1}[}\, j_{r_1}(X_1)\cdots j_{r_{j-1}}(X_{j-1}) j_\tau(Z) j_{r_j}(X_j)& \\
\cdots j_{r_n}(X_n)Vu.& \qquad (6.2)
\end{aligned}$$

*The operator $k_\tau(Z)$ is bounded with norm dominated by $\|Z\|$. The mapping $Z \mapsto k_\tau(Z)$ defines a *-homomorphism on $\mathcal{Z}$.*

Proof

Suppose that in $(\mathbf{r},\mathbf{X},u) \in \mathcal{D}$ all the r_i's are continuity points for τ. From Proposition 6.4 the identity (6.2) is satisfied for each τ_n. As $\mathbb{1}_{\tau_n\in[r_j,r_{j-1}[}$ (resp. $j_{\tau_n}(Z)$) converges strongly to $\mathbb{1}_{\tau\in[r_j,r_{j-1}[}$ (resp. $j_\tau(Z)$), as all the operators $j_{r_j}(X_j)$ are bounded, it is easy to check that $k_{\tau_n}(Z)$ converges strongly to an operator $k_\tau(Z)$ which satisfies (6.2). As the vectors $\lambda(\mathbf{r},\mathbf{X},u)$ where each r_j is a continuity point for τ form a total subset of H we get that (6.2) holds for any $(\mathbf{r},\mathbf{X},u) \in \mathcal{D}$.

From the norm estimate of $k_{\tau_n}(Z)$ and the *-homomorphism property of $k_{\tau_n}(\cdot)$ one gets, in the same way as for j_τ in Proposition 6.1, the same results for k_τ. ■

Important remarks : All what has been done here for finite stop time can be extended to any stop times in the following way.

Notice that a time $t \in \mathbb{R}^+$ defines a stop time, also denoted t, by putting $\mathbb{1}_{t\leq s} = 0$ if $s < t, I$ if $s \geq t$. For any finite stop time τ, any $n \in \mathbb{N}$, the stop time $\tau \wedge n$, as defined in Proposition 5.8 is clearly a bounded stop time, with bound n. Furthermore $\tau \wedge n$ converges to τ when n tends to $+\infty$.

If τ is any finite stop time it is easy to check that

$$\mathbb{1}_{\tau\leq n}\, j_\tau(X) = \mathbb{1}_{\tau\leq n}\, j_{\tau\wedge n}(X);$$

indeed, one checks it directly for discrete stop time and then pass to the limit.

Let τ be any stop time, even non-finite. Let ψ be an element of the range of $\mathbb{1}_{\tau<+\infty}$ that is, $\psi = \mathbb{1}_{\tau<+\infty}\psi$. Then consider the sequence $(j_{\tau\wedge n}(X)\psi)_n$. Let p be such that $||\mathbb{1}_{\tau<\infty}\psi - \mathbb{1}_{\tau<p}\psi||$ is small. Then for all $m \geq n \geq p$ we have

$$\begin{aligned}
||j_{\tau\wedge n}(X)\psi - j_{\tau\wedge m}(X)\psi|| &= ||j_{\tau\wedge n}(X)\mathbb{1}_{\tau<\infty}\psi - j_{\tau\wedge m}(X)\mathbb{1}_{\tau<\infty}\psi|| \\
&\leq ||j_{\tau\wedge n}(X)\mathbb{1}_{\tau<\infty}\psi - j_{\tau\wedge n}(X)\mathbb{1}_{\tau<p}\psi|| \\
&\quad + ||j_{\tau\wedge m}(X)\mathbb{1}_{\tau<\infty}\psi - j_{\tau\wedge m}(X)\mathbb{1}_{\tau<p}\psi|| \\
&\quad + ||j_{\tau\wedge n}(X)\mathbb{1}_{\tau<p}\psi - j_{\tau\wedge m}(X)\mathbb{1}_{\tau<p}\psi|| \\
&\leq 2||X||\,||\mathbb{1}_{\tau<\infty}\psi - \mathbb{1}_{\tau<p}\psi|| \\
&\quad + ||j_{\tau\wedge p}(X)\mathbb{1}_{\tau<p}\psi - j_{\tau\wedge p}(X)\mathbb{1}_{\tau<p}\psi|| \\
&= 2||X||\,||\mathbb{1}_{\tau<\infty}\psi - \mathbb{1}_{\tau<p}\psi||.
\end{aligned}$$

Thus the sequence $(j_{\tau\wedge n}(X)\psi)_n$ is a Cauchy sequence, therefore it is convergent. Let us call $j_\tau(X)\psi$ this limit. If ψ is any element of H then define $j_\tau(X)\psi$ to be $j_\tau(X)\mathbb{1}_{\tau<\infty}\psi$.

One easily get that $\mathbb{1}_{\tau\leq n}\, j_\tau(X)\psi = \mathbb{1}_{\tau\leq n}\, j_{\tau\wedge n}(X)\psi = j_\tau(X)\mathbb{1}_{\tau\leq n}\psi$. Thus

$$\begin{aligned}
\mathbb{1}_{\tau<\infty}\, j_\tau(X)\psi &= \lim_{n\to+\infty} \mathbb{1}_{\tau\leq n}\, j_\tau(X)\psi \\
&= \lim_{n\to\infty} j_{\tau(X)}\mathbb{1}_{\tau\leq n}\psi = j_\tau(X)\mathbb{1}_{\tau<\infty}\psi.
\end{aligned}$$

Furthermore, it is clear that $\|j_\tau(X)\psi\| \leq \|X\|\,\|\psi\|$ thus $j_\tau(X)$ defines a bounded operator on H with norm dominated by $\|X\|$. Finally, let $X, Y \in \mathcal{A}$. We have

$$\begin{aligned}
j_\tau(XY)\psi &= j_\tau(XY)\mathbb{1}_{\tau<+\infty}\psi = \lim_{n\to+\infty} j_\tau(XY)\mathbb{1}_{\tau<n}\psi \\
&= \lim_{n\to+\infty} j_{\tau\wedge n}(XY)\mathbb{1}_{\tau<n}\psi \\
&= \lim_{n\to+\infty} j_{\tau\wedge n}(X)j_{\tau\wedge n}(Y)\mathbb{1}_{\tau<n}\psi = \lim_{n\to+\infty} j_{\tau\wedge n}(X)\mathbb{1}_{\tau\leq n}\, j_{\tau\wedge n}(Y)\mathbb{1}_{\tau\leq n}\psi \\
&= \lim_{n\to+\infty} j_\tau(X)\mathbb{1}_{\tau\leq n}j_{\tau\wedge n}(Y)\mathbb{1}_{\tau\leq n}\psi = \lim_{n\to+\infty} j_\tau(X)j_\tau(Y)\mathbb{1}_{\tau\leq n}\psi \\
&= j_\tau(X)j_\tau(Y)\mathbb{1}_{\tau<+\infty}\psi = j_\tau(X)\mathbb{1}_{\tau<+\infty}\, j_\tau(Y)\mathbb{1}_{\tau<+\infty}\psi \\
&= j_\tau(X)j_\tau(Y)\psi.
\end{aligned}$$

Thus j_τ defines a *-homomorphism on $\mathcal{A}$.

The same properties clearly hold for k_τ.

7. Sufficient conditions for strong Markov property

We are giving in this section two different sufficient conditions for a Markov process $(j_t)_{t\geq 0}$ to be strong Markov.

A family $(x_t)_{t\geq 0}$ of elements of H is called a *process of vectors*. A process of vectors $(x_t)_{t\geq 0}$ is *adapted* if, for all $t \geq 0$, x_t belongs to H_t. An adapted process of vectors $(x_t)_{t\geq 0}$ is a *martingale* if for all $s \leq t$ one has $F_s\, x_t = x_s$. An adapted process of vectors $(x_t)_{t\geq 0}$ is a *regular semimartingale of vectors* if it admits a decomposition (which is always unique if it exists) as $x_t = m_t + a_t$ where $(m_t)_{t\geq 0}$ is a martingale and $a_t = \int_0^t h_s\, ds$ with $\int_0^t \|h_s\|\, ds < \infty$ and $(h_t)_{t\geq 0}$ is an adapted process of vectors. The integral $\int_0^t h_s\ ds$ is understood as the usual Hilbertian integral defined by $< f\,,\, \int_0^t h_s\, ds > \;=\; \int_0^t < f\,,\, h_s > ds$ for all $f \in H$; this clearly defines a vector in H for

$$|\int_0^t < f\,,\, h_s > ds| \leq \int_0^t |< f\,,\, h_s >|\, ds \leq \|f\| \int_0^t \|h_s\|\, ds.$$

For a process of vectors $(x_t)_{t\geq 0}$ one can wonder how to define the value x_τ of $(x_t)_{t\geq 0}$ at time τ, for any stop time τ. When τ is a discrete stop time the definition is obvious:

$$x_\tau = \sum_i 1\!\!1_{\tau=t_i} x_{t_i}.$$

But for a general stop time τ the problem is to pass to the limit on x_{τ_n} when $(\tau_n)_n$ is a sequence of discrete stop times converging to τ. The following result says that this can be realised when $(x_t)_{t\geq 0}$ is a regular semimartingale of vectors and τ is bounded.

Theorem 7.1 – *Let $(x_t)_{t\geq 0}$ be a regular semimartingale of vectors on H. Let τ be a bounded stop time. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Let $\tau_n = \tau_{E_n}$, $n \in \mathbb{N}$. Then the sequence $(x_{\tau_n})_n$ converges to a limit x_τ which is independent of the choice of the sequences $(\tau_n)_n$.*

Proof

Let T be a bound of the bounded stop time τ. Notice that for n large enough, the stop times τ_n are also bounded by T. As $(x_t)_{t\geq 0}$ is a regular semimartingale of vectors it can be written $x_t = m_t + \int_0^t h_s\, ds$ where $(m_t)_{t\geq 0}$ is a martingale. Thus

$$\begin{aligned} x_{\tau_n} &= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}[}\, [m_{t_{i+1}} + \int_0^{t_{i+1}} h_s\ ds] \\ &= \sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}[}\, F_{t_{i+1}} m_T + \sum_i \sum_{j\leq i} 1\!\!1_{\tau\in[t_i,t_{i+1}[} \int_{t_j}^{t_{j+1}} h_s\, ds \end{aligned}$$

$$
\begin{aligned}
&= F_{\tau_n} m_T + \sum_j \mathbb{1}_{\tau \geq t_j} \int_{t_j}^{t_j+1} h_s \, ds \\
&= F_{\tau_n} m_T + \sum_j \int_{t_j}^{t_j+1} \mathbb{1}_{\tau \geq t_j} h_s \, ds \\
&= F_{\tau_n} m_T + \sum_j \int_{t_j}^{t_j+1} \mathbb{1}_{\tau_n > s} h_s \, ds \\
&= F_{\tau_n} m_T + \int_0^T \mathbb{1}_{\tau_n > s} h_s \, ds.
\end{aligned}
$$

By Proposition 5.4 and Corollary 5.7, $F_{\tau_n} m_T$ converges to $F_\tau m_T$ which does not depend on $(\tau_n)_n$. Furthermore $\mathbb{1}_{\tau_n > s} h_s$ converges to $\mathbb{1}_{\tau > s} h_s$ for all but a countable set of s, thus by the dominated convergence theorem $\int_0^T \mathbb{1}_{\tau_n > s} h_s \, ds$ converges to $\int_0^T \mathbb{1}_{\tau > s} h_s \, ds$.

Finally we have proved that $(x_{\tau_n})_n$ converges to

$$
x_\tau = F_\tau m_T + \int_0^T \mathbb{1}_{\tau > s} h_s \, ds
$$

which is independent of $(\tau_n)_n$. ■

Now, recall the following characterization of Hilbertian quasimartingales.

Theorem 7.2 – *Let $(H, (F_t)_{t \geq 0})$ be a filtered Hilbert space that is, a Hilbert space H together with a right continuous increasing family $(F_t)_{t \geq 0}$ of orthogonal projections in H. Let H_t be the range of F_t, for all $t \in \mathbb{R}^+$. Let $(x_t)_{t \geq 0}$ be an adapted process of vectors (i.e. $x_t \in H_t$ for all $t \in \mathbb{R}^+$). If one has*

$$
\sup_{\mathcal{R}} \sum_i \| F_{t_i} x_{t_{i+1}} - x_{t_i} \| < +\infty,
$$

where $\mathcal{R} = \{t_i, i = 1, ..., n\}$ runs over all the partitions of a fixed bounded interval $[0, T]$, then $(x_t)_{t \geq 0}$ admits a unique decomposition as a sum of a martingale $(m_t)_{t \geq 0}$ (i.e. $F_s m_t = m_s$, $s \leq t$) and a finite variation process $(a_t)_{t \geq 0}$ (i.e. $\sup_{\mathcal{R}} \sum_i \| a_{t_{i+1}} - a_{t_i} \| < \infty$), vanishing at 0, adapted to $(H_{t-})_{t \geq 0} = (\cap_{s<t} \overline{H}_s)_{t \geq 0}$. Such a $(x_t)_{t \geq 0}$ is called a Hilbertian quasimartingale.

If $(x_t)_{t \geq 0}$ is an adapted process which satisfies

$$
\| F_s x_t - x_s \| \leq \int_s^t g(u) \, du
$$

for $s \leq t$ and a locally integrable g of then $(x_t)_{t \geq 0}$ is a Hilbertian quasimartingale whose finite variation part is of the form $a_t = \int_0^t h_u \, du$ with $\| h_u \| \leq g(u)$.

Proof

The first part of this theorem is due to Enchev ([Enc]) and one can find a nice exposition of this article in [Me2].

When $(x_t)_{t\geq 0}$ satisfies $\|F_s\, x_t - x_s\| \leq \int_s^t g(u)\, du, s \leq t \leq T$, it is clearly a Hilbertian quasimartingale and the particular form of a_t is easily obtained from Enchev's construction; indeed a_t is the limit of $\sum_i (F_{t_i} x_{t_{i+1}} - x_{t_i})$ on some appropriate sequence of partitions of $[0,T]$. ■

Corollary 7.3 – *If a quantum Markov process $(j_t)_{t\geq 0}$ is such that there exists a dense subspace $\mathcal{A}_0$ of $\mathcal{A}$ and a dense subspace $\widetilde{H}$ of H such that for all $\psi \in \widetilde{H}$, all $X \in \mathcal{A}_0$, for all $T \geq 0$ there exists a locally integrable function g such that, for all $s < t < T$ one has*

$$||F_s j_t(X)\psi - j_s(X)\psi|| \leq \int_s^t g(u)\, du$$

then $(j_t)_{t\geq 0}$ is τ-regular for every bounded stop time τ.

Proof

From Proposition 5.3 the couple $(H, (F_t)_{t\geq 0})$ is a filtered Hilbert space in the sense of Theorem 7.2 and $H_{t-} = H_t$ for all t. The process $(j_t(X)\psi)_{t\geq 0}$, which is adapted for $j_t(X)\psi = F_t\, j_t(X)\psi$, satisfies the assumptions of Theorem 7.2. Thus $(j_t(X)\psi)_{t\geq 0}$ is a regular semimartingale of vectors. Hence one can apply Theorem 7.1 to get the convergence of $(j_{\tau_n}(X)\psi)_n$ to a limit $j_\tau(X)\psi$ which depends only on τ, X and ψ. ■

A Markov process $(j_t)_{t\geq 0}$ satisfying the condition of Corollary 7.3 is said to *satisfy Enchev's condition.*

Theorem 7.4 – *Let $(j_t)_{t\geq 0}$ be a Markov process satisfying Enchev's condition. Then for any finite stop time τ the sequence $(j_{\tau\wedge n}(X))_n$ converges strongly to a bounded operator $j_\tau(X)$ whose norm is dominated by $\|X\|$, for all $X \in \mathcal{A}_0$. It thus can be extended into a bounded operator on $\mathcal{A}$. Furthermore, the mapping $X \mapsto j_\tau(X)$ is a *-homomorphism on $\mathcal{A}$.*

Proof

Let $n < m \in \mathbb{N}$. Let $\tilde{\tau}$ be a discrete stop time, based on the τ-partition $\{t_i, i = 1, ..., k\}$, which is close to τ. Let $X \in \mathcal{A}_0, \psi \in \widetilde{H}$. Then

$$\begin{aligned} &\|j_{\tau\wedge n}(X)\psi - j_{\tau\wedge m}(X)\psi\|^2 \\ &\quad \leq 3\|j_{\tau\wedge n}(X)\psi - j_{\tilde{\tau}\wedge n}(X)\psi\|^2 + 3\|j_{\tau\wedge m}(X)\psi - j_{\tilde{\tau}\wedge m}(X)\psi\|^2 \\ &\quad\quad + 3\|j_{\tilde{\tau}\wedge n}(X)\psi - j_{\tilde{\tau}\wedge m}(X)\psi\|^2. \end{aligned}$$

The process $(j_t(X)\psi)_{t\geq 0}$ is a regular semimartingale of vectors, let us denote by $x_t + \int_0^t h_s ds$ its decomposition as a sum of a martingale and an absolutely continous finite variation part. By Theorem 7.1 one has

$$\begin{aligned} &\|j_{\tau\wedge n}(X)\psi - j_{\tau\wedge m}(X)\psi\|^2 \\ &\quad \leq 3\|F_{\tau\wedge n}\, x_n + \int_0^n \mathbb{1}_{\tau\wedge n > s}\, h_s\, ds - F_{\tilde{\tau}\wedge n}\, x_n - \int_0^n \mathbb{1}_{\tilde{\tau}\wedge n > s}\, h_s\, ds\|^2 + \end{aligned}$$

$$
\begin{aligned}
&+3\|F_{\tau\wedge m}x_m + \int_0^m \mathbb{1}_{\tau\wedge m>s}\, h_s\, ds - F_{\tilde\tau\wedge m}\, x_m - \int_0^m \mathbb{1}_{\tilde\tau\wedge m>s} h_s ds\|^2 \\
&+3\|\sum_{i;n<t_i\leq m} \mathbb{1}_{\tau\in[t_i,t_{i+1}[}\, j_{t_{i+1}}(X)\psi\|^2 \\
&\leq 6\|F_n(F_\tau - F_{\tilde\tau})x_n\|^2 + 6\|F_m(F_\tau - F_{\tilde\tau})x_m\|^2 + 6\Big[\int_0^n \|(\mathbb{1}_{\tau>s} - \mathbb{1}_{\tilde\tau>s})h_s\|\, ds\Big]^2 \\
&+6\Big[\int_0^m \|(\mathbb{1}_{\tau>s} - \mathbb{1}_{\tilde\tau>s})h_s\|\, ds\Big]^2 + 3\sum_{i;n<t_i\leq m} \|\mathbb{1}_{\tau\in[t_i,t_{i+1}[}\, j_{t_{i+1}}(X)\psi\|^2 \\
&\leq 12\|(F_\tau - F_{\tilde\tau})x_k\|^2 + 12\Big[\int_0^k \|(\mathbb{1}_{\tau>s} - \mathbb{1}_{\tilde\tau>s})h_s\|\, ds\Big]^2 \\
&\qquad\qquad +3\|X\|^2 \sum_{i;n<t_i\leq m} \|\mathbb{1}_{\tau\in[t_i,t_{i+1}[}\, \psi\|^2
\end{aligned}
$$

for any $k \geq \max\{n, m\}$

$$
\leq 12\|(F_\tau - F_{\tilde\tau})x_k\|^2 + 12\Big[\int_0^k \|(\mathbb{1}_{\tau>s} - \mathbb{1}_{\tilde\tau>s})h_s\|\, ds\Big]^2 + 3\|X\|^2\|\mathbb{1}_{\tau\in]n,m]}\psi\|^2.
$$

Let $\varepsilon > 0$ be fixed. Let N_0 be such that, for all $n, m \geq N_0$ one has

$$
\|X\|^2\|\mathbb{1}_{\tau\in]n,m]}\psi\|^2 \leq \varepsilon
$$

(this is possible as τ is finite). Let n, m be fixed with $n, m \geq N_0$. Let k be fixed such that $k \geq \max\{n, m\}$. Let $\tilde\tau$ be a discrete stop time approximating τ such that $\|(F_\tau - F_{\tilde\tau})x_k\|^2 \leq \varepsilon$ and $[\int_0^k \|(\mathbb{1}_{\tau>s} - \mathbb{1}_{\tilde\tau>s})h_s\|ds]^2 \leq \varepsilon$. We then get

$$
\|j_{\tau\wedge n}(X)\psi - j_{\tau\wedge m}(X)\psi\|^2 \leq 27\,\varepsilon.
$$

This proves that $(j_{\tau\wedge n}(X)\psi)_n$ is a Cauchy sequence in H, thus it converges to a $j_\tau(X)\psi$.

All the properties of $j_\tau(X)$ are then proved in the same way as in Proposition 6.1. ■

Corollary 7.5 – *A Markov process $(j_t)_{t\geq 0}$ which satisfies Enchev's condition is a strong Markov process.* ■

This sufficient condition for $(j_t)_{t\geq 0}$ to be strong Markov applies for example very well in the case of Evans-Hudson flows on Fock space. Indeed if one considers a minimal Evans-Hudson flow $(j_t)_{t\geq 0}$ on Fock space (cf [E-H]), this means that $(j_t)_{t\geq 0}$ is Markov process which is given as a quantum stochastic integral process ([H-P]). But it is proved in [A-M], that if one applies such a process to the so-called coherent vectors of the Fock space the process of vectors obtained so is indeed a regular semimartingale of vectors. We thus get the following result.

Theorem 7.6 – *Any minimal Evans-Hudson flow on Fock space is strong Markov.* ■

Now we wish to find another sufficient condition for getting the strong Markov property, but which is expressable in terms of some property of the semigroup $(T_t)_{t\geq 0}$ and its infinitesimal generator.

Let $\mathcal{L}$ be the infinitesimal generator of the semigroup $(T_t)_{t\geq 0}$. We say that $(T_t)_{t\geq 0}$ satisfies the *S-condition* (or *stability condition*) if there exists a dense subset $\mathcal{A}_0 \subset \text{Dom}\,\mathcal{L}$ such that

(i) $\mathcal{A}_0$ is an algebra;

(ii) $\mathcal{A}_0$ is stable under $(T_t)_{t\geq 0}$;

(iii) for all $X, Y, Z \in \mathcal{A}_0$ the mapping $t \mapsto \mathcal{L}(XT_t(Y)Z)$ is locally bounded in norm.

For example **if $\mathcal{L}$ is a bounded operator then $(T_t)_{t\geq 0}$ satisfies the S-condition**.

Let $\tilde{H}$ be the set of vectors $\psi = \lambda(\mathbf{r}, \mathbf{X}, u) \in H$ such that $X_i \in \mathcal{A}_0$ for all i .

Proposition 7.7 – *Suppose $(T_t)_{t\geq 0}$ satisfies the S-condition. Then for all $T \in \mathbb{R}^+$, all $X \in \mathcal{A}_0$, all $\psi \in \tilde{H}$ there exists a constant $C > 0$ such that for all $s \leq t \leq T$ one has*

$$\|F_s j_t(X)\psi - j_s(X)\psi\| \leq C(t-s).$$

Proof

Let $\psi = \lambda(\mathbf{r}, \mathbf{Y}, u)$ be an element of $\tilde{H}$. Let $\delta = \min(r_i - r_{i+1})$. Let $T \in \mathbb{R}^+$ be fixed, let $X \in \mathcal{A}_0$. If $s \leq t \leq T$ are such that $t - s \geq \delta$ then

$$\|F_s j_t(X)\psi - j_s(X)\psi\| \leq 2\|X\|\,\|\psi\| = \frac{2\|X\|\,\|\psi\|}{\delta}\delta \leq \frac{2\|X\|\,\|\psi\|}{\delta}(t-s),$$

and the proposition is satisfied.

Thus, suppose $t-s < \delta$. Then two cases appear : there exists $k \in \{2, ..., n-1\}$ such that either $r_k \leq s < t \leq r_{k-1}$ or $r_{k+1} \leq s < r_k < t \leq r_{k-1}$.

Let us consider the first case. We have, by Lemma 3.3

$$\begin{aligned} F_s\, j_t(X)\psi &= F_s j_t(X) F_t \lambda(\mathbf{r}, \mathbf{Y}, u) \\ &= F_s\, j_t(X) j_t(T_{r_{k-1}-t} R_{Y_{k-1}} T_{r_{k-2}-r_{k-1}} R_{Y_{k-2}} \cdots \\ &\quad \cdots R_{Y_2} T_{r_1-r_2}(Y_1)) j_{r_k}(Y_k) \cdots j_{r_n}(Y_n) V u \\ &= F_s\, j_t(L_X T_{r_{k-1}-t}(\tilde{Y})) j_{r_k}(Y_k) \cdots j_{r_n}(Y_n) V u \end{aligned}$$

where L_X denotes the mapping

$$\begin{aligned} L_X : \mathcal{A} &\to \mathcal{A} \\ Y &\mapsto XY, \end{aligned}$$

and where $\tilde{Y} = R_{Y_{k-1}} T_{r_{k-1}-r_{k-2}} R_{Y_{k-2}} \cdots R_{Y_2} T_{r_1-r_2}(Y_1)$. Thus, by Lemma 3.3 again,

$$F_s\, j_t(X)\psi = j_s(T_{t-s} L_X T_{r_{k-1}-t}(\tilde{Y})) j_{r_k}(Y_k) \cdots j_{r_n}(Y_n) V u.$$

In the same way we get

$$j_s(X)\psi = j_s(L_X\; T_{r_{k-1}-s}(\tilde{Y})) j_{r_k}(Y_k) \cdots j_{r_n}(Y_n) V u.$$

This gives

$$
\begin{aligned}
&\|F_s j_t(X)\psi - j_s(X)\psi\| \\
&\quad = \|j_s(T_{t-s}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_{k-1}-s}(\tilde{Y}))j_{r_k}(Y_k)\cdots j_{r_n}(Y_n)Vu\| \\
&\quad \leq \|T_{t-s}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_{k-1}-s}(\tilde{Y})\| \, \|j_{r_k}(Y_k)\cdots j_{r_n}(Y_n)Vu\|. \qquad (7.1)
\end{aligned}
$$

Recall that for all $X \in \mathrm{Dom}\,\mathcal{L}$ one has $T_t(X) - X = \int_0^t T_s(\mathcal{L}(X))\,ds$ (as $\mathcal{L}(X) = \frac{d}{dt}|_{t=0}T_t(X)$). Thus $\|T_t(X) - X\| \leq \int_0^t \|T_s(\mathcal{L}(X))\|\,ds \leq t\|\mathcal{L}(X)\|$.

So one gets, by assumption (i) and (ii) of the S-condition,

$$
\begin{aligned}
&\|T_{t-s}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_{k-1}-s}(\tilde{Y})\| \\
&\quad \leq (t-s)\|\mathcal{L}(L_X T_{r_{k-1}-t}(\tilde{Y}))\| + \|X\|\,\|T_{r_{k-1}-t}(\tilde{Y}) - T_{t-s}T_{r_{k-1}-t}(\tilde{Y})\| \\
&\quad \leq (t-s)\big[\|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|X\|\,\|\mathcal{L}(T_{r_{k-1}-t}(\tilde{Y}))\|\big] \\
&\quad \leq (t-s)\big[\|(\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|X\|\,\|T_{r_{k-1}-t}(\mathcal{L}(\tilde{Y}))\|\big] \\
&\quad \leq (t-s)\big[\|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|X\|\,\|\mathcal{L}(\tilde{Y})\|\big] \\
&\quad \leq C(t-s) \text{ by assumption iii) of } S\text{-condition.}
\end{aligned}
$$

The term $\|j_{r_k}(Y_k)\cdots j_{r_n}(Y_n)Vu\|$ in (7.1) is dominated by

$$\max_{i=1,\ldots,n} \|j_{r_i}(Y_i)\ldots j_{r_n}(Y_n)Vu\|$$

which depends only on ψ. This concludes the first case.

Now consider the second case that is, $r_{k+1} \leq s < r_k < t \leq r_{k-1}$. One has

$$
\begin{aligned}
F_s\, j_t(X)\psi &= F_s j_t(L_X T_{r_{k-1}-t}(\tilde{Y}))j_{r_k}(Y_k)\ldots j_{r_n}(Y_n)Vu \\
&= j_s(T_{r_k-s}R_{Y_k}T_{t-r_k}L_X T_{r_{k-1}-t}(\tilde{Y}))j_{r_k}(Y_k)\cdots j_{r_n}(Y_n)Vu
\end{aligned}
$$

and

$$j_s(X)\psi = j_s(L_X T_{r_k-s}R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y}))j_{r_k}(Y_k)\cdots j_{r_n}(Y_n)Vu.$$

Thus in the same way as in the first case we have to estimate

$$\|T_{r_k-s}R_{Y_k}T_{t-r_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X\; T_{r_k-s}R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y})\|$$

which is dominated by

$$
\begin{aligned}
&\|T_{r_k-s}R_{Y_k}T_{t-r_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - T_{r_k-s}R_{Y_k}L_X T_{r_{k-1}-t}(\tilde{Y})\| \\
&\quad +\|T_{r_k-s}R_{Y_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_k-s}R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y})\| \\
&\leq \|Y_k\|\,\|T_{t-r_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_{k-1}-t}(\tilde{Y})\| + (t-s)\|\mathcal{L}(R_{Y_k}L_X T_{r_{k-1}-t}(\tilde{Y})\| \\
&\quad +\|R_{Y_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X T_{r_k-s}R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y})\| \\
&\leq (t-s)\big[\|Y_k\|\,\|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y})Y_k)\|\big] \\
&\quad +(t-s)\|X\|\,\|\mathcal{L}(R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y}))\| \\
&\quad +\|R_{Y_k}L_X T_{r_{k-1}-t}(\tilde{Y}) - L_X R_{Y_k}T_{r_{k-1}-r_k}(\tilde{Y})\| \\
&\leq (t-s)\big[\|Y_k\|\,\|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y})Y_k)\| \\
&\quad +\|X\|\,\|\mathcal{L}(T_{r_{k-1}-r_k}(\tilde{Y})Y_k)\|\big] \\
&\quad +\|X\|\,\|Y_k\|\,\|T_{r_{k-1}-t}(\tilde{Y}) - T_{t-r_k}T_{r_{k-1}-t}(\tilde{Y})\|
\end{aligned}
$$

$$\leq (t-s)\big[\|Y_k\|\,\|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y}))\| + \|\mathcal{L}(XT_{r_{k-1}-t}(\tilde{Y})Y_k)\|$$
$$+\|X\|\,\|\mathcal{L}(T_{r_{k-1}-r_k}(\tilde{Y})Y_k)\| + \|X\|\,\|Y_k\|\,\|\mathcal{L}(\tilde{Y})\|\big]$$
$$\leq C(t-s) \text{ in the same way as previously.}$$

This completes the proof. ■

Corollary 7.8 – *If $(j_t)_{t\geq 0}$ is a Markov process whose semigroup $(T_t)_{t\geq 0}$ and generator $\mathcal{L}$ satisfy the S-condition then it satisfies Enchev's condition and thus it is strong Markov.* ■

8. Extension of the time-shift to stop times

In the following $(j_t)_{t\geq 0}$ is supposed to be a strong Markov process on $\mathcal{A}$.

Theorem 8.1 – *Let τ be any stop time. Then there exists a unique contractive *-homomorphism θ_τ on $\mathcal{B}$ such that $\theta_\tau(j_t(X)) = j_{\tau+t}(X)$ and $\theta_\tau(k_s(Z)) = k_{\tau+s}(Z)$ for all $X \in \mathcal{A}, Z \in \mathcal{Z}$, $s,t \in \mathbb{R}^+$.*

Proof

For $\xi = k_{t_1}(Z_1)\cdots k_{t_n}(Z_n)j_{s_1}(X_1)\cdots j_{s_p}(X_p) \in \mathcal{B}$ consider the quantity

$$\sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,\theta_{r_{i+1}}(\xi)$$
$$= \sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,k_{t_1+r_{i+1}}(Z_1)\cdots k_{t_n+r_{i+1}}(Z_n)j_{s_1+r_{i+1}}(X_1)\cdots j_{s_p+r_{i+1}}(X_p)$$
$$= \Big(\sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,k_{t_1+r_{i+1}}(Z_1)\Big)\cdots\Big(\sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,k_{t_n+r_{i+1}}(Z_n)\Big)$$
$$\times\Big(\sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,j_{s_1+r_{i+1}}(X_1)\Big)\cdots\Big(\sum_i \mathbb{1}_{\tau\in[r_i,r_{i+1}[}\,j_{s_p+r_{i+1}}(X_p)\Big)$$
$$= k_{t_1+\tilde{\tau}}(Z_1)\cdots k_{t_n+\tilde{\tau}}(Z_n)j_{s_1+\tilde{\tau}}(X_1)\cdots j_{s_p+\tilde{\tau}}(X_p) \qquad (8.1)$$

where $\tilde{\tau}$ is the discrete stop time τ_E associated to the τ-partition $E = \{r_i; i = 1,\dots,N\}$ of $\mathbb{R}^+$.

If τ is a finite stop time the operator (8.1) converges strongly, when the diameter of E tends to 0, to the operator $k_{t_1+\tau}(Z_1)\dots k_{t_n+\tau}(Z_n)j_{s_1+\tau}(X_1)\cdots j_{s_p+\tau}(X_p)$ which we define to be $\theta_\tau(\xi)$. The operator (8.1) is equal to $\theta_{\tilde{\tau}}(\xi)$ and we clearly have that $\|\theta_{\tilde{\sigma}}(\xi)\| \leq \|\xi\|$ and that the mapping $\xi \mapsto \theta_{\tilde{\tau}}(\xi)$ is a *-homomorphism on $\mathcal{B}_0$. Thus the same holds for θ_τ and these properties can be extended to $\mathcal{B}$ by taking norm-limits.

If τ is not finite, define $\theta_\tau(\xi)$ to be $\theta_\tau(\xi)\mathbb{1}_{\tau<+\infty}$. This definition answers the statements of the theorem. Details are left to the reader. ■

Corollary 8.2 – *For every stop time τ and every $t \in \mathbb{R}^+$ we have on $\mathcal{B}$:*

$$\theta_{\tau+t} = \theta_\tau \circ \theta_t.$$

Proof

It is immediate from the construction of θ_τ for discrete stop times. One passes to the limit for general ones. ■

From now, in order to simplify the notations we make no difference between the spaces H_0 and Range (F_0). That is, we omit to write the unitary operators V and V^*.

Theorem 8.3 (Strong Markov property, general form) – *Let τ be any stop time. Let $t \in I\!R^+$. Let $\xi \in \mathcal{B}$. Then one has*

$$F_\tau \theta_{\tau+t}(\xi) F_\tau = j_\tau(F_0 \theta_t(\xi) F_0).$$

Proof

As $F_t = j_t(I)$ it is easy to check from the definition of F_τ (Proposition 5.4) that $F_\tau = j_\tau(I) = \theta_\tau(j_0(I)) = \theta_\tau(F_0)$. Thus $F_\tau \theta_{\tau+t}(\xi) F_\tau = \theta_\tau(F_0)\theta_\tau(\theta_t(\xi))\theta_\tau(F_0) = \theta_\tau(F_0\theta_t(\xi)F_0) = \theta_\tau(j_0(F_0\theta_t(\xi)F_0)) = j_\tau(F_0\theta_t(\xi)F_0)$. ■

Notice that taking $\xi = j_0(X)$ in Theorem 8.3 gives Theorem 6.2 as a simple corollary.

If τ_1 and τ_2 are two discrete stop times such that $1\!\!1_{\tau_2=r_j}$ belongs to $\mathcal{P}$ for all j then one can define $\theta_{\tau_1} \circ \theta_{\tau_2}$ on $\mathcal{P}$ by

$$\begin{aligned} \theta_{\tau_1} \circ \theta_{\tau_2}(\xi) &= \sum_i 1\!\!1_{\tau_1=t_i} \theta_{t_i}(\sum_j 1\!\!1_{\tau_2=r_j} \theta_{r_j}(\xi)) \\ &= \sum_i \sum_j 1\!\!1_{\tau_1=t_i} \theta_{t_i}(1\!\!1_{\tau_2=r_j}) \theta_{t_i+r_j}(\xi). \end{aligned}$$

Let $\{s_k; k = 1, ..., K\} = \{t_i + r_j; i = 1, ..., N; j = 1, ..., M\}$. Then

$$\theta_{\tau_1} \circ \theta_{\tau_2}(\xi) = \sum_k \Big[\sum_{i,j;t_i+r_j=s_k} 1\!\!1_{\tau_1=t_i} \theta_{t_i}(1\!\!1_{\tau_2=s_k-t_i}) \Big] \theta_{s_k}(\xi).$$

But the family of projections

$$1\!\!1_{\tau_1\circ\tau_2=s_k} \stackrel{\text{def}}{=} \sum_{i,j;\, t_i+r_j=s_k} 1\!\!1_{\tau_1=t_i} \theta_{t_i}(1\!\!1_{\tau_2=s_k-t_i})$$

clearly defines a stop time $\tau_1 \circ \tau_2$ on H ($1\!\!1_{\tau_1\circ\tau_2=s_k}$ is a projection for $1\!\!1_{\tau\le t}$ and $\theta_t(\xi)$ always commute for any $\xi \in P$; this easy result is left to the reader). In other words

$$1\!\!1_{\tau_1\circ\tau_2\le t} = \sum_{i;t_i\le t} 1\!\!1_{\tau_1=t_i} \theta_{t_i}(1\!\!1_{\tau_2\le t-t_i}).$$

We wish to define $\tau_1 \circ \tau_2$ for a more general family of stop times τ_1 and τ_2.

Define a $\mathcal{P}$*-stop time* to be any stop time τ on H which satisfies :

(i) $\mathbb{1}_{\tau\leq t} \in \mathcal{P}$ for all $t \in \mathbb{R}^+$;

(ii) $\theta_h(\mathbb{1}_{\tau\leq t}) \leq \mathbb{1}_{\tau\leq t+h}$ for all $t, h \in \mathbb{R}^+$.

We will see several examples of $\mathcal{P}$-stop times in the following sections. For τ_1 being a discrete stop time and τ_2 being a $\mathcal{P}$-stop time define

$$\mathbb{1}_{\tau_1\circ\tau_2\leq t} = \sum_{i;t_i\leq t} \mathbb{1}_{\tau_1=t_i}\theta_{t_i}(\mathbb{1}_{\tau_2\leq t-t_i}).$$

The family $(\mathbb{1}_{\tau_1\circ\tau_2\leq t})_{t\geq 0}$ clearly defines a stop time $\tau_1\circ\tau_2$ on H. We want to pass to the limit on discrete stop times approximating a general stop time τ_1.

Theorem 8.4– *Let τ_1 be any stop time. Let $(E_n)_n$ be a sequence of refining τ_1-partitions of $\mathbb{R}^+$. Let $\tau_1^n = (\tau_1)_{E_n}$, $n \in \mathbb{N}$. Let τ_2 be a $\mathcal{P}$-stop time. Then the sequence $(\tau_1^n\circ\tau_2)_n$ converges to a stop time $\tau_1\circ\tau_2$.*

Proof

For $n < m$ one has

$$\begin{aligned}
&\mathbb{1}_{\tau_1^n\circ\tau_2\leq t} - \mathbb{1}_{\tau_1^m\circ\tau_2\leq t}\\
&\quad= \sum_{i,j} \mathbb{1}_{\tau_1\in[t_i^j,t_i^{j+1}[}\,[\theta_{t_{i+1}}(\mathbb{1}_{\tau_2\leq t-t_{i+1}}) - \theta_{t_i^{j+1}}(\mathbb{1}_{\tau_2\leq t-t_i^{j+1}})]\\
&\quad= \sum_{i,j} \mathbb{1}_{\tau_1\in[t_i^j,t_i^{j+1}[}\,\theta_{t_i^{j+1}}[\theta_{t_{i+1}-t_i^{j+1}}(\mathbb{1}_{\tau_2\leq t-t_{i+1}}) - \mathbb{1}_{\tau_2\leq t-t_i^{j+1}}]\\
&\quad\leq 0 \ \text{ as } \tau_2 \text{ is a } \mathcal{P}\text{-stop time.}
\end{aligned}$$

Thus the sequence $(\mathbb{1}_{\tau_1^n\circ\tau_2\leq t})_n$ is an increasing sequence of projections, it thus converges to a projection $\mathbb{1}_{\tau_1\circ\tau_2\leq t}$. We leave to the reader to check that the projections $(\mathbb{1}_{\tau_1\circ\tau_2\leq t})_{t\geq 0}$ define a stop time $\tau_1\circ\tau_2$. ■

Remark : One should notice that, as it is defined, when τ_1 is not a discrete stop time, the limit $\tau_1\circ\tau_2$ obtained in Theorem 8.4 may depend on the choice of the approximating sequence $(\tau_1^n)_n$. Actually it seems that it should not be the case, but we are not able to prove it. Our intuition is driven to this conclusion for the following reason: the operator $\theta_{\tau_1}\circ\theta_{\tau_2}$ is not well-defined, as it is not clear that θ_{τ_2} is valued in $\mathcal{B}$ (or even in $\mathcal{P}$) and even it is not clear that θ_{τ_1} can be extended to $\mathcal{P}$, but in all the uses we make of $\tau_1\circ\tau_2$ we see that $\theta_{\tau_1\circ\tau_2}$ has all the properties we could expect from $\theta_{\tau_1}\circ\theta_{\tau_2}$.

So, in the following, when $\tau_1\circ\tau_2$ is considerated we suppose that a fixed approximating sequence $(\tau_1^n)_n$ is choosen, for example the one based on the dyadic partitions of $\mathbb{R}^+$.

A Markov process $(j_t)_{t\geq 0}$ is said to be a *shift-strong Markov process* if for every $h \geq 0$, every $X \in \mathcal{A}$, every $\mathcal{P}$-stop time τ, every approximating sequence of $\mathcal{P}$-discrete stop times $(\tau_n)_n$ the sequence $\big(\theta_h(j_{\tau_n}(X))\big)_n$ converges strongly to a limit $\theta_h(j_\tau(X))$ which does not depend on the choice of the sequence $(\tau_n)_n$.

Proposition 8.5 – *If $(j_t)_{t\geq 0}$ is a shift-strong Markov process then for every $\xi \in \mathcal{B}$, every $\mathcal{P}$-stop time τ, the operator $\theta_\tau(\xi)$ is an element of $\mathcal{P}$. Furthermore, for every $h \geq 0$, every approximating sequence of $\mathcal{P}$-discrete stop times $(\tau_n)_n$ the sequence $\theta_h(\theta_{\tau_n}(\xi))$ converges strongly to $\theta_h(\theta_\tau(\xi))$.*

Proof

First note that if $(j_t)_{t\geq 0}$ is a shift-strong Markov process it is then clear that, for any $\mathcal{P}$-stop time, $j_\tau(X)$ is an element of $\mathcal{P}$.

Now consider a discrete $\mathcal{P}$-stop time τ, fix Z in $\mathcal{Z}$ and $h \geq 0$. Then one can easily prove an analogous of Theorem 6.5 for $\theta_h(k_\tau(Z))$ that is,

$$\begin{aligned}\theta_h(k_\tau(Z))\lambda(\mathbf{r},\mathbf{X},u) = \sum_{i=1}^{n+1} &\theta_h(\mathbb{1}_{\tau\in[r_i-h,r_{i-1}-h[})j_{r_1}(X_1)\cdots j_{r_{i-1}}(X_{i-1}) \\ &\times \theta_h(j_\tau(Z))j_{r_i}(X_i)\cdots j_{r_n}(X_n)Vu \end{aligned} \tag{8.2}$$

where $\mathbb{1}_{\tau\in[a,b[}$ for $a, b \in \mathbb{R}$ is understood as $\mathbb{1}_{\tau\in[a\vee 0,b\vee 0[}$.

Consequently $k_\tau(Z)$ is an element of $\mathcal{P}$ and $\theta_h(k_\tau(Z))$ is the strong limit of $\theta_h(k_\tau(Z_n))$ for any approximating sequence $(Z_n)_n$ of Z.

Finally, in the same way as in Theorem 8.1, we get the result. ■

Theorem 8.6 – *A markov process which satisfies Enchev's condition is a shift-strong Markov process.*

Proof

First of all we have to prove that $\theta_h(F_{\tau_n})$ converges strongly to $\theta_h(F_\tau)$ for any stop time τ, any approximating sequence $(\tau_n)_n$ and any $h \in \mathbb{R}^+$. But it suffices to follow step by step the construction of F_τ in section 5 to see that it also applies to $\theta_h(F_\tau)$. In particular the characterization of the range of $\theta_h(F_\tau)$ (in order to prove that it is independent of the choice of $(\tau_n)_n$) gives

$$\mathrm{Range}(\theta_h(F_\tau)) = \{\psi \in H; \theta_h(\mathbb{1}_{\tau\leq t})\psi \in H_{t+h}, t \in \mathbb{R}^+\}.$$

Let $(j_t)_{t\geq 0}$ be a Markov process satisfying Enchev's condition that is, $j_t(X)\psi$ can be written as $j_t(X)\psi = m_t + \int_0^t h_s\,ds$ where m is a martingale. Let τ a bounded stop time (with bound T) and $h \in \mathbb{R}^+$ be fixed. Choose an approximating sequence $(\tau_n)_n$ based on a sequence of τ-partitions of $\mathbb{R}^+$ with regular steps δ_n such that for all n there exists $k_n \in \mathbb{N}$ with $k_n\delta_n = h$. We then get

$$\begin{aligned}\theta_h(j_{\tau_n}(X))\psi &= \sum_i \theta_h(\mathbb{1}_{\tau_n=t_i})j_{t_i+h}(X)\psi \\ &= \sum_i \theta_h(\mathbb{1}_{\tau_n=t_i})\big[m_{t_i+h} + \int_0^{t_i+h} h_s\,ds\big] \\ &= \sum_i \theta_h(\mathbb{1}_{\tau_n=t_i})F_{t_i+h}m_T + \sum_i \sum_{j;t_j<t_i+h} \theta_h(\mathbb{1}_{\tau_n=t_i})\int_{t_j}^{t_{j+1}} h_s\,ds\end{aligned}$$

$$= \theta_h(F_{\tau_n})m_T + \sum_j \theta_h(\mathbb{1}_{\tau_n > t_j - h}) \int_{t_j}^{t_{j+1}} h_s\, ds$$
$$= \theta_h(F_{\tau_n})m_T + \sum_j \int_{t_j}^{t_{j+1}} \theta_h(\mathbb{1}_{\tau_n > s-h}) h_s\, ds$$
$$= \theta_h(F_{\tau_n})m_T + \int_0^\infty \theta_h(\mathbb{1}_{\tau_n > s-h}) h_s\, ds.$$

It is now clear that $\theta_h(j_{\tau_n}(X))\psi$ is going to admit a limit $\theta_h(j_\tau(X))\psi$ which is independent of $(\tau_n)_n$. We then proceed in the same way as for the construction of j_τ in order to go from bounded to finite stop times. ■

Theorem 8.7 (Strong Markov property, general form II) – *Let $(j_t)_{t\geq 0}$ be a shift-strong Markov process. Then for every stop time τ_1, every $\mathcal{P}$-stop time τ_2, for all $\xi \in \mathcal{B}$, one has*

$$F_{\tau_1}\theta_{\tau_1\circ\tau_2}(\xi)F_{\tau_1} = j_{\tau_1}(F_0\theta_{\tau_2}(\xi)F_0).$$

Proof

One has that $F_{\tau_1}\theta_{\tau_1\circ\tau_2}(\xi)F_{\tau_1}$ is the strong limit of

$$\sum_{i,j;k,l;p,q}\sum_{u,v} \mathbb{1}_{\tau_1\in[r_i^j, r_i^{j+1}[} F_{r_{i+1}} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} \theta_{r_{u+1}}\big(\mathbb{1}_{\tau_2\in[r_k^l - r_u^{v+1}, r_k^{l+1} - r_u^{v+1}[}\, \theta_{r_k^{l+1} - r_u^{v+1}}(\xi)\big)$$
$$\times\, \mathbb{1}_{\tau_1\in[r_p^q, r_p^{q+1}[}\, F_{r_{p+1}}$$
$$= \sum_{k,l}\sum_{u,v} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} F_{r_{u+1}} \theta_{r_u+1}\big(\mathbb{1}_{\tau_2\in[r_k^l - r_u^{v+1}, r_k^{l+1} - r_u^{v+1}[}\, \theta_{r_k^{l+1} - r_u^{v+1}}(\xi)\big) F_{r_{u+1}}$$
$$= \sum_{k,l}\sum_{u,v} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} \theta_{r_u+1}\big(F_0\mathbb{1}_{\tau_2\in[r_k^l - r_u^{v+1}, r_k^{l+1} - r_u^{v+1}[}\, \theta_{r_k^{l+1} - r_u^{v+1}}(\xi)F_0\big)$$
$$= \sum_{u,v} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} \theta_{r_u+1}\big(\sum_{k,l} F_0\mathbb{1}_{\tau_2\in[r_k^l - r_u^{v+1}, r_k^{l+1} - r_u^{v+1}[}\, \theta_{r_k^{l+1} - r_u^{v+1}}(\xi)F_0\big)$$
$$= \sum_{u,v} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} \theta_{r_u+1}\big(\sum_{k,l} F_0\mathbb{1}_{\tau_2\in[r_k^l, r_k^{l+1}[}\, \theta_{r_k^{l+1}}(\xi)F_0\big)$$
$$= \sum_{u,v} \mathbb{1}_{\tau_1\in[r_u^v, r_u^{v+1}[} \theta_{r_u+1}\big(F_0\theta_{\tilde\tau}(\xi)F_0\big)$$

where $\tilde\tau$ is the approximation of τ based of the partition $\{r_k^l; k, l\}$

$$= \sum_{u,v} \mathbb{1}_{\tau_1\in[r_u, r_{u+1}[} \theta_{r_u+1}\big(F_0\theta_{\tilde\tau}(\xi)F_0\big).$$

As $(j_t)_{t\geq 0}$ is shift-strong, then by Proposition 8.5 we get that when $\tilde\tau$ converges to τ the last expression converges strongly to

$$\sum_{u,v} \mathbb{1}_{\tau_1\in[r_u, r_{u+1}[} \theta_{r_u+1}\big(F_0\theta_\tau(\xi)F_0\big)$$

which itself converges to

$$\theta_{\tau_1}(F_0\theta_{\tau_2}(\xi)F_0) = j_{\tau_1}(F_0\theta_{\tau_2}(\xi)F_0).$$

■

9. Martingale problem

Recall the following classical results on the generator $\mathcal{L}$ of $(T_t)_{t\geq 0}$. Parts of these results have been already used in previous sections. The domain of the generator $\mathcal{L}$ is the set of $X \in \mathcal{A}$ such that $\frac{1}{h}(T_h(X) - X)$ admits a limit (in operator norm) when h tends to 0. If $X \in \text{Dom}\,\mathcal{L}$ then $T_t(X) \in \text{Dom}\,\mathcal{L}$ and $T_t(\mathcal{L}(X)) = \mathcal{L}(T_t(X)) = \frac{d}{dt}T_t(X)$.

In the classical theory of Markov processes it is well-known that the Markov process with generator $\mathcal{L}$ solves the martingale problem associated to $\mathcal{L}$. In our non-commutative context we get a similar result.

Theorem 9.1 – *For every $X \in \text{Dom}\,\mathcal{L}$, the process*

$$M_t(X) = j_t(X) - j_0(X) - \int_0^t j_u(\mathcal{L}(X))\,du$$

is a martingale in the sense that $F_sM_t(X)F_s = M_s(X)$ for all $s \leq t$.

Furthermore, if $X \in \mathcal{A}$ is such that there exists a $Y \in \mathcal{A}$ satisfying the fact that the process

$$M'_t = j_t(X) - j_0(X) - \int_0^t j_u(Y)\,du$$

is a martingale, then $X \in \text{Dom}\,\mathcal{L}$ and $\mathcal{L}(X) = Y$.

Proof

One has, for $s \leq t$,

$$\begin{aligned}
F_sM_t(X)F_s &= j_s(T_{t-s}(X)) - j_0(X) - \int_0^t F_sj_u(\mathcal{L}(X))F_s\,du \\
&= j_s(T_{t-s}(X)) - j_0(X) - \int_0^s j_u(\mathcal{L}(X))\,du - \int_s^t j_s(T_{u-s}(\mathcal{L}(X)))\,du \\
&= j_s(T_{t-s}(X)) - j_0(X) - \int_0^s j_u(\mathcal{L}(X))\,du - \int_0^{t-s} \frac{d}{du}j_s(T_u(X))\,du \\
&= j_s(T_{t-s}(X)) - j_0(X) - \int_0^s j_u(\mathcal{L}(X))\,du - j_s(T_{t-s}(X)) + j_s(X) \\
&= M_s(X).
\end{aligned}$$

This proves the martingale property of $(M_t(X))_{t\geq 0}$.

In order to get the uniqueness part notice that

$$0 = M'_0 = F_0M'_tF_0 = j_0(T_t(X) - X) - \int_0^t j_0(T_u(Y))\,du.$$

Thus

$$\|\frac{1}{t}j_0(T_t(X)-X)-j_0(Y)\|=\frac{1}{t}\|\int_0^t j_0(T_u(Y)-Y)\,du\|\leq\frac{1}{t}\int_0^t\|T_u(Y)-Y\|\,du$$

which converges to 0 as t goes to 0. Thus $\lim_{t\to 0}\frac{1}{t}(T_t(X)-X)$ exists and is equal to Y. ■

We now recover the analogue of the usual "opérateur carré du champ" and its relation with the "angle bracket" of $(M_t(X))_{t\geq 0}$.

Theorem 9.2 – *For every $X, Y \in$ Dom $\mathcal{L}$ such that $X^*Y \in$ Dom $\mathcal{L}$, the process*

$$M_t(X^*)M_t(Y)-\int_0^t j_u(\mathcal{L}(X^*Y)-X^*\mathcal{L}(Y)-\mathcal{L}(X^*)Y)\,du,\ t\in\mathbb{R}^+$$

is a martingale.

Furthermore, if one defines

$$\langle M_.(X), M_.(Y)\rangle_t=\int_0^t j_u(\mathcal{L}(X^*Y)-X^*\mathcal{L}(Y)-\mathcal{L}(X^*)Y)\,du$$

then the following holds :

i) *$\langle M_.(X), M_.(Y)\rangle_t$ is the weak-limit of the conditionned quadratic variations*

$$\sum_i F_{t_i}\left(M_{t_{i+1}}(X^*)-M_{t_i}(X^*)\right)\left(M_{t_{i+1}}(Y)-M_{t_i}(Y)\right)F_{t_i}$$

when the diameter of the partition $\{t_i; i=1,...,n\}$ of $[0,t]$ tends to 0.

ii) *$\langle M_.(X), M_.(Y)\rangle_t$ defines a completely positive bilinear map in the sense that, for all $Y_i\in\mathcal{B}(H), X_j\in\mathcal{A}, i=1,...,N, j=1,...,M$, one has*

$$\sum_{i,j}Y_i^*\langle M_.(X_i), M_.(X_j)\rangle_t Y_j\geq 0.$$

Proof

Let $X, Y \in$ Dom $\mathcal{L}$ be such that $X^*Y \in$ Dom $\mathcal{L}$. Put

$$W_t=M_t(X^*)M_t(Y)-\int_0^t j_u(\mathcal{L}(X^*Y)-X^*\mathcal{L}(Y)-\mathcal{L}(X^*)Y)\,du.$$

Then, for any $s\leq t$ one has

$$\begin{aligned}
&F_sW_tF_s\\
&=F_s\,j_t(X^*Y)F_s-F_s\,j_t(X^*)j_0(Y)F_s-F_s\,j_t(X^*)\int_0^t j_u(\mathcal{L}(Y))\,du\,F_s\\
&\quad-F_s\,j_0(X^*)j_t(Y)F_s+F_s\,j_0(X^*Y)F_s+F_s\,j_0(X^*)\int_0^t j_u(\mathcal{L}(Y))\,du\,F_s\\
&\quad-F_s\int_0^t j_u(\mathcal{L}(X^*))\,du\,j_t(Y)F_s+F_s\int_0^t j_u(\mathcal{L}(X^*))\,du\,j_0(Y)F_s
\end{aligned}$$

$$
\begin{aligned}
&+ F_s \int_0^t j_u(\mathcal{L}(X^*))\,du \int_0^t j_u(\mathcal{L}(Y))\,du\, F_s \\
&- F_s \int_0^t j_u(\mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y)\,du\, F_s \\
&= j_s(T_{t-s}(X^*Y)) - j_s(T_{t-s}(X^*))j_0(Y) - j_s(T_{t-s}(X^*)) \int_0^s j_u(\mathcal{L}(Y))\,du \\
&- F_s\, j_t(X^*) \int_s^t j_u(\mathcal{L}(Y))\,du\, F_s - j_0(X^*)j_s(T_{t-s}(Y)) + j_0(X^*Y) \\
&+ j_0(X^*) \int_0^s j_u(\mathcal{L}(Y))\,du + j_0(X^*)j_s(T_{t-s}(Y)) - j_0(X^*)j_s(Y) \\
&- \int_0^s j_u(\mathcal{L}(X^*))\,du\, j_s(T_{t-s}(Y)) - F_s \int_s^t j_u(\mathcal{L}(X^*))\,du\, j_t(Y)F_s \\
&+ \int_0^s j_u(\mathcal{L}(X^*))\,du\, j_0(Y) + j_s(T_{t-s}(X^*))j_0(Y) - j_s(X^*)j_0(Y) \\
&+ \int_0^s j_u(\mathcal{L}(X^*))\,du \int_0^s j_u(\mathcal{L}(Y))\,du + \int_0^s j_u(\mathcal{L}(X^*))\,du\, j_s(T_{t-s}(Y)) \\
&- \int_0^s j_u(\mathcal{L}(X^*))\,du\, j_s(Y) + j_s(T_{t-s}(X^*)) \int_0^s j_u(\mathcal{L}(Y))\,du \\
&- j_s(X^*) \int_0^s j_u(\mathcal{L}(Y))\,du + F_s \int_s^t j_u(\mathcal{L}(X^*))\,du \int_s^t j_u(\mathcal{L}(Y))\,du\, F_s \\
&- \int_0^s j_u(\mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y)\,du - j_s(T_{t-s}(X^*Y)) + j_s(X^*Y) \\
&+ F_s \int_s^t j_u(X^*)j_u(\mathcal{L}(Y))\,du\, F_s + F_s \int_s^t j_u(\mathcal{L}(X^*))j_u(Y)\,du\, F_s. \\
&= W_s - F_s \int_s^t (j_u(T_{t-u}(X^*)) - j_u(X^*))j_u(\mathcal{L}(Y))\,du\, F_s \\
&-F_s \int_s^t j_u(\mathcal{L}(X^*))(j_u(T_{t-u}(Y)) - j_u(Y))\,du\, F_s \\
&+F_s \int_s^t j_u(\mathcal{L}(X^*))\,du \int_s^t j_u(\mathcal{L}(Y))\,du\, F_s \\
&= W_s - F_s \int_s^t F_u \int_u^t j_v(\mathcal{L}(X^*))\,dv\, F_u\, j_u(\mathcal{L}(Y))\,du\, F_s \\
&-F_s \int_s^t j_u(\mathcal{L}(X^*))F_u \int_u^t j_v(\mathcal{L}(Y))\,dv\, F_u\,du\, F_s \\
&+F_s \int_s^t \int_s^t j_u(\mathcal{L}(X^*))j_v(\mathcal{L}(Y))\,du\,dv\, F_s \\
&= W_s - F_s \int_s^t \int_u^t j_v(\mathcal{L}(X^*))j_u(\mathcal{L}(Y))\,dv\,du\, F_s
\end{aligned}
$$

$$-F_s \int_s^t \int_u^t j_u(\mathcal{L}(X^*))j_v(\mathcal{L}(Y))\,dv\,du\,F_s$$
$$+F_s \int_s^t \int_s^t j_u(\mathcal{L}(X^*))j_v(\mathcal{L}(Y))\,du\,dv\,F_s$$
$$= W_s.$$

This proves the martingale property.

From the martingale property of $(M_t(X))_{t\geq 0}$ we get

$$\begin{aligned}
&\sum_i F_{t_i}(M_{t_{i+1}}(X^*) - M_{t_i}(X^*))(M_{t_{i+1}}(Y) - M_{t_i}(Y))F_{t_i} \\
&\quad = \sum_i \left(F_{t_i}M_{t_{i+1}}(X^*)M_{t_{i+1}}(Y)F_{t_i} - F_{t_i}M_{t_i}(X^*)M_{t_i}(Y)F_{t_i}\right) \\
&\quad = \sum_i F_{t_i} \int_{t_i}^{t_{i+1}} j_u(\mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y)\,du\,F_{t_i}
\end{aligned}$$

by the martingale property of $(W_t)_{t\geq 0}$. Thus, the difference between this quadratic variation and $\int_0^t j_u(\mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y)\,du$ is equal to

$$\sum_i F_{t_i}\int_{t_i}^{t_{i+1}} j_u(M(X,Y))\,du\,F_{t_i} - F_{t_{i+1}}\int_{t_i}^{t_{i+1}} j_u(M(X,Y)\,du\,F_{t_{i+1}} \tag{9.1}$$

where $M(X,Y) = \mathcal{L}(X^*Y) - X^*\mathcal{L}(Y) - \mathcal{L}(X^*)Y$. Applying (9.1) to a vector φ and taking the scalar product with a vector ψ gives

$$\sum_i <\psi\,,\,[F_{t_i}\int_{t_i}^{t_{i+1}} j_u(M(X,Y))\,du\,F_{t_i} - F_{t_{i+1}}\int_{t_i}^{t_{i+1}} j_u(M(X,Y))\,du\,F_{t_{i+1}}]\varphi>$$

whose modulus is dominated by

$$\begin{aligned}
&\sum_i |<\psi\,,\,F_{t_i}\int_{t_i}^{t_{i+1}} j_u(M(X,Y))\,du\,(F_{t_{i+1}} - F_{t_i})\varphi>| \\
&\qquad + \sum_i |<\psi\,,\,(F_{t_{i+1}} - F_{t_i})\int_{t_i}^{t_{i+1}} j_u(M(X,Y))\,du\,F_{t_{i+1}}\varphi>| \\
&\quad \leq \sum_i \|\psi\|\,\|M(X,Y)\|(t_{i+1}-t_i)\|(F_{t_{i+1}} - F_{t_i})\varphi\| \\
&\qquad + \sum_i \|\varphi\|\,\|M(X,Y)\|(t_{i+1}-t_i)\|(F_{t_{i+1}} - F_{t_i})\psi\| \\
&\quad \leq \|\psi\|\,\|M(X,Y)\| \sup_{\substack{u,v\leq t\\ |v-u|\leq\delta}} \|(F_v - F_u)\varphi\|\,t \\
&\qquad + \|\varphi\|\,\|M(X,Y)\| \sup_{\substack{u,v\leq t\\ |v-u|\leq\delta}} \|(F_v - F_u)\psi\|\,t
\end{aligned}$$

where δ is the diameter of $\{t_i; i = 1,..,n\}$. By continuity of $(F_t)_{t\geq 0}$ this last quantity converges to 0 when δ tends to 0. This proves the "angle bracket" property.

The complete positivity is an easy consequence of the "angle bracket" property. ■

Consider a stop time τ which is finite, it defines a spectral measure p^τ on $I\!R^+$. That is, for every $\psi \in H$ the mapping $A \mapsto <\psi\,,\, p^\tau(A)\psi> \stackrel{\text{def}}{=} <\psi\,,\, 1\!\!1_{\tau\in A}\psi>$ defines a measure on $I\!R^+$.

In the theory of functional calculus on self-adjoint operators, the positive operator τ is defined to be $\int_0^\infty s\, p^\tau(ds)$ in the sense that Dom τ is defined to be the set of ψ such that $\int_0^\infty s^2 <\psi\,,\, p^\tau(ds)\psi>$ is finite and $\|\tau\psi\|^2$ is given by $\int_0^\infty s^2 <\psi\,,\, p^\tau(ds)\psi>$. In the same way, the domain of the operator $\tau^{1/2} = \int_0^\infty \sqrt{s}\, p^\tau(ds)$ consists of those ψ such that $\int_0^\infty s <\psi\,,\, p^\tau(ds)\psi> \,<\infty$, in which case one has $\|\tau^{1/2}\psi\|^2 = \int_0^\infty s <\psi\,,\, p^\tau(ds)\psi>$.

We are now able to present a non-commutative generalization of a well-known identity due to Dynkin in the case of classical Markov processes ([Dyn]).

Theorem 9.3 (Dynkin's localisation formula) – *Let τ be a finite stop time. Let $\psi \in H$ be such that $F_0\psi$ belongs to Dom $\tau^{1/2}$. Then, for all $X \in$ Dom $\mathcal{L}$, we have*

$$F_0\, j_\tau(X) F_0\psi = j_0(X)\psi + F_0 \int_0^\infty 1\!\!1_{\tau>s}\, j_s(\mathcal{L}(X))\, ds\, F_0\psi. \tag{9.2}$$

Proof

Let $\tilde\tau = \tau_E$ where E is the τ-partition $\{r_i; i = 1, ..., N\}$ of $I\!R^+$. Then, by Theorem 9.1 one has

$$\begin{aligned}
F_0\, j_{\tilde\tau}(X) F_0 &= \sum_i F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[}\, j_{r_{i+1}}(X) F_0 \\
&= \sum_i F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[}\, j_0(X) F_0 + \sum_i F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[}\, M_{r_{i+1}}(X) F_0 \\
&\quad + \sum_i F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[} \int_0^{r_{i+1}} j_s(\mathcal{L}(X))\, ds\, F_0 \\
&= j_0(X) + \sum_i \sum_{j\le i} F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[}\, (M_{r_{j+1}}(X) - M_{r_j}(X)) F_0 \\
&\quad + \sum_i \sum_{j\le i} F_0 1\!\!1_{\tau\in[r_i,r_{i+1}[} \int_{r_j}^{r_{j+1}} j_s(\mathcal{L}(X))\, ds\, F_0 \\
&= j_0(X) + \sum_j F_0 1\!\!1_{\tau\ge r_j}\, (M_{r_{j+1}}(X) - M_{r_j}(X)) F_0 \\
&\qquad + \sum_j F_0 \int_{r_j}^{r_{j+1}} 1\!\!1_{\tau\ge r_j}\, j_s(\mathcal{L}(X))\, ds\, F_0 \\
&= j_0(X) + \sum_j F_0 1\!\!1_{\tau\ge r_j} F_{r_j} (M_{r_{j+1}}(X) - M_{r_j}(X)) F_{r_j} F_0 \\
&\quad + \sum_i F_0 \int_{r_j}^{r_{j+1}} 1\!\!1_{\tilde\tau>s}\, j_s(\mathcal{L}(X))\, ds\;\, F_0
\end{aligned}$$

$$= j_0(X) + F_0 \int_0^\infty \mathbb{1}_{\tilde{\tau}>s}\, j_s(\mathcal{L}(X))\, ds\ F_0.$$

Thus the identity (9.2) is valid for any discrete stop time. The only obstacle in passing to the limit to finite stop times is that the integral

$$F_0 \int_0^\infty \mathbb{1}_{\tau>s}\, j_s(\mathcal{L}(X))\, ds\ F_s$$

may not be defined. Notice that if we replace τ by $\tau \wedge n$ then

$$F_0 \int_0^\infty \mathbb{1}_{\tau\wedge n>s}\, j_s(\mathcal{L}(X))\, ds\ F_0 = F_0 \int_0^n \mathbb{1}_{\tau>s}\, j_s(\mathcal{L}(X))\, ds\ F_0$$

and this last integral is well-defined for $\int_0^n \|\mathbb{1}_{\tilde{\tau}>s}\, j_s(\mathcal{L}(X))\|\, ds \leq n\|\mathcal{L}(X)\|$.

So we have to restrict ourselves to the set of $\psi \in H$ for which

$$\int_0^n F_0 \mathbb{1}_{\tau>s}\, j_s(\mathcal{L}(X)) F_0 \psi\, ds$$

admits a limit when n tends to $+\infty$.

Suppose that the operator $Y = \mathcal{L}(X)$ is positive. As one always has $Y \leq \|Y\| I$ we get $j_s(Y) \leq \|Y\| F_s$. As $j_s(Y)$ and $\mathbb{1}_{\tau>s}$ commute the operators $F_0 \mathbb{1}_{\tau>s}\, j_s(Y) F_0$ are positive operators. Thus the integral $\int_0^n < \psi\, ,\, F_0 \mathbb{1}_{\tau>s}\, j_s(Y) F_0 \psi > ds$ is positive and increasing with n. But one has

$$\int_0^n < \psi\, ,\, F_0 \mathbb{1}_{\tau>s}\, j_s(Y) F_0 \psi > ds = \int_0^n < \mathbb{1}_{\tau>s} F_0 \psi\, ,\, j_s(Y) \mathbb{1}_{\tau>s} F_0 \psi > ds$$

$$\leq \|Y\| \int_0^n < \mathbb{1}_{\tau>s} F_0 \psi\, ,\, F_s \mathbb{1}_{\tau>s} F_0 \psi > ds = \|Y\| \int_0^n < F_0 \psi\, ,\, \mathbb{1}_{\tau>s} F_0 \psi > ds.$$

Let G denote the distribution function of the measure $< F_0\psi\, ,\, p^\tau(\cdot) F_0 \psi >$ that is, $G(s) = < F_0 \psi\, ,\, \mathbb{1}_{\tau\leq s} F_0 \psi >$. Integrating by parts we have

$$\int_0^n (1 - G(s))\, ds = n(1 - G(n)) + \int_0^n s\, dG(s).$$

But it is a classical result that if $\int_0^\infty s\, dG(s)$ is finite then $\lim_{n\to\infty} n(1 - G(n)) = 0$ and $\int_0^\infty s\, dG(s) = \int_0^\infty (1 - G(s))\, ds$. Thus, we will get the convergence of $\int_0^n < F_0\psi\, ,\, \mathbb{1}_{\tau>s} F_0 \psi > ds$ provided $\int_0^\infty s < F_0 \psi\, ,\, p^\tau(ds) F_0 \psi > < \infty$. That is exactly $F_0 \psi \in \mathrm{Dom}\ \tau^{1/2}$.

As every bounded operator, such as $\mathcal{L}(X)$, is a linear combination of four positive operators, we conclude easily for the general case. ∎

Notice that in the context of classical Markov processes we have $H = L^2(\Omega)$ and $\int_0^\infty < F_0\psi\, ,\, \mathbb{1}_{\tau>s} F_0 \psi > ds = \int_0^\infty \mathbb{E}[\mathbb{E}[\psi] \mathbb{1}_{\tau>s}\, \mathbb{E}[\psi]]\, ds = \mathbb{E}[\psi]^2 \int_0^\infty P(\tau > s)\, ds = \mathbb{E}[\psi]^2 \mathbb{E}[\tau]$. So one recovers the classical Dynkin's condition : $\mathbb{E}[\tau] < +\infty$ ([Dyn]).

10. Non-commutative Dirichlet problem

In this section $\mathcal{A}$ is still supposed to be a C^*-algebra, but as we are considerating projections in the center $\mathcal{Z}$ of $\mathcal{A}$ this section applies better in the context of von Neumann algebras. But specific properties of von Neumann algebras are not used here.

As explained briefly in the introduction, in the classical theory a "good" Markov process with generator $\mathcal{L}$ solves the Dirichlet problem associated to $\mathcal{L}$. We are now going to develop the analogous theory in our non-commutative context.

In this section we suppose our Markov process $(j_t)_{t\geq 0}$ to be a shift-strong Markov process (in the sense of section 8). Recall that Markov processes that satisfy Enchev's condition belong to the class of shift-strong Markov processes.

For any projection P in $\mathcal{Z}$ define H_P to be the space of $\psi \in H$ such that
(i) $\lim_{t\to+\infty}\ \inf_{s\leq t}\ k_s(P)\psi = 0$
(ii) $\lim_{t\to 0}\ \inf_{s\leq t} k_s(P)\psi = k_0(P)\psi$.
For any projection $P \in \mathcal{Z}$ define $\mathbb{1}_{\tau(P)>t}$ to be the projection

$$\mathbb{1}_{\tau(P)>t} = \inf_{s\leq t} k_s(P).$$

The family $(\mathbb{1}_{\tau(P)>t})_{t\geq 0}$ is a family of projections on H such that $t \mapsto \mathbb{1}_{\tau(P)>t}$ is decreasing and $\mathbb{1}_{\tau(P)>t}$ commutes with $j_u(X), u \geq t, X \in \mathcal{A}$. Thus, if we take a right-continuous modification of it, it defines a stop time $\tau(P)$ on H. The space H_P is then the space of $\psi \in H$ such that ψ belongs to the range of $\mathbb{1}_{\tau(P)<+\infty}$ and $\mathbb{1}_{\tau(P)>0}\,\psi = k_0(P)\psi$.

The projection $P \in \mathcal{Z}$ has to be interpreted as the indicator function of a domain; $k_s(P)$ then stands for the event "the Markov process at time s is in the domain P"; thus $\mathbb{1}_{\tau(P)>t}$ stands for the event "for all $s \leq t$ the Markov process at time s is in P" or else "the exit-time of the process from P is strictly greater than t". This justifies intuitively that $\tau(P)$ stands for the *exit-time* of the Markov process $(j_t)_{t\geq 0}$ from P. The space H_P is the space of states in which $\tau(P)$ is finite that is, the process $(j_t)_{t\geq 0}$ exits from P in a finite time; the condition of right continuity at 0 means intuitively that in the state ψ the process $(j_t)_{t\geq 0}$ stays an infinitesimal time, at least, in P that is, it does not instantaneously jump outside P after time 0.

Lemma 10.1 – *For every projection $P \in \mathcal{Z}$, the stop time $\tau(P)$ is a $\mathcal{P}$-stop-time.*

Proof

For all s, $k_s(P)$ is an element of $\mathcal{P}$, thus so is any finite product

$$k_{t_n}(P)\cdots k_{t_n}(P).$$

Furthermore, recall that

$$\theta_h(k_{t_1}(P)\cdots k_{t_n}(P)) = k_{t_1+h}(P)\cdots k_{t_n+h}(P).$$

As $\inf_{s\leq t} k_s(P)$ is a strong limit of such finite products we get that $\mathbb{1}_{\tau(P)>t}$ belongs to $\mathcal{P}$ and

$$\theta_h(\mathbb{1}_{\tau(P)>t}) = \inf_{s\leq t} k_{s+h}(P) = \inf_{h\leq s\leq t+h} k_s(P) \geq \inf_{s\leq t+h} k_s(P) = \mathbb{1}_{\tau(P)>t+h}.$$

This proves that $\theta_h(\mathbb{1}_{\tau(P)\le t}) \le \mathbb{1}_{\tau(P)\le t+h}$. Thus $\tau(P)$ is a $\mathcal{P}$-stop time. ■

By Theorem 8.4 we can form the stop time $\tau(P)\circ\tau(Q)$ for any P, Q projections in $\mathcal{Z}$.

Proposition 10.2 – *If $P \le Q$ are projections in $\mathcal{Z}$ then $\tau(P)\circ\tau(Q) = \tau(Q)$.*

Proof

Let $(E_n)_n$ be a sequence of refining $\tau(P)$-partitions of $\mathbb{R}^+$. Let $\tau^n(P) = (\tau(P))_{E_n}$, $n \in \mathbb{N}$. From the proof of Theorem 8.4, $\tau^n(P)\circ\tau(Q)$ converges to $\tau(P)\circ\tau(Q)$. We have

$$\begin{aligned}\mathbb{1}_{\tau^n(P)\circ\tau(Q)>t} &= \sum_i \mathbb{1}_{\tau(P)\in[t_i,t_{i+1}[}\theta_{t_{i+1}}(\mathbb{1}_{\tau(Q)>t-t_{i+1}}) \\ &= \sum_i \mathbb{1}_{\tau(P)\in[t_i,t_{i+1}[} \inf_{t_{i+1}\le s\le t} k_s(Q). \qquad (10.1)\end{aligned}$$

On one hand (10.1) is dominated by

$$\sum_i \mathbb{1}_{\tau(P)\in[t_i,t_{i+1}[} \inf_{s\le t} k_s(Q) = \inf_{s\le t} k_s(Q) = \mathbb{1}_{\tau(Q)>t}.$$

On the other hand (10.1) is greater than

$$\begin{aligned}&\sum_i \mathbb{1}_{\tau(P)\in]t_i,t_{i+1}]} \inf_{\{j;\, t_{i+1}\le t_j\le t\}} k_{t_j}(Q) \\ &= \sum_i (\mathbb{1}_{\tau(P)>t_i} - \mathbb{1}_{\tau(P)>t_{i+1}}) \inf_{\{j;\, t_{i+1}\le t_j\le t\}} k_{t_j}(Q) \\ &= \sum_i \inf_{s\le t_i} k_s(P)(I - \inf_{t_i<s\le t_{i+1}} k_s(P)) \inf_{\{j;\, t_{i+1}\le t_j\le t\}} k_{t_j}(Q). \qquad (10.2)\end{aligned}$$

But for all $j \le i$ one has $\inf_{s\le t_i} k_s(P) \le k_{t_j}(P) \le k_{t_j}(Q)$ for $P \le Q$. Thus (10.2) is equal to

$$\begin{aligned}&\sum_i \inf_{s\le t_i} k_s(P)(I - \inf_{t_i<s\le t_{i+1}} k_s(P)) \inf_{\{j;\, t_j\le t\}} k_{t_j}(Q) \\ &= \inf_{\{j;\, t_j\le t\}} k_{t_j}(Q) \sum_i \mathbb{1}_{\tau(P)\in[t_i,t_{i+1}[} \\ &= \inf_{\{j;\, t_j\le t\}} k_{t_j}(Q).\end{aligned}$$

But when the partition E_n is refining this expression converges to $\inf_{s\le t} k_s(Q)$ that is, $\mathbb{1}_{\tau(Q)>t}$.

Thus we have proved that $\mathbb{1}_{\tau(P)\circ\tau(Q)>t} = \mathbb{1}_{\tau(Q)>t}$, for all $t \in \mathbb{R}^+$. ■

We can now prove the non-commutative analogue of the usual harmonicity property (1.4) or (1.4'). For every projection $P \in \mathcal{Z}$,all $X \in \mathcal{A}$ let

$$\Gamma_P(X) = F_0 j_{\tau(P)}(X) F_0.$$

Theorem 10.3 (Harmonicity property) – *If $P \le Q \in \mathcal{Z}$ then $\Gamma_P \circ \Gamma_Q = \Gamma_Q$.*

Proof

For all $X \in \mathcal{A}$ we have

$$\begin{aligned}\Gamma_P \circ \Gamma_Q(X) &= F_0\, j_{\tau(P)}(F_0\, j_{\tau(Q)}(X)F_0) = F_0\theta_{\tau(P)}(F_0\,\theta_{\tau(Q)}(X)F_0)\\ &= F_0\,\theta_{\tau(P)\circ\tau(Q)}(X)F_0 \text{ (by Theorem 8.7)}\\ &= F_0\,\theta_{\tau(Q)}(X)F_0 \text{ (by Proposition 10.2)}\\ &= F_0\, j_{\tau(Q)}(X)F_0 = \Gamma_Q(X). \qquad \blacksquare\end{aligned}$$

We can now state the non-commutative analogue of the properties (1.5) and (1.5'). That is, the non-commutative Markov process $(j_t)_{t\geq 0}$ is going to solve the non-commutative Dirichlet problem associated to $\mathcal{L}$. First of all we need a simple result which is actually a particular case of a more general proposition to be proved in the section on multiplicative cocycles.

Lemma 10.4 – *For all $X \in \mathcal{A}$, all projections $P \in \mathcal{Z}$ let*

$$T_t^P(X) = F_0\, 1\!\!1_{\tau(P)>t}\, j_t(X)F_0.$$

Then $(T_t^P)_{t\geq 0}$ is a contractive semigroup of linear maps from $\mathcal{A}$ to $\mathcal{A}$.

Proof

The only non-trivial result is the semigroup property. One has

$$\begin{aligned}T_{t+s}^P(X) &= F_0 1\!\!1_{\tau(P)>t+s}\, j_{t+s}(X)F_0\\ &= F_0 \inf_{u\leq t+s} k_u(P) j_{t+s}(X)F_0 = F_0 \inf_{k\leq t} k_u(P) \inf_{t\leq u\leq t+s} k_u(P) j_{t+s}(X)F_0\\ &= F_0 1\!\!1_{\tau(P)>t}\,\theta_t(1\!\!1_{\tau(P)>s}\, j_s(X))F_0 = F_0 1\!\!1_{\tau(P)>t}\, F_t\,\theta_t(1\!\!1_{\tau(P)>s}\, j_s(X))F_t\, F_0\\ &= F_0 1\!\!1_{\tau(P)>t}\,\theta_t(F_0 1\!\!1_{\tau(P)>s}\, j_s(X)F_0)F_0 = F_0 1\!\!1_{\tau(P)>t}\,\theta_t(T_s^P(X))F_s\\ &= F_0 1\!\!1_{\tau(P)>t}\, j_t(T_s^P(X))F_0 = T_t^P(T_s^P(X)). \qquad \blacksquare\end{aligned}$$

Theorem 10.5 (Dirichlet problem) – *Let $(j_t)_{t\geq 0}$ be a shift strong Markov process. Let P be a projection on $\mathcal{Z}$. Let $X \in \mathcal{A}$ be such that $\Gamma_P(X)$ belongs to* Dom $\mathcal{L}$. *Then on the space of $\psi \in H_P$ such that $F_0\psi$ belongs to* Dom $(\tau(P)^{1/2})$, *if we put $Y = \Gamma_P(X)$ we have*

$$P\mathcal{L}(Y) = 0 \tag{10.3}$$

$$(I-P)Y = (I-P)X. \tag{10.4}$$

Proof

We restrict ourselves to $H_P \cap \{\psi; F_0\psi \in$ Dom $(\tau(P)^{1/2})\}$. Thus by Theorem 10.3 we have for all X such that $\Gamma_P(X) \in$ Dom $\mathcal{L}$

$$F_0\, j_{\tau(P)}(\Gamma_P(X))F_0 = \Gamma_P(X) + F_0 \int_0^\infty 1\!\!1_{\tau(P)>s}\, j_s(\mathcal{L}(\Gamma_P(X)))\, ds\; F_0$$

that is,

$$\Gamma_P(\Gamma_P(X)) = \Gamma_P(X) + \int_0^\infty F_0 1\!\!1_{\tau(P)>s}\, j_s(\mathcal{L}(\Gamma_P(X)))F_0\, ds.$$

Thus, by Theorem 10.3 one has $\Gamma_P(\Gamma_P(X)) = \Gamma_P(X)$ and consequently

$$\int_0^\infty F_0 1\!\!1_{\tau(P)>s}\, j_s(\mathcal{L}(\Gamma_P(X)))F_0\, ds = 0. \tag{10.5}$$

In other words $\int_0^\infty T_s^P(\mathcal{L}(\Gamma_P(X)))ds = 0$. As it is clear that $\int_0^t T_s^P(\mathcal{L}(\Gamma_P(X)))ds$ is well-defined, then so is $\int_t^\infty T_s^P(\mathcal{L}(\Gamma_P(X)))\, ds$. But the latter is equal to

$$\begin{aligned}\int_0^\infty T_{t+s}^P(\mathcal{L}(\Gamma_P(X)))\, ds &= \int_0^\infty T_t^P(T_s^P(\mathcal{L}(\Gamma_P(X))))\, ds \quad \text{by Lemma 10.4}\\ &= T_t^P[\int_0^\infty T_s^P(\mathcal{L}(\Gamma_P(X)))\, ds] = 0 \quad \text{by (10.5)}\ .\end{aligned}$$

Thus, for all $t \in I\!\!R^+$ we have $\int_0^t F_0 1\!\!1_{\tau(P)>s}\, j_s(\mathcal{L}(\Gamma_P(X)))\, ds = 0$. Deriving in t and letting t tend to 0 we get

$$1\!\!1_{\tau(P)>0}\, j_0(\mathcal{L}(\Gamma_P(X))) = 0$$

that is

$$P\mathcal{L}(\Gamma_P(X))) = 0.$$

This proves (10.3).

Furthermore, we have

$$(I-P)\Gamma_P(X) = 1\!\!1_{\tau(P)=0}\, F_0\, j_{\tau(P)}(X)F_0 = F_0 1\!\!1_{\tau(P)=0}\, j_{\tau(P)}(X)F_0.$$

But it can be easily seen from the construction of j_τ that for any stop time τ one has $1\!\!1_{\tau=0}\, j_\tau(X) = 1\!\!1_{\tau=0}\, j_0(X)$. Thus in our case $(I-P)\Gamma_P(X) = (I-P)X$. That is, identity (10.4) is proved. ■

Example : Let $\mathcal{A}, (T_t)_{t\geq 0}, (H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0})$ be as in the previous discussions. Let $\tilde{\mathcal{A}} = L^\infty(I\!\!R^d) \otimes \mathcal{A}$. The centre $\tilde{\mathcal{Z}}$ of $\tilde{\mathcal{A}}$ is the subalgebra $L^\infty(I\!\!R^d) \otimes \mathcal{Z}$ where $\mathcal{Z}$ is the centre of $\mathcal{A}$. Let P denote the probability measure of the standard d-dimensional Brownian motion with sample trajectory $(B_t)_{t\geq 0}$. Define

$$\begin{aligned}\tilde{H} &= L^2(I\!\!R^d) \otimes L^2(P) \otimes H\\ \tilde{F}_t &= 1\!\!1 \otimes E_t \otimes F_t\\ \tilde{j}_t(G) &= \text{multiplication by } j_t(G(x+B_t)),\ G(\cdot) \in \tilde{\mathcal{A}}\end{aligned}$$

where E_t denotes the conditional expectation given $\{B_s; s \leq t\}$. Let $D \subset I\!\!R^d$ be a bounded domain with smooth boundary ∂D and let $\tau = \tau(x, D, B)$ denote the (classical) exit-time of the process $(x + B_t)_{t\geq 0}$ from D. Then the operator $\tilde{\tau}$ of multiplication by τ in $\tilde{H}$ is a finite stop time and $\tilde{j}_{\tilde{\tau}}(G)$ is the multiplication operator by the operator-valued function $j_\tau(G(x + B_\tau))$. Note that $(\tilde{H}, \tilde{F}, \tilde{j}_t)$ is a minimal Markov process over $\tilde{\mathcal{A}}$. Furthermore $\tilde{F}_0\, \tilde{j}_{\tilde{\tau}}(G)\tilde{F}_0$ is multiplication by the operator-valued function

$$\begin{aligned}H(x) &= \int F_0 j_\tau(G(x+B_\tau))F_0\, dP\\ &= \int T_\tau(G(x+B_\tau))\, dP\end{aligned}$$

which can be expressed more explicitly if the joint distribution of $(x + B_\tau, \tau)$ can be computed. If the map $x \mapsto G(x)$ is continuous from $\mathbb{R}^d$ to $\mathcal{A}$ and the generator $\mathcal{L}$ of $(T_t)_{t\geq 0}$ is bounded it is easy to see that

$$\begin{cases} \frac{1}{2}(\Delta H)(x) + \mathcal{L}(H(x)) = 0, & x \in D \\ H(y) = G(y), & y \in \partial D. \end{cases}$$

It is also of interest to note that $H(x) \geq 0$ for all $x \in D$ whenever $G(x) \geq 0$ for all $x \in \partial D$ and furthermore $H(x), x \in D$ is determined by the values of G on ∂D.

When $d = 1, D = (a, b)$, we get

$$H(x) = \int_0^\infty T_t(G(a))\mu_x(dt) + \int_0^\infty T_t(G(b))\nu_x(dt)$$

where μ_x and ν_x are positive measures satisfying for all $a < x < b,\ \alpha > 0$

$$\mu_x(\mathbb{R}^+) = \frac{b-x}{b-a},\ \nu_x(\mathbb{R}^+) = \frac{x-a}{b-a}$$

$$\int_0^\infty e^{-\alpha t}\mu_x(dt) = e^{(a-x)\sqrt{2\alpha}}\ \frac{(1 - e^{-2(b-x)\sqrt{2\alpha}})}{(1 - e^{-2(b-a)\sqrt{2\alpha}})}$$

$$\int_0^\infty e^{-\alpha t}\nu_x(dt) = e^{(b-x)\sqrt{2\alpha})}\ \frac{(1 - e^{-2(a-x)\sqrt{2\alpha}})}{(1 - e^{-2(b-a)\sqrt{2\alpha}})}.$$

11. Non-minimal Markov processes

All the constructions of sections 3 to 10 are based on the fact we are considering the *minimal* dilation $(H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0})$ of a semigroup $(T_t)_{t\geq 0}$. That is, we strongly used the property that the set $\{\lambda(\mathbf{r}, \mathbf{X}, u); (\mathbf{r}, \mathbf{X}, u) \in \mathcal{D}\}$ is total in H. But parts of these constructions can be realized in a more general context.

Let $\mathcal{A}$ be a von Neumann algebra of operators on a separable Hilbert space H_0. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of completely positive maps from $\mathcal{A}$ into itself, with $T_0 = I$. A *weak Markov process* with expectation semigroup $(T_t)_{t\geq 0}$ is a family $(H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0}, \mathcal{P}, (\theta_t)_{t\geq 0})$ such that H is a Hilbert space containing $H_0, (F_t)_{t\geq 0}$ is an increasing family of projections on H with range $(F_0) = H_0$, the j_t's are $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}(H)$, $\mathcal{P}$ is a $*$-algebra containing the C^*-algebra generated by the $j_t(X)$, $t \in \mathbb{R}^+$, $X \in \mathcal{A}$, $(\theta_t)_{t\geq 0}$ is a semigroup of $*$-endomorphisms of $\mathcal{P}$ satisfying

(i) $j_t(I) = F_t,\ t \in \mathbb{R}^+$

(ii) $F_s\, j_t(X) F_s = j_s(T_{t-s}(X)),\ s \leq t$

(iii) $j_0(X) = X$

(iv) $\theta_t(j_s(X)) = j_{s+t}(X),\ s, t \in \mathbb{R}^+,\ X \in \mathcal{A}$

(v) $F_0 Y F_0 \in j_0(\mathcal{A})$ for all $Y \in \mathcal{P}$.

It is clear that in the case of the minimal dilation of $(T_t)_{t\geq 0}$ all these properties are satisfied; indeed, the property (v) is clear from $Y \in \mathcal{B}$, thus it is valid on $\mathcal{P}$ as $\mathcal{A}$ is a von Neumann algebra i.e. it is the strong (or weak) closure of a C^*-algebra.

The definition of a strong Markov process in this context is the same as in section 6. In that case the strong Markov property (theorem 6.2) will be also true. Furthermore, Dynkin's formula (Theorem 9.3) is also valid.

12. Multiplicative cocycles

In the classical theory of Markov processes, perturbations of the semigroup by multiplicative functionals play a very important role. Here we investigate a non-commutative analogue of it and we focus on several examples, one of which giving rise to a non-commutative generalization of the Feyman-Kac's formula.

Let $(H,(F_t)_{t\geq 0},(j_t)_{t\geq 0},\mathcal{P},(\theta_t)_{t\geq 0})$ be a weak Markov process with expectation semigroup $(T_t)_{t\geq 0}$ on a von Neumann algebra $\mathcal{A}$ of operators in a Hilbert space H_0.

A family $\{M(s,t); 0\leq s\leq t<+\infty\}$ of completely positive linear maps on the algebra $\mathcal{P}$ into itself is called a *multiplicative cocycle* if the following conditions are fulfilled.

(a) $M(r,s)M(s,t)=M(r,t)$ for all $r\leq s\leq t$;

(b) $\theta_h\circ M(r,s)=M(r+h,s+h)\circ\theta_h$ for all $0\leq r\leq s<+\infty$, $h\geq 0$;

(c) $F_tM(0,t)(X)F_t=M(0,t)(F_tXF_t)$ for all $t\geq 0, X\in\mathcal{P}$;

(d) $F_0\, M(0,0)(I)F_0=j_0(P)$ for some projection $P\in\mathcal{A}$.

Our first proposition shows how a multiplicative cocycle M can be used to obtain a new semigroup $(\tilde T_t)_{t\geq 0}$ by perturbing the semigroup $(T_t)_{t\geq 0}$.

Proposition 12.1 – *Define the map* $\tilde T_t:\mathcal{A}\to\mathcal{A}$ *through the identity*

$$j_0(\tilde T_t(X))=F_0M(0,t)(j_t(X))F_0, X\in\mathcal{A},\, t\geq 0.$$

Then $(\tilde T_t)_{t\geq 0}$ *is a semigroup of completely positive maps on* $\mathcal{A}$ *with* $\tilde T_0(X)=PXP$. *If* $M(0,t)$ *is contractive for each* t *then* $(\tilde T_t)_{t\geq 0}$ *is contractive. Suppose that the map* $t\mapsto M(0,t)(j_t(X))$ *is strongly continuous. Then* $s-\lim_{t\to 0}\tilde T_t(X)=PXP,\ X\in\mathcal{A}$.

Proof

The linearity and injectivity of j_0 together with the property (v) of section 11 imply that $\tilde T_t$ is a well-defined lienar map on $\mathcal{A}$. The complete positivity of $\tilde T_t$ is immediate from the complete positivity of $M(0,t)$ and j_t. For any $X\in\mathcal{A}, s,t\geq 0$, we have

$$\begin{aligned}
j_0(\tilde T_{t+s}(X)) &= F_0\, M(0,s)\big(M(s,t+s)(j_{t+s}(X))\big)F_0\\
&= F_0\, M(0,s)\big(\theta_s(M(0,t)j_t(X))\big)F_0\\
&= F_0\, F_sM(0,s)\big(\theta_s(M(0,t)j_t(X))\big)F_s\,F_0\\
&= F_0\, M(0,s)\big(\theta_s(F_0\, M(0,t)(j_t(X))F_0)\big)F_0\\
&= F_0\, M(0,s)\big(j_s(F_0\, M(0,t)(j_t(X))F_0)\big)F_0\\
&= j_0\big(\tilde T_s(\tilde T_t(X))\big).
\end{aligned}$$

which proves the first part. The last two parts are immediate. ■

When M is a contractive multiplicative cocycle there arises naturally the problem of constructing a weak Markov process dilating the perturbed semigroup $(\tilde{T}_t)_{t\geq 0}$ of Proposition 12.1. Property (a) of the definition of multiplicative cocycles and the methods obtained in [BP1,2] yield a minimal Markov dilation $(H^M, (F_t^M)_{t\geq 0}, (j_t^M)_{t\geq 0})$ for the non-conservative completely positive evolution described by M on the *-algebra $\mathcal{P}$ with the subordination property $j_t^M(I) \leq P_t^M$. Define

$$\tilde{j}_t(X) = j_t^M \circ j_t(X),\ \tilde{F}_t = \tilde{j}_t(1) = j_t^M(F_t),\ X \in \mathcal{A}.$$

Clearly $\tilde{j}_t$ is a *-homomorphism and $\tilde{F}_t$ is a projection for each t. Furthermore, for $s < t$

$$\begin{aligned}
\tilde{F}_s \tilde{j}_t(X)\tilde{F}_s &= j_s^M(F_s)j_t^M(j_t(X))j_s^M(F_s)\\
&= j_s^M(F_s)F_s^M j_t^M(j_t(X))F_s^M j_s^M(F_s)\\
&= j_s^M(F_s)j_s^M(M(s,t)(j_t(X))j_s^M(F_s)\\
&= j_s^M(F_sM(s,t)(j_t(X))F_s)\\
&= j_s^M(\theta_s(F_0M(0,t-s)(j_{t-s}(X))F_0))\\
&= j_s^M(j_s(\tilde{T}_{t-s}(X)))\\
&= \tilde{j}_s(\tilde{T}_{t-s}(X)).
\end{aligned}$$

However $\tilde{F}_s$ is not increasing in s in general. Indeed, the operator $\tilde{F}_s\,\tilde{F}_t\,\tilde{F}_s = \tilde{j}_s(\tilde{T}_{t-s}(I))$, $s < t$ needs not even be a projection. If $\tilde{T}_t(I)$ is a projection for every t then $(\tilde{F}_t)_{t\geq 0}$ is a filtration and $(H^M, (\tilde{F}_t)_{t\geq 0}(\tilde{j}_t)_{t\geq 0})$ will be a weak Markov dilation of $(\tilde{T}_t)_{t\geq 0}$. We leave the general problem of dilating $(\tilde{T}_t)_{t\geq 0}$ open but illustrate how, in some examples, we can circumvent the difficulty by using the Fock space stochastic calculus ([Par], [Me3]).

Example 1 : Let $(H, (F_t)_{t\geq 0}, (j_t)_{t\geq 0})$ be the minimal dilation of a conservative semigroup $(T_t)_{t\geq 0}$ over $\mathcal{A}$. Choose and fix an element $B \in \mathcal{A}$ and define $m(s,t)$ by the differential equation

$$\frac{d}{dt}m(s,t) = m(s,t)j_t(B),\ t \geq s \tag{12.1}$$

with the initial condition $m(s,s) = I$. Choose $\mathcal{P}$ to be the intrinsic path algebra defined in section 4 and define

$$M(s,t)(Y) = m(s,t)Ym(s,t)^*,\ Y \in \mathcal{P},\ 0 \leq s \leq t < +\infty. \tag{12.2}$$

It follows from (12.1) that $m(s,t)$ can be expressed as

$$m(s,t) = I + \sum_{n=1}^{\infty} \int_{s<s_1<\cdots<s_n<t} j_{s_1}(B)\cdots j_{s_n}(B)\,ds_1\cdots ds_n \tag{12.3}$$

where the right hand side converges in operator norm. Since $\theta_h(j_t(B)) = j_{t+h}(B)$ it follows from (12.3) and a change of variables in the integration that

$$\theta_h(m(s,t)) = m(s+h,t+h)$$

and hence

$$\theta_h(M(s,t)) = M(s+h,t+h)\circ\theta_h.$$

It is a consequence of (12.1) that $m(r,s)m(s,t) = m(r,t)$ and hence

$$M(r,s)M(s,t) = M(r,t)$$

for all $r \le s \le t$.

Thus $M(s,t)$ defined by (12.2) is, indeed a multiplicative cocycle. If $-(B+B^*) \ge 0$ then $-j_t(B+B^*) \ge 0$ and equation (12.1) implies that $m(s,t)$ is a contractive operator and hence M defined by (12.2) is a contractive multiplicative cocyle.

In this case too we do not have a method of constructing a Markov dilation. We proceed to look at a modification of (12.1) by considering

$$\frac{d}{dt}m(s,t) = -\frac{1}{2}m(s,t)k_t(Z), \quad m(s,s) = I,\ t \ge s$$

where Z is a non-negative element in the centre $\mathcal{Z}$ of $\mathcal{A}$. Owing to the commutativity of the family $(k_t(Z))_{t\ge 0}$ we have

$$m(s,t) = \exp(\frac{1}{2}\int_s^t k_r(Z)\,dr)$$

and (12.2) takes the form

$$M(s,t)(Y) = \exp(-\frac{1}{2}\int_s^t k_r(Z)\,dr)Y\ \exp(-\frac{1}{2}\int_s^t k_r(Z)\,dr).$$

Denote by $(T_t^Z)_{t\ge 0}$ the semigroup $(\tilde{T}_t)_{t\ge 0}$ of Proposition 13.1 when M is as above. As $k_s(Z)$ commutes with $j_t(X)$ for $s \le t$, $X \in \mathcal{A}$ we have

$$j_0(T_t^Z(X)) = F_0\, j_t(X)\exp(-\int_s^t k_r(Z)\,dr)F_0.$$

To construct a Markov dilation of $(T_t^Z)_{t\ge 0}$ we shall use the methods of Fock space stochastic calculus. To this end consider the Hilbert space $\hat{H} = H\otimes\Gamma(L^2(\mathbb{R}^+))$ and the Fock-adapted selfadjoint operator-valued process $(N_t)_{t\ge 0}$ satisfying the stochastic differential equation

$$dN_t = d\Lambda_t + \sqrt{k_t(Z)}(dA_t + dA_t^+) + k_t(Z)dt,\ N_0 = 0.$$

Since $(k_t(Z))_{t\ge 0}$ is a commuting family of bounded nonnegative operators it follows that $(N_t)_{t\ge 0}$ can, indeed, be chosen to be a commuting family of nonnegative selfadjoint operators where the spectrum of each N_t is contained in the set $\{0,1,...\}$. Define the projection-valued Fock-adapted process

$$P_t = \mathbb{1}_{\{0\}}(N_t).$$

It follows from quantum Ito's formula that

$$dP_t = -P_t\,dN_t,\quad P_0 = I. \tag{12.4}$$

Denote by E_t the projection on the subspace $H\otimes\Gamma(L^2([0,t]))\otimes\Phi_{[t} \subset \hat{H}$ where $\Phi_{[t}$ is the Fock-vacuum in $\Gamma(L^2([t,\infty[))$ and put

$$\hat{j}_t(X) = j_t(X)P_t\,E_t,\quad \hat{F}_t = F_t\,E_t \tag{12.5}$$

where $j_t(X)$ and F_t are to be understood as their respective ampliations to $\hat{H}$. It is clear that the correspondence $t \mapsto \hat{F}_t$ is a strongly continuous monotonic increasing projection-valued map. Since P_t is made up of the operators $k_s(Z)$ and the Fock space creation, annihilation and conservation operators $A_s^+, A_s, \Lambda_s, s \le t$ it follows that P_t commutes with $j_t(X)$. Thus $\hat{\jmath}_t$ defined by (12.5) is a *-homomorphism and $\hat{\jmath}_t(I) \le F_t\, E_t$.

Proposition 12.2 – *The triple $(\hat{H}, (\hat{F}_t)_{t\ge 0}, (\hat{\jmath}_t)_{t\ge 0})$ is a weak Markov process subordinated to the filtration $\hat{F}$ and having expectation semigroup $(T_t^Z)_{t\ge 0}$.*

Proof

For $s \le t$ one has

$$\begin{aligned} \hat{F}_s\, \hat{\jmath}_t(X)\hat{F}_s &= F_s\, E_s\, j_t(X) P_t\, E_t\, F_s\, E_s \\ &= F_s\, j_t(X) E_s\, P_t\, E_s\, F_s \end{aligned} \tag{12.6}$$

where (12.4) implies

$$\begin{aligned} E_s\, P_t\, E_s &= E_s\, P_s\, E_s - \int_s^t E_s\, P_r\, k_r(Z) E_s\, dr \\ &= E_s\, P_s\, E_s - \int_s^t E_s\, P_r\, E_s\, k_r(Z)\, dr. \end{aligned}$$

Thus

$$E_s\, P_t\, E_s = E_s\, P_s \exp(-\int_s^t k_r(Z)\, dr) E_s \quad \text{for } t \ge s.$$

Substituting this in (12.6) we get

$$\begin{aligned} \hat{F}_s\, \hat{\jmath}_t(X)\hat{F}_s &= E_s\, F_s\, j_t(X) \exp(-\int_s^t k_r(Z) dr) F_s\, P_s\, E_s \\ &= E_s\, \theta_s(F_0 j_{t-s}(X) \exp(-\int_0^{t-s} k_r(Z)\, dr) F_0) P_s\, E_s \\ &= j_s(T_{t-s}^Z(X)) P_s\, E_s \\ &= \hat{\jmath}_s(T_{t-s}^Z(X)) \end{aligned}$$

which proves the claim. ∎

We now evaluate the generator of the semigroup $(T_t^Z)_{t\ge 0}$ and thereby we get an analogue of the classical Feynman-Kac perturbation of the Laplacian by the operator of multiplication by a negative function.

Theorem 12.3 (Feynman-Kac's formula) – *Let Z be a nonnegative element of $\mathcal{Z}$. Then the generator of the perturbed semigroup $(T_t^Z)_{t\ge 0}$ given by*

$$j_0(T_t^Z(X)) = F_0 \ \exp(-\int_0^t k_r(Z)\, dr) j_t(X) F_0, \quad X \in \mathcal{A}$$

is $\mathcal{L} - L_Z$ where $\mathcal{L}$ is the original generator of $(T_t)_{t\geq 0}$ and L_Z is the operator of multiplication by Z.

Proof

Note that, by definition

$$\begin{aligned} j_0(T_t^Z)(X)) &= F_0\, j_t(X)\exp(-\int_0^t k_r(Z)\,dr)F_0 \\ &= s - \lim_{n\to+\infty} F_0 j_t(X)\exp(-\frac{t}{n}\sum_{j=1}^n k_{jt/n}(Z))F_0 \\ &= s - \lim_{n\to+\infty} F_0(\prod_{j=1}^n k_{jt/n}(\exp(-\frac{-t}{n}Z)))j_t(X)F_0 \\ &= s - \lim_{n\to+\infty} j_0((L_{\exp(\frac{-t}{n}Z)}T_{t/n})^n(X)). \end{aligned}$$

where L_Y denotes the left multiplication by Y. If $\mathcal{L}$ denotes the generator of $(T_t)_{t\geq 0}$ it now follows that the generator of $(T_t^Z)_{t\geq 0}$ is $\mathcal{L} - L_Z$ with the same domain as $\mathcal{L}$. ■

Example 2 : Consider the minimal dilation $(H,(F_t)_{t\geq 0},(j_t)_{t\geq 0})$ of a conservative semigroup $(T_t)_{t\geq 0}$ on a von Neumann algebra $\mathcal{A}$. Let P be a projection in the centre $\mathcal{Z}$ of $\mathcal{A}$. Consider the exit time $\tau(P)$ associated to P (cf section 10). Let

$$M(s,t)(Y) = \left[\inf_{s\leq r\leq t} k_r(P)\right] Y \left[\inf_{s\leq r\leq t} k_r(P)\right],\ Y\in\mathcal{P}, 0\leq s\leq t<\infty.$$

where $\mathcal{P}$ is the usual intrinsic path algebra. It is clear that M thus defined is a contractive multiplicative cocycle. Put

$$\hat{j}_t(X) = \inf_{0\leq r\leq t} k_r(P) j_t(X) = \mathbb{1}_{\tau(P)>t}\, j_t(X).$$

Then $(H,(F_t)_{t\geq 0},(\hat{j}_t)_{t\geq 0})$ is clearly a Markov dilation of the expectation semigroup $(T_t^P)_{t\geq 0}$ defined by

$$j_0\, T_t^P(X)) = F_0\, \hat{j}_t(X) F_0.$$

Note that $\hat{j}_t(I)\leq F_t$ for all t.

We have already met the semigroup $(T_t^P)_{t\geq 0}$ in Lemma 10.4. The Markov process $(\hat{j}_t)_{t\geq 0}$ can be considered as the Markov process $(j_t)_{t\geq 0}$ but "killed" when it hits the "boundary" of P.

Example 3 : Choosing $(H,(F_t)_{t\geq 0},(j_t)_{t\geq 0})$ as before, consider a quantum stochastic differential equation on the Fock space $\hat{H} = H\otimes\Gamma(L^2(\mathbb{R}^t;\mathbb{C}^n))$ of the form

$$\begin{aligned} dW(s,t) = W(s,t)[\sum_{i=1}^n (j_t(-L_i)\,dA_i^+(t) + j_t(L_i^*)\,dA_i(t)) + \\ + j_t(iH - \frac{1}{2}\sum_{i=1}^n L_i^* L_i)dt] \end{aligned} \qquad (12.7)$$

$$W(s,s) = 1,\ t \geq s$$

where $H, L_i \in \mathcal{A}, i = 1, ..., n$ and H is selfadjoint. By a generalization ([Att]) of the standard quantum stochastic calculus ([H-P]) there exist isometric operators $W(s,t)$ in $H \otimes \Gamma(L^2(I\!R^+; \mathbb{C}^n))$ satisfying (12.7) and the relation

$$W(r,t) = W(r,s)W(s,t) \text{ for } r \leq s \leq t.$$

Define

$$\hat{j}_t(X) = W(0,t)j_t(X)W(0,t)^*,\ \hat{F}_t = F_t\, E_t$$

where E_t is defined as the projection on the subspace $H \otimes \Gamma(L^2([0,t]; \mathbb{C}^n)) \otimes \Phi_{[t}$ (cf example 1). Then $(\hat{F}_t)_{t\geq 0}$ is a filtration and for $t \geq s$

$$\begin{aligned}\hat{F}_s\, \hat{j}_t(X)\hat{F}_s &= F_s\, E_s W(0,t)j_t(X)W(0,t)^* F_s\, E_s \\ &= F_s W(0,s)E_s W(s,t)j_t(X)W(s,t)^* E_s W(0,s)^* F_s \\ &= F_s W(0,s)M(s,t)(j_t(X))E_s W(0,s)^* F_s\end{aligned}$$

where

$$M(s,t)(Y) = Y + \sum_{k=1}^{\infty} \int_{s<s_1<\cdots<s_k<t} \mathcal{L}_{s_1}\mathcal{L}_{s_2} \cdots \mathcal{L}_{s_k}(Y)\, ds_1 \cdots ds_k \qquad (12.8)$$

and

$$\begin{aligned}\theta_h(\mathcal{L}_s(Y)) = i[j_{s+h}(H), Y] - \frac{1}{2} \sum_{i=1}^{n} (&j_{s+h}(L_i^* L_i)\theta_h(Y) + \theta_h(Y)j_{s+h}(L_i^* L_i) \\ &- 2\, j_{s+h}(L_i^*)\theta_h(Y)j_{s+h}(L_i) \\ = \mathcal{L}_{s+h}(\theta_h(Y)).&\end{aligned}$$

Substituting this in the n-term of the right hand side of (12.9) and making a change of variables in the integrations we conclude that

$$\theta_h M(s,t) = M(s+h, t+h) \circ \theta_h.$$

This together with (12.8) implies that $(M(s,t))_{s\leq t}$ is indeed, a multiplicative cocycle and $(\hat{H}, (\hat{F}_t)_{t\geq 0}, (\hat{j}_t)_{t\geq 0})$ is a Markov dilation of the semigroup $(\tilde{T}_t)_{t\geq 0}$ given by Proposition 12.1.

We shall now evaluate the generator of $(\tilde{T}_t)_{t\geq 0}$. To this end we first observe that

$$\begin{aligned}F_s &\int_{s<s_1<\cdots<s_k<t} \mathcal{L}_{s_1} \cdots \mathcal{L}_{s_k}\, ds_1 \cdots ds_k\, F_s \\ &= \theta_s\Big(\int_{0<s_1<\cdots<s_k<t-s} F_0\, \mathcal{L}_{s_1-s} \cdots \mathcal{L}_{s_k-s}(j_{t-s}(X))F_0\, ds_1 \cdots ds_k\Big) \\ &= \theta_s\Big(\int_{0<s_1<\cdots<s_k<t-s} T_{s_1} K T_{s_2-s_1} K \cdots T_{s_k-s_{k-1}} K T_{t-s-s_k}(X)\, ds_1 \cdots ds_k\Big)\end{aligned}$$

where $K(X) = i[H,X] - \frac{1}{2} \sum_{i=1}^{n}(L_i^* L_i X + X L_i^* L_i - 2L_i^* X L_i)$, $X \in \mathcal{A}$. Thus

$$\begin{aligned}&F_s M(s,t)(j_t(X))F_0 \\ &\quad = j_s(T_{t-s}(X)) +\end{aligned}$$

$$+\sum_{k=1}^{\infty}\int_{0<s_1<\cdots<s_k<t-s} T_{s_1}KT_{s_2-s_1}K\cdots KT_{t-s-s_k}(X)ds_1\cdots ds_k$$
$$= j_s(\tilde{T}_{t-s}(X))$$

where, thanks to Dyson's perturbation series expansion, the semigroup $(\tilde{T}_t)_{t\geq 0}$ has generator $\mathcal{L}+K$ with domain same as that of $\mathcal{L}$, the generator of $(T_t)_{t\geq 0}$.

Example 4 : Consider $(H,(F_t)_{t\geq 0},(j_t)_{t\geq 0})$ as before. Let $Z\in\mathcal{Z}$ with $Z^{-1}\in\mathcal{Z}$. Define

$$M(s,t)(\xi) = k_s(Z)k_t(Z^{-1})\xi\, k_s(Z^*)k_t(Z^{*^{-1}}).$$

It is clear that $M(s,t)$ is a multiplicative cocycle and

$$F_0\, M(0,t)(j_t(X))F_0 = F_0\, ZZ^*T_t(ZZ^*)^{-1}X)F_0.$$

Thus the perturbed semigroup $(\tilde{T}_t)_{t\geq 0}$ is given by

$$\tilde{T}_t(X) = |Z|^2T_t(|Z|^{-2}X).$$

The perturbed semigroup is contractive if and only if

$$\tilde{T}_t(I) = |Z|^2T_t(|Z|^{-2}) \leq I \text{ for all } t$$

or, equivalently,

$$T_t(|Z|^{-2}) \leq |Z|^{-2} \text{ for all } t,$$

i.e. $|Z|^{-2}$ is an excessive element for $(T_t)_{t\geq 0}$.

The semigroup $(T_t)_{t\geq 0}$ is conservative if and only if $T_t(|Z|^{-2}) = |Z|^{-2}$, i.e. $|Z|^{-2}$ is harmonic for $(T_t)_{t\geq 0}$.

A similar construction is obtained by taking

$$M(s,t)(Y) = j_s(B^{-1})j_t(B)Yj_t(B^*)j_s(B^{*^{-1}}), Y\in\mathcal{P}$$

where $B, B^{-1}\in\mathcal{A}$. Then $M(\cdot,\cdot)$ is a multiplicative cocycle. Furthermore

$$j_0(\hat{T}_t(X)) = F_0M(0,t)(j_t(X))F_0$$
$$= F_0\, j_0(B^{-1})j_t(B)j_t(X)j_t(B^*)j_0(B^{*-1})F_0.$$

Thus

$$\hat{T}_t(X) = B^{-1}T_t(BXB^*)B^{*^{-1}}.$$

$\hat{T}_t$ is contractive if $T_t(BB^*)\leq BB^*$, i.e. BB^* is excessive for $(T_t)_{t\geq 0}$. It is conservative if BB^* is harmonic.

We thus get two extensions of the classical Doob's transform. It is an open problem to develop a boundary theory which would enable us to construct the associated Markov process along classical lines ([Dy2]).

13. Additive cocycles

We now investigate a particular family of additive functionals which give rise to a perturbation of the semigroup by a random time change.

Let $(H,(F_t)_{t\geq 0},(j_t)_{t\geq 0})$ be the usual minimal dilation of a conservative semigroup $(T_t)_{t\geq 0}$ on a von Neumann algebra $\mathcal{A}$. Let $\mathcal{Z}$ be the centre of $\mathcal{A}$. Let $\mathcal{P}$ be the intrinsic path algebra.

Let $(L_t)_{t\geq 0}$ be a strongly continuous, strictly increasing commuting family of positive operators in $\mathcal{P}$ such that

(i) L_t commutes with $j_u(X)$, $u \geq t$, $X \in \mathcal{A}$

(ii) $\theta_h(L_t) = L_{t+h} - L_h$.

An example of such a family is obtained, for example, by putting

$$L_t = \int_0^t k_r(Z)dr$$

for some $Z > 0$ in $\mathcal{Z}$.

For every $a \in I\!R^+$ define the stop time τ_a by

$$1\!\!1_{\tau_a > t} = 1\!\!1_{[0,a]}(L_t)$$

in the usual sense of the functional calculus on selfadjoint operators.

Lemma 13.1 – *For every $a \in I\!R^+, \tau_a$ is a $\mathcal{P}$-stop time.*

Proof

For any polynomial function f on $I\!R^+$ it is clear that $f(L_t)$ belongs to $\mathcal{P}$ and that $\theta_h(f(L_t)) = f(\theta_h(L_t))$. If f is a continuous function with compact support on $I\!R^+$ then, by Weierstrass theorem, it can be uniformly approximated by a sequence of polynomials $(f_n)_n$. Thus $f_n(L_t)$ converges strongly to $f(L_t)$ and $\theta_h(f_n(L_t)) = f_n(\theta_h(L_t))$ converges strongly to $f(\theta_h(L_t))$. Thus $f(L_t)$ belongs to $\mathcal{P}$ and $\theta_h(f(L_t)) = f(\theta_h(L_t))$. Now if f is the indicator function of some interval it can be approximated by a decreasing sequence of continuous functions with compact support. By monotone convergence theorem we get that also $f(L_t) \in \mathcal{P}$ and $\theta_h(f(L_t)) = f(\theta_h(L_t))$. This implies that $1\!\!1_{\tau_a > t} \in \mathcal{P}$ and

$$\begin{aligned}\theta_h(1\!\!1_{\tau_a > t}) &= 1\!\!1_{[0,a]}(\theta_h(L_t)) = 1\!\!1_{[0,a]}(L_{t+h} - L_h)\\ &\geq 1\!\!1_{[0,a]}(L_{t+h}) \ \text{ for } \ L_{t+h} - L_h \leq L_{t+h}\\ &= 1\!\!1_{\tau_a > t+h}.\end{aligned}$$

This proves that τ_a is a $\mathcal{P}$-stop time for every $a \in I\!R^+$. ■

Thus, by Theorem 8.4, we can form the stop time $\tau_a \circ \tau_b$ for every $a, b \in I\!R^+$.

Proposition 13.2 – *For every $a, b \in I\!R^+$ one has $\tau_a \circ \tau_b = \tau_{a+b}$.*

Proof

The operator $1\!\!1_{\tau_a \circ \tau_b \leq t}$ is the strong limit, when the partition $\{t_i; i = 1, ..., n\}$ is refining, of

$$\begin{aligned}&\sum_{i; t_i \leq t} 1\!\!1_{\tau_a \in]t_i, t_{i+1}]} \theta_{t_{i+1}}(1\!\!1_{\tau_b \leq t - t_{i+1}})\\ &\quad = \sum_{i; t_i \leq t} (1\!\!1_{\tau_a > t_i} - 1\!\!1_{\tau_a > t_{i+1}}) \theta_{t_{i+1}}(1\!\!1_{L_{t - t_{i+1}} > b})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i;t_i\le t} (\mathbb{1}_{L_{t_i}\le a} - \mathbb{1}_{L_{t_{i+1}}\le a})\mathbb{1}_{L_t - L_{t_{i+1}}>b}\\
&= \sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_t - L_{t_{i+1}}>b}\\
&= \sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_t - L_{t_{i+1}}>b}\,\mathbb{1}_{L_t>b+a}\\
&= \mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_t - L_{t_i}>b}\\
&\quad + \mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}(\mathbb{1}_{L_t - L_{t_{i+1}}>b} - \mathbb{1}_{L_t - L_{t_i}>b})\\
&= \mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\mathbb{1}_{L_{t_{i+1}}>a}\\
&\quad - \mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t} \mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{b+L_{t_i}<L_t\le b+L_{t_{i+1}}}. \qquad (13.1)
\end{aligned}
$$

Note that we are taking some liberty in our notations, but all this can be justified in the following way : the operators L_t are commuting and selfadjoint, so there exists one spectral measure P on $\mathbb{R}^t$ such that $L_t = \int_0^\infty \varphi(t,x)P(dx)$ for some function φ on $\mathbb{R}^+\times\mathbb{R}^+$. Furthermore, for each x, the function $\varphi(\cdot,x)$ has the same properties as $(L_t)_{t\ge 0}$ i.e. it is continuous, strictly increasing and strictly positive. Furthermore, for any bounded measurable function f on $\mathbb{R}^+$ one has $f(L_t) = \int_0^\infty f(\varphi(t,x))P(dx)$. So all what is written in terms of the L_t's can be transfered in the same way in terms of the $\varphi(t,x)'s$ where x is fixed.

A consequence of that is that if $L_t > b+a$ (i.e. $\varphi(t,x) > b+a$ for all x) as L_T is increasing continuous there exists a i such that $L_{t_i}\le a$ and $L_{t_{i+1}>a}$. Consequently the operator $\mathbb{1}_{L_t>b+a}\ \sum_{i;t_i\le t}\mathbb{1}_{L_t\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}$ is equal to $\mathbb{1}_{L_t>b+a}$. Thus (13.1) is equal to

$$\mathbb{1}_{L_t>b+a} - \mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t}\mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_{t_i}<L_t-b\le L_{t_{i+1}}} \qquad (13.2)$$

Let $T = a+b+h$ for some $h>0$, one has

$$
\begin{aligned}
&\mathbb{1}_{L_t>b+a}\sum_{i;t_i\le t}\mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_{t_i}<L_t-b\le L_{t_{i+1}}}\\
&\qquad = \mathbb{1}_{T\ge L_t>b+a}\sum_{i;t_i\le t}\mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_{t_i}<L_t-b\le L_{t_{i+1}}}\\
&\qquad\quad + \mathbb{1}_{L_t>T}\sum_{i;t_i\le t}\mathbb{1}_{L_{t_i\le a}}\,\mathbb{1}_{L_{t_{i+1}}>a}\,\mathbb{1}_{L_{t_i}<L_t-b\le L_{t_{i+1}}} \qquad (13.3)
\end{aligned}
$$

The second term of the right hand side of (13.3) is dominated by

$$\mathbb{1}_{L_t>T}\sum_{i;t_i\le t}\mathbb{1}_{L_{t_i}\le a}\,\mathbb{1}_{L_{t_{i+1}}\ge a+h}$$

$$\leq \mathbb{1}_{L_t>T} \sum_{i;t_i\leq t} \mathbb{1}_{L_{t_{i+1}}-L_{t_i}\geq h}\cdot$$

Coming back to the spectral representation of L_t we have to consider

$$\mathbb{1}_{\varphi(t,x)>T} \sum_{i;t_i\leq t} \mathbb{1}_{\varphi(t_{i+1},x)-\varphi(t_i,x)\geq h}\cdot \tag{13.4}$$

But $\varphi(\cdot,x)$ is continuous, thus uniformely continuous on $[0,t]$. Consequently, when $\sup_i |t_{i+1}-t_i|$ tends to 0 the expression (13.4) tends to 0.

Consider the first term of the right hand side of (13.3), it is equal to

$$\mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \mathbb{1}_{[0,a]}(L_{t_i})\mathbb{1}_{]a,h+a]}(L_{t_{i+1}})\mathbb{1}_{L_{t_i}<L_t-b\leq L_{t_{i+1}}}$$

$$= \mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \mathbb{1}_{L_{t_i}=0}\, \mathbb{1}_{]a,h+a]}(L_{t_{i+1}})\mathbb{1}_{L_{t_i}<L_t-b\leq L_{t_{i+1}}}$$

$$+ \mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \sum_{n=0}^{N} \sum_{n=0}^{M} \mathbb{1}_{]a-\frac{n+1}{N}a\,,\,a-\frac{n}{N}a]}(L_{t_i})$$

$$\times \mathbb{1}_{]a+\frac{m}{M}h\,,\,a+\frac{m+1}{M}h]}(L_{t_{i+1}})\mathbb{1}_{L_{t_i}<L_t-b\leq L_{t_{i+1}}}\cdot \tag{13.5}$$

The first term of the right hand side is dominated by $\sum_{i;h_i\leq t} \mathbb{1}_{L_{t_{i+1}}-L_{t_i}>a}$ thus applying the same argument as for (13.4) this quantity converges to 0 when the diameter of the partition tends to 0. Consider the second term of the right hand side of (13.5), it is equal to

$$\mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \mathbb{1}_{]a-\frac{1}{N}a\,,\,a]}(L_{t_i})\mathbb{1}_{]a\,,\,a+\frac{1}{M}h]}(L_{t_i})\mathbb{1}_{L_{t_i}<L_t-b\leq L_{t_{i+1}}}$$

$$+ \mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \sum_{\substack{n,m=0\\(n,m)\neq(0,0)}} \mathbb{1}_{]a-\frac{n+1}{N}a\,,\,a-\frac{n}{N}a]}(L_{t_i})$$

$$\times \mathbb{1}_{]a+\frac{m}{M}h\,,\,a+\frac{m+1}{M}h]}(L_{t_{i+1}})\mathbb{1}_{L_{t_i}<L_t-b\leq L_{t_{i+1}}}$$

$$\leq \mathbb{1}_{h+a\geq L_t-b>a}\Big(\sum_{i;t_i\leq t} \mathbb{1}_{]a-\frac{1}{N}a\,,\,a]}(L_{t_i})\mathbb{1}_{]a\,,\,a+\frac{1}{M}h]}(L_{t_{i+1}})\Big)\mathbb{1}_{a-\frac{1}{N}a<L_t-b\leq a+\frac{1}{M}h}$$

$$+ \mathbb{1}_{h+a\geq L_t-b>a} \sum_{i;t_i\leq t} \sum_{\substack{n,m=0\\(n,m)\neq(0,0)}}^{N,M} \mathbb{1}_{L_{t_{i+1}}-L_{t_i}>\frac{1}{N}a\wedge\frac{1}{M}h}\cdot$$

The second term converges to 0 as previously. If the diameter of the partition $\{t_i; i\}$ is small enough, the first term is equal to

$$\mathbb{1}_{h+a\geq L_t-b>a}\, \mathbb{1}_{a-\frac{1}{N}a<L_t-b\leq a+\frac{1}{M}h}\cdot \tag{13.6}$$

If $L_t-b>a$ it is clear that $\mathbb{1}_{L_t-b\leq a+\frac{1}{M}h}$ converges to 0 when M tends to $+\infty$. Furthermore, this last expression (13.6) is independent of the partition $\{t_i, i = 1,...,n\}$.

Thus, for a fixed M, letting the diameter of the partition $\{t_i, i=1,...,n\}$ tend to 0 and then letting M tend to 0 we have proved that

$$\mathbb{1}_{\tau_a\circ\tau_b\leq t} = \mathbb{1}_{L_t>b+a} = \mathbb{1}_{\tau_{a+b}\leq t}\cdot \qquad \blacksquare$$

From now on we assume $(j_t)_{t\geq 0}$ to be a shift-strong Markov process.

One can define a new Markov process by random time change : $a \mapsto \tau_a$. Put $\hat{F}_a = F_{\tau_a}, \hat{j}_a = j_{\tau_a}$, define

$$\hat{T}_a(X) = \hat{F}_0\, \hat{j}_a(X)\hat{F}_0 = F_0\, j_{\tau_a}(X)F_0.$$

Then by the strong Markov property (Theorem 8.7) it is clear that $(\hat{T}_a)_{a\geq 0}$ is a contractive semigroup of completely positive linear maps on $\mathcal{A}$. It is also clear that $(H, (\hat{F}_a)_{a\geq 0}, (\hat{j}_a)_{a\geq 0})$ is a dilation of $(\hat{T}_a)_{a\geq 0}$ into a weak Markov process.

In the case where $L_t = \int_0^t k_r(Z)dr$ for some positive element$Z$ in $\mathcal{Z}$ we shall compute the generator of $(\hat{T}_a)_{a\geq 0}$ and thereby get a non-commutative analogue of the usual perturbation of the Laplacian by multiplication by a strictly positive function.

Lemma 13.3– *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup of operators with generator A in a Banach space χ satisfying*

$$\|T_t\| \leq \alpha e^{-\varepsilon t}\ \forall\, t > 0$$

for some $\alpha, \varepsilon > 0$. Then $0 \in R(A)$ and

$$A^{-1} = -\int_0^\infty T_t\, dt.$$

Proof

Let $u \in$ Dom (A). Put $S = \int_0^\infty T_t dt$. Then $\|S\| \leq \varepsilon^{-1}$ and

$$\begin{aligned} SAu &= \int_0^\infty T_t\, Au\, dt = \int_0^\infty \frac{d}{dt} T_t\, u\, dt \\ &= \lim_{N\to\infty} T_N\, u - u = -u \quad (\text{for } \|T_N\, u\| \leq \alpha e^{-\varepsilon N}\|u\|). \end{aligned}$$

On the other hand

$$\begin{aligned} A\int_0^N T_t\, u\, dt &= \lim_{h\to 0} \int_0^N \frac{T_{t+h} - T_t}{h}\, u\, dt \\ &= \lim_{h\to 0} \int_0^N T_t \left(\frac{T_h - I}{h}\right) u\, dt \\ &= \int_0^N T_t\, A\, u\, dt = T_N\, u - u. \end{aligned}$$

Thus $A\int_0^\infty T_t\, u\, dt = -u$ as A is closed. In other words $ASu = -u$ for $u \in$ Dom(A). Since S is bounded and A is closed we have $ASu = -u$ for all $u \in \chi$. Thus $-S = A^{-1}$. ■

Lemma 13.4– *Let τ be a stop time and let u be in the range of $\mathbb{1}_{\tau<+\infty}$. Then*

$$\int_0^\infty s < u\, ,\, \tau(ds)u > = \int_0^\infty < u\, ,\, \mathbb{1}_{\tau>s} u > ds. \tag{13.7}$$

Proof

Let $G_s = < u\, ,\, \mathbb{1}_{\tau\leq s} u >$ and $\beta = \lim_{s\to+\infty} G_s = \|u\|^2$. Suppose that the left hand side of (13.7) is finite. Then

$$0 = \lim_{N\to+\infty} \int_N^\infty s\, dG_s \geq \lim_{N\to+\infty} N(\beta - G_N)$$

and integration by parts yields

$$\int_0^N s\, dG_s = -\int_0^N s\, d(\beta - G_s) = -N(\beta - G_N) + \int_0^N (\beta - G_s)\, ds.$$

Letting N tend to $+\infty$ we obtain (13.7).

Conversely let the right hand side of (13.7) be finite. Then

$$\infty > \int_0^\infty (\beta - G_s)\, ds = \sum_{n=0}^\infty \int_n^{n+1} (\beta - G_s)\, ds \geq \sum_{n=0}^\infty (\beta - G_{n+1}).$$

Hence $\lim_{N\to\infty} N(\beta - G_N) = 0$. Integration by parts yields

$$\int_0^N (\beta - G_s)\, ds = N(\beta - G_N) + \int_0^N s\, dG_s.$$

Letting N tend to $+\infty$ we obtain (13.7). ■

Lemma 13.5 – *Let $Z \in \mathcal{Z}, Z \geq 0$. Denote by $(T_t^Z)_{t\geq 0}$ the contraction semigroup with the Feyman-Kac perturbed generator $\mathcal{L} - L_Z$ where $\mathcal{L}$ is the generator of T_t. Suppose that $u \in H_0$ satisfies*

$$\int_0^\infty < u\, ,\, T_s^Z(I) u > ds < \infty.$$

Then u is in the range of $\mathbb{1}_{\tau_a<\infty}$ for every $a > 0$ and

$$\int_0^s s < u\, ,\, \tau_a(ds) u > ds < \infty.$$

Proof

We have

$$\begin{aligned}
&\int_0^\infty e^{-a} \int_0^\infty < u\, ,\, \mathbb{1}_{\tau_a > s} u > ds\, da \\
&= \int_0^\infty \int_0^\infty e^{-a} < u\, ,\, \mathbb{1}_{[0,a]}(L_s) u > da\, ds \\
&= \int_0^\infty < u\, ,\, e^{-L_s} u > ds \\
&= \int_0^\infty < u\, ,\, T_s^Z(I) u > ds < \infty
\end{aligned}$$

by hypothesis. Hence by Fubini's theorem

$$\int_0^\infty < u\, ,\, \mathbb{1}_{\tau_a > s} u > ds < \infty \quad \text{for almost all } a.$$

Since the integrand here is increasing in a it follows that $\int_0^\infty < u\, ,\, \mathbb{1}_{\tau_a > s} u > ds < \infty$ for every $a > 0$.

In particular $\lim_{s\to+\infty} < u\, ,\, \mathbb{1}_{\tau_a > s} u > = 0$ and hence u is in the range of $\mathbb{1}_{\tau_a < \infty}$ for every $a > 0$. Now by Lemma 13.4 the required result follows. ■

Corollary 13.6 – *In Lemma 13.5 let $Z \geq c > 0$ for some constant c. Then u is in the range of $\mathbb{1}_{\tau_a < \infty}$ and*

$$\int_0^\infty s < u\, ,\, \tau_a(ds) u > < \infty \quad \textit{for all} \quad a > 0, u \in H_0.$$

Proof

We have

$$\begin{aligned} \|T_s^2(I)\| &\leq \sup_{\|u\|=1} < u\, ,\, F_0 e^{-\int_0^s k_r(z)\, dr} F_0 u > \\ &\leq e^{-cs} \end{aligned}$$

and hence

$$\int_0^\infty < u\, ,\, T_s^Z(I) u > ds \leq c^{-1} \|u\|^2 < \infty. \qquad ■$$

Theorem 13.7 – *Let $\hat{\mathcal{L}}$ denote the generator of the semigroup $(\hat{T}_a)_{a\geq 0}$ associated with the additive cocycle $L_t = \int_0^t k_r(Z)\, dr$ where $Z \in \mathcal{Z}$, $Z \geq 0$. Let $u \in H_0$ satisfy*

$$\int_0^\infty < u\, ,\, T_s^Z(1) u > ds < \infty.$$

Then for any $\beta > 0, X \in$ $Dom(\mathcal{L})$, we have

$$< u\, ,\, -(\hat{\mathcal{L}} - \beta)^{-1}(X) u > = -\beta^{-1} < u\, ,\, X_u > - \beta^{-1} \int_0^\infty < u\, ,\, T_t^{\beta Z}(\mathcal{L}(X)) u > dt. \tag{13.8}$$

If there exists a positive constant c such that $Z \geq cI$ then

$$\hat{\mathcal{L}} = L_Z^{-1} \mathcal{L}.$$

Proof

By Lemma 13.5 and Dynkin's formula (Theorem 9.3) we have for $X \in \mathrm{Dom}\mathcal{L}$

$$< u\, ,\, (F_0 j_{\tau_a}(X) F_0 - j_0(X)) u > = \int_0^\infty < u\, ,\, F_0 \mathbb{1}_{[0,a]}(L_t) j_t(\mathcal{L}(X)) F_0 u > dt.$$

Multiplying both sides by $c^{-\beta a}$ and integrating with respect to a in $I\!R^+$ we get from an application of the Feynman-Kac formula

$$\begin{aligned} < u\, ,\, -(\hat{\mathcal{L}} - \beta)^{-1}(X) u > - \beta^{-1} < u\, ,\, X u > &= \beta^{-1} \int_0^\infty < u\, ,\, e^{-\beta L_t} j_t(\mathcal{L}(X)) u > dt \\ &= \beta{-}1 \int_0^\infty < u\, ,\, T_t^{\beta Z}(\mathcal{L}(X)) u > dt. \end{aligned}$$

which proves the first part.

If $Z \geq c > 0$ then by Corollary 13.6 the identity (13.8) holds for all $u \in H_0$. As in the proof of Corollary 13.6 one has

$$\begin{aligned} \|T_t^{\beta Z}(X)\| &= \|F_0 e^{\beta \int_0^t k_r(Z)\,dr} j_t(X) F_0\| \\ &\leq \|X\| e^{-\beta c t}. \end{aligned}$$

By Lemma 13.3 one has

$$\int_0^\infty T_t^{\beta Z}\,dt = -(\mathcal{L} - \beta L_Z)^{-1}.$$

Thus, for all $X \in \mathrm{Dom}(\mathcal{L})$

$$(\hat{\mathcal{L}} - \beta)^{-1}(X) = -\beta^{-1} X + \beta^{-1} (\mathcal{L} - \beta L_Z)^{-1} \mathcal{L}(X)$$

which implies

$$(\hat{\mathcal{L}} - \beta)^{-1}(X) = (\mathcal{L} - \beta L_Z)^{-1} L_Z(X) = (L_Z^{-1}\mathcal{L} - \beta)^{-1}(X).$$

In other words $\hat{\mathcal{L}} = L_Z^{-1}\mathcal{L}$. ■

When $\mathcal{A} = L^\infty(\mathbb{R}^d)$ and $\mathcal{L}$ is the generator of the standard Brownian motion process in $\mathbb{R}^d$ and Z is multiplication by a Borel measurable function φ satisfying $0 < a \leq \varphi(x) \leq b$, the construction yields the Markov process with generator $\frac{1}{2}\varphi(x)^{-1}\Delta$. The additive functional used in the construction is $\int_0^t \varphi(x + B_s)\,ds$.

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