FROM $(n+1)$-LEVEL ATOM CHAINS TO n-DIMENSIONAL NOISES

Stéphane ATTAL and Yan PAUTRAT

This article is dedicated to the memory of Paul-André MEYER

Abstract

In quantum physics, the state space of a countable chain of $(n+1)$-level atoms becomes, in the continuous field limit, a Fock space with multiplicity n. In a more functional analytic language, the continuous tensor product space over $I\!R^+$ of copies of the space $\mathbb{C}^{n+1}$ is the symmetric Fock space $\Gamma_s(L^2(I\!R^+;\mathbb{C}^n))$. In this article we focus on the probabilistic interpretations of these facts. We show that they correspond to the approximation of the n-dimensional normal martingales by means of obtuse random walks, that is, extremal random walks in $I\!R^n$ whose jumps take exactly $n+1$ different values. We show that these probabilistic approximations are carried by the convergence of the basic matrix basis $a^i_j(p)$ of $\otimes_{I\!N}\mathbb{C}^{n+1}$ to the usual creation, annihilation and gauge processes on the Fock space.

I. Introduction

In functional analysis, the tensor product of a family of Hilbert spaces indexed by a continuous set, is a well-understood notion (see the very complete book [Gui]) which leads to notions such as “Fock spaces” or “symmetric space associated to a measured space”.

A physical interpretation of those continuous tensor product spaces consists in considering them as the continuous field limit of a countable chain of quantum system state spaces (such as a spin chain, for example).

The interesting point in these constructions is that, for all $n \in I\!N$, the continuous tensor product space

$$\bigotimes_{I\!R^+} \mathbb{C}^{n+1}$$

is the symmetric Fock space $\Gamma_s(L^2(I\!R^+;\mathbb{C}^n))$. In a more physical language, the continuous field limit of the state space of a countable chain of $(n+1)$-level atoms is a Fock space with multiplicity n. A rigourous setting in which such an approximation is made true is developped in [At1].

Both the spaces $\otimes_{I\!N}\mathbb{C}^{n+1}$ and $\Gamma_s(L^2(I\!R^+;\mathbb{C}^n))$ admit natural probabilistic interpretations. Indeed, the Fock space $\Gamma(L^2(I\!R^+;\mathbb{C}^n))$ admits natural probabilistic interpretations in terms of n-dimensional normal martingales, such as n-dimensional Brownian motion, n-dimensional Poisson process, n-dimensional

Azéma martingales ... (cf [A-E] and [At2]). The aim of this article is to understand how the approximation of $\Gamma(L^2(\mathbb{R}^+;\mathbb{C}^n))$ by means of spaces $\otimes_{\mathbb{N}}\mathbb{C}^{n+1}$ can be interpreted in probabilistic terms.

The structure of the space $\otimes_{\mathbb{N}}\mathbb{C}^{(n+1)}$ suggests that we are dealing with random walks whose jumps are taking $(n+1)$ different values.

In this article we show that the key point of this approximation is the notion of *obtuse random walks*, developped in [A-E]. They are the centered and normalized random variables in $\mathbb{R}^n$ which take exactly $(n+1)$ different values.

These obtuse random variables are naturally associated to an algebraic object called *sesqui-symmetric 3-tensor* and the associated random walk satisfies a *discrete-time structure equation.* This structure equation allows us to represent the multiplication operators by this random walk in terms of some basic operators of $\otimes_{\mathbb{N}}\mathbb{C}^{n+1}$.

Considering the approximation of the Fock space $\Gamma(L^2(\mathbb{R}^+;\mathbb{C}^n))$ by means of spaces $\otimes_{\mathbb{N}}\mathbb{C}^{n+1}$, we obtain the approximation of a continuous-time normal martingale. The sesqui-symmetric 3-tensor Φ then converges to a so-called *doubly-symmetric 3-tensor* which is the key of the structure equation describing the probabilistic behaviour of that normal martingale (jumps, continuous and purely discontinuous parts...).

This article is organized in the following way: in section two we introduce the state space of the atom chain and the associated operators. In section three, we describe obtuse random walks in $\mathbb{R}^n$, their structure equations and their representations as operators on the state space of the atom chains. In section four we introduce Fock space and its quantum stochastic calculus, and the relation of these objects with the atom chains. In section five we describe structure equations for normal martingales and the information given by these equations in a special case. In section six we put together all of our tools and prove convergence in law of random walks to well-identified normal martingales. In section seven we review some explicit and illustrative examples.

II. The structure of the atom chain

We here introduce the mathematical structure and notations associated to the space $\otimes_{\mathbb{N}}\mathbb{C}^{n+1}$. As the reader will easily see, this only means choosing a particular basis for the vectors and for the operators on that space. The physical-like terminology that we here use time to time is not necessary for the sequel, it is just informative (though it is pertinent and really used in articles such as [A-P]).

Consider the space $\mathbb{C}^{n+1}$ in which we choose an orthonormal basis denoted by $\{\Omega, X^1, \ldots, X^n\}$. This space and this particular choice of an orthonormal basis physically represent either a particle with n excited states X^i and a ground state Ω, or a site which is either empty (Ω) or occupied by a type i particle (X^i). We often write X^0 for Ω when we need unified notations, but it is important in the sequel to distinguish one of the basis states.

Together with this basis of $\mathbb{C}^{n+1}$ we consider the following natural basis of $\mathcal{L}(\mathbb{C}^{n+1}) = M_{n+1}(\mathbb{C})$:

$$a^i_j X^k = \delta_{ki} X^j,$$

for all $i, j, k = 0, \ldots, n$. With these notations the operator a^0_j corresponds, up to a sign factor, to classical fermionic creation operator for the particle X^j; indeed, we have $a^0_j \Omega = X^j$ and $(a^0_j)^2 = 0$. The operator a^j_0 corresponds to its associated annihilation operator. The operator a^i_j exchanges a i-level state with a j-level state particle.

We now consider a chain of copies of this system, like a chain of $(n+1)$-level atoms. That is, we consider the Hilbert space

$$T\Phi = \bigotimes_{i \in \mathbb{N}} \mathbb{C}^{n+1}$$

made of a countable tensor product, indexed by $\mathbb{N}$, of copies of $\mathbb{C}^{n+1}$. By this we mean that a natural orthonormal basis of $T\Phi$ is described by the family

$$\{X_A; A \in \mathcal{P}_n\}$$

where

– $\mathcal{P}_n$ is the set of finite subset $A = \{(n_1, i_1), \ldots, (n_k, i_k)\}$ of $\mathbb{N} \times \{1, \ldots, n\}$ such that the n_i's are two by two different. Another way to describe the set $\mathcal{P}_n$ is to identify it to the set of sequences $(A_k)_{k \in \mathbb{N}}$ with values in $\{0, \ldots, n\}$, but taking only finitely many times a value different from 0.

– X_A denotes the vector

$$\Omega \otimes \ldots \otimes \Omega \otimes X^{i_1} \otimes \Omega \otimes \ldots \otimes \Omega \otimes X^{i_2} \otimes \ldots$$

of $T\Phi$, where X^{i_1} appears in the copy number n_1, X^{i_2} appears in the copy n_2,... When A is seen as a sequence $(A_k)_{k \in \mathbb{N}}$ as above, then X_A is advantageously written $\otimes_k X_{A_k}$.

The physical meaning of this basis is easy to understand: we have a chain of sites, indexed by $\mathbb{N}$; on each site there is an atom in the ground state or an atom at energy level 1... The above basis vector X_A specifies that there is an atom at level i_1 in the site n_1, an atom at level i_2 in the site $n_2 \ldots$, all the other sites being at the ground state. The space $T\Phi$ is what we shall call the $(n+1)$-*level atom chain*.

We denote by $a^i_j(k)$ the natural ampliation of the operator a^i_j to $T\Phi$ which acts as a^i_j on the copy number k of $\mathbb{C}^{n+1}$ and as the identity on the other copies.

Note, for information only, that the operators $a^i_j(k)$ form a basis of the algebra $\mathcal{B}(T\Phi)$ of bounded operators on $T\Phi$. That is, the von Neumann algebra generated by the $a^i_j(k)$, $i, j = 0, \ldots, n$, $k \in \mathbb{N}$, is the whole of $\mathcal{B}(T\Phi)$ (for $T\Phi$ admits no subspace which is non trivial and invariant under this algebra).

III. Obtuse random walks in $\mathbb{R}^n$

We now abandon for a while this structure in order to concentrate on the probabilistic and algebraic structure of the obtuse random variables. The space $T\Phi$ will come back naturally when describing the obtuse random walks.

Let X be a random variable in $\mathbb{R}^n$ which takes exactly $n+1$ different values $v_1, \ldots, v_{n+1}$ with respective probability $\alpha_1, \ldots, \alpha_{n+1}$ (all different from 0 by hypothesis). We assume, for simplicity, that X is defined on its canonical space $(A, \mathcal{A}, P)$, that is, $A = \{1, \ldots, n+1\}$, $\mathcal{A}$ is the σ-field of subsets of A, the probability measure P is given by $P(\{i\}) = \alpha_i$ and X is given by $X(\{i\}) = v_i$, for all $i = 1, \ldots, n+1$.

Such a random variable X is called *centered and normalized* if $\mathbb{E}[X] = 0$ and $\mathrm{Cov}(X) = I$.

A family of elements $v_1, \ldots, v_{n+1}$ of $\mathbb{R}^n$ is called an *obtuse system* if

$$< v_i\,,\, v_j > = -1$$

for all $i \neq j$.

We consider the coordinates $X^1, \ldots, X^n$ of X in the canonical basis of $\mathbb{R}^n$, together with the random variable Ω on $(A, \mathcal{A}, P)$ which is deterministic and always equal to 1.

We put $\widetilde{X}^i$ to be the random variable $\widetilde{X}^i(j) = \sqrt{\alpha_j}\, X^i(j)$ and $\widetilde{\Omega}(j) = \sqrt{\alpha_j}$. For any element $v = (a_1, \ldots, a_n)$ of $\mathbb{R}^n$ we put $\widehat{v} = (1, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$.

The following proposition is rather straightforward and left to the reader.

Proposition 1. – *The following assertions are equivalent.*

i) X is centered and normalized.

ii) The $(n+1) \times (n+1)$-matrix $(\widetilde{\Omega}, \widetilde{X}^1, \ldots, \widetilde{X}^n)$ is unitary.

iii) The $(n+1) \times (n+1)$-matrix $(\sqrt{\alpha_1}\,\widehat{v}_1, \ldots, \sqrt{\alpha_{n+1}}\,\widehat{v}_{n+1})$ is unitary.

iv) The family $v_1, \ldots, v_{n+1}$ is an obtuse system of $\mathbb{R}^n$ and

$$\alpha_i = \frac{1}{1 + ||v_i||^2}.$$

Let T be a 3-tensor in $\mathbb{R}^n$, that is, a linear mapping from $\mathbb{R}^n$ to $M_n(\mathbb{R})$. We write T^{ij}_k for the coefficients of T in the canonical basis of $\mathbb{R}^n$, that is,

$$(T(x))_{i,j} = \sum_{k=1}^n T^{ij}_k x_k.$$

Such a 3-tensor T is called *sesqui-symmetric* if

i) $(i,j,k) \longmapsto T^{ij}_k$ is symmetric

and

ii) $(i,j,l,m) \longmapsto \sum_k T^{ij}_k T^{lm}_k + \delta_{ij}\delta_{lm}$ is symmetric.

Theorem 2 – *If X is a centered and normalized random variable in $\mathbb{R}^n$, taking exactly $n+1$ values, then there exists a sesqui-symmetric 3-tensor T such that*

$$X \otimes X = I + T(X). \qquad (1)$$

Proof

By Proposition 1, the matrix $(\sqrt{\alpha_1}\,\widehat{v}_1,\dots,\sqrt{\alpha_{n+1}}\,\widehat{v}_{n+1})$ is unitary. In particular the matrix $(\widehat{v}_1,\dots,\widehat{v}_{n+1})$ is invertible and so is its adjoint matrix. But the latter is the matrix whose columns are the values of the random variables $\Omega, X_1,\dots,X_n$. As a consequence, these $n+1$ random variables are linearly independent. They thus form a basis of $L^2(A,\mathcal{A},P)$ for it is a $n+1$ dimensional space.

The random variable X^iX^j belongs to $L^2(A,\mathcal{A},P)$ and can thus be written as

$$X^iX^j=\sum_{k=0}^n T_k^{ij}X^k,$$

for some real coefficients T_k^{ij}, $k=0,\dots,n$, $i,j=1,\dots n$, where X^0 denotes Ω. The fact that $\mathbb{E}[X^k]=0$ and $\mathbb{E}[X^iX^j]=\delta_{ij}$ implies $T_0^{ij}=\delta_{ij}$. This gives the representation (1).

The fact that the 3-tensor T associated to the above coefficients T_k^{ij}, $i,j,k=1,\dots n$, is sesqui-symmetric is an easy consequence of the fact that the expressions X^iX^j are symmetric in i,j and $X^i(X^jX^k)=(X^iX^j)X^k$ for all i,j,k. We leave it to the reader. ■

There is actually a natural bijection between the set of sesqui-symmetric 3-tensors and the set of obtuse random variables. This is a result obtained in [A-E], Theorem 2, pp. 268-272, which is far from obvious but which we shall not really need here.

Theorem 3. – *The formulas*

$$S=\{x\in\mathbb{R}^n;\, x\otimes x=I+T(x)\}.$$

and

$$T(x)=\sum_{y\in S}p_y<y\,,\,x>y\otimes y,$$

where $p_x=1/(1+||x||^2)$, *define a bijection between the set of sesqui-symmetric 3-tensor* T *on* $\mathbb{R}^n$ *and the set of obtuse systems* S *in* $\mathbb{R}^n$. ■

Now we wish to consider the random walks which are induced by obtuse systems. That is, on the probability space $(A^{\mathbb{N}},\mathcal{A}^{\otimes\mathbb{N}},P^{\otimes\mathbb{N}})$, we consider a sequence $(X(n))_{n\in\mathbb{N}}$ of independent random variables with the same law as a given centered normalized random variable X.

Recalling the notations of section II, for any $A\in\mathcal{P}_n$, we define the random variable

$$X_A=\prod_{(p,i)\in A}X^i(p)$$

with the convention

$$X_\emptyset=\mathbb{1}.$$

Proposition 4. – *The family $\{X_A; A \in \mathcal{P}_n\}$ is an orthonormal basis of the space $L^2(A^{I\!N}, \mathcal{A}^{\otimes I\!N}, P^{\otimes I\!N})$.*

Proof

For any $A, B \in \mathcal{P}_n$ we have

$$< X_A \,,\, X_B > = I\!E[X_A X_B] = I\!E[X_{A\Delta B}]I\!E[X^2_{A\cap B}]$$

by the independence of the $X(p)$. For the same reason, the first term $I\!E[X_{A\Delta B}]$ gives 0 unless $A\Delta B = \emptyset$, that is $A = B$. The second term $I\!E[X^2_{A\cap B}]$ is then equal to $\prod_{(p,i)\in A} I\!E[X^i(p)^2] = 1$. This proves the orthonormal character of the family $\{X_A; A \in \mathcal{P}_n\}$.

Let us now prove that it generates a dense subspace of $L^2(A^{I\!N}, \mathcal{A}^{\otimes I\!N}, P^{\otimes I\!N})$. Had we considered random walks indexed by $\{0, \dots, N\}$ instead of $I\!N$, the X_A, $A \subset \{0, \dots, N\}$ would have formed an orthonormal basis of $L^2(A^N, \mathcal{A}^{\otimes N}, P^{\otimes N})$, for their dimensions are equal. Now a general element f of $L^2(A^{I\!N}, \mathcal{A}^{\otimes I\!N}, P^{\otimes I\!N})$ can be easily approached by a sequence $(f_N)_N$ such that $f_N \in L^2(A^N, \mathcal{A}^{\otimes N}, P^{\otimes N})$, for all N, by taking conditional expectations on the trajectories of X up to time N. ■

For every obtuse random variable X, we thus obtain a Hilbert space

$$T\!\Phi(X) = L^2(A^{I\!N}, \mathcal{A}^{\otimes I\!N}, P^{\otimes I\!N}),$$

with a natural orthonormal basis $\{X_A; A \in \mathcal{P}_n\}$ which emphasizes the independence of the $X(p)$'s. In particular there is a natural isomorphism between all the spaces $T\!\Phi(X)$ which consists in identifying the associated bases. In the same way, all these canonical spaces $T\!\Phi(X)$ of obtuse random walks are naturally isomorphic to the atom chain $T\!\Phi$ of previous section (again by identifying their natural orthonormal bases).

Of course this identification of Hilbert spaces does not mean much for the moment: in particular, it loses all the probabilistic properties of the random variables $X^i(p)$, be it individual (the law) or collective (probabilistic independence) properties.

The only way to recover the full probabilistic information on $X^i(p)$ in the Hilbert space formalism associated to $T\!\Phi$ is to consider the *multiplication operator* by $X^i(p)$ instead of the Hilbert space element $X^i(p)$. Indeed, if we know the representation in $T\!\Phi$ of the operator $\mathcal{M}_{X^i(p)}$ of multiplication by $X^i(p)$ on $T\!\Phi(X)$, we know everything about the random variable $X^i(p)$ and its relation with other random variables. The above idea is what makes quantum probabilistic tools relevent for the study of classical probability; following this idea, the next theorem is one of the keys of this article. It is what allows us to translate probabilistic properties into theoretic language, showing that *all* the obtuse random walks in $I\!R^n$ can be represented in a single space $T\!\Phi$ with very economical means: linear combinations of the operators $a^i_j(p)$.

Theorem 5. – *Let X be an obtuse random variable, let $(X(p))_{p\in\mathbb{N}}$ be the associated random walk on the canonical space $T\Phi(X)$. Let T be the sesqui-symmetric 3-tensor associated to X. Let U be the natural unitary isomorphism from $T\Phi(X)$ to $T\Phi$. Then, for all $p\in\mathbb{N}, i=\{1,\dots,n\}$ we have*

$$U\mathcal{M}_{X_i(p)}U^* = a_i^0(p) + a_0^i(p) + \sum_{j,l=1}^n T_i^{jl}\, a_l^j(p).$$

Proof

It suffices to compute the action of $\mathcal{M}_{X_i(p)}$ on the basis elements X_A, $A\in\mathcal{P}_n$. Denote by "$(p,.)\notin A$" the claim "for no i does (p,i) belong to A". Then, by Theorem 1, there exists a sesquisymmetric tensor T on $\mathbb{R}^n$ such that

$$\begin{aligned}
X_i(p)X_A &= \mathbb{1}_{(p,\cdot)\notin A}X_i(p)X_A + \sum_{j=1}^n \mathbb{1}_{(p,j)\in A}X_i(p)X_A \\
&= \mathbb{1}_{(p,\cdot)\notin A}X_{A\cup\{(p,i)\}} + \sum_{j=1}^n \mathbb{1}_{(p,j)\in A}X_i(p)X_j(p)X_{A\setminus\{(p,j)\}} \\
&= \mathbb{1}_{(p,\cdot)\notin A}X_{A\cup\{(p,i)\}} + \sum_{j=1}^n \mathbb{1}_{(p,j)\in A}\left(\delta_{ij} + \sum_l T_l^{ij}X_l(p)\right)X_{A\setminus\{(p,j)\}} \\
&= \mathbb{1}_{(p,\cdot)\notin A}X_{A\cup\{(p,i)\}} + \mathbb{1}_{(p,i)\in A}X_{A\setminus(p,i)} + \sum_{j=1}^n\sum_{l=1}^n \mathbb{1}_{(p,j)\in A}T_l^{ij}X_{A\setminus\{(p,j)\}\cup\{(p,i)\}}
\end{aligned}$$

and we immediately recognize the formula for

$$a_i^0(p)X_A + a_0^i(p)X_A + \sum_{p,l} T_l^{ij}a_l^j(p)X_A.$$

■

Let us now return to quantum probabilistic structures and describe the Fock space structure and its approximation by the atom chain.

IV. Approximation of the Fock space by atom chains

We recall the structure of the bosonic Fock space Φ and its basic structure (see *e.g.* [At3] or [Pau] for details).

Let $\Phi = \Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}^n))$ be the symmetric (bosonic) Fock space over the space $L^2(\mathbb{R}^+;\mathbb{C}^n)$. We shall here give a very efficient presentation of that space, the so-called *Guichardet interpretation* of the Fock space.

Let $I=\{1,\dots,n\}$ and $\mathcal{P}$ be the set of finite subsets $\{(s_1,i_1),\dots,(s_k,i_k)\}$ of $\mathbb{R}^+\times I$ such that the s_i are mutually distinct. Then $\mathcal{P}=\cup_k\mathcal{P}(k)$ where $\mathcal{P}(k)$ is the set of k-elements subsets of $\mathbb{R}^+\times I$. By ordering the $\mathbb{R}^+$-part of the elements of $\sigma\in\mathcal{P}(k)$, the set $\mathcal{P}(k)$ can be identified to the increasing simplex $\Sigma_k=\{0<t_1<\dots<t_k\}\times I$ of $\mathbb{R}^k\times I$. Thus $\mathcal{P}(k)$ inherits a measured space

structure from the product of Lebesgue measure on $\mathbb{R}^k$ and the counting measure on I. This also gives a measure structure on $\mathcal{P}$ if we specify that on $\mathcal{P}_0 = \{\emptyset\}$ we put the measure $\delta_\emptyset$. Elements of $\mathcal{P}$ are often denoted by σ, the measure on $\mathcal{P}$ is denoted by $d\sigma$. The σ-field obtained this way on $\mathcal{P}$ is denoted $\mathcal{F}$.

We identify any element $\sigma \in \mathcal{P}$ with a family $\{\sigma_1, \dots, \sigma_n\}$ of (two by two disjoint) subsets of $\mathbb{R}^+$ where

$$\sigma_i = \{s \in \mathbb{R}^+; (s,i) \in \sigma\}.$$

For a $s \in \mathbb{R}^+$ we denote by $\{s\}_i$ the element $\sigma = \{\emptyset, \dots, \emptyset, \{s\}, \emptyset, \dots \emptyset\}$ of $\mathcal{P}$ (where $\{s\}$ is at the i-th position.

The *Fock space* Φ is the space $L^2(\mathcal{P}, \mathcal{F}, d\sigma)$. An element f of Φ is thus a measurable function $f : \mathcal{P} \to \mathbb{C}$ such that

$$||f||^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma < \infty \ .$$

One can define, in the same way, $\mathcal{P}_{[a,b]}$ and $\Phi_{[a,b]}$ by replacing $\mathbb{R}^+$ with $[a,b] \subset \mathbb{R}^+$. There is a natural isomorphism between $\Phi_{[0,t]} \otimes \Phi_{[t,+\infty[}$ given by $h \otimes g \mapsto f$ where $f(\sigma) = h(\sigma \cap [0,t]) \, g(\sigma \cap (t, +\infty[)$. Define also $\mathbb{1}$ to be the *vacuum vector*, that is, $\mathbb{1}(\sigma) = \delta_\emptyset(\sigma)$.

Define $\chi^i_t \in \Phi$ by

$$\chi_t(\sigma) = \begin{cases} \mathbb{1}_{[0,t]}(s) & \text{if } \ \sigma = \{s\}_i \\ 0 & \text{otherwise.} \end{cases}$$

Then χ_t belongs to $\Phi_{[0,t]}$. We even have $\chi^i_t - \chi^i_s \in \Phi_{[s,t]}$ for all $s \le t$. This last property allows to define a so-called *Itô integral* on Φ. Indeed, let $(g^i_t)_{t \ge 0}$ be families in Φ, for $i = 1, \dots, n$, such that

i) $t \mapsto \|g^i_t\|$ is measurable,

ii) $g^i_t \in \Phi_{[0,t]}$ for all t,

iii) $\int_0^\infty \|g^i_t\|^2 \, dt < \infty$,

then one defines $\sum_i \int_0^\infty g^i_t \, d\chi^i_t$ to be the limit in Φ of

$$\sum_{i=1}^n \sum_{j=0}^\infty \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} P_{t_j} g^i_s \, ds \otimes \left(\chi^i_{t_{j+1}} - \chi^i_{t_j} \right) \tag{3}$$

where P_t is the orthogonal projection onto $\Phi_{[0,t]}$ and $\{t_j, \ j \in \mathbb{N}\}$ is a partition of $\mathbb{R}^+$ which is understood to be refining and to have its diameter tending to 0. Note that $\frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} P_{t_j} g_s \, ds$ belongs to $\Phi_{[0,t_j]}$, which explains the tensor product symbol in (3).

We get that $\sum_i \int_0^\infty g^i_t \, d\chi^i_t$ is an element of Φ with

$$\left\| \sum_i \int_0^\infty g_t \, d\chi_t \right\|^2 = \sum_i \int_0^\infty \left| g^i_t \right|^2 \, dt \ . \tag{4}$$

Let $f \in L^2(\mathcal{P}_n)$; one can easily define the *iterated Itô integral* on Φ.

$$I_n(f) = \int_{\mathcal{P}_n} f(\sigma) \, d\chi^{i_1}_{t_1} \dots d\chi^{i_n}_{t_n}$$

by iterating the definition of the Itô integral. We use the following notation:

$$I_n(f) = \int_{\mathcal{P}_n} f(\sigma)\ d\chi_\sigma$$

which we extend, in an obvious way, to any $f \in \Phi$. We then have the following important representation.

Theorem 6. – *Any element f of Φ admits an* abstract chaotic representation

$$f = \int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma$$

with

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2\ d\sigma$$

and an abstract predictable representation

$$f = f(\emptyset)\mathbb{1} + \sum_i \int_0^\infty D_t^i f\ d\chi_t^i$$

with

$$\|f\|^2 = |f(\emptyset)|^2 + \sum_i \int_0^\infty \|D_s^i f\|^2\ ds$$

where $[D_s^i f](\sigma) = f(\sigma \cup \{s\}_i)\mathbb{1}_{\sigma \subset [0,s[}$.

Let us now recall the definitions of the basic noise operators $a_j^i(t)$, $i, j = 0, \ldots, n$, on Φ. They are respectively defined by

$$\begin{aligned}
[a_i^0(t)f](\sigma) &= \sum_{s \in \sigma_i \cap [0,t]} f(\sigma \setminus \{s\}_i), \\
[a_0^i f](\sigma) &= \int_0^t f(\sigma \cup \{s\}_i)\ ds, \\
[a_j^i f](\sigma) &= \sum_{s \in \sigma_i \cap [0,t]} f(\sigma \setminus \{s\}_i \cup \{s\}_j)
\end{aligned}$$

for $i, j \neq 0$ and

$$a_0^0(t) = tI.$$

There is a good common domain to all these operators, namely

$$\mathcal{D} = \left\{ f \in \Phi\ ;\ \int_{\mathcal{P}} |\sigma|\ |f(\sigma)|^2\ d\sigma < \infty \right\}.$$

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_p < \cdots\}$ be a partition of $\mathbb{R}^+$ and $\delta(\mathcal{S}) = \sup_i |t_{i+1} - t_i|$ be the diameter of $\mathcal{S}$. For fixed $\mathcal{S}$, define $\Phi_p = \Phi_{[t_p, t_{p+1}]}$, $i \in \mathbb{N}$. We then have $\Phi \simeq \otimes_{p \in \mathbb{N}} \Phi_p$ (with respect to the stabilizing sequence $(\mathbb{1})_{p \in \mathbb{N}}$).

For all $p\in\mathbb{N}$, define for $i,j\neq 0$

$$
\begin{aligned}
X^i(p) &= \frac{\chi^i_{t_{p+1}}-\chi^i_{t_p}}{\sqrt{t_{p+1}-t_p}}\in\Phi_p\ ,\\
a^i_0(p) &= \frac{a^i_0(t_{p+1})-a^i_0(t_p)}{\sqrt{t_{p+1}-t_p}}P_{1]}\ ,\\
a^i_j(p) &= P_{1]}\big(a^i_j(t_{p+1})-a^i_j(t_p)\big)P_{1]}\ ,\\
a^0_j(p) &= P_{1]}\frac{a^0_j(t_{p+1})-a^0_j(t_p)}{\sqrt{t_{p+1}-t_p}}\ ,
\end{aligned}
$$

where $P_{1]}$ is the orthogonal projection onto $L^2(\mathcal{P}_1)$ and where the above definition of $a^0_i(p)$ is understood to be valid on Φ_p only, with $a^0_i(p)$ being the identity operator I on the others Φ_q's (the same is automatically true for a^i_0, a^i_j). We put $a^0_0(p)=I$.

Proposition 7. – *We have*

$$
\begin{cases} a^i_0(p)X^j(p)=\delta_{ij}\mathbb{1}\\ a^i_0\mathbb{1}=0\end{cases}
$$

$$
\begin{cases} a^i_j(p)X^k(p)=\delta_{ik}X^j(p)\\ a^i_j\mathbb{1}=0\end{cases}
$$

$$
\begin{cases} a^0_j(p)X^i(p)=0\\ a^0_j(p)\mathbb{1}=X^j(p).\end{cases}
$$

■

Thus the action of the operators a^i_j on the $X^i(p)$ is similar to the action of the corresponding operators on the atom chain of section two. We are now going to construct the atom chain inside Φ. We are still given a fixed partition $\mathcal{S}$. Define $T\Phi(\mathcal{S})$ to be the space of vectors $f\in\Phi$ which are of the form

$$
f=\sum_{A\in\mathcal{P}_N} f(A)X_A
$$

(with $\|f\|^2=\sum_{A\in\mathcal{P}_N}|f(A)|^2<\infty$).

The space $T\Phi(\mathcal{S})$ is thus clearly identifiable to the atom chain $T\Phi$; the operators $a^i_j(p)$ act on $T\Phi(\mathcal{S})$ exactly in the same way as the corresponding operators on $T\Phi$. We have completely embedded the toy Fock space into the Fock space. Let $\mathcal{S}=\{0=t_0<t_1<\cdots<t_p<\cdots\}$ be a fixed partition of $\mathbb{R}^+$. The space $T\Phi(\mathcal{S})$ is a closed subspace of Φ. We denote by $P_{\mathcal{S}}$ the operator of orthogonal projection from Φ onto $T\Phi(\mathcal{S})$.

We are now going to prove that the Fock space Φ and its basic operators $a^i_j(t)$ can be approached by the toy Fock spaces $T\Phi(\mathcal{S})$ and their basic operators $a^i_j(p)$.

We are given a sequence $(\mathcal{S}_p)_{p\in\mathbb{N}}$ of partitions which are getting finer and finer and whose diameter $\delta(\mathcal{S}_p)$ tends to 0 when p tends to $+\infty$. Let $T\Phi(p)=T\Phi(\mathcal{S}_p)$ and let P_p be the orthogonal projector onto $T\Phi(\mathcal{S}_p)$, for all $p\in\mathbb{N}$.

Theorem 8.–

i) For every $f\in\Phi$ there exists a sequence $(f_p)_{p\in\mathbb{N}}$ such that $f_p\in\mathrm{T}\Phi(p)$, for all $p\in\mathbb{N}$, and $(f_p)_{p\in\mathbb{N}}$ converges to f in Φ.

ii) For all i,j let

$$\varepsilon_{ij}=\frac{1}{2}(\delta_{0i}+\delta_{0j}).$$

If $\mathcal{S}_p=\{0=t_0^p<t_1^p<\cdots<t_k^p<\cdots\}$, then for all $t\in\mathbb{R}^+$, the operators

$$\sum_{k;t_k^p\le t}(t_{k+1}^p-t_k^p)^{\varepsilon_{ij}}a_j^i(k)$$

converge strongly on $\mathcal{D}$ to $a_j^i(t)$.

Proof

i) As the $\mathcal{S}_p$ are refining then the $(P_p)_p$ form an increasing family of orthogonal projection in Φ. Let $P_\infty=\vee_p P_p$. Clearly, for all $s\le t$, all i we have that $\chi_t^i-\chi_s^i$ belongs to $\mathrm{Ran}P_\infty$. But by the construction of the Itô integral and by Theorem 5, we have that the $\chi_t^i-\chi_s^i$ generate Φ. Thus $P_\infty=I$. Consequently if $f\in\Phi$, the sequence $f_p=P_pf$ satisfies the statements.

ii) The convergence of $\sum_{k,t_k^p\le t}(t_{k+1}^p-t_k^p)^{\varepsilon_{ij}}a_j^i(k)$ to $a_j^i(t)$ is clear from the definitions when $i\ne 0$. Let us check the case of a_i^0. We have, for $f\in\mathcal{D}$

$$\Big[\sum_{k;t_k^p\le t}\sqrt{t_{k+1}^p-t_k^p}\,a_i^0(k)f\Big](\sigma)=\sum_{k;t_k^p\le t}\mathbb{1}_{|\sigma\cap[t_k^p,t_{k+1}^p]|=1}\sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}f(\sigma\setminus\{s\}).$$

Put $t^p=\inf\left\{t_k^p\in\mathcal{S}_p\ ;t_k^p\ge t\right\}$. We have

$$\begin{aligned}
&\Big\|\sum_{k;t_k^p\le t}\sqrt{t_{k+1}^p-t_k^p}\,a_i^0(k)-a_i^0(t)f\Big\|^2\\
&=\int_{\mathcal{P}}\Big|\sum_{k;t_k^p\le t}\mathbb{1}_{|\sigma\cap[t_k^p,t_{k+1}^p]|=1}\sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}f(\sigma\setminus\{s\})-\sum_{s\in\sigma\cap[0,t]}f(\sigma\setminus\{s\})\Big|^2\,d\sigma\\
&\le 2\int_{\mathcal{P}}\Big|\sum_{s\in\sigma\cap[t,t^p]}f(\sigma\setminus\{s\})\Big|^2\,d\sigma\\
&\qquad+2\int_{\mathcal{P}}\Big|\sum_{k;t_k^p\le t}\mathbb{1}_{|\sigma\cap[t_k^p,t_{k+1}^p]|\ge 2}\times\sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}f(\sigma\setminus\{s\})\Big|^2\,d\sigma.
\end{aligned}$$

For any fixed σ, the terms inside each of the integrals above converge to 0 when p tends to $+\infty$. Furthermore we have, for large enough p ,

$$\begin{aligned}
\int_{\mathcal{P}}\Big|\sum_{s\in\sigma\cap[t,t^p]}f(\sigma\setminus\{s\})\Big|^2\,d\sigma &\le\int_{\mathcal{P}}|\sigma|\sum_{\substack{s\in\sigma\\ s\le t+1}}|f(\sigma\setminus\{s\})|^2\,d\sigma\\
&=\int_0^{t+1}\int_{\mathcal{P}}(|\sigma|+1)|f(\sigma)|^2\,d\sigma\,ds\\
&\le(t+1)\int_{\mathcal{P}}(|\sigma|+1)|f(\sigma)|^2\,d\sigma
\end{aligned}$$

which is finite for $f\in\mathcal{D}$;

$$\begin{aligned}
\int_{\mathcal P}\Big|\sum_{k;t_k^p\le t}\mathbb{1}_{|\sigma\cap[t_k^p,t_{k+1}^p]|\ge 2}\sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}f(\sigma\setminus\{s\})\Big|^2\,d\sigma&\\
\le\int_{\mathcal P}\Big(\sum_{k;t_k^p\le t}\mathbb{1}_{|\sigma\cap[t_k^p,t_{k+1}^p]|\ge 2}\Big|\sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}f(\sigma\setminus\{s\})\Big|\Big)^2\,d\sigma&\\
\le\int_{\mathcal P}\Big(\sum_{k;t_k^p\le t}\ \sum_{s\in\sigma\cap[t_k^p,t_{k+1}^p]}|f(\sigma\setminus\{s\})|\Big)^2\,d\sigma&\\
=\int_{\mathcal P}\Big(\sum_{\substack{s\in\sigma\\ s\le t^p}}|f(\sigma\setminus\{s\})|\Big)^2\,d\sigma&\\
=\int_{\mathcal P}|\sigma|\sum_{\substack{s\in\sigma\\ s\le t^p}}|f(\sigma\setminus\{s\})|^2\,d\sigma&\\
\le(t+1)\int_{\mathcal P}(|\sigma|+1)\big|f(\sigma)\big|^2\,d\sigma&
\end{aligned}$$

in the same way as above. So we can apply Lebesgue's theorem. This proves *ii)*.

■

V. Multidimensional structure equations

Let us recall some basic facts about normal martingales in $\mathbb{R}^n$; except for Theorem 13, all the statements in this section are taken from [A-E].

In the same way as the Fock space $\Phi=\Gamma(L^2(\mathbb{R}^+;\mathbb{C}))$ admits probabilistic interpretations in terms of one-dimensional normal martingales (see [At3]), the multiple Fock space $\Phi=\Gamma(L^2(\mathbb{R}^+;\mathbb{C}^n))$ admits probabilistic interpretations in terms of multidimensional normal martingales. The point here is that the extension of the notion of normal martingale, structure equation...to the multidimensional case is not so immediate. Some interesting algebraic structures appear.

A martingale $X=(X^1,\dots,X^n)$ with values in $\mathbb{R}^n$ is called *normal* if $X_0=0$ and if, for all i and j, the process $X_t^iX_t^j-\delta_{ij}t$ is a martingale. This is equivalent to saying that

$$\langle X^i,X^j\rangle_t=\delta_{ij}t$$

for all $t\in\mathbb{R}^+$, or else this is equivalent to saying that the process

$$[X^i,X^j]_t-\delta_{ij}t$$

is a martingale.

A normal martingale $X=(X^1\dots X^n)$ in $\mathbb{R}^n$ is said to *satisfy a structure equation* if each of the martingales $[X^i,X^j]_t-\delta_{ij}t$ is a stochastic integral with respect to X:

$$[X^i,X^j]_t=\delta_{ij}t+\sum_{k=1}^n\int_0^t T_k^{ij}(s)\,dX_s^k$$

where the T_k^{ij} are predictable processes.

Any family $\left\{A_k^{ij}\ ;\ i,j,k \in \{1\dots n\}\right\}$ of real numbers is identified to a 3-tensor, that is, a linear map A from $\mathbb{R}^n$ to $\mathbb{R}^n \otimes \mathbb{R}^n$ by

$$(Ax)_{ij} = \sum_{k=1}^n A_k^{ij} x_k\ .$$

Such a family is said to be *diagonalizable in some orthonormal basis* if there exists an orthonormal basis $\{e^1 \dots e^n\}$ of $\mathbb{R}^n$ for which

$$Ae^k = \lambda_k e^k \otimes e^k$$

for all $k = 1\dots n$ and for some *eigenvalues* $\lambda_1 \dots \lambda_n \in \mathbb{R}$.

A family $\left\{A_k^{ij}\ ;\ i,j,k \in \{1\dots n\}\right\}$ is called *doubly symmetric* if

i) $(i,j,k) \mapsto A_k^{ij}$ is symmetric on $\{1\dots n\}^3$ and

ii) $(i,j,i',j') \mapsto \sum_{k=1}^n A_k^{ij} A_k^{i'j'}$ is symmetric on $\{1\dots n\}^4$.

Theorem 9. – *For a family $\left\{A_k^{ij}\ ;\ i,j,k \in \{1\dots n\}\right\}$ of real numbers, the following assertions are equivalent.*

i) A is doubly symmetric.

ii) A is diagonalizable in some orthonormal basis.

This means that the condition of being doubly symmetric is the exact extension to 3-tensors of the symmetry property for matrices (2-tensors): it is the necessary and sufficient condition for being diagonalisable in some orthonormal basis.

A family $\{x^1 \dots x^k\}$ of elements of $\mathbb{R}$ is called *orthogonal family* if the x^i are all different from 0 and are two by two orthogonal.

Theorem 10. – *There is a bijection between the doubly symmetric families A of $\mathbb{R}^n$ and the orthogonal families Σ which is given by*

$$Af = \sum_{x\in\Sigma} \frac{1}{\|x\|^2} \langle x, f\rangle\, x \otimes x$$

and

$$\Sigma = \{x \in \mathbb{R}^n \setminus \{0\}\ ;\ Ax = x \otimes x\}\,.$$

These algebraic preliminaries are used to determine the behaviour of the multidimensional normal martingales.

Theorem 11. – *Let X be a normal martingale in $\mathbb{R}^n$ satisfying a structure equation*

$$[X^i, X^j]_t = \delta_{ij} t + \sum_{k=1}^n \int_0^t T_k^{ij}(s)\ dX_s^k.$$

Then for a.a. (t,ω) *the family* $\{T_k^{ij}(s,\omega)\ ;\ i,j,k=1\ldots n\}$ *is doubly symmetric. If* $\Sigma_t(\omega)$ *is its associated orthogonal system and if* $\pi_t(\omega)$ *denotes the orthogonal projection onto* $(\Sigma_t(\omega))^\perp$, *then the continuous part of* X *is given by*

$$X_t^{c,i} = \sum_{j=1}^n \int_0^t \pi_s^{ij}\ dX_s^j;$$

the jumps of X *happen only at totally inaccessible times and they satisfy*

$$\Delta X_t(\omega) \in \Sigma_t(\omega)\ .$$

We can now study a basic example. The simplest case occurs when T is constant in t. Contrarily to the unidimensional case, this situation is already rather rich.

Proposition 12. – *Let* T *be a doubly symmetric family on* $I\!R^n$. *Let* Σ *be its associated orthogonal system. Let* B *be a Brownian motion with values in the Euclidian space* $\Sigma^\perp$. *For each* $x \in \Sigma$, *let* N^x *be a Poisson process with intensity* $\|x\|^{-2}$. *We assume* B *and all the* N^x *to be independent. Then the martingale*

$$X_t = B_t + \sum_{x\in\Sigma}(N_t^x - \|x\|_t^{-2})x$$

satisfies the constant coefficient structure equation

$$[X^i, X^j]_t = \delta_{ij}t + \sum_{k=1}^n T_k^{ij}\ X_t^k\ .$$

Conversely, every normal martingale which is solution of the above equation has the same law as X.

Finally, let us recall a particular case of a theorem proved in [At2], which has the advantage of not needing the introduction of quantum stochastic integrals and of being sufficient for our purpose.

Theorem 13. – *Let* X *be a normal martingale in* $I\!R^n$ *which satisfies a structure equation of the above form :*

$$[X^i, X^j]_t = \delta_{ij}t + \sum_{k=1}^n T_k^{ij}\ X_t^k\ .$$

Then $(X_t)_t$ *possesses the chaotic representation property. Furthermore, the space* $L^2(\Omega,\mathcal{F},P)$, *where* $(\Omega,\mathcal{F},P)$ *is the canonical space associated with* $(X_t)_t$, *is naturally isomorphic to* Φ, *by identification of the chaotic expansion of* f *with the element* $\widetilde{f}$ *of* Φ *whose abstract chaotic expansion has the same coefficients.*

Within this identification the operator of multiplication by X_t^k *is equal to*

$$\mathcal{M}_{X_t^k} = a_k^0(t) + a_0^k(t) + \sum_{i,j=1}^n T_k^{ij} a_j^i(t).$$

VI Convergence to normal martingales

Now we can close the circle under the form of a kind of commutative diagram and establish some convergence theorem.

Starting from an obtuse random random variable X depending on a parameter $h \in \mathbb{R}^+$, with associated sesqui-symmetric tensor T, we associate a random walk $(X_p)_{p\in\mathbb{N}}$of i.i.d. random variables with the same law as X. By Theorem 2, the renormalized random walk

$$\widetilde{X}_n = \sqrt{h}\, X_n$$

satisfies the discrete time structure equation

$$\widetilde{X} \otimes \widetilde{X} = hI + \widetilde{T}(\widetilde{X})$$

where $\widetilde{T}_k^{ij} = \sqrt{h}\, T_k^{ij}$. The tensor $\widetilde{T}$ is also sesqui-symmetric but with the relation

ii') $(i,j,l,m) \longmapsto \sum_k T_k^{ij} T_k^{lm} + h\delta_{ij}\delta_{lm}$ is symmetric.

Theorem 5 shows that the associated multiplication operator by $\widetilde{X}$ is given by

$$U\mathcal{M}_{\widetilde{X}_i(k)}U^* = \sqrt{h}(a_i^0(k) + a_0^i(k)) + \sum_{j,l=1}^{n} \widetilde{T}_i^{jl}\, a_l^j(k).$$

By Proposition 7 we can embed this situation inside the Fock space Φ and we get a family of operators on Φ such that

$$\sum_{k\leq[t/h]} U\mathcal{M}_{\widetilde{X}_i(k)}U^*$$

converges strongly on $\mathcal{D}$ to

$$X_t = a_i^0(t) + a_0^i(t) + \sum_{j,l=1}^{n} S_i^{jl} a_l^j(t)$$

where $S_i^{jl} = \lim_{h\to 0} \widetilde{T}_i^{jl}$, by Theorem 8. Because of the relation ii') above, the limit tensor S is automatically doubly-symmetric.

Thus by Theorem 13, the operators X_t are the canonical multiplication operators by a normal martingale, solution of the structure equation

$$[X^i, X^j]_t = \delta_{ij}t + \sum_{k=1}^{n} S_k^{ij}\ X_t^k\ .$$

From the above we see that only the coefficients $\widetilde{T}_k^{ij}$ which admit a limit S_k^{ij}, when $h \to 0$, contribute to the limit normal martingale $(X_t)_{t\geq 0}$. This means that only the coefficients T_k^{ij} which have a dominant term of order $1/\sqrt{h}$ will contribute non-trivialy to the limit. A smaller dominant term gives 0 in the limit and a larger dominant term will not admit a limit.

If the obtuse random variable X is given one direction for which its probability is of order h, then, by Proposition 1 iv), the length of the jump in that direction is of order $1/\sqrt{h}$. The associated tensor will then get terms T_k^{ij} of order $1/\sqrt{h}$ too (Theorem 3). Thus in the limit this terms will participate to the tensor S. By Proposition 12, these terms S_k^{ij} will participate to the Poisson-type behaviour of the normal martingale.

In the same way one gets easily conviced that the directions of X which are visited with a probability of constant order, or of bigger order than h will contribute to the diffusive part of the martingale.

Note that, in order to understand the above discussion in probabilistic terms it is not necessary to pass throught the representation in terms of creation and annihilation operators. One can directly approach a normal martingale in $\mathbb{R}^n$ by some obtuse random walks (this has be achieved explicitely in [Tav]). But this was not our purpose here to detail this approximation. We just wanted to show how it is naturally related to the approximation of the Fock space by state spaces of $(n+1)$-level atom chains.

We have already a convergence of the random walk to a normal martingale of which the law is given by Theorem 12. Yet this strong convergence of multiplication operators is not easy to translate into probabilistic language, because determining which random variables in $L^2(\Omega, \mathcal{F}, P)$ are sent to $\mathcal{D}$ by identification is not an easy problem (it amounts to studying the integrability properties of the chaotic expansion of random variables).

Obtaining the convergence in law in the above framework requires some more work. This is what the next theorem does.

Theorem 14. – *With the above notations, the random variable*

$$\sqrt{h}\sum_{n=1}^{[t/h]} X_n$$

converges to X_t in law, for almost all t.

Proof

Developping all the details of this proof would need much more serious and longer developments which are not compatible with the length and the spirit of this article. This is the reason why we adopt a more concise style in the following proof.

Choose α in $\mathbb{R}^n$ with coordinates $\alpha_1, \ldots, \alpha_n$ and denote by $M(\alpha, h, k)$ the operator of multiplication by

$$\langle \alpha, \sqrt{h}\sum_{p=1}^{k} X_p\rangle_{\mathbb{R}^n}$$

on $L^2(A^{\otimes \mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$. Then denote by u_k the operator $\exp(i\,M(\alpha,h,k))$. The family u_k satisfies the equations

$$\begin{cases} u_{k+1} = e^{i\langle \alpha, \sqrt{h} X_{k+1}\rangle} u_k \\ u_0 = I. \end{cases}$$

Now, by a slight adaptation of Theorem 19 in [A-P] (the terms D_{ij} have to be allowed to converge as h goes to zero instead of being constant, but this is immediate from the proof), the sequence $u_{[t/h]}$, seen as a sequence of operators on Φ, converges strongly to U_t for almost all t, where (U_s) solves

$$\begin{cases} dU_s = \sum_{j,l=0}^n L_l^j U_s \, da_l^j(s) \\ U_0 = I \end{cases}$$

with

$$\begin{aligned} L_0^0 &= -\frac{1}{2} W^* W \\ L_l^0 &= W_l \\ L_0^j &- (W^* S)_j \\ L_l^j &= S_l^j - I \end{aligned}$$

where

$$\begin{aligned} S &= \exp(iD) \\ W &= (S-I)/D\,\alpha \end{aligned}$$

and D is the $n \times n$ matrix with coefficients

$$D_{jl} = \sum_{i=1}^n \alpha_i T_i^{jl}.$$

Note that we do not assume that D is invertible; if it is not then $(S-I)/D$ is still defined by its series expansion.

Such an equation, called a Hudson-Parthasarathy equation, always has a solution made of unitary operators U_s, and such an equation is unique (for such results we refer the reader to [Par]).

On the other hand, consider the operator V_t which is the value for $s = 1$ of the unitary semigroup $e^{isM(\alpha,t)}$ where $M(\alpha,t)$ is the operator of multiplication by X_t (on the space $L^2(\Omega)$ identified with Φ). Remember that $M(\alpha,t)$ has the representation

$$\sum_{i=1}^n \alpha_i \Big(a_i^0(t) + a_0^i(t) + \sum_{j,l=1}^n T_i^{jl}\, a_l^j(t) \Big)$$

on $\mathcal{D}$. We now wish to apply Vincent-Smith's formula (see [ViS]) for $e^{iM(\alpha,t)}$, but Vincent-Smith's result would require $M(\alpha,t)$ to belong to the class of regular semimartingales (as in [At4]); here it would need the additional property of being bounded.

Yet in our case it is easy to prove *a posteriori* that the formula holds; this is done by an usual trick. Checking that a formal application of Vincent-Smith's formula for $\exp(isM(\alpha,t))$ gives a quantum stochastic integral which is a bounded operator, a strongly continuous semigroup of the parameter s, and that it is unitary, its generator can then be computed and shown to be equal $M(\alpha, t)$ on some good domain (*e.g.* the space of coherent vectors). Since $M(\alpha, t)$, as a linear combination of the fundamental operators $a_l^j(t)$, is known to be essentially selfadjoint on the exponential domain, Stone's theorem proves the validity of the integral representation on Φ for all $\exp(isM(\alpha, t))$.

Let us therefore apply Vincent-Smith's formula for $\exp(isM(\alpha, t))$. It simplifies greatly since the operator $\langle \alpha, X_t \rangle_{I\!R^n}$, as a scalar multiplication, commutes with all other coefficients. We obtain therefore the equation

$$dV_s = \sum_{j,l=1}^{n} K_l^j V_s \, da_l^j(s)$$

with

$$\begin{aligned} K_l^0 &= \int_0^1 \exp\big(i(1-u)D\big)\alpha \, du \\ K_0^j &= \int_0^1 \alpha^* \exp\big(iuD\big) \, du \\ K_l^j &= \int_0^1 \exp\big(i(1-u)D\big) D \, du \\ K_0^0 &= \int_0^1 \int_0^1 u \, \alpha^* \exp\big(iu(1-v)D\big)\alpha \, du \, dv \\ &= \int_0^1 \alpha^* \exp\big(iuD\big)\alpha \, du \end{aligned}$$

and it is clear that the K_l^j equal the corresponding L_l^j. Since $V_0 = U_0$ we obtain the equality $U_t = V_t$ for all t.

On the other hand, for all α, the operator $M(\alpha, h, k)$ is selfadjoint on $T\Phi(h)$. By Nelson's theorem, there exists a dense set of vectors f such that

$$u_k f = \sum_{j \in I\!N} \frac{i^j M(\alpha, h, k)^j}{j\,!} f.$$

When seen on the $L^2(\Omega, \mathcal{F}, P)$, this means that almost everywhere, the equality

$$(u_k f)(\omega) = \exp(i\langle \alpha, \sqrt{h} \sum_{p=1}^{k} X_p(\omega) \rangle_{I\!R^n}) \, f(\omega)$$

holds for all f in the previously specified dense set. Note that this equality is only apparently trivial because of our identification.

The almost everywhere equality can therefore be extended to all f in $T\Phi(h)$, all k. Similarly we can obtain the almost everywhere equality

$$(U_t f)(\omega) = \exp(i\langle \alpha, X_t(\omega) \rangle_{I\!R^n})$$

for all f in Φ.

The above equalities and the strong convergence prove in particular that

$$\mathbb{E}\big(\exp(i\langle\alpha, \sum_{p=1}^{[t/h]} X_p\rangle_{\mathbb{R}^n})\big) = \langle\Omega, u_{[t/h]}\Omega\rangle_{\mathrm{T}\Phi(h)}$$

converges to

$$\langle\Omega, U_t\Omega\rangle_\Phi = \mathbb{E}\big(\exp(i\langle\alpha, X_t\rangle_{\mathbb{R}^n})\big)$$

as h tends to zero. This holds for all α, so that the conclusion holds.

■

VII. Some approximations of 2-dimensional noises

We end this article by computing some simple and illustrative examples in the case $n = 2$.

We consider, in the case $n = 2$, an obtuse random variable X which takes the values $v_1 = (a, 0)$, $v_2 = (b, c)$ and $v_3 = (b, d)$ with respective probabilities p, q, r. In order that X be obtuse we put

$$a = \sqrt{1/p - 1},\ b = -1/a,\ c = \sqrt{1/q - 1 - b^2},\ d = -\sqrt{1/r - 1 - b^2}.$$

Let us call S this set of values for X and p_s the probability associated to $s \in S$. The associated sesqui-symmetric 3-tensor T is given by

$$T(v) = \sum_{s\in S} p_s < s, x > s \otimes s.$$

For example, in the case $p = 1/2$, $q = 1/3$ and $r = 1/6$ we get $a = 1$, $b = -1$, $c = 1$ and $d = -2$. The tensor T is then given by

$$T(v) = \begin{pmatrix} 0 & -y \\ -y & -x - y \end{pmatrix}$$

if $v = (x, y)$. Thus the multiplication operator by X_1 is equal to

$$X_1 = a_0^1 + a_1^0 - a_2^2$$

and the multiplication operator by X_2 is equal to

$$X_2 = a_0^2 + a_2^0 - (a_2^1 + a_1^2 + a_2^2).$$

Now we consider a random walk $(X(k))_{k\geq 0}$ made of independent copies of this random variable X, with time step h. In the framework of the Fock space approximation described above, the operator

$$\sum_{k;kh\leq t} \sqrt{h}\, X_1(k)$$

converges, both in the sense of convergence of multiplication operators and in law, to

$$a_1^0(t) + a_0^1(t)$$

and the operator

$$\sum_{k;kh\le t} \sqrt{h}\, X_2(k)$$

converges to

$$a^0_2(t) + a^2_0(t).$$

This means that the limit process $X(t)$ is a 2-dimensional Brownian motion. Indeed, the above representation shows that the associated doubly-symmetric tensor Φ is null and thus X satisfies the structure equation

$$\begin{aligned} d[X_1, X_1]_t &= dt \\ d[X_1, X_2]_t &= 0 \\ d[X_2, X_2]_t &= dt \end{aligned}$$

which is exactly the structure equation verified by two independent Brownian motions.

It is clear, that whatever the values of p, q, r are, if they are independent of the time step parameter h, we will always obtain a 2-dimensional Brownian motion as a limit of this random walk.

When some of the probabilities p, q or r depend on h the behaviour is very different. Let us follow two examples.

In the case $p = 1/2$, $q = h$ and $r = 1/2 - h$ we get

$$a = 1,\ b = -1,\ c = \frac{1}{\sqrt{h}} + O(h^{1/2}),\ d = -2\sqrt{h} + o(h^{3/2}).$$

For the tensor T we get

$$T(v) = \begin{pmatrix} 0 + o(h^{5/2}) & -y + o(h^2) \\ -y + o(h^2) & -\frac{y}{\sqrt{h}} - x + o(h^{1/2}) \end{pmatrix}.$$

The multiplication operators are then given by

$$X_1 = a^1_0 + a^0_1 - a^2_2 + O(h^2)$$

and

$$X_2 = a^2_0 + a^0_2 - (a^1_2 + a^2_1) + \frac{1}{\sqrt{h}} a^2_2 + O(h^{1/2}).$$

In the same limit as above we thus obtain the operators

$$a^1_0(t) + a^0_1(t)$$

and

$$a^2_0(t) + a^0_2(t) - a^2_2(t).$$

This means that the coordinate $X_1(t)$ is a Brownian motion and $X_2(t)$ is an independent Poisson process, with intensity 1 and directed upwards. Indeed, the associated tensor Φ is given by

$$\Phi(v) = \begin{pmatrix} 0 & 0 \\ 0 & -y \end{pmatrix}$$

and the associated structure equation is

$$\begin{aligned} d[X_1, X_1]_t &= dt \\ d[X_1, X_2]_t &= 0 \\ d[X_2, X_2]_t &= dt + dX_2(t) \end{aligned}$$

which is the structure equation of the process we described.

The last example we shall treat is the case $p = 1 - 2h$, $q = r = h$. We get, for the dominating terms

$$a = \sqrt{2}\sqrt{h},\ b = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}},\ c = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}},\ d = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}},$$

and

$$\begin{aligned} X_1 &= a_0^1 + a_1^0 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}}a_2^2 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}}a_1^1 \\ X_2 &= a_0^2 + a_2^0 - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{h}}(a_2^1 + a_1^2). \end{aligned}$$

The limit process is then solution of the structure equation

$$\begin{aligned} d[X_1, X_1]_t &= dt - \frac{1}{\sqrt{2}}\, dX_1(t) \\ d[X_1, X_2]_t &= -\frac{1}{\sqrt{2}}\, dX_2(t) \\ d[X_2, X_2]_t &= dt - \frac{1}{\sqrt{2}}\, dX_1(t). \end{aligned}$$

The associated tensor is easy to diagonalise and one finds the eigenvectors

$$(-1/\sqrt{2}, 1/\sqrt{2}) \ \text{ and } \ (-1/\sqrt{2}, -1/\sqrt{2}).$$

The limit process is made of two independent Poisson processes, with intensity 2 and respective direction (-1,1) and (-1,-1).

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Stéphane Attal
Institut Fourier, U.M.R. 5582
Université de Grenoble I, BP 74
38402 St-Martin d'Hères cedex, France

Yan Pautrat
McGill mathematics and statistics
805 Sherbrooke West
Montreal, QC, H3A 2K6, Canada