CLASSICAL AND QUANTUM STOCHASTIC CALCULUS
Survey article

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Abstract

We focus on some connections between classical and quantum stochastic calculus. It is shown how these calculus can be great sources of ideas and developments for each other.

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Introduction

It is really a great pleasure for me to be given the opportunity to write a survey article about the connections between classical and quantum stochastic calculus throught my work.

It is clear to anyone involved in Quantum Probability, or at least who has attended one of our meetings, that this field is far from beeing reducible to a simple non-commutative extension of classical probability theory. Our work does not consist in finding an exhaustive list of non-commutative analogues of the classical theorems. If Quantum Probability is very interesting it is because it is a transversal subject. It finds its fundamental axioms in quantum physics and the connections with domains such as quantum mechanics, quantum field theory, quantum optics, scattering theory are very deep. For a large number of my collegues Quantum Probability is also part of functional analysis (C^*-algebras and von Neumann algebras theory, non-commutative geometry, quantum groups,...). But since I have started to work in Quantum Probability, 4 years ago, I have so far been mainly interested in studying the connections between classical and quantum stochastic calculus in both directions. These connections do exist, they have been driving me all along my work and they have always been a great source of ideas.

Before describing the contents of this article I have to warn the reader that this article does not pretend to be a state of the art on the subject. It is just a very personal (and unmodest) point of view on it, based on what I know best that is, my work. I just hope it to make classical and quantum readers beeing convinced of the use and the interest of both fields.

The article is divided into two main chapters. The first chapter is devoted to the contributions of classical stochastic calculus to the quantum one. The second chapter is devoted to the contributions of quantum stochastic calculus to the classical one.

In the first section of the first chapter we show how the usual Ito calculus can be translated in an intrinsic way on Fock space. This abstract Ito calculus gives rise to an extension of Hudson-Parthasarathy's notion of adaptedness for operators. As a consequence one can give a new definition of quantum stochastic integrals. This new definition extends all the previous ones (Hudson-Parthasarathy, Belavkin-Lindsay, Attal-Meyer); it has all the advantages of the previous ones without any of the inconvenients. This definition is shown to be maximal and to completely solve the problem of Attal-Meyer's equations (which is the most probabilistic point of view on quantum stochastic integrals).

In the second section, we show some consequences of Attal-Meyer's point of view on quantum stochastic integrals. These developments are in the direction of developing a quantum stochastic calculus which is as easy to manipulate as the classical one. We identify an algebra on quantum semimartingales. On this algebra we define quantum square and angle brackets for quantum semimartingales. Functional quantum Ito formulas are obtained.

In a third section we show how the probabilistic idea of getting the chaotic expansion of a random variable by iterating infinitely many times its previsible representation can give an idea for getting the Maassen kernel of an operator. This idea applies very well to Hilbert-Schmidt operators.

In the fourth and last section of the first chapter, we present Enchev's Hilbertian extension of the concept of quasimartingales. His theorem is they key of many applications in quantum probability, above all when quantum stop times are involved. We see two applications: one shows that one can stop a large class of vector and operator processes with quantum stop times, the other one shows that every minimal Evans-Hudson flows are quantum strong Markov processes.

The second chapter consists into two sections. The first section presents the quantum stochastic calculus as a natural extension of the classical one. We show how the concepts developed in the first two sections of chapter I extend and unify the classical ones in the probabilistic interpretations of the Fock space (that is, in the case of normal martingales admitting the chaotic representation property).

The second section is devoted to Wiener space endomorphisms. That is, some transformations of the Wiener space which preserve the Wiener measure. Such transformations, when lifted to the space of square integrable functionals of the Brownian motion, can be very naturally studied through quantum stochastic calculus. One obtains an algebraic characterization of these endomorphisms. These methods apply to more general endomorphisms such as endomorphisms of the Poisson process, the Azéma martingales,...

Notations

When one writes an article on quantum stochastic calculus he has to make a choice. He can either choose the full generality by working on the symmetric Fock space with initial space and countable multiplicity, or try to make his exposition as clear as possible, with simple notations, and choose to deal with the multiplicity one symmetric Fock space. I have made the second choice for this article. There are two reasons for that. First, this article is an expositary one, it is supposed to strengthen the links between classical and quantum stochastic calculus; so flooding the reader with cumbersome notations does not help to understand the ideas. Secondly, though most of the results presented here already exist in the countable multiplicity case, this is not true for all of them; some of the most recent ones (such as [A-L]) are still under work as the finite multiplicity extension is easy but the infinite one is not. Consequently, I only deal with the multiplicity one symmetric Fock space.

All vector spaces are on the complex field $\mathbb{C}$. Note that unless it is not clear, the norm and the scalar product in Hilbert spaces are denoted $||\cdot||$ and $<\cdot,\cdot>$

respectively, without specifying which space it refers to. The scalar products $(x,y)\mapsto <x\,,\,y>$ are linear in y and antilinear in x.

Let Φ be the boson (or symmetric) Fock space over $L^2(\mathbb{R}^+)$ that is,

$$\Phi = \Gamma(L^2(\mathbb{R}^+)) \stackrel{\text{def}}{=} \oplus_{n\in\mathbb{N}}\, L^2(\mathbb{R}^+)^{\odot n}$$

where $\odot$ denotes symmetric tensor products. So elements of Φ are of the form $f=\oplus_n f_n$ where f_0 is a scalar, where f_n is a symmetric square integrable function on $(\mathbb{R}^+)^n$ for $n\geq 1$ and where

$$||f||^2_\Phi = \sum_n ||f_n||^2_{L^2(\mathbb{R}^+)^{\odot n}}.$$

For any $s\leq t$ in $\mathbb{R}^+$ we put $\Phi_{t]}=\Gamma(L^2([0,t]))$, $\Phi_{[s,t]}=\Gamma(L^2([s,t]))$, and $\Phi_{[t}=\Gamma(L^2([t,+\infty[))$.

There exists another very useful description of the Fock space: the *Guichardet symmetric space*. Let $\mathcal{P}$ denote the finite power set of $\mathbb{R}^+$ that is, the set of finite subsets of $\mathbb{R}^+$ (elements of $\mathcal{P}$ are denoted with small greek letters $\sigma,\tau,\omega,\ldots$). Thus $\mathcal{P}$ admits a disjoint partition: $\mathcal{P}=\cup_n\mathcal{P}_n$ where $\mathcal{P}_0=\{\emptyset\}$ and, for $n\geq 1$, $\mathcal{P}_n$ is the set of n-elements subsets of $\mathbb{R}^+$. For $n\geq 1$ the set $\mathcal{P}_n$ can be identified with the increasing simplex $\{(t_1,\ldots,t_n)\in\mathbb{R}^n;\, 0<t_1<\ldots<t_n\}$ and then the restriction of the Lebesgue measure on $\mathbb{R}^n$ induces a measure on $\mathcal{P}_n$. By declaring the measure on $\mathcal{P}_0$ to be the unit one, we obtain a σ-finite measure on $\mathcal{P}$ called the *symmetric measure* on $\mathbb{R}^+$ ([Gui]). The element of volume of this measure is denoted $d\sigma, d\tau, d\omega,\ldots$ Elements of $L^2(\mathcal{P})$ are thus functions $f:\sigma\mapsto f(\sigma)$ such that $\int_\mathcal{P}|f(\sigma)|^2\,d\sigma<\infty$. The isomorphism between Φ and $L^2(\mathcal{P})$ is now clear by defining $f_n(t_1,\ldots,t_n)=f(\sigma)$ when $\sigma=\{t_1,\ldots,t_n\}$ and by noticing that the norm of each of the spaces Φ and $L^2(\mathcal{P})$ coincide.

So from now on the Fock space Φ is not distinguished from $L^2(\mathcal{P})$; elements of Φ are square integrable (class of) functions on $\mathcal{P}$. For a $f\in\Phi$ the family of values $\{f(\sigma);\,\sigma\in\mathcal{P}\}$ is called the *chaotic expansion* of f.

We need a few more notations in the Guichardet space language. For $t\in\mathbb{R}^+$ define $\mathcal{P}_{t]}$ to be the set of $\sigma\in\mathcal{P}$ such that $\sigma\subset[0,t]$, let $\mathcal{P}_{[t}$ be the set of $\sigma\in\mathcal{P}$ such that $\sigma\subset[t,\infty[$. For $\sigma\in\mathcal{P}$ and $t\in\mathbb{R}^+$ put $\sigma\cup t$ to be the set $\sigma\cup\{t\}$; in the same way $\sigma\setminus t$ is the set $\sigma\setminus\{t\}$. If $\sigma\neq\emptyset$ then $\vee\sigma$ denotes $\max\sigma$ and σ_- denotes $\sigma\setminus\vee\sigma$; finally $\wedge\sigma$ denotes $\min\sigma$. For $s\leq t$ and $\sigma\in\mathcal{P}$ we put $\sigma_{t)}=\sigma\cap[0,t[$, $\sigma_{(s,t)}=\sigma\cap]s,t[$ and $\sigma_{(t}=\sigma\cap]t,+\infty[$. Finally, for $\sigma\in\mathcal{P}$, $\#\sigma$ denotes the cardinal of the set σ.

In this context we often use the following combinatoric lemma (cf [L-P]).

$\int$-Lemma – *Let g be a positive (resp. integrable) measurable function from $\mathcal{P}\times\mathcal{P}$ to $\mathbb{C}$. Then $G:\sigma\mapsto\sum_{\alpha\subset\sigma}g(\alpha,\sigma\setminus\alpha)$ defines a positive (resp. integrable) measurable function on $\mathcal{P}$ satisfying*

$$\int_\mathcal{P} G(\sigma)\,d\sigma = \int_\mathcal{P}\int_\mathcal{P} g(\alpha,\beta)\,d\alpha\,d\beta. \qquad \blacksquare$$

From this lemma it is easy to check that for all $t \in \mathbb{R}^+$ the mapping

$$\begin{array}{rcl} U \ : \ \Phi_{t]} \otimes \Phi_{[t} & \longrightarrow & \Phi \\ f \otimes g & \longmapsto & \Big(\sigma \mapsto f(\sigma_{t)})\, g(\sigma_{(t})\Big) \end{array}$$

extends to a unitary isomorphism between $\Phi_{t]} \otimes \Phi_{[t}$ and Φ. So from now on we omit the mapping U, we do not distinguish between the spaces $\Phi_{t]} \otimes \Phi_{[t}$ and Φ; we also identify $\Phi_{t]}$ and $\Phi_{[t}$ to subspaces of Φ. In the same way, for all $s \leq t$, the spaces $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ and Φ are isomorphic.

A family of subpaces of Φ is going to be of great interest: the spaces of *coherent vectors*. Consider a dense subspace $\mathcal{M}$ of $L^2(\mathbb{R}^+)$; it is said to be *admissible* if $\mathcal{M}$ is stable under multiplication by $1\!\!1_{[0,t]}$ for all t. For a vector u in an admissible $\mathcal{M}$ one defines the associated coherent vector $\varepsilon(u)$ in Φ by $\varepsilon(u)(\sigma) = \prod_{s\in\sigma} u(s)$, where as usual the empty product equals 1. In this way we obtain a Fock space element with $||\varepsilon(u)||^2 = \exp(||u||^2)$. One can also check that in the tensor product structure $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ we have $\varepsilon(u) = \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})$, where $u_{t]}$ denotes $u1\!\!1_{[0,t]}$ and $u_{[t}$ denotes $u1\!\!1_{[t,+\infty[}$. The vector space generated by the $\varepsilon(u)$ when u runs over $\mathcal{M}$ is denoted $\mathcal{E}(\mathcal{M})$; it is a dense subspace of Φ.

This particular property of coherent vectors with respect to the continuous tensor product structure of Φ is the key point in Hudson-Parthasarthy's treatment of quantum stochastic calculus ([H-P]). For a fixed $t \in \mathbb{R}^+$ they define an operator H from Φ to Φ to be *t-adapted* if $\mathrm{Dom}\, H$ contains $\mathcal{E}(\mathcal{M})$ and if it satisfies for all $u \in \mathcal{M}$:

i) $H\varepsilon(u_{t]})$ belongs to $\Phi_{t]}$

ii) $H\varepsilon(u) = [H\varepsilon(u_{t]})] \otimes \varepsilon(u_{[t})$.

An *adapted process* of operators is a family $(H_t)_{t\geq 0}$ of operators from Φ to Φ such that H_t is t-adapted for all t and such that the mapping $t \mapsto H_t\, \varepsilon(u)$ is strongly measurable for all $u \in \mathcal{M}$.

Let us recall Hudson-Parthasarathy's definitions of quantum stochastic integrals. Consider four adapted processes of operators H, K, L, M on Φ, defined on $\mathcal{E}(\mathcal{M})$. In [H-P] it is proved that if, for all $u \in \mathcal{M}$, all $t \geq 0$ one has

$$\int_0^t |u(s)|^2 ||H_s\, \varepsilon(u_{s]})||^2\, ds + \int_0^t ||K_s\, \varepsilon(u_{s]})||^2\, ds + \int_0^t |u(s)|\, ||L_s\, \varepsilon(u_{s]})||\, ds + \int_0^t ||M_s\, \varepsilon(u_{s]})||\, ds < \infty$$

then there exists a unique adapted process of operators $(T_t)_{t\geq 0}$ denoted

$$T_t = \int_0^t H_s\, d\Lambda_s + \int_0^t K_s\, dA_s^\dagger + \int_0^t L_s\, dA_s + \int_0^t M_s\, ds$$

satisfying

$$< \varepsilon(v)\, ,\, T_t\, \varepsilon(u) > = \int_0^t \overline{v}(s) u(s) < \varepsilon(v)\, ,\, H_s\, \varepsilon(u) > + \overline{v}(s) < \varepsilon(v)\, ,\, K_s\, \varepsilon(u) > + u(s) < \varepsilon(v)\, ,\, L_s\, \varepsilon(u) > + < \varepsilon(v)\, ,\, M_s\, \varepsilon(u) > ds$$

for all $u, v \in \mathcal{M}$, all $t \geq 0$. These integrals are the quantum stochastic integrals of H, K, L, M with respect to the quantum noises $(\Lambda_t)_{t\geq 0}, (A_t^\dagger)_{t\geq 0}, (A_t)_{t\geq 0}$ and to the time $(tI)_{t\geq 0}$. These integrators are respectively called the conservation, creation, annihilation and time processes. The definition of Λ_t (*resp.* $A_t^\dagger, A_t, tI$) can be recovered by taking H_s (*resp.* K_s, L_s, M_s) to be I for $s \leq t$, 0 for $s > t$ and the other three processes to vanish. For more details on Hudson-Parthasarathy's quantum stochastic calculus one can refer to the original article [H-P], to the books [Me1], [Par], or to the course [Bia].

I From classical to quantum stochastic calculus

I.1 Ito calculus approch to quantum stochastic calculus

This section presents quantum stochastic calculus under a new point of view: we take the quantum stochastic calculus defined in [A-L] as starting point that is, as the most general definition. We then recover all the previous definitions and theorems as consequences.

I.1.1 Ito calculus on Fock space

We here introduce some of the fundamental tools for this article. They consist into a Fock space transcription of classical probabilistic objects. It is well-known to quantum probabilists that probabilistic operations such as Skorohod integral, stochastic derivative, Skorohod isometry formula are not only probabilistic, they can be translated in an intrinsic way in the Fock space. It is less common to them that Ito integral, predictable representation property and Ito isometry formula can also be adapted in an intrinsic language to Fock space. This new approch has been developed in [A-L] and all details can be found there (it has to be noticed that the intrinsic expression of the Ito integral was already mentionned in [Ma1]).

We are going to define operators on Φ. First some well-known ones. For all $f \in \Phi$, all $t \in \mathbb{R}^+$ and all $\sigma \in \mathcal{P}$ define

$$[P_t f](\sigma) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma) f(\sigma).$$

Then $\sigma \mapsto [P_t f](\sigma)$ is cleary square integrable and $P_t f$ defines an element of Φ. The operator P_t can be easily seen to be the orthogonal projection from Φ onto $\Phi_{t]}$.

For all f in Φ, all t in $\mathbb{R}^+$ and all $\sigma \in \mathcal{P}$ define

$$[\nabla_t f](\sigma) = f(\sigma \cup t).$$

For this mapping to be square integrable in σ one needs f to belong to a certain proper subdomain of Φ. There is a useful and natural common domain for all these operators ∇_t which is $\mathrm{Dom}\,\sqrt{N} \stackrel{\mathrm{def}}{=} \{f \in \Phi;\ \int_{\mathcal{P}} \#\sigma |f(\sigma)|^2\, d\sigma < \infty\}$. The operators ∇_t are often known as *Malliavin gradient* operators.

Let $x_. = (x_t)_{t\geq 0}$ be a *vector process* in Φ that is, a measurable mapping $(\sigma, t) \mapsto x(\sigma, t)$ on $\mathcal{P} \times I\!\!R^+$ such that for all t the mapping $\sigma \mapsto x_t(\sigma) \stackrel{\text{def}}{=} x(\sigma, t)$ is square integrable. For such a process define

$$[\mathcal{S}(x_.)](\sigma) = \sum_{s\in\sigma} x_s(\sigma\backslash s)$$

where as usual the empty sum is equal to 0. When this mapping is square integrable in σ one says that $(x_t)_{t\geq 0}$ is *Skorohod integrable* and the Fock space vector $\mathcal{S}(x_.)$ is called the *Skorohod integral* of $(x_t)_{t\geq 0}$.

All these operators are well-known to quantum probabilists as well as classical ones (see section II.1.2 for a probabilistic interpretation of these operators). But the following ones are less common to quantum probabilists whereas they have well-known probabilistic counterparts.

For every $t \in I\!\!R^+$, $f \in \Phi$, $\sigma \in \mathcal{P}$ define the quantity

$$[D_t f](\sigma) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma) f(\sigma \cup t).$$

From the $\int$-Lemma we easily get the following identity.

Lemma I.1 – *For all $f \in \Phi$ one has*

$$\int_0^\infty \int_{\mathcal{P}} \left|[D_t f](\sigma)\right|^2 d\sigma\, dt = ||f||^2 - |f(\emptyset)|^2. \qquad \blacksquare$$

Lemma I.1 implies in particular that for all $f \in \Phi$, almost all $t \in I\!\!R^+$, the mapping $\sigma \mapsto [D_t f](\sigma)$ is square integrable. Thus for all $f \in \Phi$, almost all t, $D_t f$ is a well-defined element of Φ. Here a short discussion is needed. For each fixed t the linear application D_t cannot be considered as an operator on Φ, only what one can say is that for all $f \in \Phi$, almost all (depending on f) $t \in I\!\!R^+$, the mapping $D_t f$ is defined as a vector in Φ. It is more rigorous to define the operator

$$\begin{array}{rcccc} D & : & L^2(\mathcal{P}) & \longrightarrow & L^2(I\!\!R^+ \times \mathcal{P}) \\ & & f & \longmapsto & \big((t,\sigma) \mapsto [D_t f](\sigma)\big). \end{array}$$

From Lemma I.1 the operator D is a partial isometry and the almost everywhere defined operators D_t are sections of D. Another important remark is that it can be seen from the definitions that $D_t = P_t \nabla_t$, or more rigorously if one defines

$$\begin{array}{rcccc} P & : & L^2(I\!\!R^+ \times \mathcal{P}) & \longrightarrow & L^2(I\!\!R^+ \times \mathcal{P}) \\ & & (x_t)_{t\geq 0} & \longmapsto & (P_t x_t)_{t\geq 0} \end{array}$$

$$\begin{array}{rcccc} \nabla & : & \mathrm{Dom}\sqrt{N} \subset \Phi & \longrightarrow & L^2(I\!\!R^+ \times \mathcal{P}) \\ & & f & \longmapsto & (\nabla_t f)_{t\geq 0} \end{array}$$

we have $D = \overline{P\nabla}$ that is, D is the closure of $P\nabla$. This closure property is fundamental because, though the ∇_t's are defined on a proper subdomain of Φ and D_t is equal to $P_t \nabla_t$ on this domain, we have that the D_t's are **defined everywhere**. This property of beeing defined on the whole Fock space for D_t instead of an arbitrary subdomain for ∇_t is going to be fundamental.

In the following, when dealing with the operators D_t we write "for almost all" without mentioning that this "almost all" depends on a vector $f \in \Phi$. It may even happen that we omit to mention the "almost all".

Note the simple but important property which is an immediate consequence of the definitions:

$$D_u P_t = \begin{cases} 0 & \text{for almost all } u \geq t \\ D_u & \text{for almost all } u < t. \end{cases} \tag{I.1}$$

Let $x. = (x_t)_{t\geq 0}$ be a vector process in Φ. We say that $x.$ is *Ito integrable* if

i) $x_t \in \Phi_{t]}$ for all t,

ii) $\int_0^\infty ||x_t||^2 \, dt < \infty$.

If $x.$ is a Ito integrable process on Φ we define the following mapping on $\mathcal{P}$

$$[I(x.)](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \\ x_{\vee\sigma}(\sigma-) & \text{otherwise.} \end{cases}$$

Another application of the $\not{\int}$-Lemma gives the following.

Lemma I.2 – *For every Ito integrable process* $(x_t)_{t\geq 0}$ *we have*

$$\int_{\mathcal{P}} \left|[I(x.)](\sigma)\right|^2 \, d\sigma = \int_0^\infty ||x_t||^2 \, dt. \qquad \blacksquare$$

That is, for any Ito integrable process $(x_t)_{t\geq 0}$ the mapping $I(x.)$ defines an element of Φ called the *Ito integral* of $(x_t)_{t\geq 0}$ and satisfying the *isometry formula*

$$\left|\left|I(x.)\right|\right|^2 = \int_0^\infty ||x_t||^2 \, dt.$$

Note that the Ito integral operator coincide with the Skorohod integral operator restricted to Ito integrable processes.

It is also worth noticing that the Ito integral operator $I(\cdot)$ actually corresponds to an integration with respect to some curve in Φ. Indeed, for all $t \in \mathbb{R}^+$ define the vector $\chi_t \in \Phi$ by

$$\begin{cases} \chi_t(\sigma) = 0 & \text{if } \#\sigma \neq 1, \\ \chi_t(\{s\}) = \mathbb{1}_{[0,t]}(s). \end{cases}$$

It is clear that χ_t belongs to $\Phi_{t]}$ for all $t \in \mathbb{R}^+$ but one can check that $(\chi_t)_{t\geq 0}$ is furthermore an "independent increment" vector process that is, for $s \leq t$ the increment $\chi_t - \chi_s$ belongs to $\Phi_{[s,t]}$. Let $(y_t)_{t\geq 0}$ be a Ito integrable process. Suppose first that $(y_t)_{t\geq 0}$ is a step process that is, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ converging to $+\infty$ and such that $y_t = y_{t_n}$ for $t_n \leq t < t_{n+1}$. Then define a mapping on $\mathcal{P}$ by

$$\left[\int_0^\infty y_t \otimes d\chi_t\right](\sigma) \stackrel{\text{def}}{=} \sum_n \left[y_{t_n} \otimes (\chi_{t_{n+1}} - \chi_{t_n})\right](\sigma)$$

(recall that y_{t_n} belongs to $\Phi_{t_n]}$ and $\chi_{t_{n-1}} - \chi_{t_n}$ belongs to $\Phi_{[t_n}$). It can be easily computed that

$$\left[\int_0^\infty y_t \otimes d\chi_t\right](\sigma) = [I(y.)](\sigma).$$

Thus, for step Ito integrable processes $(y_t)_{t\geq 0}$ the vector $\int_0^\infty y_t \otimes d\chi_t$ coincides with the Ito integral $I(y.)$. By the isometry formula we get

$$\left|\left|\int_0^\infty y_t \otimes d\chi_t\right|\right|^2 = ||I(y.)||^2 = \int_0^\infty ||y_t||^2\,dt$$

and we can easily extend the integral to all Ito integrable processes $(y_t)_{t\geq 0}$. Hence the Ito integral $I(y.)$ actually corresponds to an integration with respect to the curve $(\chi_t)_{t\geq 0}$ in Φ. From now on $I(y.)$ is denoted $\int_0^\infty y_t\,d\chi_t$. Note that we have dropped the $\otimes$ symbol.

From the definition of the operators D_t and Lemma I.1 it is clear that for any $f \in \Phi$ the vector process $(D_t f)_{t\geq 0}$ is Ito integrable. Let us compute $\int_0^\infty D_t f\,d\chi_t$. We have

$$\begin{aligned}\left[\int_0^\infty D_t f\,d\chi_t\right](\sigma) &= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ [D_{\vee\sigma} f](\sigma-) & \text{otherwise}\end{cases} \\ &= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ \mathbb{1}_{\mathcal{P}_{\vee\sigma]}}(\sigma-) f(\sigma- \cup \vee\sigma) & \text{otherwise}\end{cases} \\ &= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma) & \text{otherwise.}\end{cases}\end{aligned}$$

Thus noticing that $[P_0 f](\sigma) = \mathbb{1}_{\sigma=\emptyset} f(\emptyset)$ ($P_0 f$ can then be identified to a complex number) we easily conclude to the following fundamental property.

Theorem I.3 (Fock space predictable representation property) – *For all $f \in \Phi$ we have the representation*

$$f = P_0 f + \int_0^\infty D_t f\,d\chi_t,$$

with

$$||f||^2 = |P_0 f|^2 + \int_0^\infty ||D_t f||^2\,dt$$

and for all $g \in \Phi$

$$< f\,,\, g > = \overline{P_0 f}\, P_0 g + \int_0^\infty < D_t f\,,\, D_t g > dt. \qquad \blacksquare$$

I.1.2 Adaptedness revisited

The above Ito calculus on Fock space is the main ingredient for developing a new definition of adaptedness for operators. The new definition has the advantage of beeing very simply expressed, of getting freed from the coherent vector spaces and of applying to a very large class of different domains.

Let us forget Hudson-Parthasarathy's adaptedness for a while and try to redefine it. Let t be fixed in $\mathbb{R}^+$. Let $\mathcal{D}$ be a domain in Φ that is, a subspace of Φ. We say that $\mathcal{D}$ is a *t-adapted domain* if $f \in \mathcal{D}$ implies $P_t f \in \mathcal{D}$ and $D_u f \in \mathcal{D}$ for almost all $u \geq t$.

One can immediatly wonder what kind of spaces are satisfying these stability properties. The answer is that all the space usually considered in quantum stochastic calculus are adapted in this sense:

- the whole Fock space is of course t-adapted for all t;

- the coherent vector spaces $\mathcal{E}(\mathcal{M})$ are t-adapted domains for all t if and only if $\mathcal{M}$ is admissible;

- the so-called "space of finite particles" $\Phi_F \stackrel{\text{def}}{=} \{f \in \Phi;$ for some $n \in \mathbb{N}$, $f(\sigma) = 0$ when $\#\sigma > n\}$ is t-adapted for all t;

- the space of Maassen-Meyer test vectors that is, the set of $f \in \Phi$ satisfying

i) $f(\sigma) = 0$ if $\sigma \not\subset [0, T]$ for some T,

ii) $|f(\sigma)| \leq CM^{\#\sigma}$ for some $C, M \geq 0$,

is t-adapted for all t;

- the Fock scales $\Phi(a) \stackrel{\text{def}}{=} \{f \in \Phi; \int_{\mathcal{P}} a^{\#\sigma} |f(\sigma)|^2 \, d\sigma < \infty\}$ for $a \geq 1$ are all t-adapted for any t.

So adapted domains constitute a much larger class than the coherent ones; this class includes all the spaces usually considered in quantum stochastic calculus.

We can now introduce a new definition of adaptedness for operators. Let t be fixed in $\mathbb{R}^+$. Let H be an operator from Φ to Φ with domain $\mathcal{D}$. The operator H is *t-adapted* if

i) $\mathcal{D}$ is a t-adapted domain

ii) on $\mathcal{D}$ one has

$$\begin{cases} P_t H = H P_t \\ D_u H = H D_u \qquad \text{for a.a. } u \geq t. \end{cases}$$

Once again let us precise that the last identity has to be understood in the following way: for all $f \in \mathcal{D}$, almost all $u \geq t$ one has $D_u H f = H D_u f$.

There are several equivalent ways of defining this new notion of adaptedness. We here present only one. For a $\sigma = \{t_1 < \ldots < t_n\} \in \mathcal{P}$ we denote by D_σ the operator $D_{t_1} \ldots D_{t_n}$ and when $\sigma = \emptyset$ the operator D_σ is the identity operator I.

Proposition I.4 – *Let t be fixed in $\mathbb{R}^+$. An operator H on Φ with domain $\mathcal{D}$ is t-adapted if and only if*

i) *$\mathcal{D}$ is a t-adapted domain,*

ii) *for all $f \in \Phi$, a.a. $\sigma \in \mathcal{P}$ one has*

$$[Hf](\sigma) = [HP_t D_{\sigma_{(t}} f](\sigma_{t)}).$$

Proof

Suppose H is t-adapted. Take $\sigma \in \mathcal{P}$. If $\sigma_{(t} = \emptyset$ then $\sigma \subset [0, t]$ and

$$[HP_t D_{\sigma_{(t}} f](\sigma_{t)}) = [HP_t f](\sigma) = [P_t H f](\sigma) = [Hf](\sigma).$$

If $\sigma_{(t}$ is not empty then assume it is of the form $\sigma_{(t} = \{t_1 < \ldots < t_n\}$. We have

$$\begin{aligned} [Hf](\sigma) &= [D_{t_n} H f](\sigma \backslash t_n) = [H D_{t_n} f](\sigma \backslash t_n) \\ &\ \vdots \\ &= [H D_{t_1} \ldots D_{t_n} f](\sigma \backslash \{t_1, \ldots, t_n\}) \\ &= [H D_{\sigma_{(t}} f](\sigma_{t)}) = [P_t H D_{\sigma_{(t}} f](\sigma_{t)}) = [H P_t D_{\sigma_{(t}} f](\sigma_{t)}). \end{aligned}$$

This proves the proposition in one direction.

Conversly, suppose H satisfies properties i) and ii). Then one has

$$[P_tHf](\sigma) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma)[Hf](\sigma) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma)[HP_tD_{\sigma_{(t}}f](\sigma_{t)}) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma)[HP_tf](\sigma).$$

But by (I.1) we also have

$$[HP_tf](\sigma) = [HP_tD_{\sigma_{(t}}P_tf](\sigma) = 1\!\!1_{\sigma_{(t}=\emptyset}[HP_tf](\sigma_{t)}) = 1\!\!1_{\mathcal{P}_{t]}}(\sigma)[HP_tf](\sigma).$$

This proves $P_tHf = HP_tf$. Furthermore, for a.a. $u \geq t$ we have

$$\begin{aligned}[D_uHf](\sigma) &= 1\!\!1_{\mathcal{P}_{u]}}(\sigma)[Hf](\sigma \cup u) = 1\!\!1_{\mathcal{P}_{u]}}(\sigma)[HP_tD_{(\sigma\cup u)_{(t}}f]((\sigma\cup u)_{t)})\\ &= 1\!\!1_{\mathcal{P}_{u]}}(\sigma)[HP_tD_{\sigma_{(t}}D_uf](\sigma_{t)}) = 1\!\!1_{\mathcal{P}_{u]}}(\sigma)[HD_uf](\sigma).\end{aligned}$$

But we also have, by (I.1),

$$[HD_uf](\sigma) = [HP_tD_{\sigma_{(t}}D_uf](\sigma_{t)}) = 1\!\!1_{\mathcal{P}_{u]}}(\sigma)[HP_tD_{\sigma_{(t}}D_uf](\sigma_{t)})$$

which finally equals $1\!\!1_{\mathcal{P}_{u]}}(\sigma)[HD_uf](\sigma)$. We have proved that $D_uHf = HD_uf$ for a.a. $u \geq t$. ■

Of course this new notion of adaptedness would be of no interest if it were not coinciding with Hudson-Parthasarathy's one when restricted to coherent vectors.

Proposition I.5 – *Let $\mathcal{M}$ be a dense subspace of $L^2(\mathbb{R}^+)$. Let H be an operator defined on $\mathcal{E}(\mathcal{M})$. Then $\mathcal{E}(\mathcal{M})$ is a t-adapted domain for all t if and only if $\mathcal{M}$ is admissible. The operator H is t-adapted if and only if it is t-adapted in Hudson-Parthasarathy's sense.*

Proof

The adaptedness property of the domain is easy to get since $P_t\,\varepsilon(u) = \varepsilon(u_{t]})$ and $D_t\,\varepsilon(u) = u(t)\varepsilon(u_{t]})$. It is also easy to check that in both of the two adaptedness definitions we have $P_t\,H = HP_t$. Now, in any case let us compute $[H\varepsilon(u_{t]})]\otimes \varepsilon(u_{[t})$ for $u \in \mathcal{M}$. We have

$$\begin{aligned}\big[[H\varepsilon(u_{t]})]\otimes \varepsilon(u_{[t})\big](\sigma) &= [H\varepsilon(u_{t]})](\sigma_{t)})\,[\varepsilon(u_{[t})](\sigma_{(t}) = [H\varepsilon(u_{t]})](\sigma_{t)})\prod_{s\in\sigma_{(t}} u(s)\\ &= \prod_{s\in\sigma_{(t}} u(s)\,[HP_t\,\varepsilon(u_{\wedge\sigma_{(t]}})](\sigma_{t)}) = [HP_tD_{\sigma_{(t}}\varepsilon(u)](\sigma_{t)}).\end{aligned}$$

One concludes easily. ■

Together with the notion of adaptedness for domains and operators there exist conditional expectations for domains and operators. Let $\mathcal{D}$ be any domain in Φ. Let t be fixed in $\mathbb{R}^+$. Define

$$\mathbb{E}_t(\mathcal{D}) \stackrel{\text{def}}{=} \{f \in \Phi;\ P_tD_\sigma f \in \mathcal{D} \text{ for a.a. } \sigma \in \mathcal{P}_{[t}\}.$$

Then $\mathbb{E}_t(\mathcal{D})$ is a good candidate for beeing the conditional expectation of the domain $\mathcal{D}$ at time t.

Proposition I.6 – *For all $t \in \mathbb{R}^+$, and every domain $\mathcal{D}$ in Φ we have that*

i) $\mathbb{E}_t(\mathcal{D})$ *is a* u*-adapted domain for all* $u \geq t$*;*

ii) $\mathbb{E}_t(\mathbb{E}_s(\mathcal{D})) = \mathbb{E}_{s\wedge t}(\mathcal{D})$ *for all* s, t*;*

iii) *if* $\mathcal{D}$ *is a* t*-adapted domain then* $\mathcal{D} \subset \mathbb{E}_t(\mathcal{D})$*.*

The proofs are straightforward from the definitions. ■

Let H be any operator on Φ, with domain $\operatorname{Dom} H$. Define the operator $\mathbb{E}_t(H)$ on Φ whose domain is

$$\operatorname{Dom} \mathbb{E}_t(H) \stackrel{\text{def}}{=} \{f \in \mathbb{E}_t(\operatorname{Dom} H);\ \sigma \mapsto [HP_tD_{\sigma_{(t}}f](\sigma_{t)}) \in L^2(\mathcal{P})\}$$

and whose value on its domain is given by

$$[\mathbb{E}_t(H)f](\sigma) = [HP_tD_{\sigma_{(t}}f](\sigma_{t)}).$$

We now give a list of the main properties of this operator conditional expectation. We do not give any proofs as they require nothing else than applying the definitions and Propositions I.4, I.6 (cf [A-L]).

Proposition I.7 – *For any operator* H *on* Φ*, any* t *in* $\mathbb{R}^+$*, we have that*

i) $\mathbb{E}_t(H)$ *is a* u*-adapted operator for all* $u \geq t$*;*

ii) $\mathbb{E}_t(\mathbb{E}_s(H)) = \mathbb{E}_{s\wedge t}(H)$ *for all* s, t*;*

iii) H *is a* t*-adapted operator if and only if* $\operatorname{Dom} H$ *is* t*-adapted and* H *is a restriction of* $\mathbb{E}_t(H)$*;*

iv) *if* H *and* $\mathbb{E}_t(H)$ *are densely defined then* $\mathbb{E}_t(H)^*$ *is* t*-adapted and* $\mathbb{E}_t(H)^*$ *extends* $\mathbb{E}_t(H^*)$*.* ■

I.1.3 Quantum stochastic integrals revisited

The new definition of adaptedness, the associated conditional expectation of operators and the extension property iii) of Proposition I.7 are the key points for revisiting Hudson-Parthasarthy's definition of quantum stochastic integrals and extending them. There are many different ways of presenting these extensions. For this article I have choosen a new one which is different from the original approch ([A-L]).

From Hudson-Parthasarathy's stochastic calculus, or from many other ways, one can write explicit formulas for the action of the fundamental noises on a vector f of Φ. For example the creation process $(A_t^\dagger)_{t\geq 0}$ is given by

$$[A_t^\dagger f](\sigma) = \sum_{\substack{s\in\sigma \\ s\leq t}} f(\sigma \setminus s) \tag{I.2}$$

and the domain of $A_t^\dagger$ is exactely the set of f such that the above expression is square integrable in σ. In the same way the annihilation, conservation and time processes are respectively given by

$$[A_t f](\sigma) = \int_0^t f(\sigma \cup s)\, ds \tag{I.3}$$

$$[\Lambda_t f](\sigma) = \sum_{\substack{s\in\sigma \\ s\le t}} f(\sigma) \tag{I.4}$$

$$[tIf](\sigma) = \int_0^t f(\sigma)\, ds. \tag{I.5}$$

Note that the expressions (I.4) and (I.5) can be simplified but it is useful to keep them under this form.

Let $(H_t)_{t\geq 0}$ be an adapted process of operators. Assume that $(H_t)_{t\geq 0}$ is a step process that is, it is constant on the intervals of a partition $\{t_i;\ i\in \mathbb{N}\}$ of $\mathbb{R}^+$. Let us consider the annihilation integral case for example. If we want to define the quantum stochastic integral $\int_0^\infty H_s\, dA_s$ as in [H-P] we look at the Riemann sums $\sum_i H_{t_i}(A_{t_{i+1}} - A_{t_i})$. Note that the product $H_{t_i}(A_{t_{i+1}} - A_{t_i})$ is actually not a composition of operators but a tensor product of operators. Indeed, as our adaptedness coincides with Hudson-Parthasarathy's one we have that H_{t_i} is of the form $K\otimes I$ in the tensor product $\Phi = \Phi_{t_i]}\otimes \Phi_{[t_i}$ and $A_{t_{i+1}} - A_{t_i}$ is of the form $I\otimes K'$.

Now, as our conditional expectation extends the domain of already adapted operators we shall better look at

$$\sum_i \mathbb{E}_{t_i}(H_{t_i})(A_{t_{i+1}} - A_{t_i}) = \sum_i (A_{t_{i+1}} - A_{t_i})\mathbb{E}_{t_i}(H_{t_i})$$

if we want to enlarge the domain of the stochastic integral. We obtain

$$\begin{aligned}
\sum_i [(A_{t_{i+1}} - A_{t_i})\mathbb{E}_{t_i}(H_{t_i})f](\sigma) &= \sum_i \int_{t_i}^{t_{i+1}} [\mathbb{E}_{t_i}(H_{t_i})f](\sigma\cup s)\, ds \\
&= \sum_i \int_{t_i}^{t_{i+1}} [H_{t_i}P_{t_i}D_{(\sigma\cup s)_{(t_i}} f]((\sigma\cup s)_{t_i)})\, ds \\
&= \sum_i \int_{t_i}^{t_{i+1}} [H_{t_i}P_{t_i}D_{\sigma_{(t_i}\cup s} f](\sigma_{t_i)})\, ds \\
&= \sum_i \int_{t_i}^{t_{i+1}} [H_{t_i}P_{t_i}D_{\sigma_{(t_i,s)}} D_s D_{\sigma_{(s}} f]((\sigma_{s)})_{t_i)})\, ds \\
&= \sum_i \int_{t_i}^{t_{i+1}} [H_{t_i}D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds \\
&= \sum_i \int_{t_i}^{t_{i+1}} [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds \\
&= \int_0^\infty [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds.
\end{aligned}$$

The latter expression does no longer depend on the fact that $(H_t)_{t\geq 0}$ is a step process and we can keep it as a new definition of the annihilation integral $\int_0^\infty H_s\, dA_s$. Let us be more precise. For a given adapted process of operators $(H_t)_{t\geq 0}$ we consider the quantity

$$[A(H)f](\sigma) \stackrel{\text{def}}{=} \int_0^\infty [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds. \tag{I.6}$$

We define the annihilation integral of $(H_t)_{t\geq 0}$ to be the operator $A(H)$ whose domain is the set of $f \in \Phi$ such that f belongs to $\cap_s \mathbb{E}_s(\mathrm{Dom}\, H_s)$, such that $s \mapsto [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})$ is Lebesgue integrable for a.a. σ and such that the above expression $[A(H)f](\sigma)$ is square integrable in σ. The value of $A(H)$ on its domain is then given by (I.6).

The creation, conservation and time integrals of $(H_t)_{t\geq 0}$ are defined following the same procedure. We obtain respectively

$$[A^\dagger(H)f](\sigma) \stackrel{\text{def}}{=} \sum_{s\in\sigma} [H_s P_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds \tag{I.7}$$

$$[\Lambda(H)f](\sigma) \stackrel{\text{def}}{=} \sum_{s\in\sigma} [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds \tag{I.8}$$

$$[T(H)f](\sigma) \stackrel{\text{def}}{=} \int_0^\infty [H_s P_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds \tag{I.9}$$

with the obvious corresponding definitions for domains and values.

This new definition of quantum stochastic integrals can seem rather complicated and difficult to work with. The main interest of this definition is that it gives very satisfactory answers to several problems in quantum stochastic calculus. The first point is that this definition extends all the previous ones (Hudson-Parthasarathy (denoted H-P), Belavkin-Lindsay (denoted B-L, see [Bel] and [Lin]) and Attal-Meyer (denoted A-M, see [A-M]). It keeps all the advantages of the previous definitions without any of their inconvenients. Let us first show how our definition extends B-L's definition and thus H-P's one.

For a given adapted process of operators $(H_t)_{t\geq 0}$ define the B-L domains :

$$\mathrm{Dom}_{BL}\Lambda(H) = \{f \in \mathcal{F}; f \in \mathrm{Dom}\sqrt{N},\ \nabla_s f \in \mathrm{Dom}\, H_s \text{ for all } s \text{ and } s \mapsto H_s \nabla_s f \text{ is Skorohod integrable}\}$$

$$\mathrm{Dom}_{BL}A^\dagger(H) = \{f \in \mathcal{F}; f \in \mathrm{Dom}\, H_s \text{ for all } s \text{ and } s \mapsto H_s f \text{ is Skorohod integrable}\}$$

$$\mathrm{Dom}_{BL}A(H) = \{f \in \mathcal{F}; f \in \mathrm{Dom}\sqrt{N},\ \nabla_s f \in \mathrm{Dom}\, H_s \text{ for all } s \text{ and } \int_0^\infty ||H_s \nabla_s f||\, ds < \infty\}$$

$$\mathrm{Dom}_{BL}T(H) = \{f \in \mathcal{F}; f \in \mathrm{Dom}\, H_s \text{ for all } s \text{ and } \int_0^\infty ||H_s f||\, ds < \infty\}.$$

Theorem I.8 – *Let $(H_t)_{t\geq 0}$ be an adapted process of operators on Φ. Then*

i) $\mathrm{Dom}_{BL}\,\Lambda(H)$ *is included in* $\mathrm{Dom}\,\Lambda(H)$ *and on the smallest domain one has* $\Lambda(H)f = \mathcal{S}(H.\nabla.f)$;

ii) $\mathrm{Dom}_{BL}\,A^\dagger(H)$ *is included in* $\mathrm{Dom}\,A^\dagger(H)$ *and on the smallest domain one has* $A^\dagger(H)f = \mathcal{S}(H.f)$;

iii) $\mathrm{Dom}_{BL}\, A(H)$ *is included in* $\mathrm{Dom}\, A(H)$ *and on the smallest domain one has* $A(H)f = \int_0^\infty H_s \nabla_s f\, ds$;

iv) $\mathrm{Dom}_{BL}\, T(H)$ *is included in* $\mathrm{Dom}\, T(H)$ *and on the smallest domain one has* $T(H)f = \int_0^\infty H_s f\, ds$.

Proof

An element $f \in \mathcal{F}$ belongs to $\mathrm{Dom}_{BL}\, A^\dagger(H)$ if and only if f belongs to $\mathrm{Dom}\, H_s$ for all s and $\int_{\mathcal{P}} \left|\sum_{s\in\sigma}[H_s f](\sigma\backslash s)\right|^2 d\sigma$ is finite. But by adaptedness this is equivalent to

$$\int_{\mathcal{P}} \Big|\sum_{s\in\sigma}[H_s P_s D_{\sigma_{(s}} f](\sigma_{s)})\Big|^2 d\sigma < \infty.$$

This clearly proves that $\mathrm{Dom}_{BL}\, A^\dagger(H)$ is included in $\mathrm{Dom}\, A^\dagger(H)$ and that the two definitions of the creation integral coincide on the smallest domain.

In the case of the conservation integral, if $f \in \mathrm{Dom}\, \nabla$ and $\nabla_s f \in \mathrm{Dom}\, H_s$ for all s, we have $[H_s \nabla_s f](\sigma\backslash s) = [H_s P_s D_{\sigma_{(s}} \nabla_s f](\sigma_{s)})$. But the operators ∇_s and D_u commute for a.a. $u \geq s$; this property can be easily seen from the definitions of both operators, but it is interesting to note that it is a consequence of the fact that ∇_s is a s-adapted operator in A-L's sense. This gives $[H_s P_s D_{\sigma_{(s}} \nabla_s f](\sigma_{s)}) = [H_s P_s \nabla_s D_{\sigma_{(s}} f](\sigma_{s)}) = [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})$. We conclude in the same way as for the creation integral.

The case of the annihilation and time integrals are treated almost in the same way. ■

As one knows that B-L's definition extends H-P's one (cf [Lin] or also [Bia]) we obtain the following (a direct proof can be found in [A-L] Theorem 7.1).

Corollary I.9 – *Let* $(H_t)_{t\geq 0}$ *be an adapted process of operators defined on a coherent vector space* $\mathcal{E}(\mathcal{M})$ *for an admissible* $\mathcal{M}$. *Then* $\mathcal{E}(\mathcal{M})$ *is included in the domains of* $\Lambda(H), A^\dagger(H), A(H)$ *and* $T(H)$. *On* $\mathcal{E}(\mathcal{M})$, *the quantum stochastic integrals* $\Lambda(H), A^\dagger(H), A(H)$ *and* $T(H)$ *coincide with the corresponding H-P's integrals.* ■

I.1.4 Complete solution of A-M's equations

The most satisfactory consequence of A-L's extension of quantum stochastic integrals is the way it solves all the open problems which were occuring in A-M's definition. Let us recall about this latter.

In [A-M] is developped an extension of Hudson-Parthasarathy's quantum stochastic calculus. This extension has an important probabilistic source of inspiration. The underlying idea is as follows. Let f be an element of Φ, let $(H_t)_{t\geq 0}$ be an adapted process of operators on Φ. Let us consider formally the quantum stochastic integral $T = \int_0^\infty H_s\, dX_s$ where dX_s denotes any of the four basic integrands $d\Lambda_t, dA_t^\dagger, dA_t$ or dt. For all $t \in \mathbb{R}^+$ let $f_t = P_t f$ and let $T_t = \int_0^t H_s\, dX_s$. The vector process $(f_t)_{t\geq 0}$ is a vector martingale in Φ (in the sense $P_s f_t = f_s$ for

$s \le t$), and one has the representation

$$f_t = P_0 f + \int_0^t D_s f \, d\chi_s.$$

The process $(T_t)_{t\ge0}$ is an operator martingale when $dX = d\Lambda, dA^\dagger$ or dA (in the sense $\mathbb{E}_s(T_t) = T_s$ for $s \le t$), it is a time integral when $dX_t = dt$; hence $(T_t)_{t\ge0}$ can be considered as an operator semimartingale on Φ. When applying the operator semimartingale $(T_t)_{t\ge0}$ to the vector martingale $(f_t)_{t\ge0}$ we formally ask the result to be a vector semimartingale in Φ which satisfies a Ito-like formula

$$\begin{aligned} d(T_t f_t) &= (dT_t) f_t + T_t(df_t) + (dT_t)(df_t) \\ &= (H_t \, dX_t) f_t + T_t(D_t f \, d\chi_t) + (H_t \, dX_t)(D_t f \, d\chi_t). \end{aligned}$$

But in the tensor product structure $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ this gives

$$\begin{aligned} d(T_t f_t) &= (H_t \otimes dX_t)(f_t \otimes 1) + (T_t \otimes I)(D_t f \otimes d\chi_t) + (H_t \otimes dX_t)(D_t f \otimes d\chi_t) \\ &= H_t f_t \otimes dX_t 1 + T_t D_t f \otimes d\chi_t + H_t D_t f \otimes dX_t \, d\chi_t. \end{aligned}$$

We now need a formal table giving the values of $dX_t \, 1$ and $dX_t d\chi_t$ in the four cases. This table can be obtain by computing $(X_{t+h} - X_t)(\chi_{t+h} - \chi_t)$ with the formulas (I.2)-(I.5) and passing to the limit when h tends to 0. We obtain

$$\begin{cases} dA_t^\dagger \, 1 = d\chi_t & dA_t^\dagger \, d\chi_t = 0 \\ d\Lambda_t \, 1 = 0 & d\Lambda_t \, d\chi_t = d\chi_t \\ dA_t \, 1 = 0 & dA_t \, d\chi_t = dt \\ dt \, 1 = dt & dt \, d\chi_t = 0. \end{cases}$$

This finally gives

$$d(T_t f_t) = T_t D_t f \, d\chi_t + \begin{cases} H_t P_t f \, d\chi_t & \text{if } X_t = A_t^\dagger \\ H_t D_t f \, d\chi_t & \text{if } X_t = \Lambda_t \\ H_t D_t f \, dt & \text{if } X_t = A_t \\ H_t P_t f \, dt & \text{if } X_t = tI. \end{cases} \tag{I.10}$$

Note the following general result. If H is a t-adapted operator on Φ and f is a vector of Φ we have, for a.a. $u \ge t$, $D_u H(f - P_t f) = H D_u (f - P_t f) = H D_u f$ and, for a.a. $u \le t$, $D_u H(f - P_t f) = D_u (I - P_t) H f = 0$. Furthermore we have $P_0 H(f - P_t f) = P_0 (I - P_t) H f = 0$. By the Fock space predictable representation property (Theorem I.3) we conclude to the identity

$$H(f - P_t f) = \int_t^\infty H D_u f \, d\chi_u.$$

Adding this remark to (I.10) leads to the following equation

$$T_t f = \int_0^\infty T_{s\wedge t} D_s f \, d\chi_s + \begin{cases} \int_0^t H_s P_s f \, d\chi_s & \text{if } X_s = A_s^\dagger \\ \int_0^t H_s D_s f \, d\chi_s & \text{if } X_s = \Lambda_s \\ \int_0^t H_s D_s f \, ds & \text{if } X_s = A_s \\ \int_0^t H_s P_s f \, ds & \text{if } X_s = sI. \end{cases} \tag{I.11}$$

This equation is the basis of A-M's definition of quantum stochastic integrals. Indeed, one defines the process $(T_t)_{t\geq 0}$ to be the stochastic integral process $\int_0^t H_s\,dX_s$ on a domain $\mathcal{D}$ if for all $f \in \mathcal{D}$ equation (I.11) is meaningful (from the domain and integrability point of view) and holds true.

This presentation of the quantum stochastic integrals has many advantages. First, it is shown in [A-M] that it extends Hudson-Parthasarathy's definition that is, an adapted process $(T_t)_{t\geq 0}$ is solution of equation (I.11) on a coherent vectors domain if and only if it is the Hudson-Parthasarathy stochastic integral process $\int_0^t H_s\,dX_s$. Secondly, this new definition admits no arbitrary restriction in the domains of the operators. Indeed, in some good cases, equation (I.11) may be valid for all $f \in \Phi$; this is not the case with H-P's or B-L's definitions because, even if all the operators involved are very good, their quantum stochastic integrals have their domains limitated to $\mathcal{E}(\mathcal{M})$ or $\mathrm{Dom}\,\sqrt{N}$ respectively. This means that the A-M's integrals are the first one to allow some quantum stochastic integrals to be defined everywhere on Φ. As a consequence a large family of quantum stochastic integrals are composable and the quantum Ito formula can be obtained for actual composition of operators. This leads to the identification of an algebra of quantum semimartingales ([At1]) and many further developments that are explained in the next section. Thirdly (and this is a rather personal point of view), A-M's definition uses Ito integrals and the "derivatives" D_t on Φ instead of the Skorohod integrals and the Malliavin gradient as in B-L; Ito stochastic is more pleasant to use as it leads to nicer isometry formulas than Skorohod calculus.

But A-M's definition suffers of one very important defect: it is implicit. We have lost the explicitness of B-L's definition. Indeed equation (I.11) gives the process $(T_t)_{t\geq 0}$ as a solution of a kind of stochastic differential equation. Thus, in full generality, if one is given an adapted process $(H_t)_{t\geq 0}$ one does not know neither if (I.11) admits a solution, nor if the solution is unique, nor on which maximal domain it is valid. Only some sufficient conditions are given in [A-M].

This long discussion has for only target to focus the reader's attention on the important contribution of A-L's integrals to solve the above defects of A-M's integrals. Indeed, it appears that A-M's equations (I.11) **always** admit a solution which is **unique** and we are able to identify the **maximal** domain of the solution; this solution is A-L's corresponding stochastic integral. Let us now express all this in more rigorous terms.

For a given adapted process of operators $(H_t)_{t\geq 0}$ we have the A-L's stochastic integrals $\Lambda(H), A^\dagger(H), A(H)$ and $T(H)$ with their maximal domain as defined in previous subsection. It is useful here to slightly restrict these domains. The restricted domain of $\Lambda(H)$ (*resp.* $A^\dagger(H)$) is the intersection of its maximal domain with the set of $f \in \Phi$ satisfying $\int_0^\infty ||H_s D_s f||^2\,ds < \infty$ (*resp.* $\int_0^\infty ||H_s P_s f||^2\,ds < \infty$). The restricted domain of $A(H)$ (*resp.* $T(H)$) is the intersection of its maximal domain with the set of $f \in \Phi$ satisfying $\int_{\mathcal{P}} \left[\int_0^\infty \left|[H_s D_s f](\sigma)\right|ds\right]^2 d\sigma < \infty$ (*resp.* $\int_{\mathcal{P}} \left[\int_0^\infty \left|[H_s P_s f](\sigma)\right|ds\right]^2 d\sigma < \infty$).

In the following theorem we make use of the word *maximal* for operator processes which satisfy some properties; this maximality has to be understood for the following partial order on operator processes: two operator processes $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are such that $(S_t)_{t\geq 0} \subset (T_t)_{t\geq 0}$ if $S_t \subset T_t$ for all t in the sense of operators extension.

Theorem I.10 – *Let $(H_t)_{t\geq 0}$ be an adapted process of operators on Φ. Let $\mathcal{L}(H)$ denote the set of adapted processes of operators $(T_t)_{t\geq 0}$ on Φ satisfying*

1) $f \in \operatorname{Dom} T_t$ *if and only if* i) $D_s f \in \operatorname{Dom} T_{s\wedge t}$ *for a.a.* s;
 ii) $D_s f \in \operatorname{Dom} H_s$ *for a.a.* $s \leq t$;
 iii) $\int_0^t ||T_s D_s f||^2 + ||H_s D_s f||^2 \, ds < \infty$;

2) *for all* $f \in \operatorname{Dom} T_t$ *one has*

$$T_t f = \int_0^\infty T_{s\wedge t} D_s f \, d\chi_s + \int_0^t H_s D_s f \, d\chi_s.$$

Then $\mathcal{L}(H)$ contains one and only one maximal element which is the A-L stochastic integral process $(\Lambda_t(H))_{t\geq 0}$ with its restricted domain.

Let $\mathcal{A}^\dagger(H)$ denote the set of adapted processes of operators $(T_t)_{t\geq 0}$ on Φ satisfying

1) $f \in \operatorname{Dom} T_t$ *if and only if* i) $D_s f \in \operatorname{Dom} T_{s\wedge t}$ *for a.a.* s;
 ii) $P_s f \in \operatorname{Dom} H_s$ *for a.a.* $s \leq t$;
 iii) $\int_0^t ||T_s D_s f||^2 + ||H_s P_s f||^2 \, ds < \infty$;

2) *for all* $f \in \operatorname{Dom} T_t$ *one has*

$$T_t f = \int_0^\infty T_{s\wedge t} D_s f \, d\chi_s + \int_0^t H_s P_s f \, d\chi_s.$$

Then $\mathcal{A}^\dagger(H)$ contains one and only one maximal element which is the A-L stochastic integral process $(A_t^\dagger(H))_{t\geq 0}$ with its restricted domain.

Let $\mathcal{A}(H)$ denote the set of adapted processes of operators $(T_t)_{t\geq 0}$ on Φ satisfying

1) $f \in \operatorname{Dom} T_t$ *if and only if* i) $D_s f \in \operatorname{Dom} T_{s\wedge t}$ *for a.a.* s;
 ii) $D_s f \in \operatorname{Dom} H_s$ *for a.a.* $s \leq t$;
 iii) $\int_0^t ||T_s D_s f||^2 \, ds < \infty$;
 iv) $(H_s D_s f)_{s\leq t}$ *is Lebesgue integrable*;

2) *for all* $f \in \operatorname{Dom} T_t$ *one has*

$$T_t f = \int_0^\infty T_{s\wedge t} D_s f \, d\chi_s + \int_0^t H_s D_s f \, ds.$$

Then $\mathcal{A}(H)$ contains one and only one maximal element which is the A-L stochastic integral process $(A_t(H))_{t\geq 0}$ with its restricted domain.

Let $\mathcal{T}(H)$ denote the set of adapted processes of operators $(T_t)_{t\geq 0}$ on Φ satisfying

1) $f \in \operatorname{Dom} T_t$ *if and only if* i) $D_s f \in \operatorname{Dom} T_{s\wedge t}$ *for a.a.* $s \leq t$;
   ii) $P_s f \in \operatorname{Dom} H_s$ *for a.a.* $s \leq t$;
   iii) $\int_0^t ||T_s D_s f||^2 \, ds < \infty$;
   iv) $(H_s P_s f)_{s\leq t}$ *is Lebesgue integrable*;

2) *for all* $f \in \operatorname{Dom} T_t$ *one has*

$$T_t f = \int_0^\infty T_{s\wedge t} D_s f \, d\chi_s + \int_0^t H_s P_s f \, ds.$$

Then $\mathcal{T}(H)$ contains one and only one maximal element which is the A-L stochastic integral process $(T_t(H))_{t\geq 0}$ with its restricted domain.

Proof

We are going to present the proof in the case of the annihilation integral only as it contains all the difficulties. Let us prove that the A-L stochastic integral $A(L)$ is an element of $\mathcal{A}(H)$. We have

$$\begin{aligned}
&\int_0^\infty \mathbb{1}_{\mathcal{P}_{t]}}(\sigma)[H_s D_s D_{(\sigma\cup t)_{(s}} f]\big((\sigma\cup t)_{s)}\big) \, ds \\
&\quad = \int_0^\infty \mathbb{1}_{\mathcal{P}_{t]}}(\sigma)\mathbb{1}_{[0,t]}(s)[H_s D_s D_{\sigma_{(s}} D_t f](\sigma_{s)}) + \mathbb{1}_{\mathcal{P}_{t]}}(\sigma)\mathbb{1}_{]t,\infty[}(s)[H_s D_s f](\sigma\cup t) \, ds \\
&\quad = \int_0^\infty \mathbb{1}_{[0,t]}(s)[H_s D_s D_{\sigma_{(s}} D_t f](\sigma_{s)}) + \mathbb{1}_{]t,\infty[}(s)[D_t H_s D_s f](\sigma) \, ds. \qquad \text{(I.12)}
\end{aligned}$$

We have that f belongs to $\operatorname{Dom} A(H)$ if and only if the left hand side of (I.12), which is equal to $[D_t A(H) f](\sigma)$, is square integrable in both σ and t. Thus f belongs to the restricted domain of $A(H)$ if and only if each of the two terms of the right hand side are square integrable in both σ and t. But the right hand side is then equal to $[A_t(H) D_t f + \int_t^\infty D_t H_s D_s f \, ds](\sigma)$ which is actually equal to $[A_t(H) D_t f + D_t \int_0^\infty H_s D_s f \, ds](\sigma)$. Furthermore we have

$$P_0 A(H) f = P_0 \int_0^\infty H_s D_s f \, ds.$$

All this together with the Fock space predictable representation property gives the annihilation A-M equation for $A(H)$. We have proved that the A-L integral process $(A_t(H))_{t\geq 0}$ on its restricted domain is an element of the set $\mathcal{A}(H)$ of solutions to the annihilation A-M equation.

Now let us prove that any element of $\mathcal{A}(H)$ is included in $(A_t(H))_{t\geq 0}$. Let $(T_t)_{t\geq 0}$ be any element of $\mathcal{A}(H)$. Let f be an element of $\operatorname{Dom} T_t$ then by 2) one has, for any $\sigma = \{t_1 < t_2 < \ldots < t_n\} \in \mathcal{P}$

$$[T_t f](\sigma) = [T_t f](t_1, \ldots, t_n) = [T_{t_n \wedge t} D_{t_n} f](t_1, \ldots, t_{n-1}) + \int_0^t [H_s D_s f](t_1, \ldots, t_n)\, ds.$$

But $D_{t_n} f$ is an element of $\operatorname{Dom} T_{t_n \wedge t}$, hence one can apply the same kind of identity to the term $[T_{t_n \wedge t} D_{t_n} f](t_1, \ldots, t_{n-1})$. Iterating this procedure we get

$$\begin{aligned}[T_t f](t_1, \ldots, t_n) = [T_{t_1 \wedge t} D_{t_1} \ldots D_{t_n} f](\emptyset) + \int_0^{t_2 \wedge t} [H_s D_s D_{t_2} \ldots D_{t_n} f](t_1)\, ds \\ + \ldots + \int_0^t [H_s D_s f](t_1, \ldots, t_n)\, ds.\end{aligned}$$

From the equation 2) giving $T_t g$ for a general $g \in \operatorname{Dom} T_t$ we always have $[T_t g](\emptyset) = \int_0^t [H_s D_s g](\emptyset)\, ds$. Putting $t_0 = 0$ and $t_{n+1} = t$ we finally get

$$[T_t f](t_1, \ldots, t_n) = \sum_{i=0}^{n} \int_0^{t_{i+1} \wedge t} [H_s D_s D_{t_{i+1}} \ldots D_{t_n} f](t_1, \ldots t_i)\, ds.$$

Note that in the latter identity, because $H_s D_s g$ is always an element of $\Phi_{s]}$, the integral between 0 and $t_{i+1} \wedge t$ is actually an integral between $t_i \wedge t$ and $t_{i+1} \wedge t$; this holds for all $i \in \{0, \ldots, n\}$. Consequently, the integrals appearing in the previous sum are based on disjoint and complementary intervals. This finally gives

$$[T_t f](\sigma) = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1} \wedge t} [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds = \int_0^t [H_s D_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds. \qquad \blacksquare$$

We have proved that A-L's definition of quantum stochastic integrals is the unique and maximal solution of A-M's equations.

I.2 Quantum semimartingales

I.2.1 An algebra of quantum semimartingales

In A-L's definition (or A-M's as we know that they are equivalent) of quantum stochastic integrals there may happen that a process $(T_t)_{t \geq 0}$ of bounded operators admits a representation of the form

$$T_t = \int_0^t H_s\, d\Lambda_s + \int_0^t K_s\, dA_s^\dagger + \int_0^t L_s\, dA_s + \int_0^t M_s\, ds$$

on the whole of the Fock space Φ. Hence two such processes can be composed and we obtain the same Ito-like integration by part formula as in [H-P].

Theorem I.11 – *Let*

$$T_t = \int_0^t H_s\, d\Lambda_s + \int_0^t K_s\, dA_s^\dagger + \int_0^t L_s\, dA_s + \int_0^t M_s\, ds$$

$$T'_t = \int_0^t H'_s\, d\Lambda_s + \int_0^t K'_s\, dA_s^\dagger + \int_0^t L'_s\, dA_s + \int_0^t M'_s\, ds$$

be two processes of bounded operators whose integral representation is valid on all Φ. Then the process $(T_tT'_t)_{t\geq 0}$ also admits an integral representation on all Φ and this representation is given by

$$T_tT'_t = \int_0^t [H_sT'_s + T_sH'_s + H_sH'_s]\, d\Lambda_s + \int_0^t [K_sT'_s + T_sK'_s + H_sK'_s]\, dA^\dagger_s$$
$$+ \int_0^t [L_sT'_s + T_sL'_s + L_sH'_s]\, dA_s + \int_0^t [M_sT'_s + T_sM'_s + L_sK'_s]\, ds. \quad \text{(I.13)}$$

Proof

To get this Ito formula it suffices to write the A-M equations satisfied by T'_tf for a $f \in \Phi$ and then to develop the A-M equation for $T_tT'_tf$. It is almost straightforward and it is not worth developing pages of computations here as this Ito formula is finally just an extension of the well-known Hudson-Parthasarathy's one (cf [A-L]). ■

This Ito formula allows to identify a useful class of adapted processes of operators. Define $\mathcal{S}$ to be the space of adapted processes $(T_t)_{t\geq 0}$ of bounded operators on Φ admitting an integral representation

$$T_t = \int_0^t H_s\, d\Lambda_s + \int_0^t K_s\, dA^\dagger_s + \int_0^t L_s\, dA_s + \int_0^t M_s\, ds$$

on a coherent vector space $\mathcal{E}(\mathcal{M})$, with all the operators H_s, K_s, L_s, M_s beeing bounded and satisfying:

$s \mapsto ||M_s||$ is locally integrable,

$s \mapsto ||K_s||$ and $s \mapsto ||L_s||$ are locally square integrable,

$s \mapsto ||H_s||$ is locally bounded.

Theorem I.12 – *Every element $(T_t)_{t\geq 0}$ of $\mathcal{S}$ has its integral representation which can be extended to all Φ. The space $\mathcal{S}$ is a $*$-algebra under composition and adjoint mapping.*

Proof

To show that the integral representation of $(T_t)_{t\geq 0}$ can be extended to all Φ we use the A-M point of view. We know that $(T_t)_{t\geq 0}$ satisfies

$$T_tf = \int_0^\infty T_{s\wedge t}D_sf\, d\chi_s + \int_0^t H_sD_sf\, d\chi_s + \int_0^t K_sP_sf\, d\chi_s + \int_0^t L_sD_sf\, ds$$
$$+ \int_0^t M_sP_sf\, ds \quad \text{(I.14)}$$

for all $f \in \mathcal{E}(\mathcal{M})$. It is easy to check that $t \mapsto ||T_t||$ has to be locally bounded (indeed, this is true for $t \mapsto ||\int_0^t M_s\, ds||$ and $T_\cdot - \int_0^\cdot M_s\, ds$ is an operator martingale). Let us denote by $B_t(f)$ the right hand side of (I.14). We have

$$||B_t(f)||^2 \leq 5\Big[\sup_{s\leq t} ||T_s||^2 + \sup_{s\leq t} ||H_s||^2 + \int_0^t ||K_s||^2\, ds + \int_0^t ||L_s||^2\, ds$$

$$+ \left[\int_0^t ||M_s|| \, ds \right]^2 \Big] ||f||^2. \qquad \text{(I.15)}$$

Let $f \in \Phi$ be fixed. Take a sequence $(f_n)_n$ of elements of $\mathcal{E}(\mathcal{M})$ converging to f. We have that $T_t f_n$ converges to $T_t f$ and, from (I.15), that $B_t(f_n)$ converges to $B_t(f)$. Thus (I.14) is valid for all $f \in \Phi$. This means that the integral representation of $(T_t)_{t\geq 0}$ is valid on all Φ.

Any two elements $(T_t)_{t\geq 0}$ and $(T'_t)_{t\geq 0}$ of $\mathcal{S}$ are thus composable. The quantum Ito formula (I.13) gives the coefficient of the integral representation of $(T_t T'_t)_{t\geq 0}$. For example the coefficient of ds is $T_s M'_s + M_s T'_s + L_s K'_s$. As $||T_\cdot||$ and $||T'_\cdot||$ are locally bounded, as $||M_\cdot||$ and $||M'_\cdot||$ are locally integrable, as $||L_\cdot||$ and $||K'_\cdot||$ are locally square integrable we get that the coefficient of ds is locally integrable. In the same way for the others coefficients we get that $(T_t T'_t)_{t\geq 0}$ is an element of $\mathcal{S}$.

The stability of $\mathcal{S}$ under the adjoint mapping is clear since the adjoint process $(T^*_t)_{t\geq 0}$ admits on $\mathcal{E}(\mathcal{M})$ the representation

$$T^*_t = \int_0^t H^*_s \, d\Lambda_s + \int_0^t L^*_s \, dA^\dagger_s + \int_0^t K^*_s \, dA_s + \int_0^t M^*_s \, ds. \qquad \blacksquare$$

We dispose of a very usefull algebra of quantum semimartingales $\mathcal{S}$ on which a quantum stochastic calculus can be developed. But defining $\mathcal{S}$ as previously leads to inconvenients. Indeed, this definition is based on the integral representation of $(T_t)_{t\geq 0}$ and the regularity of its coefficients. These properties are difficult to check in general; it would be worth having a characterization of $\mathcal{S}$ based on the process $(T_t)_{t\geq 0}$ alone. This is achieved in the following way.

An adapted process of bounded operators $(T_t)_{t\geq 0}$ is a *regular semimartingale of operators* if there exists an absolutely continuous measure μ on $\mathbb{R}^+$ such that for all $f \in \mathcal{E}(\mathcal{M})$ (for some coherent space $\mathcal{E}(\mathcal{M})$), all $r < s < t$ one has

$$||T_t P_r f - T_s P_r f||^2 \leq ||P_r f||^2 \mu([s,t]) \qquad \text{(I.16)}$$

$$||T^*_t P_r f - T^*_s P_r f||^2 \leq ||P_r f||^2 \mu([s,t]) \qquad \text{(I.17)}$$

$$||P_s T_t P_r f - T_s P_r f|| \leq ||P_r f|| \, \mu([s,t]). \qquad \text{(I.18)}$$

Theorem I.13 – *An adapted process of bounded operators $(T_t)_{t\geq 0}$ is an element of the algebra $\mathcal{S}$ if and only if it is a regular semartingale of operators.*

Proof

If $(T_t)_{t\geq 0}$ is an element of $\mathcal{S}$, norm estimates such as (I.15) easily give that $(T_t)_{t\geq 0}$ is a regular semimartingale. Conversly if $(T_t)_{t\geq 0}$ is a regular semimartingale of operators the inequality (I.18) actually means that the vector process $(T_t P_r f)_{t\geq r}$ is a Hilbertian quasimartingale in the sense on Enchev (cf section I.4). That is, for $r < s < t$, one has the representation

$$T_t P_r f - T_s P_r f = m_t - m_s + \int_s^t \xi_u \, du$$

for a vector martingale $(m_t)_{t\geq 0}$ and an adapted process of vectors $(\xi_u)_{u\geq r}$ satisfying $||\xi_u|| \leq ||P_r f|| \mu'(u)$, where μ' is the derivative of μ. We thus have

$$P_s T_t P_r f - T_s P_r f = \int_s^t P_s \xi_u \, du.$$

If $(T_t)_{t\geq 0}$ had an integral representation we would have from A-M's equation

$$P_s T_t P_r f - T_s P_r f = \int_s^t P_s M_u P_r f \, du.$$

Hence we can identify the time coefficients M_t of $(T_t)_{t\geq 0}$. The norm of M_t is dominated by $\mu'(t)$ so it is integrable. Subtracting $\int_0^t M_s \, ds$ to T_t we obtain a martingale of operators satisfying the first two inequalities (I.16) and (I.17). That is, we obtain a regular martingale in the sense of Parthasarathy-Sinha ([PS1]). Combining their result and Meyer's treatment of it ([Me2]) we get the complete integral representation of $(T_t)_{t\geq 0}$ with the good estimates for the coefficients. The process $(T_t)_{t\geq 0}$ is an element of $\mathcal{S}$. ■

I.2.2 Quantum square and angle brackets

The algebra $\mathcal{S}$ is a good departure point for developing a quantum stochastic calculus close to the classical one. For any elements T, T' of $\mathcal{S}$ one can define two new adapted processes, *the integrals of* T *with respect to* T' :

$$\int_0^t T_s \, dT'_s \stackrel{\text{def}}{=} \int_0^t T_s H'_s \, d\Lambda_s + \int_0^t T_s K'_s \, dA^\dagger_s + \int_0^t T_s L'_s \, dA_s + \int_0^t T_s M'_s \, ds$$

$$\int_0^t dT'_s \, T_s \stackrel{\text{def}}{=} \int_0^t H'_s T_s \, d\Lambda_s + \int_0^t K'_s T_s \, dA^\dagger_s + \int_0^t L'_s T_s \, dA_s + \int_0^t M'_s T_s \, ds.$$

These processes are not in general elements of $\mathcal{S}$. Indeed, they admit an integral representation with coefficients satisfying the same norm conditions as elements of $\mathcal{S}$, but the operators $\int_0^t T_s \, dT'_s$ and $\int_0^t dT'_s \, T_s$ themselves have no reason to be bounded. So, in general, the integrals $\int T \, dT'$ and $\int dT' \, T$ are defined only on $\mathcal{E}(\mathcal{M})$. But note that, because of the norm properties of their coefficients, their integral representation can always be extended to the whole of their domain.

We define $\mathcal{S}'$ to be the space of adapted processes $(T_t)_{t\geq 0}$ on Φ, admitting an integral representation on $\mathcal{E}(\mathcal{M})$ such that the coefficients of the representation satisfy the same conditions as in the definition of $\mathcal{S}$. The only difference between the definitions of $\mathcal{S}'$ and $\mathcal{S}$ is that in $\mathcal{S}'$ the process $(T_t)_{t\geq 0}$ *is not necessarily made of bounded operators.* Thus $\mathcal{S}$ is a subspace of $\mathcal{S}'$ (actually, surprising as it may be, there exists a very simple and natural bijection between $\mathcal{S}$ and $\mathcal{S}'$, cf [At2]).

We have seen that if $(T_t)_{t\geq 0}$ and $(T'_t)_{t\geq 0}$ are element of $\mathcal{S}$ we have that $\int T \, dT'$ and $\int dT' \, T$ are element of $\mathcal{S}'$. It also appears that $\mathcal{S}'$ is a natural space for defining the quantum square and angle brackets operator processes.

Let $(T_t)_{t\geq 0}$ and $(T_t)'_{t\geq 0}$ be two elements of $\mathcal{S}'$. Define the *square bracket* of T and T' to be the operator process

$$[T, T']_t \stackrel{\text{def}}{=} \int_0^t H_s H'_s \, d\Lambda_s + \int_0^t H_s K'_s \, dA^\dagger_s + \int_0^t L_s H'_s \, dA_s + \int_0^t L_s K'_s \, ds.$$

Define the *angle bracket* of T and T' to be the operator process

$$\langle T, T' \rangle_t \stackrel{\text{def}}{=} \int_0^t L_s K'_s \, ds.$$

It is clear that $[T, T']$ and $\langle T, T' \rangle$ are both elements of $\mathcal{S}'$.

In chapter II we relate these quantum brackets to the usual brackets of classical stochastic calculus. But from the definitions one can immediatly check that the quantum brackets satisfy the usual properties of the classical square and angle brackets. This is expressed in the following proposition. Note that any element of $\mathcal{S}'$ is the sum of an operator martingale (the sum of the $d\Lambda$, $dA^\dagger$ and dA terms) and a "finite variation" part (the time integral).

Proposition I.14 – *Let S, S', S'' be elements of $\mathcal{S}'$, let T be an element of $\mathcal{S}$. We have the following properties for their non-commutative brackets.*

i) *$[S, S']$ and $\langle S, S' \rangle$ depend only on the martingale part of S and S'.*

ii) *If S and S' are (operator) martingales in $\mathcal{S}$, then $SS' - \langle S, S' \rangle$ and $SS' - [S, S']$ are also martingales.*

iii) *The quantum brackets are associative that is, for example $[S, [S', S'']] = [[S, S'], S'']$.*

iv) *We have $[\int T \, dS, S'] = \int T \, d[S, S']$ and $[S', \int dS \, T] = \int d[S', S] \, T$. The same identities hold for the angle bracket.*

v) *We have the following adjoint relations*

$$[S, S']^* = [S'^*, S^*]$$
$$\langle S, S' \rangle^* = \langle S'^*, S^* \rangle. \qquad \blacksquare$$

There is one more important point to be discussed. In classical stochastic calculus the square bracket of two semimartingales is the limit of the associated quadratic variations, the angle bracket is also the limit of the quadratic variation but conditionned at proper times. One can wonder if the same properties hold for the quantum brackets. Furthermore, the algebra $\mathcal{S}$ admits an intrinsic characterization (Theorem I.13) that is, a characterization which depends on the processes $(T_t)_{t \geq 0}$ themselves and not on their integral representation. It is thus disappointing to have a definition of the quantum brackets which is based on the integral representation. Getting a characterization of the brackets through limits of quadratic variations makes the connection with the classical theorems and at the same time gives an intrinsic characterization of the quantum brackets.

Theorem I.15 – *Let $(S_t)_{t \geq 0}$ and $(T_t)_{t \geq 0}$ be two elements of $\mathcal{S}$. Let t be fixed in $\mathbb{R}^+$, let $\{0 = t_0^n < t_1^n < \ldots < t_n^n = t\}$ be a sequence of subdivisions of $[0, t]$ whose diameter tends to 0 when n tends to $+\infty$. Let $\mathcal{M}$ be any admissible subspace of $L^2_{lb}(\mathbb{R}^+)$, the space of locally bounded square integrable functions on $\mathbb{R}^+$.*

Then, on $\mathcal{E}(\mathcal{M})$, the square bracket $[T,S]_t$ is the weak limit when n tends to $+\infty$ of the quadratic variations

$$\sum_i (T_{t^n_{i+1}} - T_{t^n_i})(S_{t^n_{i+1}} - S_{t^n_i})$$

and, on Φ, the angle bracket $\langle T,S\rangle_t$ is the weak limit of the conditioned quadratic variations

$$\sum_i P_{t^n_i}(T_{t^n_{i+1}} - T_{t^n_i})(S_{t^n_{i+1}} - S_{t^n_i})P_{t^n_i}.$$

The proof for the angle bracket is straightforward, whereas there are rather long and complicated estimates to be developed in order to prove the convergence of the quadratic variations to the square bracket. It is not interesting to develop them here and we refer to the original reference [At1]. ■

I.2.3 Functional quantum Ito formulas

With the development of the bracket language the integration by part formula (I.13) can be written in a simple way. Indeed, for all S and T elements of $\mathcal{S}$ we have

$$S_tT_t = \int_0^t S_s\,dT_s + \int_0^t dS_s\,T_s + [S,T]_t. \tag{I.19}$$

As $\mathcal{S}$ is an algebra we can first consider polynomial functionals of elements of $\mathcal{S}$. Working by induction on (I.19) we obtain the following result.

Proposition I.16 – *Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}$. Let $n \in I\!\!N$ be fixed. Then one has*

$$T_t^n = \int_0^t H_n(s)\,d\Lambda_s + \int_0^t K_n(s)\,dA_s^\dagger + \int_0^t L_n(s)\,dA_s + \int_0^t M_n(s)\,ds \tag{I.20}$$

where

$$\begin{aligned}
H_n(s) &= (T_s + H_s)^n - T_s^n \\
K_n(s) &= \sum_{p+q=n-1} T_s^p K_s (T_s + H_s)^q \\
L_n(s) &= \sum_{p+q=n-1} (T_s + H_s)^p L_s\, T_s^q \\
M_n(s) &= \sum_{p+q=n-1} T_s^p M_s\, T_s^q + \sum_{p+q+r=n-2} T_s^p L_s (T_s + H_s)^q K_s\, T_s^r.
\end{aligned}$$

■

We thus dispose of a quantum Ito formula for polynomial functions. The natural question now is whether the algebra $\mathcal{S}$ is stable under analytic functions. The answer is positive and this result is due to G. Vincent-Smith [ViS]. For getting this result he uses the functional calculus on operators for analytic functions. Let us recall it.

Let T be a bounded operator on a Hilbert space. For all λ in the resolvent set of T, let $R_\lambda(T)$ denote the resolvent of T at the point λ. Let f be an analytic function on the disc $D(0,R)$ where $R > ||T||$. Then the operator $f(T)$ is defined by

$$f(T) = \oint_\gamma f(\lambda) R_\lambda(T)\, d\lambda$$

where γ is the circle $C(0,r)$ with $R > r > ||T||$ and $\oint_\gamma$ is $\frac{1}{2\pi i}$ times the contour integral round γ.

Theorem I.17 – *Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}$. Let $T \in I\!R^+$ be fixed. Let $\rho = \max\{||T_t||, ||T_t + H_t||; t \leq T\}$. Let f be an analytic function of $D(0,R)$ for some $R > \rho$. Then $(f(T_t))_{t\geq 0}$ is an element of $\mathcal{S}$ and its integral representation for $t \leq T$ is given by*

$$f(T_t) = f(0) + \int_0^t H_f(s)\, d\Lambda_s + \int_0^t K_f(s)\, dA_s^\dagger + \int_0^t L_f(s)\, dA_s + \int_0^t M_f(s)\, ds \quad \text{(I.21)}$$

where

$$\begin{aligned}
H_f(s) &= f(T_s + H_s) - f(T_s) \\
K_f(s) &= \oint_\gamma f(\lambda)\big(R_\lambda(T_s)\, K_s\, R_\lambda(T_s + H_s)\big)\, d\lambda \\
L_f(s) &= \oint_\gamma f(\lambda)\big(R_\lambda(T_s + H_s)\, L_s\, R_\lambda(T_s)\big)\, d\lambda \\
M_f(s) &= \oint_\gamma f(\lambda)\big(R_\lambda(T_s)\, M_s\, R_\lambda(T_s)\big)\, d\lambda \\
&\quad + \oint_\gamma f(\lambda)\big(R_\lambda(T_s)\, L_s\, R_\lambda(T_s + H_s)\, K_s\, R_\lambda(T_s)\big)\, d\lambda
\end{aligned}$$

and γ is the circle $C(0,r)$ with $R > r > \rho$.

Proof

Let us describe in few words Vincent-Smith's proof. One observes that formula (I.21) coincides with (I.20) for f beeing a polynomial function. One obtains the general formula by passing to the limit on polynomial approximation of analytic functions. Using the usual norm estimates on the resolvants we get that the coefficients of (I.21) satisfy the condition for $(f(T_t))_{t\geq 0}$ to be an element of $\mathcal{S}$. ∎

When operators are self-adjoint it is known that functional calculus can be developed further. Let T be a self-adjoint bounded operator on a Hilbert space. Let $f : I\!R \longrightarrow I\!R$ be a Lebesgue integrable function whose Fourier transform $\widehat{f}$ is also Lebesgue integrable. Then one defines $f(T)$ by

$$f(T) = \int_{I\!R} \widehat{f}(p) e^{ipT}\, dp.$$

Define the space $C^{2+}_{\text{loc}} \stackrel{\text{def}}{=} \{f \in L^1(I\!R); p^2 \widehat{f}(p) \in L^1(I\!R)\}$. Then for function f in C^{2+}_{loc} Vincent-Smith also obtain a very satisfactory result about the algebra $\mathcal{S}$. We give the result without proof.

Theorem I.18 – *Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}$ made of self-adjoint operators. Let*

$$T_t = \int_0^t H_s\, d\Lambda_s + L_s\, dA_s + L_s^*\, dA_s^\dagger + M_s\, ds$$

be its integral representation (the operators H_s and M_s have to be self-adjoint). Let f be an element of C^{2+}_{loc}. Then the process $(f(T_t))_{t\geq 0}$ is an element of $\mathcal{S}$ made of self-adjoint operators and whose integral representation is given by

$$T_t = \int_0^t H_f(s)\, d\Lambda_f(s) + L_f(s)\, dA_s + L_f^*(s)\, dA_s^\dagger + M_f(s)\, ds$$

where

$$\begin{aligned}
H_f(s) &= f(T_s + H_s) - f(T_s)\\
L_f(s) &= \int_{\mathbb{R}} ip\widehat{f}(p)\Big\{\int_0^1 e^{ip(1-u)T_s} L_s\, e^{ipu(T_s+H_s)}\, du\Big\}\, dp\\
M_f(s) &= \int_{\mathbb{R}} ip\widehat{f}(p)\Big\{\int_0^1 e^{ip(1-u)T_s} M_s\, e^{ipuT_s}\, du\Big\}\, dp\\
&\quad + \int_{\mathbb{R}} ip\widehat{f}(p)\Big\{\int_0^1\int_0^1 ue^{ip(1-u)T_s} L_s\, e^{ipu(1-v)(T_s+H_s)} L_s^*\, e^{ipuvT_s}\, du\, dv\Big\}\, dp. \blacksquare
\end{aligned}$$

I.3 Chaotic expansion of operators

This section is almost independent from the rest of the article. Some parts are rather technical and can be skiped at first reading.

I.3.1 Maassen kernels as iterated integral representations

The second kind of probabilistic contribution to quantum stochastic calculus we want to present concerns Maassen kernel representation of operators on Fock space.

Maassen kernels are an alternative to integral representation of operator on Fock space. They have been defined under their first form with two arguments by Maassen [Ma2] and under their definitive form with three arguments by Meyer [Me3]. Maassen kernels are to the quantum stochastic calculus as what chaotic expansions are to the classical one. That is, a Maassen kernel is *formally* an operator T on Φ which admits a representation as a series of iterated non-commutative stochastic integrals of *scalar* operators with respect to the creation, annihilation and conservation processes. Using the same kind of Guichardet notation as before, this can be written

$$T = \int_{\mathcal{P}^3} \widehat{T}(\alpha,\beta,\gamma)\, dA_\alpha^\dagger\, d\Lambda_\beta\, dA_\gamma$$

where for $X = A^\dagger, \Lambda$ or A and for $\tau = \{t_1,\ldots,t_n\}$ the notation dX_τ means $dX_{t_1}\ldots dX_{t_n}$.

This has no rigourous meaning, in particular this form does no longer respect the adaptedness of the integrated processes and the convergence of the series has to be studied. But one can formally describe how such an operator acts on a vector of the Fock space. It suffices to determine formally the action of an operator of the form $dA^\dagger_\alpha\, d\Lambda_\beta\, dA_\gamma$ on an element f of the Fock space of the form $d\chi_{t_1}\dots d\chi_{t_n}$. One obtains (cf [Me3]) that the image Tf of a vector $f\in\Phi$ under T has the following chaotic expansion

$$[Tf](\sigma)=\int_{\mathcal{P}}\sum_{\alpha+\beta+\gamma=\sigma}\widehat{T}(\alpha,\beta,\mu)f(\mu+\beta+\gamma)\,d\mu. \tag{I.22}$$

Although this is not rigourous, the latter identity is for some "good" operators and some "good" vectors. This is the rigourous definition of Maassen and Meyer. An operator T from Fock space Φ to itself is said to have a *Maassen kernel* if there exists a set-function $\widehat{T}$ on $\mathcal{P}^3$ (in fact $\widehat{T}$ needs only to be defined on pairwise disjoint α,β,γ in $\mathcal{P}^3$) such that for a dense subset of vectors f of Φ one has (I.22) to be well-defined and to hold.

In his article Maassen develops the theory of such kernels. It is very satisfactory as under some conditions they form a $*$-algebra of operators (which are not all bounded operators). The formulas for composition and adjoint are simple. Computations with such operators are very simple and explicit. But this theory suffers from an important defect: one does not know any general criterion for a given operator T on Φ to admit such a kernel representation. One knows only a list of examples (operators of multiplication by a random variable in some probabilistic interpretation of Φ, some solutions of quantum stochastic differential equations,...). This lake of criterion makes the theories of quantum stochastic integrals and Maassen kernels beeing disconected, whereas they should not be. What we want to present here is an idea which is hoped to give a satisfactory criterion for an operator to admit a kernel. For the moment this idea applies only to a class of operators, the Hilbert-Schmidt operators on Φ, but there is some hope that the A-L point of view on quantum stochastic integral may make the idea apply to a larger class of operators. Let us explain the idea and its probabilistic inspiration.

Consider for example the Wiener space $(\Omega,\mathcal{F},P)$ and its canonical Brownian motion $(W_t)_{t\geq 0}$. One knows that square integrable functionals of $(W_t)_{t\geq 0}$ admit a chaotic expansion. That is, any random variable f in $L^2(\Omega)$ can be represented as a series of iterated stochastic integral of deterministic functions with respect to $(W_t)_{t\geq 0}$. That is, a representation of the form

$$f=\mathbb{E}[f]+\sum_{n=1}^{\infty}\int_{0<t_1<\dots<t_n} f_n(t_1,\dots,t_n)\,dW_{t_1}\,\dots\,dW_{t_n}.$$

One way for proving the chaotic expansion property of square integrable Wiener functionals is to iterate the predictable representation property of $(W_t)_{t\geq 0}$. Indeed, the Brownian motion has the predictable representation property that is, every element f of $L^2(\Omega)$ can be written

$$f=\mathbb{E}[f]+\int_0^\infty \Psi_s\,dW_s,$$

where Ψ is a predictable process in $L^2(\Omega,\mathcal{F},P)$. So, for almost all s, Ψ_s is an element of $L^2(\Omega, \mathcal{F}, P)$. Then one can apply the predictable representation property again and get

$$\Psi_s = \mathbb{E}[\Psi_s] + \int_0^s \Psi'_{s,u}\, dW_u,$$

for a predictable process $(\Psi'_{s,u})_{u\leq s}$. Inserting this identity in the representation of f, one obtains

$$f = \mathbb{E}[f] + \int_0^\infty \mathbb{E}[\Psi_s]\, dW_s + \int_0^\infty \int_0^s \Psi'_{s,u}\, dW_u\, dW_s.$$

Repeating the procedure we obtain

$$\begin{aligned} f = \mathbb{E}[f] &+ \int_0^\infty \mathbb{E}[\Psi_s]\, dW_s + \int_0^\infty \int_0^s \mathbb{E}[\Psi'_{s,u}]\, dW_u\, dW_s \\ &+ \int_0^\infty \int_0^s \int_0^u \Psi''_{s,u,v}\, dW_v\, dW_u\, dW_s. \end{aligned}$$

One can iterate this operation arbitrarily many times. One then obtains two terms in the representation of f: a sum of iterated stochastic integrals of *deterministic* functions and an iterated stochastic integral of some predictable process. The first term constitutes the beginning of the chaotic expansion of f, the second term tends to 0 when one iterates the procedure. In this way we obtain the chaotic representation of f.

We want to apply the same idea to the non-commutative case. For this section we shall change our notations in order to simplify the equations. The conservation process $(\Lambda_t)_{t\geq 0}$ shall be denoted $(A_t^\circ)_{t\geq 0}$, the annihilation process $(A_t)_{t\geq 0}$ is denoted $(A_t^-)_{t\geq 0}$, the creation process remains unchanged. Suppose that there exists a family $\mathcal{I}$ of operators T on Φ which admit an integral representation

$$T = \lambda I + \sum_{\varepsilon\in\{\circ,\dagger,-\}} \int_0^\infty H_s^\varepsilon\, dA_s^\varepsilon$$

on a certain domain $\mathcal{D}$ and such that all the coefficients H_s^ε belong to the same family $\mathcal{I}$. Then one can represent each of the operators H_s^ε in the same way:

$$H_s^\varepsilon = \lambda_s^\varepsilon I + \sum_{\varepsilon'\in\{\circ,\dagger,-\}} \int_0^s H_{s,u}^{\varepsilon,\varepsilon'}\, dA_u^{\varepsilon'}.$$

Coming back to the representation of T we get

$$T = \lambda I + \sum_{\varepsilon\in\{\circ,\dagger,-\}} \int_0^\infty \lambda_s^\varepsilon I\, dA_s^\varepsilon + \sum_{\varepsilon,\varepsilon'\in\{\circ,\dagger,-\}} \int_0^\infty \int_0^s H_{s,u}^{\varepsilon,\varepsilon'}\, dA_u^{\varepsilon'}\, dA_s^\varepsilon.$$

One can iterate this procedure infinitely many times. We see that at step n the representation of T can be decomposed into two parts: the first part consists in a sum of iterated integrals of *scalar operators* with respect to the three quantum noises, the second part is a sum of order n iterated integrals of some operator processes. Mimicking the probabilistic case, our hope is that the second term should tend to 0 when n goes to $+\infty$. What would remain is the expression of T as a series of iterated quantum stochastic integrals of scalar operators; that is, the *non-commutative chaotic expansion* of T. But from this expression it is not difficult to recover the Maassen kernel of the operator, it is just a question of reordering and of some analysis to make the series converging (cf [A-H]).

I.3.2 The case of Hilbert-Schmidt operators

The idea developed above applies very well to Hilbert-Schmidt operators. It is not very difficult to show that Hilbert-Schmidt operators admit a Maassen kernel, independently of our procedure. This result was already proved in [HLP]. Let us show a simple proof.

Theorem I.19 – *Let H be an Hilbert-Schmidt operator from Φ into Φ. Then there exists a mapping $\widehat{H}$ from $\mathcal{P}^3$ into $\mathbb{C}$ such that for all $f \in \Phi$, all $\sigma \in \mathcal{P}$ we have*

$$[Hf](\sigma) = \int_{\mathcal{P}} \sum_{\alpha+\beta+\gamma=\sigma} \widehat{H}(\alpha, \beta, \mu) f(\mu + \beta + \gamma)\, d\mu.$$

Proof

As Φ is isomorphic to the space $L^2(\mathcal{P})$, where $\mathcal{P}$ is endowed with the σ-finite measure described previously, H is then a Hilbert-Schmidt operator from $L^2(\mathcal{P})$ to $L^2(\mathcal{P})$. Therefore, H admits a kernel representation that is, there exists a mapping φ from $\mathcal{P}^2$ into $\mathbb{R}$ such that $\int_{\mathcal{P}^2} \varphi(\alpha, \beta)^2\, d\alpha\, d\beta < \infty$ and such that for all $f \in L^2(\mathcal{P})$

$$[Hf](\sigma) = \int_{\mathcal{P}} \varphi(\sigma, \mu) f(\mu)\, d\mu.$$

Or else

$$[Hf](\sigma) = \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} 1\!\!1_{\{\sigma \backslash \alpha = \emptyset\}}\, \varphi(\alpha, \mu) f(\mu + \sigma \backslash \alpha)\, d\mu.$$

The Möbius inversion formula gives $\sum_{\beta \subset \gamma} (-1)^{|\beta|} = 1\!\!1_{\{\gamma = \emptyset\}}$ for all $\gamma \in \mathcal{P}$. So if one puts $\widehat{H}(\alpha, \beta, \gamma) = (-1)^{|\beta|}\, \varphi(\alpha, \gamma)$ we get

$$\begin{aligned}
[Hf](\sigma) &= \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} \sum_{\beta \subset \sigma \backslash \alpha} (-1)^{|\beta|}\, \varphi(\alpha, \mu) f(\mu + \sigma \backslash \alpha)\, d\mu \\
&= \int_{\mathcal{P}} \sum_{\sigma \subset \alpha} \sum_{\beta \subset \sigma \backslash \alpha} \widehat{H}(\alpha, \beta, \mu) f(\mu + \sigma \backslash \alpha)\, d\mu \\
&= \int_{\mathcal{P}} \sum_{\alpha+\beta+\gamma=\sigma} \widehat{H}(\alpha, \beta, \mu) f(\mu + \beta + \gamma)\, d\mu. \qquad \blacksquare
\end{aligned}$$

So Hilbert-Schmidt operators do admit a Maassen kernel. In [At3] it is proved that the integral representation iteration procedure works perfectly well in this context; that is, one can prove that every Hilbert-Schmidt operator admits a Maassen kernel by iterating its integral representation. We recover the kernel described above, but this latter can be completely described from the operator itself. Let us resume the main results of this article.

A martingale $(H_t)_{t\geq 0}$ of operators from Φ to Φ is an *Hilbert-Schmidt martingale* if, for all $t \in \mathbb{R}^+$, the operator H_t, restricted to $\Phi_{t]}$, is an Hilbert-Schmidt operator. We denote by $||\cdot||_{HS}$ the Hilbert-Schmidt norm of operators.

Proposition I.20 – *Let $t \in [0, +\infty]$ be fixed, let H_t be a Hilbert-Schmidt operator from $\Phi_{t]}$ to $\Phi_{t]}$. Then H_t admits a non-commutative stochastic integral representation, in the extended sense on all Φ, of the form*

$$H_t = P_0[H_t 1]I - \int_0^t H_s^{\circ}\, dA_s^{\circ} + \int_0^t H_s^{-}\, dA_s^{-} + \int_0^t H_s^{\dagger}\, dA_s^{\dagger},$$

where $(H_s^{\circ})_{s\leq t}$ is the martingale associated to H_t and where, for $\varepsilon = -, \dagger$, for almost all $s \leq t$ the operator H_s^{ε} is Hilbert-Schmidt from $\Phi_{s]}$ to $\Phi_{s]}$ and satisfies $\int_0^t ||H_s^{\varepsilon}||_{HS}^2\, ds \leq ||H_t||_{HS}^2$.

Proof

This proposition is taken from [PS1]. It is an application of their integral representation theorem for regular martingale of operators. Note that Hilbert-Schmidt martingales are elements of the algebra $\mathcal{S}$ of regular semimartingales. ■

From this proposition one sees that the family $\mathcal{I}$ of Hilbert-Schmidt operators on Φ is a family which satisfies the properties described in previous subsection that is, it is a family of representable operators such that the coefficients of the representation all belong to $\mathcal{I}$. So one can apply our iteration procedure.

Let $E_n = \{\dagger, \circ, -\}^n$. For an element $E = (\varepsilon_1, \ldots, \varepsilon_n)$ of E_n we denote by $n_{\circ}(E)$ the number of elements ε_i which are equal to "$\circ$". In the following we use families of operators $H_{t_n \ldots t_1}^{\varepsilon_n \ldots \varepsilon_1}$, indexed by $E_n \times \mathcal{P}_n$. We use the following notation:

$$\sum_{E \in E_n} \int_{\mathcal{P}_n} H_{\mu}^{E}\, dA_{\mu}^{E} \stackrel{\text{def}}{=} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{\dagger, \circ, -\}} \int_0^{\infty} \int_0^{t_n} \cdots \int_0^{t_2} H_{t_n \ldots t_1}^{\varepsilon_n \ldots \varepsilon_1}\, dA_{t_1}^{\varepsilon_1} \ldots dA_{t_n}^{\varepsilon_n}.$$

By iteration of Proposition I.20 one easily obtains the following.

Proposition I.21 – *Let H be a Hilbert-Schmidt operator from Φ to Φ. For all $N \in \mathbb{N}^*$, H admits on all Φ an integral representation of the form*

$$H = \sum_{n=0}^{N-1} \sum_{E \in E_n} \int_{\mathcal{P}_n} (-1)^{n_{\circ}(E)} P_0[H_{\mu}^{E} 1]I\, dA_{\mu}^{E} + \sum_{E \in E_N} \int_{\mathcal{P}_N} (-1)^{n_{\circ}(E)} H_{\mu}^{E}\, dA_{\mu}^{E}.$$

Furthermore, for all $\mu = (t_1 < \ldots < t_N) \in \mathcal{P}_N$, all $E \in E_N$, the operator H_{μ}^{E} is a Hilbert-Schmidt operator from $\Phi_{t_1]}$ to $\Phi_{t_1]}$. We have the estimate $\int_{\mathcal{P}_N \cap [0,T]^N} ||H_{\mu}^{E}||_{HS}^2\, d\mu \leq T^N ||H||_{HS}^2$ for all $E \in E_N$, all $0 < T < \infty$. ■

In this representation we immediatly distinguish the begining of the non-commutative chaotic expansion of H (the first term of the right hand side) from the remainder (the second term). Let $\mathcal{E}_{lb}^{c}$ denote the coherent space $\mathcal{E}(\mathcal{M})$ where $\mathcal{M}$ is the space of locally bounded, compact supported elements of $L^2(\mathbb{R}^+)$.

Proposition I.22 – *Let H be a Hilbert-Schmidt operator from Φ to Φ. Let $N \in I\!N^*$, let*

$$R_N = \sum_{E\in E_N} \int_{\mathcal{P}_N} (-1)^{n_\circ(E)} H^E_\mu \, dA^E_\mu$$

be the remainder of the integral representation (iterated N times) of H given by Proposition I.21. Then for all $f, g \in \mathcal{E}^c_{lb}$ the quantity $< g\,,\, R_N f >$ converges to 0 when N tends to $+\infty$.

Thus, in the sense of this weak convergence, one has

$$H = \sum_{n=0}^{\infty} \sum_{E\in E_n} \int_{\mathcal{P}_n} I\!E[H^E_\mu 1\!\!1] I \, dA^E_\mu .$$

The proof is easy from the estimates at the end of Proposition I.21. ■

We have finally written the non-commutative chaotic expansion of H.

But the article [At3] does not end at this result. Indeed, the above weak convergence is not sufficient for proving that a Hilbert-Schmidt operator H admits a Maassen kernel on all Φ. Actually one has to go back to Proposition I.20, take a f in Φ and compute Hf with A-M's equations. We get an integral representation of any Hilbert-Schmidt operator applied to any vectors. As the coefficients of the representation are also made of Hilbert-Schmidt operators applied to vectors of Φ, one can iterate the procedure. While we iterate we see the chaotic expansion of Hf appearing and a remainder. The most difficult and tiresome part consists in proving that the remainder converges to 0. We thus get the complete chaotic expansion of Hf. A combinatoric rearrangement allows to prove that it is of the form of a Maassen kernel applied to f. Anyway, it is not worth developing that here; I just wanted to illustrate how this probabilistic inspiration could work and help to find some new operators admitting a Maassen kernel.

I.4 Hilbertian quasimartingales and quantum stop times

I.4.1 Enchev's Hilbertian quasimartingales

In [Enc], O. Enchev extends to the Hilbertian context the usual notion of quasimartingales. One can find a nice exposition of his article in [Me4]. Let us recall the main result.

Theorem I.23 – *Let $(H, (P_t)_{t\geq 0})$ be a filtered Hilbert space that is, a Hilbert space H together with a right continuous increasing family $(P_t)_{t\geq 0}$ of orthogonal projections in H. Let H_t be the range of P_t, for all $t \in I\!R^+$. Let $(x_t)_{t\geq 0}$ be an adapted process of vectors (i.e. $x_t \in H_t$ for all $t \in I\!R^+$) such that $x_0 = 0$. If one has $\sup_{\mathcal{R}} \, \Sigma_i ||P_{t_i} x_{t_{i+1}} - x_{t_i}|| < +\infty$, where $\mathcal{R} = \{t_i, i = 1, ..., n\}$ runs over all the partitions of a fixed bounded interval $[0,T]$, then $(x_t)_{t\geq 0}$ admits a unique decomposition as a sum of a martingale $(m_t)_{t\geq 0}$ (i.e. $P_s \, m_t = m_s$, $s \leq t$) and a finite* *variation process* $(a_t)_{t\geq 0}$ *(i.e.* $\sup_{\mathcal{R}} \Sigma_i ||a_{t_{i+1}} - a_{t_i}|| < \infty$*), vanishing at 0, adapted to* $(H_{t-})_{t\geq 0} = (\cap_{s<t} H_s)_{t\geq 0}$*. Such a* $(x_t)_{t\geq 0}$ *is called a Hilbertian quasimartingale.*

If $(x_t)_{t\geq 0}$ *is an adapted process which satisfies* $||P_s\, x_t - x_s|| \leq \int_s^t g(u)\, du$ *for* $s \leq t$ *and a locally integrable* g*, then* $(x_t)_{t\geq 0}$ *is a Hilbertian quasimartingale whose finite variation part is of the form* $a_t = \int_0^t h_u\, du$ *with* $||h_u|| \leq g(u)$*.* ■

Applied to the Fock space context this theorem gives the following.

Corollary I.24 – *Let* $(x_t)_{t\geq 0}$ *be a vector process in* Φ *such that* $x_t \in \Phi_{t]}$ *for all* $t \in \mathbb{R}^+$ *and such that* $x_0 = 0$*. The following assertions are then equivalent.*

i) *There exists a locally integrable function* g *on* $\mathbb{R}^+$ *such that for all* $s \leq t$ *one has* $||P_s\, x_t - x_s|| \leq \int_s^t g(u)\, du$*.*

ii) *The vector process* $(x_t)_{t\geq 0}$ *admits a unique representation as*

$$x_t = \int_0^t \xi_s\, d\chi_s + \int_0^t h_s\, ds$$

for adapted vector processes $(\xi_t)_{t\geq 0}$ *and* $(h_t)_{t\geq 0}$ *satisfying* $\int_0^t ||\xi_s||^2\, ds < \infty$ *and* $\int_0^t ||h_s||\, ds < \infty$ *for all* t*.*

Proof

The Hilbert space Φ together with the family of projections $(P_t)_{t\geq 0}$ is a filtered Hilbert space, with a continuous filtration. From Theorem I.23, every adapted vector process $(x_t)_{t\geq 0}$ in Φ which satisfies i) admits a unique representation as $x_t = m_t + \int_0^t h_s\, ds$ where $(h_t)_{t\geq 0}$ is such as described in ii) and where $(m_t)_{t\geq 0}$ is a vector martingale with $m_0 = 0$. The Fock space predictable representation property implies that for all t we have $m_t = \int_0^t D_s m_t\, d\chi_s$. But note that for a.a. $s \leq t$ and $r \geq 0$ we have $D_s m_t = D_s P_t m_{t+r} = D_s m_{t+r}$. So the mapping $r \mapsto D_s m_{s+r}$ is constant and its value can be denoted $D_s m_{s+}$. We finally have the representation $m_t = \int_0^t D_s m_{s+}\, d\chi_s$. This gives the corollary in one direction. The converse direction is trivial from the Ito isometry formula and the usual norm estimate on time integrals. ■

Corollary I.25 – *Let* $(T_t)_{t\geq 0}$ *be any adapted process of operators on* Φ *which admits an integral representation*

$$T_t = \int_0^t H_s\, d\Lambda_s + \int_0^t K_s\, dA_s^\dagger + \int_0^t L_s\, dA_s + \int_0^t M_s\, ds$$

on a domain $\mathcal{D}$*. For any* $f \in \mathcal{D}$ *define* $f_t = P_t f$*,* $t \in \mathbb{R}^+$*. Then the vector process* $(T_t\, f_t)_{t\geq 0}$ *is a Enchev quasimartingale on* Φ*.*

Proof

From A-M's equations we have that for all $f \in \mathcal{D}$

$$T_t f_t = \int_0^t [T_s D_s f + H_s D_s f + K_s P_s f]\, d\chi_s + \int_0^t [L_s D_s f + M_s P_s f]\, ds.$$

We conclude by Corollary I.24. ■

This Hilbertian generalization of the classical notion of quasimartingale is clearly interesting by itself, but it is also of great interest in the problems of stopping vector or operator processes with quantum stop times.

I.4.2 Quantum stop times and quasimartingales

The classical notion of stop time on a filtered probability space admits a natural non-commutative extension. On Fock space for example they have been first studied by R.L. Hudson in [Hud] then by K.R. Parthasarathy and K.B. Sinha in [PS2]. Their idea for extending the notion of stop time is the following. A classical stop time τ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is determined by the family of indicator fuctions $\mathbb{1}_{\tau\leq t}$, $t \in \mathbb{R}^+$. These indicator functions form a right continuous increasing family of projection operators on the Hilbert space $L^2(\Omega)$ that is, a spectral measure; they furthermore satisfy an adaptation property. The non-commutative generalization of the notion of stop times is thus as follows.

A (quantum) *stop time* τ on Φ is a spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the space of orthogonal projections on Φ and such that for all t the operator $\tau([0,t])$ is a t-adapted operator.

In the following we adopt a probabilistic-like notation: for any Borel subset $A \subset \mathbb{R}^+ \cup \{+\infty\}$, the operator $\tau(A)$ is denoted $\mathbb{1}_{\tau\in A}$; in the same way $\tau(\{t\})$ is denoted $\mathbb{1}_{\tau=t}$, the operator $\tau([0,t])$ is denoted by $\mathbb{1}_{\tau\leq t}$, etc...

A stop time τ is *finite* if $\mathbb{1}_{\tau=+\infty} = 0$. It is *bounded* by $T \in \mathbb{R}^+$ if $\mathbb{1}_{\tau\leq T} = I$. A point t in $\mathbb{R}^+$ is a *continuity point* for τ if $\mathbb{1}_{\tau=t} = 0$. Note that, unless $\tau \equiv 0$, the point 0 is always a continuity point for τ. It is also easy to check that the set of points $t \in \mathbb{R}^+$ which are not continuity points for τ is at most countable.

If τ and τ' are two stop times on Φ, one says that $\tau \leq \tau'$ if for all $t \in \mathbb{R}^+$ one has $\mathbb{1}_{\tau\leq t} \geq \mathbb{1}_{\tau'\leq t}$ (in the usual sense of comparison of two projections).

A stop time τ is *discrete* if there exists a finite set $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ such that $\mathbb{1}_{\tau\in E} = I$.

A sequence of stop times $(\tau_n)_n$ is said to *converge* to a stop time τ if, for all continuity point t for τ, the operators $\mathbb{1}_{\tau_n\leq t}$ converge strongly to $\mathbb{1}_{\tau\leq t}$.

A *sequence of refining τ-partitions* is a sequence $(E_n)_n$ of partitions $E_n = \{0 \leq t_1^n < t_2^n < \cdots < t_{i_n}^n < +\infty\}$ of $\mathbb{R}^+$ such that

i) all the t_j^i are continuity points for τ;

ii) $E_n \subseteq E_{n+1}$ for all n;

iii) the diameter, $\max\{t_{i+1}^n - t_i^n\,; i = 1, ..., i_n\}$, of E_n tends to 0 when n tends to $+\infty$;

iv) $t_{i_n}^n$ tends to $+\infty$ when n tends to $+\infty$.

The following result is taken from [PS2], Proposition 3.3 and from [Me5].

Proposition I.26 – *Let τ be any stop time. Then there exists a sequence $(\tau_n)_n$ of discrete stop times such that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau$ and $(\tau_n)_n$ converges to τ.*

Proof

Let $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define a spectral measure τ_E by

$$\tau_E(\{t_i\}) = \begin{cases} \mathbb{1}_{\tau < t_1} & \text{if } i = 1, \\ \mathbb{1}_{\tau \in [t_{i-1}, t_i[} & \text{if } 1 < i \leq n-1, \end{cases}$$
$$\tau_E(\{t_n\}) = \mathbb{1}_{\tau \geq t_{n-1}}.$$

The spectral measure τ_E clearly defines a discrete stop time on Φ and $\tau_E \geq \tau$. Taking a sequence $(E_n)_n$ of refining τ-partitions of $\mathbb{R}^+$ gives the required sequence $(\tau_n)_n = (\tau_{E_n})_n$. ∎

The theory of quantum stop times admits many interesting developments that can not be discussed here. We just present results where Enchev's quasimartingales appear to be helpful.

One of the main problem with quantum stop times is to identify the class of vector and operator processes that can be stopped. That is, for a given stop time τ, on what kind of vector process $(x_t)_{t\geq 0}$ (*resp.* operator process $(X_t)_{t\geq 0}$) can we reasonably define the value x_τ (*resp.* X_τ) of the process at time τ?

When τ is a discrete stop time there is a natural definition for x_τ obtained by mimicking the classical definition:

$$x_\tau = \sum_i \mathbb{1}_{\tau = t_i} x_{t_i}$$

where the t_i's constitute the support of the discrete spectral measure τ. For a general quantum stop time τ, one can think of Proposition I.26 and approximate τ by a sequence of discrete stop times. We are led to consider the convergence of

$$x_{\tau_n} = \sum_i \mathbb{1}_{\tau \in [t_i, t_{i+1}[} x_{t_{i+1}}$$

when the diameter of the partition $\{t_i; i = 1, \ldots, n\}$ tends to 0. This problem has been studied in details in [PS2] and they obtain the convergence for all vector processes of the form $x_t = m_t \otimes y_t$, where $(m_t)_{t\geq 0}$ is a complete vector martingale on Φ (that is, $m_t = P_t m$ for a $m \in \Phi$) and where $(y_t)_{t\geq 0}$ is a *process in the future* (that is, $y_t \in \Phi_{[t}$ for all t). This result is extended in [A-S] to the case where $(x_t)_{t\geq 0}$ is of the form $x_t = z_t \otimes y_t$, where $(z_t)_{t\geq 0}$ is a Enchev quasimartingale on Φ. But before proving this extension and describing its application to stopping operator processes, let us show that Enchev's quasimartingales on Φ are easy to stop. Let $z_t = \int_0^t \xi_s d\chi_s + \int_0^t h_s\, ds$, $t \in \mathbb{R}^+$, be a Enchev quasimartingale.

Let τ be a finite stop time. The vector process $(z_t)_{t\geq 0}$ is said to be τ-*integrable* if it satisfies

$$\int_0^\infty ||\mathbb{1}_{\tau > s}\xi_s||^2\, ds + \int_0^\infty ||\mathbb{1}_{\tau > s} h_s||\, ds < \infty.$$

Proposition I.27 – *Let $z_t = \int_0^t \xi_s\, d\chi_s + \int_0^t h_s\, ds$, $t \in \mathbb{R}^+$, be a Enchev quasimartingale. Let τ be a finite quantum stop time such that $(z_t)_{t\geq 0}$ is τ-integrable.*

Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Put $\tau_n = \tau_{E_n}$, for all $n \in \mathbb{N}$. Then the sequence $(z_{\tau_n})_n$ converges in Φ to a vector z_τ which is given by

$$z_\tau = \int_0^\infty \mathbb{1}_{\tau>s}\, \xi_s\, d\chi_s + \int_0^\infty \mathbb{1}_{\tau>s}\, h_s\, ds.$$

Proof

Suppose first that τ is bounded by $T \in \mathbb{R}^+$. We have

$$\begin{aligned}
z_{\tau_n} &= \sum_i \mathbb{1}_{\tau_n=t_i}\, z_{t_i} = \sum_i \mathbb{1}_{\tau_n=t_i} \Big[\int_0^{t_i} \xi_s\, d\chi_s + \int_0^{t_i} h_s\, ds\Big] \\
&= \sum_i \sum_{j<i} \mathbb{1}_{\tau_n=t_i} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, d\chi_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\
&= \sum_j \mathbb{1}_{\tau_n>t_j} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, d\chi_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\
&= \sum_j \Big[\int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n>t_j} \xi_s\, d\chi_s + \int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n>t_j} h_s\, ds\Big]
\end{aligned}$$

(by boundedness and t_j-adaptedness of $\mathbb{1}_{\tau_n>t_j}$)

$$= \int_0^T \mathbb{1}_{\tau_n>s} \xi_s\, d\chi_s + \int_0^T \mathbb{1}_{\tau_n>s} h_s\, ds.$$

When n tends to $+\infty$ the right hand side of the above identity converges to

$$\int_0^T \mathbb{1}_{\tau>s} \xi_s\, d\chi_s + \int_0^T \mathbb{1}_{\tau>s} h_s\, ds.$$

This proves the proposition for bounded stop times. To conclude for any finite stop times one approximates such stop times by a sequence of bounded stop times. ■

Note that as τ defines a spectral measure we have, from the above proposition, $z_\tau = \int \mathbb{1}_{\tau\in ds}\, z_s$ in the sense of spectral measure integrals. We can now state [A-S]'s main result.

Theorem I.28 – *Let $z_t = \int_0^t \xi_s d\chi_s + \int_0^t h_s\, ds$, $t \in \mathbb{R}^+$, be a Enchev quasimartingale. Let $(y_t)_{t\geq 0}$ be a vector process, adapted to the future and bounded in norm. Let τ be a finite stop time such that $(z_t)_{t\geq 0}$ is τ-integrable. Let $w_t = z_t \otimes y_t$, $t \in \mathbb{R}^+$. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Put $\tau_n = \tau_{E_n}$, $n \in \mathbb{N}$. Then the sequence $(w_{\tau_n})_n$ converges in Φ to a vector w_τ which is given by*

$$w_\tau = \int \mathbb{1}_{\tau\in ds}\, [P_s z_\tau] \otimes y_s.$$

Proof

The complete proof of this theorem, given in [A-S], is rather long but one can easily give, at least formally, the main idea of it. As we have said before, Parthasarathy and Sinha have proved in [PS2] that integrals of the form $\int \mathbb{1}_{\tau\in ds}[P_s m]\otimes y_s$ always converge. Thus so does $\int \mathbb{1}_{\tau\in ds}[P_s z_\tau]\otimes y_s$ which is equal to $\int \mathbb{1}_{\tau\in ds}[P_s \int \mathbb{1}_{\tau\in du}\, z_u]\otimes y_s$. As it is s-adapted (or not far) the operator $\mathbb{1}_{\tau\in ds}$ commutes with P_s, so the latter integral equals $\int[P_s \mathbb{1}_{\tau\in ds} \int \mathbb{1}_{\tau\in du}\, z_u]\otimes y_s$. As $\mathbb{1}_{\tau\in ds}\mathbb{1}_{\tau\in du}$ is equal to $\mathbb{1}_{u=s}\mathbb{1}_{\tau\in ds}$ we obtain $\int[P_s \mathbb{1}_{\tau\in ds} z_s]\otimes y_s$ that is,

$$\int \mathbb{1}_{\tau\in ds}\, z_s\otimes y_s. \qquad \blacksquare$$

This result is an extension of Parthasarathy-Sinha's one but it also has an application to the problem of stopping operator processes. Indeed, for a given quantum stop time τ we which to define X_τ for an operator process $(X_t)_{t\geq 0}$. Of course, as in the case of vector processes the definition of X_τ is clear when τ is a discrete stop time. But one has to pay attention to the fact that in this case there are three natural ways of defining X_τ:

the left-stopping : $\tau\circ X=\sum_i \mathbb{1}_{\tau=t_i} X_{t_i}$

the right-stopping : $X\circ\tau=\sum_i X_{t_i}\,\mathbb{1}_{\tau=t_i}$

the two-sided-stopping : $\tau\circ X\circ\tau=\sum_i \mathbb{1}_{\tau=t_i}\, X_{t_i}\,\mathbb{1}_{\tau=t_i}$.

As previously we approximate τ by a sequence of discrete stop times and wonder if one can pass to the limit. The point with Theorem I.28 is that it gives a good answer for left-stopping and right-stopping.

Theorem I.29 – *Let $(X_t)_{t\geq 0}$ be an adapted process of operators on Φ. Suppose that for all $u\in\mathcal{M}$ the process $\left(X_t\,\varepsilon(u_{t]})\right)_{t\geq 0}$ is a Enchev quasimartingale on Φ. Let τ be a finite stop time such that, for all $u\in\mathcal{M}$, the process $\left(X_t\,\varepsilon(u_{t]})\right)_{t\geq 0}$ is τ-integrable. Then the left stopping $\tau\circ X$ converges strongly on $\mathcal{E}(\mathcal{M})$.*

Proof

By Theorem I.28 we have that the quantity

$$\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}(X_{t_{i+1}}\varepsilon(u_{t_{i+1}]}))\otimes\varepsilon(u_{[t_{i+1}})$$

admits a limit when the diameter of the τ-partition $\{t_i;\, i=1,...,n\}$ tends to 0. But this quantity is also equal to

$$\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}(X_{t_{i+1}}\varepsilon(u))=\Big[\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}X_{t_{i+1}}\Big]\varepsilon(u).$$

Thus, the Riemann sums associated to the left-stopping of X converge. $\blacksquare$

The class of operator processes which satisfy the conditions of Theorem I.29 is very large, as proves the following.

Corollary I.30 – *Let*

$$X_t = \int_0^t H_s \, d\Lambda_s + \int_0^t K_s \, dA_s^\dagger + \int_0^t L_s \, dA_s + \int_0^t M_s \, ds$$

be any adapted process of operators which admits an integral representation on a coherent space $\mathcal{E}(\mathcal{M})$. Let τ be any finite stop time on Φ such that

$$\int_0^\infty 1\!\!1_{\tau > s} H_s \, d\Lambda_s + \int_0^\infty 1\!\!1_{\tau > s} K_s \, dA_s^\dagger + \int_0^\infty 1\!\!1_{\tau > s} L_s \, dA_s + \int_0^\infty 1\!\!1_{\tau > s} M_s \, ds \quad \text{(I.23)}$$

is well-defined on $\mathcal{E}(\mathcal{M})$. Then the left-stopping $\tau \circ X$ converges strongly on $\mathcal{E}(\mathcal{M})$ and $\tau \circ X$ is given by (I.23). *Furthermore, if the adjoint process $(X_t^*)_{t \geq 0}$ also admits an integral representation we get the analogous result for the right-stopping $X \circ \tau$.*

Proof

Because of Corollary I.25 we know that if $(X_t)_{t \geq 0}$ admits an integral representation on $\mathcal{E}(\mathcal{M})$ then $(X_t \, \varepsilon(u_{t]}))_{t \geq 0}$ is a Enchev quasimartingale. The condition that (I.23) defines a quantum stochastic integral on $\mathcal{E}(\mathcal{M})$ is the same as saying that the quasimartingale $(X_t \, \varepsilon(u_{t]}))_{t \geq 0}$ is τ-integrable. We conclude easily for the left-stopping.

For the right-stopping problem, it suffices to note that $(\tau \circ X)^* = X^* \circ \tau$. ■

I.4.3 Quantum strong Markov process

In this last subsection we quickly introduce the notion of *quantum Markov process* as developed in [B-P], and the notion of quantum *strong* Markov process as developed in [A-P]. We are again going to see the importance of Enchev's quasimartingales for stopping processes with quantum stop times.

Let $\mathcal{A}$ be a unital C^*-algebra of bounded operators on a complex Hilbert space H_0. Let $(T_t)_{t \geq 0}$ be a semigroup of contractive, unital and completely positive maps from $\mathcal{A}$ into itself. Recall that complete positivity means $\sum_{i,j} X_i^* T_t(Y_i^* Y_j) X_j \geq 0$ for all $X_i \in \mathcal{B}(H_0), \ Y_i \in \mathcal{A}, \ i = 1, ..., n$.

Theorem I.31 ([B-P]) – *There exists a Hilbert space H, an increasing family $(F_t)_{t \geq 0}$ of projection operators on H, a family of *-homomorphisms $j_t : \mathcal{A} \to \mathcal{B}(H), t \geq 0$ and a unitary isomorphism V from H_0 onto the range of F_0 satisfying the following properties :*

i) $j_t(I) = F_t$, *for all* $t \geq 0$;

ii) $F_s j_t(X) F_s = j_s(T_{t-s}(X))$ *for any* $s \leq t, \ X \in \mathcal{A}$;

iii) $j_0(X) V = V X$ *for all* $X \in \mathcal{A}$;

iv) *the set* $\{j_{t_1}(X_1) ... j_{t_n}(X_n) V u; t_1 > t_2 > \cdots > t_n \geq 0, X_i \in \mathcal{A}, i \in \{1, ..., n\}, n \geq 1, u \in H_0\}$ *is total in* H. ■

This theorem is the quantum analogue of Kolmogorov's construction of a Markov process for given semigroup and initial measure. Indeed, the C^*-algebra

$\mathcal{A}$ is the quantum extension of the algebra of bounded functions on the state space, the projections F_t play the role of conditional expectations $I\!E[\,\cdot\,|\mathcal{F}_t]$ and j_t is the extension of the algebra morphism $(x_t)_{t\geq 0}\,:\, f\mapsto f(x_t)\circ I\!E[\,\cdot\,|\mathcal{F}_t]$ made from the Markov process $(x_t)_{t\geq 0}$ acting on the algebra of bounded functions $\mathcal{A}$. Identity ii) is just the Markov property.

This theorem gives a notion of quantum Markov processes. Such a quantum Markov process is said to be *minimal* when it satisfies condition iv). The next interesting step is to try to define quantum *strong* Markov processes. We need a notion of quantum stop time in this context. As in previous section a *stop time* τ on H is a spectral measure on $I\!R^+\cup\{+\infty\}$ with values in the set of orthogonal projection operators on H and such that

$$1\!\!1_{\tau\leq t}\, j_u(X)=j_u(X)1\!\!1_{\tau\leq t}\quad \text{for all } u\geq t,\ X\in\mathcal{A}. \tag{I.24}$$

The commutation relation (I.24) replaces the adaptedness condition in our context. It expresses that $1\!\!1_{\tau\leq t}$ does not interfere with the future "trajectories" of the Markov process. In other words "knowing that the event of stopping at time τ has occured before time t does not affect the Markov process after time t". In the language of physicists this is called the *non-demolition property.* Note that in the context of commutative Markov processes the condition (I.24) exactly expresses the fact that $(\tau\leq t)$ is a $\mathcal{F}_t$-measurable event.

If one wants to deal with the notion of strong Markov property one needs to be able to define j_τ for stop times τ. As in previous subsection, the problem lies in passing to the limit on expressions like

$$\sum_i 1\!\!1_{\tau\in[t_i,t_{i+1}[}\, j_{t_{i+1}}(X).$$

Note that for $\Psi\in H$ the vector processes $j_t(X)\Psi$ are always adapted to the filtration $(F_t)_{t\geq 0}$, thus the same idea as in previous subsection applies: when the vector processes $(j_t(X)\Psi)_{t\geq 0}$ are Enchev's quasimartingales on H it is proved in [A-P] that j_τ can be defined and that $(j_t)_{t\geq 0}$ is a quantum strong Markov process in the sense that it satisfies the quantum strong Markov property

$$F_\tau j_{\tau+t}(X)F_\tau = j_\tau(T_t(X)).$$

In [E-H] have been considered the so-called Evans-Hudson flows; that is, solutions of some quantum stochastic differential equations on the Fock space Φ. It is known that such flows give rise to quantum Markov processes. But as such flows are made of quantum stochastic integrals on Φ, because of the above criterion for the quantum strong Markov property, and because of Corollary I.25 we have that **every minimal Evans-Hudson flow is a quantum strong Markov process**.

II From quantum to classical stochastic calculus

This chapter is devoted to the contributions of quantum stochastic calculus to classical stochastic calculus. We first present the so-called "probabilistic interpretations of the Fock space". We see that quantum stochastic calculus is an extension and a unification of classical stochastic calculus in several different contexts such as Brownian motion, Poisson process, Azema martingales,... The second section is devoted to showing that quantum stochastic calculus is a natural language for studying Wiener space endomorphisms, and also other "chaotic spaces" endomorphisms.

II.1 Extension and unification of classical stochastic calculus

II.1.1 Probabilistic interpretations of Fock space

Consider a (classical) martingale $(x_t)_{t\geq 0}$ on its canonical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. This martingale is said to be *normal* if the process $(x_t^2 - t)_{t\geq 0}$ is also a martingale; in other words, if the angle bracket process $(\langle x, x\rangle_t)_{t\geq 0}$ is such that $\langle x, x\rangle_t = t$ for all t.

A normal martingale $(x_t)_{t\geq 0}$ is said to have the *predictable representation property* if every random variable f in $L^2(\Omega)$ admits a representation as the sum of its expectation and stochastic integral of some predictable process with respect to $(x_t)_{t\geq 0}$:

$$f = \mathbb{E}[f] + \int_0^\infty \xi_s \, dx_s.$$

For a normal martingale $(x_t)_{t\geq 0}$ it is possible to define iterated stochastic integrals such as

$$\int_{0<t_1<\ldots<t_n<\infty} f_n(t_1, \ldots, t_n) \, dx_{t_1} \, \ldots \, dx_{t_n}$$

for some square integrable (deterministic) functions f_n on the increasing simplex $\Sigma_n \stackrel{\text{def}}{=} \{(t_1, \ldots, t_n) \in \mathbb{R}^n; 0 < t_1 < \ldots < t_n\}$ (cf [Me6]). We define the *chaotic space* of $(x_t)_{t\geq 0}$, denoted $CS(x)$, to be the space of random variables f in $L^2(\Omega)$ which can be written as series of such iterated integrals:

$$f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_{0<t_1<\ldots<t_n<\infty} f_n(t_1, \ldots, t_n) \, dx_{t_1} \, \ldots \, dx_{t_n}. \qquad \text{(II.1)}$$

The decomposition of f as such a series is called the *chaotic expansion* of f. Recall that two iterated integrals that are not of the same order of iteration are orthogonal for the $L^2(\Omega)$ scalar product; so the norm of f in $L^2(\Omega)$ is

$$||f||^2 = |\mathbb{E}[f]|^2 + \sum_{n=1}^{\infty} \int_{0<t_1<\ldots<t_n<\infty} |f_n(t_1, \ldots, t_n)|^2 \, dt_1 \, \ldots \, dt_n.$$

When $CS(x)$ is the whole of $L^2(\Omega)$ one says that $(x_t)_{t\geq 0}$ has the *chaotic representation property.*

Let $(x_t)_{t\geq 0}$ be a normal martingale with the predictable representation property. The difference $[x, x]_t - \langle x, x\rangle_t$ between the square bracket of x and its angle bracket is always a martingale. Thus, from the predictable representation property, there exists a predictable process $(\Psi_t)_{t\geq 0}$ in $L^2(\Omega)$ such that

$$[x,x]_t - \langle x,x\rangle_t = \int_0^t \Psi_s \, dx_s.$$

In other words

$$d[x,x]_t = dt + \Psi_t \, dx_t.$$

This equation is called the *structure equation* of $(x_t)_{t\geq 0}$ (cf [Em1]). These equations appear to be very useful for identifying properties of the process $(x_t)_{t\geq 0}$. Depending on the form of $(\Psi_t)_{t\geq 0}$ we recover several well-known cases (cf [Em1]):

if $\Psi_t \equiv 0$ for all t then $(x_t)_{t\geq 0}$ is the Brownian motion,

if $\Psi_t \equiv -1$ for all t then $(x_t)_{t\geq 0}$ is the compensated Poisson process,

if $\Psi_t = \beta x_{t-}$ then $(x_t)_{t\geq 0}$ is the Azema martingale with coefficient β.

Each of the two first cases together with the third one when $\beta \in [-2,0]$ have the chaotic representation property ([Em1]).

Whatever is the case we consider, there is a strong relation between normal martingales with the predictable representation property and the Fock space. Indeed, note that a square integrable function f_n on the increasing simplex Σ_n can be seen as a square integrable *symmetric* function on $(\mathbb{R}^+)^n$ that is, an element of $L^2(\mathbb{R}^+)^{\odot n}$. Thus the chaotic space $CS(x)$ of x is isomorphic to $\oplus_n L^2(\mathbb{R}^+)^{\odot n}$ that is, the symmetric Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+))$. The isomorphism $J : CS(x) \to \Phi = L^2(\mathcal{P})$ is given by

$$[Jf](\sigma) = f_n(t_1, \ldots, t_n) \qquad \big([Jf](\emptyset) = \mathbb{E}[f]\big)$$

where $\sigma = \{t_1 < \ldots < t_n\} \in \mathcal{P}$ and where f has his chaotic expansion given by (II.1). Note that the isomorphism J depends on the martingale $(x_t)_{t\geq 0}$.

This is the reason why $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (x_t)_{t\geq 0}, CS(x)\big)$ (or simply $(x_t)_{t\geq 0}$) is called a *probabilistic interpretation* of the Fock space Φ.

II.1.2 Interpretation of the Ito calculus

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (x_t)_{t\geq 0}, CS(x)\big)$ be a *probabilistic interpretation* of the Fock space Φ. All the operators P_t, D_t, ∇_t, the Skorohod and the Ito integrals on Φ such as described in section I.1 admits an interpretation on $L^2(\Omega)$ as well-known probabilistic operators. Let $J : CS(x) \to \Phi$ be the isomorphism between $CS(x)$ and Φ.

First of all, one has $P_t = J \circ E_t \circ J^{-1}$ where E_t is the operator of conditional expectation $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$; by the way the space $\Phi_{t]} = \mathrm{Im} P_t$ is isomorphic to $L^2(\Omega, \mathcal{F}_t, P) \cap CS(x)$.

Let f be a random variable in $CS(x)$. We know that there exists a predictable process $(\xi_t(f))_{t\geq 0}$ in $L^2(\Omega)$ such that $f = E[f] + \int_0^\infty \xi_t(f)\, dx_t$. One can see ξ_t as an a.e. defined operator on $L^2(\Omega)$. From this point of view ξ_t is nothing but the probabilistic interpretation of the operator D_t; that is, $D_t = J \circ \xi_t \circ J^{-1}$.

By analogy, one can wonder what is the probabilistic interpretation of the operator ∇_t. It is the Malliavin's gradient, or stochastic derivative along the element $f(s) = s \wedge t$ of the Cameron-Martin space (see [N-Z] for example).

Thus the operator $\mathcal{S}$, which is the adjoint of $(\nabla_t)_{t\geq 0}$ (cf [A-L] for example), is interpreted on $L^2(\Omega)$ as the operator of Skorohod integral with respect to $(x_t)_{t\geq 0}$ (in the probabilistic context, see [G-T] for the adjoint relation between $\mathcal{S}$ and ∇, see [Sko] for the definition of the Skorohod integral).

The operator of Ito integration, which is the Skorohod integral restricted to adapted processes, thus corresponds to the usual Ito integral with respect to $(x_t)_{t\geq 0}$.

The vector process $(\chi_t)_{t\geq 0}$ in Φ, with respect to which the Fock space Ito integral is an actual integral, interprets on $L^2(\Omega)$ as the normal martingale $(x_t)_{t\geq 0}$ itself that is, $Jx_t = \chi_t$.

Theorem I.3, when interpreted in $L^2(\Omega)$, only expresses the predictable representation property of $(x_t)_{t\geq 0}$ and the isometry formula for the Ito integral.

The property $D_t = P_t \nabla_t$ for a.a. t is interpreted as a Fock space extension of Clark's formula ([Cla]).

In this way all the operators introduced in section I.1 can be interpreted as well-known operators coming from the stochastic calculus when the Fock space is interpreted as the chaotic space of some normal martingales. In fact, one should think the other way round. Probabilistic operations such as Ito integration, Skorohod integration, Malliavin gradient, predictable representation, etc... can be expressed in terms of the chaotic expansion of the random variables only. In their definition they do not use any specific property of the normal martingale involved except the chaotic representation property and the Ito isometry formula (which is the same for all the probabilistic interpretation for $< x, x >_t = t$, $\forall t$). Hence they can be translated into intrinsic operators on the Fock space which is a kind of abstract chaotic space in the sense that it contains only the abstract structure of the chaotic representation property and of the Ito isometry formula.

II.1.3 Extension of classical stochastic calculus

In sections I.1 and I.2 we have developed a non-commutative stochastic calculus on the Fock space Φ. Indeed, we have developed the notions of adapted processes, stochastic integrals, semimartingales, square and angle brackets, Ito formula, etc... in the Fock space context. We are going to see that these notion are the non-commutative extensions of the corresponding classical ones and that they unify the different probabilistic interpretations of Φ.

There are two ingredients that are linking classical and quantum stochastic calculus. The first one is that random variables in a probabilistic interpretation of Φ are particular operators on Φ. Indeed, let $(x_t)_{t\geq 0}$ be a probabilistic interpretation of Φ. Let f be a random variable in $CS(x)$. Then f defines a self-adjoint operator $\mathfrak{M}_f$ on $CS(x)$, the *operator of multiplication* by f defined by

$$\begin{array}{cccc} \mathfrak{M}_f & : & \operatorname{Dom}\mathfrak{M}_f \subset CS(x) & \longrightarrow & CS(x) \\ & & g & \longmapsto & fg \end{array}$$

with $\operatorname{Dom}\mathfrak{M}_f = \{g \in CS(x); fg \in CS(x)\}$. Throught the isomorphism $J : CS(x) \longrightarrow \Phi$ the operator $\mathfrak{M}_f$ translates into an operator on Φ : for all $f \in \Phi$ define the operator

$$\begin{array}{cccc} \mathfrak{M}_f & : & \operatorname{Dom}\mathfrak{M}_f \subset \Phi & \longrightarrow & \Phi \\ & & g & \longmapsto & J[(J^{-1}f)\,(J^{-1}g)] \end{array}$$

where $\operatorname{Dom}\mathfrak{M}_f = \{g \in \Phi; (J^{-1}f)\,(J^{-1}g) \in CS(x)\}$. In other words, we translate the Fock space vectors f and g in the probabilistic interpretation, we multiply them together in the probabilistic interpretation (when the product stays in $CS(x)$) and then we translate the result back to the Fock space.

Another crucial way of understanding this is that each probabilistic interpretation x of Φ defines a product on Φ. Indeed for $f, g \in \Phi$ define fg to be $J[(J^{-1}f)\,(J^{-1}g)]$. Each probabilistic interpretation defines its own product on Φ for the classical integration by part formula makes use of the square brackets. Indeed, two different probabilistic interpretations of Φ have their angle brackets in common, but not their square brackets: the coefficients Ψ of the structure equations make the difference.

The second ingredient is A-M's point of view on quantum stochastic calculus. Indeed, we are going to see that this point of view on quantum stochastic calculus is the one which is the closest to classical stochastic calculus.

Theorem II.1 – *Let $(x_t)_{t\geq 0}$ be a probabilistic interpretation of Φ. Let $d[x,x]_t = dt + \Psi_t\, dx_t$ be the associated structure equation. Let f be an element of Φ. Then the operator $\mathfrak{M}_f$ on Φ admits an integral representation on $\operatorname{Dom}\mathfrak{M}_f$. This integral representation is given by*

$$\mathfrak{M}_f = \mathbb{E}[f]I + \int_0^\infty \mathfrak{M}_{D_s f}\,(dA_s^\dagger + dA_s) + \int_0^\infty \mathfrak{M}_{D_s f}\,\mathfrak{M}_{\Psi_s}\, d\Lambda_s. \tag{II.2}$$

Proof

Let g be an element of Φ such that $(J^{-1}f)(J^{-1}g) \in CS(x)$. Let $J^{-1}f = \mathbb{E}[f] + \int_0^\infty \xi_s(f)\, dx_s$ and $J^{-1}g = \mathbb{E}[g] + \int_0^\infty \xi_s(g)\, dx_s$ be their predictable representation. Let $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ be respectively the martingales $(P_t f)_{t\geq 0}$ and $(P_t g)_{t\geq 0}$. By the classical integration by part formula we have

$$\begin{aligned}(J^{-1}f)(J^{-1}g) &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty \xi_s(f) g_{s-}\, dx_s + \int_0^\infty \xi_s(g)\, f_{s-}\, dx_s \\ &\quad + \int_0^\infty \xi_s(f)\xi_s(g)\, d[x,x]_s \\ &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty \xi_s(f) g_{s-}\, dx_s + \int_0^\infty \xi_s(g)\, f_{s-}\, dx_s\end{aligned}$$

$$+\int_0^\infty \xi_s(f)\xi_s(g)\Psi_s\,dx_s + \int_0^\infty \xi_s(f)\xi_s(g)\,ds. \qquad \text{(II.3)}$$

Here we have to discuss an important technical point. Let $(y_t)_{t\geq 0}$ be a Ito integrable vector process in Φ. As $(y_t)_{t\geq 0}$ is strongly measurable, the family $(J^{-1}y_t)_{t\geq 0}$ of random variables admits a measurable version. Because of the Ito integrability conditions on $(y_t)_{t\geq 0}$ we easily check that the predictable projection of the process $(J^{-1}y_t)_{t\geq 0}$ is a modification of it. So when we have said that the Ito integral operator on Φ interprets as the Ito integral with respect to $(x_t)_{t\geq 0}$ in $L^2(\Omega)$ we meant

$$J^{-1}\Big(\int_0^\infty y_t\,d\chi_t\Big) = \int_0^\infty J^{-1}y_t\,dx_t$$

where the process $(J^{-1}y_t)_{t\geq 0}$ in the latter integral is actually the predictable projection of it (for simplicity we keep the same notation for both of the two process). In our example above, as $(\xi_t(f))_{t\geq 0}$ is a predictable process and as $(g_t)_{t\geq 0}$ is a martingale, we have that the process $(\xi_t(f)g_{t-})_{t\geq 0}$ is the predictable projection of the process $(\xi_t(f)g_t)_{t\geq 0}$. Thus, we have

$$\int_0^\infty \xi_t(f)g_{t-}\,dx_t = J^{-1}\Big(\int_0^\infty J\big(\xi_t(f)g_t\big)\,d\chi_t\Big).$$

So, applying the isomorphism J to (II.3), we get

$$\begin{aligned}
fg &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty J(\xi_s(f)g_s)\,d\chi_s + \int_0^\infty J(f_s\,\xi_s(g))\,d\chi_s \\
&\quad + \int_0^\infty J(\xi_s(f)\xi_s(g)\Psi_s)\,d\chi_s + \int_0^\infty J(\xi_s(f)\xi_s(g))\,ds \\
\mathfrak{M}_f\,g &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty \mathfrak{M}_{D_sf}\,Jg_s\,d\chi_s + \int_0^\infty \mathfrak{M}_{P_sf}\,J\xi_s(g)\,d\chi_s \\
&\quad + \int_0^\infty \mathfrak{M}_{D_sf}\mathfrak{M}_{\Psi_s}\,J\xi_s(g)\,d\chi_s + \int_0^\infty \mathfrak{M}_{D_sf}\,J\xi_s(g)\,ds \\
\mathfrak{M}_f\,g &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty \mathfrak{M}_{D_sf}\,P_sg\,d\chi_s + \int_0^\infty \mathfrak{M}_{P_sf}\,D_sg\,d\chi_s \\
&\quad + \int_0^\infty \mathfrak{M}_{D_sf}\,\mathfrak{M}_{\Psi_s}\,D_sg\,d\chi_s + \int_0^\infty \mathfrak{M}_{D_sf}\,D_sg\,ds.
\end{aligned}$$

It is easy to check that $\mathfrak{M}_{P_sf} = \mathbb{E}_s[\mathfrak{M}_f]$ in the sense of operator conditional expectations. So if we denote by T the operator $\mathfrak{M}_f$ and by $(T_t)_{t\geq 0}$ its associated operator martingale (that is, $T_t = \mathbb{E}_t[T]$) we then have

$$\begin{aligned}
Tg &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty T_sD_sg\,d\chi_s + \int_0^\infty \mathfrak{M}_{D_sf}\,\mathfrak{M}_{\Psi_s}\,D_sg\,d\chi_s \\
&\quad + \int_0^\infty \mathfrak{M}_{D_sf}\,P_sg\,d\chi_s + \int_0^\infty \mathfrak{M}_{D_sf}\,D_sg\,ds.
\end{aligned}$$

That is exactely the same as saying that

$$T = \mathbb{E}[f]I + \int_0^\infty \mathfrak{M}_{D_sf}\,(dA_s^\dagger + dA_s) + \int_0^\infty \mathfrak{M}_{D_sf}\mathfrak{M}_{\Psi_s}\,d\Lambda_s$$

in the sense of A-M's equations. ■

In particular we have that if $(x_t)_{t\geq 0}$ is a probabilistic interpretation of Φ, with coefficient $(\Psi_t)_{t\geq 0}$ in its structure equation, then

$$d\mathfrak{M}_{x_t} = dA_t^\dagger + dA_t + \mathfrak{M}_{\Psi_t}\, d\Lambda_t.$$

Hence we recover that the Brownian motion is represented in Φ by $A_t^\dagger + A_t$; the compensated Poisson process is $A_t^\dagger + A_t + \Lambda_t$; the coefficient β Azéma martingale is the unique solution of $dX_t = dA_t^\dagger + dA_t + \beta X_t\, d\Lambda_t$.

In this way, we see that quantum stochastic calculus on Fock space is at the same time an extension of classical stochastic calculus (as classical random variables are particular quantum stochastic integrals), but it also unifies it in the sense that Fock space stochastic calculus includes all the probabilistic interpretations of Φ (Brownian motion, Poisson process, Azéma martingales,...) in one single context, in one single calculus.

With the same kind of computations and the same technical remark as in Theorem II.1, we obtain the following result (cf [At1], Proposition 9).

Proposition II.2 – *Let $(x_t)_{t\geq 0}$ be a probabilistic interpretation of Φ. Let $(z_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ be semimartingales in this interpretation. Suppose that the operator processes $(\mathfrak{M}_{z_t})_{t\geq 0}$ and $(\mathfrak{M}_{y_t})_{t\geq 0}$ are elements of the space $\mathcal{S}'$ defined in section I.2. Then the quantum square bracket (resp. quantum angle bracket) of $\mathfrak{M}_z$ and $\mathfrak{M}_y$ is the multiplication operator by the classical angle bracket (resp. classical angle bracket) of z and y; that is,*

$$[\mathfrak{M}_z, \mathfrak{M}_y]_t = \mathfrak{M}_{[z,y]_t} \qquad \text{and} \qquad \langle \mathfrak{M}_z, \mathfrak{M}_y\rangle_t = \mathfrak{M}_{\langle z,y\rangle_t}.$$ ■

We end this subsection on an interesting remark which relates some natural operators coming from classical stochastic calculus to the quantum stochastic integrals. This exemple is treated in details in [A-M].

Consider a given probabilistic interpretation $(x_t)_{t\geq 0}$ of Φ. Let $\lambda \in I\!R$. Let $(h_t)_{t\geq 0}$ be a bounded predictable process. Let $(k_t)_{t\geq 0}$ be a process of the form $k_t = \int_0^t p_s\, ds$, $t \in I\!R^+ \cup \{+\infty\}$. Let $(m_t)_{t\geq 0}$ and $(n_t)_{t\geq 0}$ be complete martingales with predictable representation $m_t = c + \int_0^t \mu_s\, dx_s$ and $n_t = c' + \int_0^t \nu_s\, dx_s$ with the random variables μ_s an ν_s beeing bounded.

Let $L_b^2(\Omega)$ be the subspace of bounded random variables in $L^2(\Omega)$. Under all these conditions one can define on $L_b^2(\Omega)$ the following five basic operators:

$$\begin{array}{llll} E_\lambda & : f & \longmapsto & \lambda I\!E[f] \\ I_h & : f & \longmapsto & \int_0^t h_s\, df_s \\ J_n & : f & \longmapsto & \int_0^t f_s\, dn_s \\ T_k & : f & \longmapsto & \int_0^t f_s\, dk_s \\ C_m & : f & \longmapsto & \langle f\,,\, m\rangle_t, \end{array}$$

where $(f_t)_{t\geq 0}$ is the process $(P_t f)_{t\geq 0}$. As an easy consequence of A-M's equations we obtain.

Proposition II.3 – *The adapted operator $T = E_\lambda + I_h + J_n + T_k + C_m$ admits an integral representation on $L^2_b(\Omega)$. This representation is given by*

$$T = \lambda I + \int_0^\infty (\mathfrak{M}_{h_s} - T_s)\, d\Lambda_s + \int_0^\infty \mathfrak{M}_{\nu_s}\, dA_s^+ + \int_0^\infty \mathfrak{M}_{\mu_s}\, dA_s^- + \int_0^\infty \mathfrak{M}_{p_s}\, ds$$

where $(T_t)_{t\geq 0}$ is the operator martingale associated to T. ■

This result is interesting because it gives a one to one correspondance between four basic operators on a probabilistic intepretation and the four types of quantum stochastic integrals on Φ. This gives an idea of what kind of classical operators each of the quantum stochastic integrals is supposed to extend.

II.1.4 Extension of the classical Ito formula

As we have seen in section I.2, the quantum integration by part formula for elements of the algebra $\mathcal{S}$ is

$$X_t Y_t = \int_0^t X_s\, dY_s + \int_0^t dX_s\, Y_s + [X, Y]_t. \tag{II.4}$$

The classical one is

$$x_t\, y_t = \int_0^t x_{s-}\, dy_s + \int_0^t y_{s-}\, dx_s + [x, y]_t. \tag{II.5}$$

The only difference lies in the fact that in the classical case x_{s-} and y_{s-} are appearing instead of X_s and Y_s in the quantum one. But the same technical remark as in Theorem II.1 applies here. That is, when X and Y are taken to be multiplication operators by some classical semimartingales, applying the isomorphism J^{-1} to (II.4) gives (II.5) (cf [At1]).

But one can wonder if functional Ito formulas such as in Theorem I.18 coincide with the classical Ito formula. This was already known for polynomial functions in [At1], but in [ViS] the complete result for C^{2+}_{loc} functions is obtained. We reproduce here his theorem and his nice proof.

Theorem II.4 – *In every probabilistic interpretation of the Fock space the quantum Ito formula (Theorem I.17) for self-adjoint regular quantum semimartingales and $f \in C^{2+}_{loc}$ implies the classical Ito formula:*

$$\begin{aligned} f(z_t) = f(z_0) &+ \int_0^t f'(z_{s-})\, dz_s + \frac{1}{2}\int_0^t f''(z_{s-})\, d\langle z^c, z^c\rangle_s \\ &+ \sum_{0\leq s\leq t} \big(f(z_s) - f(z_{s-}) - f'(z_{s-})\Delta z_s\big) \end{aligned}$$

where z^c denotes the continuous part of z.

Proof

Let $z_t = \int_0^t \xi_s\, dx_s + \int_0^t k_s\, ds$, $t \geq 0$, be a semimartingale in the probabilistic interpretation $(x_t)_{t\geq 0}$. Recall that $z_t^c = \int_0^t \mathbb{1}_{\{\Psi_s=0\}\cup\{\xi_s=0\}} \xi_s\, dx_s$ (cf [ViS]). Assume that $\mathfrak{M}_{z_t} = \int_0^t \mathfrak{M}_{\xi_s}\,(dA_s + dA_s^\dagger + \mathfrak{M}_{\Psi_s}\,d\Lambda_s) + \int_0^t \mathfrak{M}_{k_s}\,ds$, $t \geq 0$, is an element of the algebra $\mathcal{S}$. Note that for $f \in C^{2+}_{loc}$, for $x, h \in I\!R$ we have

$$Af(x,h) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} ip\widehat{f}(p) \int_0^1 e^{ip(x+uh)}\,du\,dp = \begin{cases} \frac{f(x+h)-f(x)}{h} & \text{if } h \neq 0 \\ f'(x) & \text{if } h = 0 \end{cases}$$

$$\begin{aligned} Bf(x,h) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (ip)^2 \widehat{f}(p) \int_0^1 \int_0^1 u e^{ip(x+uvh)}\,du\,dv\,dp \\ &= \begin{cases} \frac{f(x+h)-f(x)-hf'(x)}{h^2} & \text{if } h \neq 0 \\ \frac{1}{2}f''(x) & \text{if } h = 0. \end{cases} \end{aligned}$$

As in our case all the coefficients of the integral representation of $\mathfrak{M}_{z_t}$ commute with each other, the coefficients of the integral representation of $f(\mathfrak{M}_{z_t})$ given by Theorem I.18 are:

$$\begin{aligned} H_f(s) &= f(\mathfrak{M}_{z_s} + \mathfrak{M}_{\xi_s}\mathfrak{M}_{\Psi_s}) - f(\mathfrak{M}_{z_s}) \\ L_f(s) &= Af(\mathfrak{M}_{z_s}, \mathfrak{M}_{\xi_s}\mathfrak{M}_{\Psi_s})\mathfrak{M}_{\xi_s} \\ M_f(s) &= \mathfrak{M}_{k_s} f'(\mathfrak{M}_{z_s}) + \mathfrak{M}_{\xi_s}\mathfrak{M}_{\xi_s} Bf(\mathfrak{M}_{z_s}, \mathfrak{M}_{\xi_s}\mathfrak{M}_{\Psi_s}). \end{aligned}$$

Applying the integral representation of $f(\mathfrak{M}_{z_t})$ to the vector 1 with the help of A-M's equations, applying the isomorphism J^{-1} to the result, using the same kind of technical remark as in Theorem II.1 (cf [At1], p. 319), we get

$$\begin{aligned} f(z_t) &= f(z_0) + \int_0^t Af(z_{s-}, \xi_s\Psi_s)\xi_s\,dx_s + \int_0^t f'(z_s)k_s\,ds + \int_0^t Bf(z_{s-}, \xi_s\Psi_s)\xi_s^2\,ds \\ &= f(z_0) + \int_0^t f'(z_{s-})\xi_s\,dx_s + \int_0^t f'(z_s)k_s\,ds \\ &\quad + \int_0^t 1\!\!1_{\{\Psi_s=0\}\cup\{\xi_s=0\}} Bf(z_{s-}, \xi_s\Psi_s)\xi_s^2\Psi_s\,dx_s \\ &\quad + \int_0^t 1\!\!1_{\{\xi_s\Psi_s\neq 0\}} Bf(z_{s-}, \xi_s\Psi_s)\xi_s^2\Psi_s\,dx_s + \int_0^t Bf(z_s, \xi_s\Psi_s)\xi_s^2\,ds \\ &= f(z_0) + \int_0^t f'(z_{s-})\xi_s\,dz_s + \frac{1}{2}\int_0^t f''(z_{s-})\,d\langle z^c, z^c\rangle_s \\ &\quad + \int_0^t 1\!\!1_{\{\xi_s\Psi_s\neq 0\}} Bf(z_{s-}, \xi_s\Psi_s)\,(\xi_s^2\Psi_s\,dx_s + \xi_s^2\,ds). \end{aligned}$$

Since x^d (the purely discontinuous part of x) is equal to $1\!\!1_{\{\Psi_s\neq 0\}}x$ (cf [Em1]) it follows that $[x,x]$ is constant between jumps and $\Psi_s\,dx_s + ds = 0$ on $\{\Psi_s \neq 0\}$. Therefore the final integral becomes

$$\sum_{0\leq s\leq t} \frac{f(z_{s-} + \xi_s\Psi_s) - f(z_{s-}) + \xi_s\Psi_s f'(z_{s-})}{\xi_s\Psi_s} = \sum_{0\leq s\leq t} \big(f(z_s) - f(z_{s-}) - f'(z_{s-})\Delta z_s\big).$$

■

II.2 Representation of Wiener space endomorphisms

In this section we present the results of [At4] and their extension in [At5] and [AE1]. The problem consists in looking at transformations of the Wiener space which transform the canonical Brownian motion into another Brownian motion. For some of them we are able to give a complete algebraic characterization of such transformations. This is obtained with the help of quantum stochastic calculus.

II.2.1 The martingale preserving endomorphisms

Let $(\Omega, \mathcal{F}, P)$ be the Wiener space, let $(W_t)_{t\geq 0}$ be the canonical Brownian motion, let $(\mathcal{F}_t)_{t\geq 0}$ be its natural filtration, let E_t denote the conditional expectation operator $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$, $t \in \mathbb{R}^+$.

A measurable mapping $\widetilde{T} : \Omega \to \Omega$ is an *endomorphism* if it preserves the Wiener measure P. In this case one defines the random variables $\widetilde{W_t}$ by $\widetilde{W_t}(\omega) = W_t(\widetilde{T}\omega)$. The process $(\widetilde{W_t})_{t\geq 0}$ is then a Brownian motion.

The endomorphism $\widetilde{T}$ is *martingale-preserving* if $\widetilde{W}$ is a Brownian motion for the filtration $(\mathcal{F}_t)_{t\geq 0}$. In this case $\widetilde{W}$ is a martingale for $(\mathcal{F}_t)_{t\geq 0}$, with angle bracket $\langle \widetilde{W}, \widetilde{W}\rangle_t = t$. Thus, from the Brownian motion predictable representation property, the process $(\widetilde{W_t})_{t\geq 0}$ admits a representation

$$\widetilde{W} = \int_0^t k_s \, dW_s$$

where $(k_t)_{t\geq 0}$ is a predictable process satisfying $k_t^2(\omega) = 1$ for almost all (ω, t). It is clear that there exists a bijection between the set of martingale-preserving endomorphism $\widetilde{T}$ and the set of predictable process $(k_t)_{t\geq 0}$ satisfying $k_t^2(\omega) = 1$ for almost all (ω, t).

Let $\widetilde{T}$ be an endomorphism of Ω. One can associate to $\widetilde{T}$ an operator T on $L^2(\Omega, \mathcal{F}, P)$ by $Tf \stackrel{\text{def}}{=} f \circ \widetilde{T}$. This operator satisfies:

i) T is an isometry (T is unitary if and only if $\widetilde{T}$ is invertible),

ii) $T(fg) = (Tf)(Tg)$ for all $f, g \in L^2(\Omega)$ such that $fg \in L^2(\Omega)$.

It is easy to check that $\widetilde{T}$ is martingale preserving if and only if T satisfies

iii) $TE_t = E_tT$ for all $t \in \mathbb{R}^+$.

The Brownian motion possesses the chaotic representation property, so the space $L^2(\Omega, \mathcal{F}, P)$ is isomorphic to the Fock space Φ. In this section we omit the isomorphism J between $L^2(\Omega, \mathcal{F}, P)$ and Φ and we make no difference between the two spaces. The only point to be precised is that when we speak of multiplication of two vectors f, g of Φ we mean the multiplication of the corresponding random variables in the Brownian interpretation of Φ. This is the only operation here which is not intrinsic to Φ.

So we dispose of an operator T on Φ which is an isometry, which respects Wiener multiplication and which commutes with all the operators P_t (recall that E_t interprets as P_t on Φ). We can form the operator martingale associated to T that is, $T_t \stackrel{\text{def}}{=} \mathbb{E}_t[T]$, $t \in \mathbb{R}^+$, in the sense of section I.1 operator conditional expectations.

It is easy to check that T_t is the operator associated to the martingale-preserving endomorphism $\widetilde{T}_t$ defined by

$$\widetilde{T}_t W_s = \int_0^s k_u^t \, dW_u, \quad \text{where } k_u^t = \begin{cases} k_u & \text{if } u \leq t \\ 1 & \text{if } u > t. \end{cases}$$

In other words, $\widetilde{T}_t$ is the same endomorphism as $\widetilde{T}$ up to time t and it is the identity endomorphism after time t.

Note that property i) for the operator T is a simple consequence of properties ii) and iii). Indeed, if T satisfies ii) and iii) we have $P_0(T1) = TP_0 1 = T1$ so $T1$ is an element of $\Phi_{0]} = \mathbb{C}$. Furthermore $T(1) = T(11) = (T1)(T1)$ thus $T1$ equals either 0 or 1. But as for all $f \in \Phi$ we have $Tf = T(1f) = (T1)(Tf)$ we must have $T1 = 1$ if the operator T is not identically 0. Now, because the scalar product in Φ identifies as the $L^2(\Omega)$ usual scalar product we have

$$\begin{aligned} < Tf\,, Tg > &= P_0[(\overline{Tf})(Tg)] = P_0[T(\overline{f}g)] = TP_0[\overline{f}g] \\ &= P_0[\overline{f}g]T1 = P_0[\overline{f}g] = < f\,, g >. \end{aligned}$$

This proves that T is an isometry. Consequently we have to consider operators on Φ which satisfy properties ii) and iii).

Note the following important remark. Every operator associated to a martingale-preserving endomorphism satisfies properties ii) and iii). But the converse is not a priori true. Indeed, if the operator T were furthermore unitary then together with property ii) this would imply (cf [Cho]) that T is the operator associated to an endomorphism $\widetilde{T}$ of Ω; the condition iii) would then imply that $\widetilde{T}$ is martingale-preserving. But in our case the fact that T satisfying ii) and iii) is only an isometry does not allow to say that it is associated to an endomorphism of Ω. Actually, the characterization we are going to get through quantum stochastic calculus is going to prove that conditions ii) and iii) are not only necessary but also *sufficient* for a Fock space operator T to be the operator associated to a martingale-preserving endomorphism. This conclusion is obtained with the help of quantum stochastic calculus.

We first focus on operators satisfying the commutation property iii).

Theorem II.5 – *Let T be a bounded operator on Φ. The following assertions are equivalent.*

i) $T\,P_t = P_t\,T$, *for all* $t \in \mathbb{R}^+$.

ii) *There exists an adapted process* $(H_t)_{t \geq 0}$ *of bounded operators on* Φ *and a* $\lambda \in \mathbb{C}$ *such that we have on all* Φ

$$T = \lambda I + \int_0^\infty H_s \, d\Lambda_s.$$

Proof

This theorem is proved in [At6]. It is a consequence of Parthasarathy-Sinha's representation theorem on regular martingales. Or better, in the context of this article, let T be an operator satisfying i). Consider the operator martingale $(T_t)_{t\geq 0}$ associated to T. This process can be seen to be an element of the algebra $\mathcal{S}$ of section I.2 as the commutation relations with the P_t's imply that $(T_t)_{t\geq 0}$ satisfy (I.16) and (I.17) (of course it satisfies (I.18) as it is an operator martingale) for the null measure μ. The fact that the measure μ is the null one implies that the dA and $dA^\dagger$ coefficients of the representation of T vanish. This gives ii). The converse is immediate from A-M's equation. ■

We can now look at the multiplication-preserving property.

Theorem II.6 – *Let T be a bounded operator on Φ which commutes with the P_t, $t \in \mathbb{R}^+$ and such that $T1 = 1$. Let $T = I + \int_0^\infty H_s\, d\Lambda_s$ be its integral representation on all Φ (Theorem II.5). Let $(T_t)_{t\geq 0}$ be the associated operator martingale. The following assertions are equivalent.*

i) *$T(fg) = (Tf)(Tg)$ for all $f, g \in \Phi$ such that $fg \in \Phi$.*

ii) *For almost all t, we have $H_t = \mathfrak{M}_{H_t 1} \circ T_t$ and $H_t 1$ is valued in $\{0, -2\}$.*

iii) *For almost all t, for all $f, g \in \Phi$ such that $fg \in \Phi$, we have*

$$H_t(fg) = -\frac{1}{2}(H_t f)(H_t g),$$

and, for almost all $s \leq t$,

$$H_t W_s = (H_t 1) \int_0^s (1 + H_u 1)\, dW_u.$$

Proof

The complete proof of this theorem is given in [At4], Theorem 4. It is rather long and we present here a shortened version.

Suppose that T satisfies i). Let η_t denote $T_t + H_t$. Let $f \in \Phi$. From A-M's equation we have

$$Tf = \mathcal{P}_0 f + \int_0^\infty \eta_t D_t f\, dW_t.$$

Suppose f^2 belongs to Φ, the Ito formula (either the quantum or the classical one, this has no importance anymore) gives

$$(P_t f)^2 = (P_0 f)^2 + 2\int_0^t (D_s f)(P_s f)\, dW_s + \int_0^t (D_s f)^2\, ds.$$

Thus we easily get

$$T((P_t f)^2) = (P_0 f)^2 + 2\int_0^t \eta_s((D_s f)(P_s f))\, dW_s + \int_0^t T((D_s f)^2)\, ds \quad \text{(II.6)}$$

$$(TP_t f)^2 = (P_0 f)^2 + 2\int_0^t (\eta_s D_s f)(TP_s f)\, dW_s + \int_0^t (\eta_s D_s f)^2\, ds. \quad \text{(II.7)}$$

If T satisfies i) one has $T((P_t f)^2) = (TP_t f)^2$ so identifying (II.6) and (II.7) gives in particular $(\eta_t D_t f)^2 = T_t((D_t f)^2)$ for almost all t. Polarising we get

$(\eta_t D_t f)(\eta_t D_t g) = T_t((D_t f)(D_t g))$. Putting $f = W_{t+1}$ so that $D_t f = 1$ we get $(\eta_t 1)(\eta_t D_t g) = T(D_t g)$. If one also puts g to be W_{t+1} we get $(\eta_t 1)^2 = T_t 1 = 1$. Thus $\eta_t D_t g = (\eta_t 1)(T(D_t g))$. It is not difficult to conclude that $\eta_t g = (\eta_t 1) T_t g$ for all $g \in \Phi$, thus $H_t = \mathfrak{M}_{H_t 1} \circ T_t$. The condition $(\eta_t 1)^2 = 1$ implies that $H_t 1$ is valued in $\{-2, 0\}$. This proves that i) implies ii).

Assume ii) is satisfied. It is clear then that $\eta_t = \mathfrak{M}_{\eta_t 1} \circ T_t$ and that $(\eta_t 1)^2 = 1$. Putting this in (II.6) and (II.7), polarizing, we get

$$\begin{aligned} T(fg) &= P_0 f\, P_0 g + \int_0^\infty (\eta_t 1) T((D_t f)(P_t g))\, dW_t + \\ &\quad + \int_0^\infty (\eta_t 1) T((D_t g)(P_t f))\, dW_t + \int_0^\infty T((D_t f)(D_t g))\, dt \\ (Tf)(Tg) &= P_0 f\, P_0 g + \int_0^\infty (\eta_t 1)[(T D_t f)(T P_t g)]\, dW_t \\ &\quad + \int_0^\infty (\eta_t 1)[(T D_t g)(T P_t g)]\, dW_t + \int_0^\infty (T D_t g)(T D_t f)\, dt. \end{aligned}$$

Let us call $P(n, m)$ the property "$T(fg) = (Tf)(Tg)$ holds for all $f \in L^2(\mathcal{P}_n)$ and all $g \in L^2(\mathcal{P}_m)$". We see from the previous two identities that $P(n, m)$ is true if $P(n, m-1), P(n-1, m)$ and $P(n-1, m-1)$ are (for if $f \in L^2(\mathcal{P}_n)$ then $D_t f \in L^2(\mathcal{P}_{n-1})$). As $P(0, m)$ and $P(n, 0)$ are obviously true one concludes easily. We have proved that i) is equivalent to ii).

Assume that ii) is satisfied. We have

$$\begin{aligned} (H_t f)(H_t g) &= [(H_t 1) T_t f][(H_t 1) T_t g] = [(1 + H_t 1 - 1) T_t f][(1 + H_t 1 - 1) T_t g] \\ &= [(1 + H_t 1) T_t f][(1 + H_t 1) T_t g] - (T_t f)[(1 + H_t 1) T_t g] \\ &\quad - [(1 + H_t 1) T_t f](T_t g) + (T_t f)(T_t g) \\ &= 2\,(T_t f)(T_t g) - 2\ (1 + H_t 1)[(T_t f)(T_t g)] \\ &= -2(H_t 1) T_t(fg) = -2 H_t(fg). \end{aligned}$$

This gives the first condition of iii) on H_t. The second one is an easy consequence of A-M's equation. We have that ii) implies iii).

Finally, assume iii) is satisfied. We have $(H_t 1)^2 = -2 H_t 1$ thus $H_t 1$ is valued in $\{-2, 0\}$. Let us prove that $H_t f = (H_t 1) T_t f$ for all f. This is true for $f = c \in \mathbb{C}$. Now, let $f \in \Phi$ be such that $H_t D_s f = (H_t 1) T_t D_s f$ for $s \leq t$. We *formally* have

$$\begin{aligned} H_t P_t f &= H_t \int_0^t D_s f\, dW_s = \int_0^t H_t(D_s f\, dW_s) \\ &= -\frac{1}{2} \int_0^t (H_t D_s f)\,(H_t\, dW_s) = -\frac{1}{2} \int_0^t (H_t D_s f)\,(H_t 1)(1 + H_s 1)\, dW_s \\ &= -\frac{1}{2} (H_t 1)^2 \int_0^t (T_s D_s f)(1 + H_s 1)\, dW_s \\ &= (H_t 1) \int_0^t (1 + H_s 1)(T_s D_s f)\, dW_s \end{aligned}$$

$$= (H_t 1) \int_0^t (T_s + H_s) D_s f \, dW_s$$
$$= (H_t 1) T_t \int_0^t D_s f \, dW_s \qquad \text{(by A-M's equation)}$$
$$= (H_t 1) T_t P_t f.$$

We conclude by induction on n by taking $f \in L^2(\mathcal{P}_n)$. ■

We can now state the final characterization of martingale-preserving endomorphisms of the Wiener space.

Theorem II.7 – *Let T be a non null, bounded operator on Φ. The following assertions are equivalent.*

i) *T is the operator associated to a martingale-preserving endomorphism $\widetilde{T}$ of the Wiener space.*

ii) *The operator T commutes with the operators P_t, $t \in \mathbb{R}^+$, and it satisfies $T(fg) = (Tf)(Tg)$ for all $f, g \in \Phi$ such that $fg \in \Phi$.*

When these conditions are satisfied the image $(\widetilde{W}_t)_{t \geq 0}$ of the Brownian motion $(W_t)_{t \geq 0}$ under $\widetilde{T}$ is given by $\widetilde{W}_t = \int_0^t k_s \, dW_s$ where $k_s = 1 + H_s 1$ and $(H_t)_{t \geq 0}$ is the $d\Lambda$-coefficient in the representation of T.

Proof

We know that i) implies ii). Suppose ii) is satisfied. Thus condition iii) of Theorem II.6 is satisfied. Consequently, one has $(1 + H_s 1)^2 = 1$. Let us put $k_s = 1 + H_s 1$ for all s. We are then given a predictable process $(k_t)_{t \geq 0}$ satisfying $k_t^2 = 1$ for all t. Let $\widetilde{T}$ be the martingale preserving endomorphism associated to $(k_t)_{t \geq 0}$. Let T' be the operator on Φ associated to $\widetilde{T}$. From Theorems II.5 and II.6 we know that there exists an adapted process of bounded operators $(H'_t)_{t \geq 0}$ such that $T' = I + \int_0^\infty H'_s \, d\Lambda_s$ on all Φ and $(H'_t)_{t \geq 0}$ satisfies the conditions iii) of Theorem II.6. By A-M's equation we have $T' W_t = \int_0^t (1 + H'_s 1) \, dW_s$ but it is also equal to $\int_0^t k_s \, dW_s$, thus $1 + H'_s 1 = k_s$ for almost all s. This means that $H_s 1$ and $H'_s 1$ coincide for a.a. s. Because both H and H' satisfy Theorem II.6 iii) we have that $H_t W_s$ and $H'_t W_s$ coincide for a.a. $s \leq t$. As they both respect the Wiener multiplication H and H' coincide on polynomial functions of the Brownian motion. As they are both bounded operators they coincide on all Φ. ■

In conclusion, with the help of quantum stochatic calculus we have obtained an algebraic characterization of martingale-preserving endomorphisms. This characterization does not seem obvious to be obtained without the help of quantum stochastic calculus.

II.2.2 The kernel mystery

We have seen that martingale-preserving endomorphisms when lifted to the Fock space always admit an integral representation. One can wonder if they admit a Maassen kernel representation. The problem appears still open to me, even if I have the strong feeling that they do admit one. Indeed, there do formally exists such a kernel but it is very complicated and analytic problems are thus rather difficult to overcome.

Anyway, we can look at Maassen kernels which satisfy the commutation and the multiplication property. The following result is a combinaison of results from [At6] and [At5]. They are given without proof here.

Theorem II.8 – *Let T be a non-vanishing bounded operator from Φ to Φ which admits a Maassen kernel $\widehat{T}$. The following assertions are equivalent.*

i) *The operator T is the operator associated to a martingale-preserving endomorphism of the Wiener space.*

ii) *The kernel of T satisfies the following two properties :*

a) $\widehat{T}(\alpha,\beta,\gamma)=0$ *unless* $\vee(\alpha+\beta+\gamma)$ *is an element of* β,

b) for a dense set of vectors f, for all $\alpha,\beta,\gamma\in\mathcal{P}^3$, one has

$$\int_{\mathcal{P}}\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \gamma_1+\gamma_3=\gamma}}\widehat{T}(\,\alpha_1\,,\,\alpha_2+\beta\,,\,\gamma_1+\mu\,)\,f(\mu+\alpha_2+\alpha_3+\gamma_3)\,d\mu=$$

$$=\int_{\mathcal{P}}\sum_{\substack{\alpha_1+\alpha_3=\alpha\\ \gamma_1+\gamma_2+\gamma_3=\gamma}}[Tf](\alpha_1+\gamma_1+\gamma_2+\mu)\,\widehat{T}(\,\mu+\alpha_3\,,\,\beta+\gamma_2\,,\,\gamma_3\,)\,d\mu.$$

In any case the chaotic expansion of the predictable process $(k_t)_{t\geq 0}$ associated to the endomorphism is given by $\widehat{k_t}(\alpha)=\mathbb{1}_{\alpha=\emptyset}+\widehat{T}(\alpha,\{t\},\emptyset)$, $\alpha\in\mathcal{P}$. ■

It is the combinaison of the previous section theorems that gives this result. But this results contains a mystery. Indeed, properties a) and b) are sufficient for the kernel to be a martingale-preserving endomorphism. Furthermore the chaotic expansion of the associated predictable process $(k_t)_{t\geq 0}$ depends only on the family of values $\widehat{T}(\alpha,\{t\},\emptyset)$. As this predictable process determines the endomorphism, this means that the kernel values $\widehat{T}(\alpha,\beta,\gamma)$ are completely determined by those of $\widehat{T}(\alpha,\{t\},\emptyset)$. This means that there exists an algebraic relation, which has to be deduced from identities a) and b), which gives the values $\widehat{T}(\alpha,\beta,\gamma)$ in terms of the $\widehat{T}(\alpha,\{t\},\emptyset)$'s. Finding this formula is still an open problem and seems rather complicated, but we know that this formula exists. Describing it would be of great interest. Indeed, given the chaotic expansions of the k_t's we would be able to describe completely the associated endomorphism $\widetilde{T}$. This should enable us to answer to several questions such as characterizing when on $(k_t)_{t\geq 0}$ the endomorphism $\widetilde{T}$ is invertible.

II.2.3 Some more general endomorphisms

The technics developed in subsection II.2.1 allow to consider much more general types of endomorphisms. In [At5] some more general endomorphisms of the Wiener space are characterized.

Let $\widetilde{T}$ be an endomorphism of the Wiener space. Let $(\widetilde{W_t})_{t\geq 0}$ be the Brownian motion, image of $(W_t)_{t\geq 0}$ under $\widetilde{T}$. One says that the endomorphism $\widetilde{T}$ is *adapted* if the process $(\widetilde{W_t})_{t\geq 0}$ is adapted to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(W_t)_{t\geq 0}$. An adapted endomorphism is *regular* the process $(\widetilde{W_t})_{t\geq 0}$ admits a representation of the form

$$\widetilde{W_t} = \widetilde{T}W_t = \int_0^t k_s\, dW_s + \int_0^t h_s\, ds.$$

Note that in the case where $(h_t)_{t\geq 0}$ is the null process one recovers martingale-preserving endomorphisms.

Let T be the operator from Φ to Φ associated to an adapted endomorphism $\widetilde{T}$ that is, the operator defined by $Tf = f\circ\widetilde{T}$, $f \in \Phi$. This operator T has the following properties:

i) it is an isometry (it is unitary if and only if $\widetilde{T}$ is invertible)

ii) $T(fg) = (Tf)\,(Tg)$, for $f, g \in \Phi$ such that $fg \in \Phi$

iii) it preserves the spaces $\operatorname{Im} P_t$, for all $t \in \mathbb{R}^+$ (or equivalently $T\,P_t = P_t\,T\,P_t$, $t \in \mathbb{R}^+$).

The results and the techniques for characterizing regular endomorphisms are almost the same as for the martingale-preserving ones. The differences are that the isometry property is not a consequence of ii) and iii) anymore, and that an operator satisfying iii) is not always representable as quantum stochastic integrals (a counter example is given in [At5]). We obtain the following characterization.

Theorem II.9 – *Let T be a non null bounded operator from Φ to Φ. The following assertions are equivalent.*

i) *T is the operator associated to a regular endomorphism of the Wiener space.*

ii) *The operator T admits an integral representation on all Φ, it preserves the spaces $\operatorname{Im}\mathbb{E}_t$, $t \in \mathbb{R}^+$, it is an isometry of Φ and it satisfies $T(fg) = (Tf)(Tg)$ for all $f, g \in \Phi$ such that $fg \in \Phi$.*

iii) *The operator T admits an integral representation on all Φ of the form*

$$T = I + \int_0^\infty H_s^\circ\, d\Lambda_s + \int_0^\infty H_s^-\, dA_s$$

where:

- $H_t^\varepsilon = \mathfrak{M}_{H_t^\varepsilon 1} \circ T_t$, *for almost all t, all $\varepsilon \in \{\circ, -\}$,*
- $H_t^\circ 1$ *is valued in $\{0, -2\}$ for almost all t,*
- $(H_t^-)^*\, T_t = 0$ *for almost all t.*

When any of these conditions is satisfied, the regular endomorphism $\widetilde{T}$ associated to T is given by

$$\widetilde{T}W_t = \int_0^t k_s\, dW_s + \int_0^t h_s\, ds$$

where $k_s = 1 + H_s^\circ 1$ and $l_s = H_s^- 1$. ■

Note two important remarks. The new type of condition $(H_t^-)^* T_t = 0$ is the same as saying (in the probabilistic point of view) that the random variable l_t is orthogonal to $\operatorname{Im} T_t$. It is also the condition which gives the isometry property of T. Another consequence of this theorem is that all the operators associated to regular endomorphisms are again representable on all Φ as quantum stochastic integrals.

In [AE1] the same kind of regular endomorphisms are considered but in different probabilistic interpretations than the Wiener space. If x is a probabilistic interpretation of Φ with coefficient Ψ in its structure equation we say that Ψ is *in the first chaios* of x if Ψ_t is of the form $C + \int_0^t h(s)\, dx_s$ for some $c \in \mathbb{C}$ and some deterministic function h on $\mathbb{R}^+$. One says that a structure equation has the *property of uniqueness in law of its solutions* if all solutions of this equation have the same law. There are four examples of normal martingales whose coefficient Ψ is in the first chaos and which possess the uniqueness in law property: the Brownian motion, the compensated Poisson process, the Azéma martingales and the case where Ψ is a deterministic function of time.

Theorem II.10 – *Let $(x_t)_{t\geq 0}$ be a probabilistic interpretation of Φ with structure equation $d[x,x]_t = dt + \Psi_t\, dx_t$ with Ψ beeing in the first chaos of x. Assume that this structure equation has the property of uniqueness in law of its solutions. Let T be a non-vanishing bounded operator from Φ to Φ. The following assertions are equivalent.*

i) *T is the operator associated to a regular endomorphism of the probabilistic interpretation $(x_t)_{t\geq 0}$.*

ii) *The operator T admits an integral representation on all Φ, it preserves the spaces $\operatorname{Im} \mathbb{E}_t$, $t \in \mathbb{R}^+$, it is an isometry of Φ and it satisfies $T(fg) = (Tf)(Tg)$ for all $f, g \in \Phi$ such that $fg \in \Phi$ (and where the products are products in the $(x_t)_{t\geq 0}$-interpretation).*

iii) *The operator T admits an integral representation on all Φ of the form*

$$T = I + \int_0^\infty H_s^\circ \, d\Lambda_s + \int_0^\infty H_s^- \, dA_s$$

where:

- $H_t^\varepsilon = \mathfrak{M}_{H_t^\varepsilon 1} T_t$, *for almost all t, all $\varepsilon \in \{\circ, -\}$,*
- $H_t^\circ 1$ *is valued in $\{0, -2\}$, for almost all t,*
- *if $k_t \stackrel{\text{def}}{=} 1 + H_t^\circ 1$ and $l_t \stackrel{\text{def}}{=} H_t^- 1$ then $k_t^2 \Psi_t = k_t T\Psi_t$ and $k_t^2 = 1 + l_t T\Psi_t$,*
- $(H_t^-)^* T_t = 0$, *for almost all t.*

When any of these conditions is satisfied, the regular endomorphism $\widetilde{T}$ associated to T is given by

$$\widetilde{T}W_t = \int_0^t k_s \, dW_s + \int_0^t h_s \, ds. \qquad \blacksquare$$

Note the new type of conditions, depending on the coefficient Ψ of the structure equation: $k_t^2\Psi_t = k_tT\Psi_t$ and $k_t^2 = 1 + l_tT\Psi_t$.

When we are in the Brownian case that is, $\Psi_t = 0$ we recover Theorem II.9.

When we are in the compensated Poisson case that is, $\Psi_t = -1$ we have $k_t^2 = k_t$ and $k_t^2 = 1 - l_t$. Thus k_t is valued in $\{0, 1\}$ and l_t is valued in $\{-1, 0\}$ respectively. But as l_t is orthogonal to $\mathrm{Im}T_t$ we have in particular $\mathbb{E}[l_t] = 0$. This is impossible for a $\{-1, 0\}$-valued random variable unless l_t is the null random variable. In this case k_t is identically 1 and T has to be the identity operator. In conclusion the only regular endomorphism of the compensated Poisson process is the identity. This is actually not a surprising result.

In the case where Ψ is made of deterministic functions we get in the same way that: k_t has to be 1 and l_t has to be 0 for those t such that $\Psi_t \neq 0$; k_t is any random variable such that $k_t^2 = 1$ for those t such that $\Psi_t = 0$.

The most difficult case is the case of Azéma martingales. It is proved in [AE1] that if x is an Azéma martingale with coefficient β, the only possibility for the process

$$x'_t = \int_0^t k_s\,dx_s + \int_0^t l_s\,ds$$

to be an Azéma martingales with coefficient β is that $l_t = 0$ for a.a. t, $k_t^2 = 1$ for a.a. t and $t \mapsto k_t$ is constant during the excursions of x. The proof for this result is purely *probabilistic* and uses fine properties of Azéma martingales. As the properties iii) in Theorem II.10 completely characterise such endomorphisms, I think there should exist a purely *algebraic* proof of the previous property by applying properties iii) to the case $\Psi_t = \beta\chi_t$. For the moment I have not been able to perform this algebraic proof.

II.2.4 Levy transform as a counter-example to Q.S.D.E.

In [At7] are considered quantum stochastic differential equations with adapted coefficients. That is, equations of the form

$$dU_t = H_tU_t\,d\Lambda_t + K_tU_t\,dA_t^\dagger + L_tU_t\,dA_t + M_tU_t\,dt$$

where all the coefficients H, K, L, M are made of adapted processes of bounded operators. In [At7] sufficient conditions are given for such an equation to admit a solution and for the solution to be unique. This kind of quantum stochastic differential equations differs from the kind of equations that are usually considered. Indeed, usually the coefficient are not time-dependent and above all they are not adapted. That is, they act trivially on some "initial space" (cf [H-P], [Par] or [Me1]). I do not intend to detail this theory here but I just want to point out, for those who are aware about this subject, an interesting example coming from classical stochastic calculus.

Let us considered much simpler equations:

$$dU_t = (Y_t - I)U_t\,d\Lambda_t.$$

It is well-know, in the usual case (that is when $(Y_t)_{t\geq 0}$ is made of operators acting on some initial space only) that the solution $(U_t)_{t\geq 0}$ of such an equation is a process of unitary operators if and only if $(Y_t)_{t\geq 0}$ is. This property fails in the adapted case. Let us show a counter-example coming from Wiener space endomorphisms.

Consider the martingale-preserving endomorphism, known as the "Levy transform", given by

$$\widetilde{W_t} = \int_0^t \mathrm{sgn}(W_s)\, dW_s$$

where $\mathrm{sgn}(W_s)$ denotes the sign of W_s. From Tanaka's formula we have that $\widetilde{W_t} = |W_t| - L_t$ where $(L_t)_{t\geq 0}$ is the local time at 0 of the Brownian motion $(W_t)_{t\geq 0}$. Thus the natural filtration of $(\widetilde{W_t})_{t\geq 0}$ is included in the natural filtration of $(|W_t|)_{t\geq 0}$ which is itself strictly included in $(\mathcal{F}_t)_{t\geq 0}$. Hence this endomorphism is not invertible. The associated operator T on Φ is an isometry but it is not unitary. Let $(T_t)_{t\geq 0}$ be the operator martingale associated to T. This martingale is also made of non-unitary operators. But from previous results such as Theorem II.7 we have

$$T_t = I + \int_0^t (K_t - I) T_t \, d\Lambda_s$$

where for each $t \in I\!R^+$, K_t is the operator $\mathfrak{M}_{k_t}$ for $k_t = \mathrm{sgn}(W_t)$. Thus the K_t's are unitary operators (as $K_t^* = K_t$ and $K_t^2 = I$). We have indeed an equation of the form $dU_t = (Y_t - I)U_t \, d\Lambda_t$ with the Y_t's beeing unitary and the solution $(U_t)_{t\geq 0}$ beeing made of non-unitary operators.

Conclusion

My hope is that it is clear from all the results developed along this article that the connections between classical and quantum stochastic are very deep. Each one cannot be developed if the other one is ignored.

We have seen that the key points that are linking the two calculus are the Ito calculus on Fock space, the A-M point of view on quantum stochastic integrals and the structure equations. About structure equations, it should be mentionned here that in [AE2], structure equations for multidimensional normal martingales are studied. In this case it appears that the multidimensional coefficient Ψ cannot be of any kind. Indeed, before asking for some analytical conditions that makes the structure equation admitting a solution, the coordinates of the coefficient Ψ must satisfy some algebraic relations. It is worth mentioning that the discovery of the necessity of these algebraic relations comes from quantum stochastic calculus; even though the article [AE2] is a purely probabilistic one.

In order to conclude I want to point out that the link between Fock space and classical stochastic calculus is the chaotic representation property. But M. Emery proves in [Em2] that there exists a normal martingale which has the predictable representation property without having the chaotic one. That's the first known counter-example which shows that predictable and the chaotic representation properties are not equivalent.

We know that the Fock stochastic calculus needs the chaotic representation property. But it appears through A-L's characterization of adaptedness and through A-M's equations that the only important operators are the P_t's and the D_t's; that is, operators that are involved only in the predictable representation property. So a natural question is: is it possible to develop a quantum stochastic calculus in the predictable representation property context?

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