Statistical properties of Pauli matrices going through noisy channels

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Abstract

We study the statistical properties of the triplet $(\sigma_x, \sigma_y, \sigma_z)$ of Pauli matrices going through a sequence of noisy channels, modeled by the repetition of a general, trace-preserving, completely positive map. We show a non-commutative central limit theorem for the distribution of this triplet, which shows up a 3-dimensional Brownian motion in the limit with a non-trivial covariance matrix. We also prove a large deviation principle associated to this convergence, with an explicit rate function depending on the stationary state of the noisy channel.

1 Introduction

In quantum information theory one of the most important question is to understand and to control the way a quantum bit is modified when transmitted through a quantum channel. It is well-known that realistic transmission channels are not perfect and that they distort the quantum bit they transmit. This transformation of the quantum state is represented by the action of a completely positive map. These are the so-called noisy channels.

The purpose of this article is to study the action of the repetition of a general completely positive map on basic observables. Physically, this model can be thought of as the sequence of transformations of small identical

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pieces of noisy channels on a qubit. It can also be thought of as a discrete approximation of the more realistic model of a quantum bit going through a semigroup of completely positive maps (a Lindblad semigroup).

As basic observables, we consider the triplet $(\sigma_x, \sigma_y, \sigma_z)$ of Pauli matrices. Under the repeated action of the completely positive map, they behave as a 3-dimensional quantum random walk. The aim of this article is to study the statistical properties of this quantum random walk.

Indeed, for any initial density matrix ρ_{in}, we study the statistical properties of the empirical average of the Pauli matrices in the successive states $\Phi^n(\rho_{in}), n \geq 0$ where Φ is some completely positive and trace-preserving map describing our quantum channel. Quantum Bernoulli random walks studied by Biane in [1] corresponds to the case where Φ is the identity map. Biane [1] proved an invariance principle for this quantum random walk when $\rho_{in} = \frac{1}{2}I$.

This article is organized as follows. In section two we describe the physical and mathematical setup. In section three we establish a functional central limit theorem for the empirical average of the quantum random walk associated to the Pauli matrices generalizing Biane's result [1]. This central limit theorem involves a 3-dimensional Brownian motion in the limit, whose covariance matrix is non-trivial and depends explicitly on the stationary state of the noisy channel. In section four, we apply our central limit theorem to some explicit cases, in particular to the King-Ruskai-Szarek-Werner representation of completely positive and trace-preserving maps in $M_2(\mathbb{C})$. This allows us to compute the limit Brownian motion for the most well-know quantum channels: the depolarizing channel, the phase-damping channel, the amplitude-damping channel. Finally, in the last section, a large deviation principle for the empirical average is proved.

2 Model and notations

Let $M_2(\mathbb{C})$ be the set of 2×2 matrices with complex coefficients. The set of 2×2 self-adjoint matrices forms a four dimensional real vector subspace of $M_2(\mathbb{C})$. A convenient basis $\mathcal{B}$ is given by the following matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the traditional Pauli matrices, they satisfy the commutation relations: $[\sigma_x, \sigma_y] = 2i\sigma_z$, and those obtained by cyclic permutations of σ_x, σ_y, σ_z. A state on $M_2(\mathbb{C})$ is given by a density matrix (i.e. a positive semi-definite matrix with trace one) which we will suppose to be of the form

$$\rho = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & 1-\alpha \end{pmatrix}$$

where $0 \leq \alpha \leq 1$ and $|\beta|^2 \leq \alpha(1-\alpha)$. The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

Let $M_1,\ M_2, \ldots,\ M_k, \ldots$ be infinitely many copies of $M_2(\mathbb{C})$. For each given state ρ, we consider the algebra

$$\mathcal{M}_\rho = M_1 \otimes M_2 \otimes \ldots \otimes M_k \otimes \ldots$$

where the product is taken in the sense of W^*-algebra with respect to the product state

$$\omega = \rho \otimes \Phi(\rho) \otimes \Phi^2(\rho) \otimes \ldots \otimes \Phi^k(\rho) \otimes \ldots.$$

Our main hypothesis is the following. We assume that for any state ρ, the sequance $\Phi^n(\rho)$ converges to a stationary state ρ_∞, which we write as

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \overline{\beta}_\infty & 1-\alpha_\infty \end{pmatrix}$$

where $0 \leq \alpha_\infty \leq 1$ and $|\beta_\infty|^2 \leq \alpha_\infty(1-\alpha_\infty)$.

Put

$$v_1 = 2\,\mathrm{Re}(\beta_\infty), v_2 = -2\,\mathrm{Im}(\beta_\infty), v_3 = 2\alpha_\infty - 1.$$

For every $k \geq 1$, we define

$$x_k = I \otimes \ldots \otimes I \otimes (\sigma_x - v_1\ I) \otimes I \otimes \ldots$$

$$y_k = I \otimes \ldots \otimes I \otimes (\sigma_y - v_2\ I) \otimes I \otimes \ldots$$

$$z_k = I \otimes \ldots \otimes I \otimes (\sigma_z - v_3\ I) \otimes I \otimes \ldots$$

where each $(\sigma_. - v_.\ I)$ appears on the k^{th} place.
For every $n \geq 1$, put

$$X_n = \sum_{k=1}^{n} x_k,\ Y_n = \sum_{k=1}^{n} y_k,\ Z_n = \sum_{k=1}^{n} z_k$$

with initial conditions

$$X_0 = Y_0 = Z_0 = 0.$$

The integer part of a real t is denoted by $[t]$. To each process we associate a continuous time normalized process denoted by

$$X_t^{(n)} = n^{-1/2} X_{[nt]},\ Y_t^{(n)} = n^{-1/2} Y_{[nt]},\ Z_t^{(n)} = n^{-1/2} Z_{[nt]}.$$

3 A central limit theorem

The aim of our article is to study the asymptotical properties of the quantum process $(X_t^{(n)}, Y_t^{(n)}, Z_t^{(n)})$ when n goes to infinity. This process being truly non-commutative, there is no hope to obtain an asymptotic behaviour in the classical sense.

For any polynomial $P = P(X_1, X_2, \ldots, X_m)$ of m variables, we denote by $\widehat{P}$ the *totally symmetrized polynomial* of P obtained by symmetrizing each monomial in the following way:

$$X_{i_1} X_{i_2} \ldots X_{i_k} \longrightarrow \frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(k)}}$$

where S_k is the group of permutations of $\{1, \ldots, k\}$.

Theorem 3.1. *Assume that*

$$(\mathbf{A}) \qquad\qquad \Phi^n(\rho) = \rho_\infty + o(\frac{1}{\sqrt{n}}).$$

Then, for any polynomial P of $3m$ variables, for any $(t_1, \ldots, t_m)$ such that $0 \le t_1 < t_2 < \ldots < t_m$, the following convergence holds:

$$\lim_{n \to +\infty} w\left[\widehat{P}(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \ldots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)})\right]$$
$$= \mathbb{E}\left[P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \ldots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)})\right]$$

where $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \ge 0}$ is a three-dimensional centered Brownian motion with covariance matrix Ct, where

$$C = \begin{pmatrix} 1 - v_1^2 & -v_1 v_2 & -v_1 v_3 \\ -v_1 v_2 & 1 - v_2^2 & -v_2 v_3 \\ -v_1 v_3 & -v_2 v_3 & 1 - v_3^2 \end{pmatrix}.$$

Remark :
Theorem 3.1 has to be compared with the quantum central limit theorem obtained in [5] and [9]. In our case, the state under which the convergence holds does not need to be an infinite tensor product of states. We also give here a functional version of the central limit theorem. Finally, in [5] (see Remark 3 p.131), the limit is described as a so-called quasi-free state in quantum mechanics. We prove in Theorem 3.1 that the limit is real Gaussian for the class of totally symmetrized polynomials.

Proof:
Let $m \geq 1$ and $(t_0, t_1, \dots, t_m)$ such that $t_0 = 0 < t_1 < t_2 < \dots < t_m$. The polynomial $P(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \dots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)})$ can be rewritten as a polynomial function Q of the increments: $X_{t_1}^{(n)}$, $Y_{t_1}^{(n)}$, $Z_{t_1}^{(n)}$, $X_{t_2}^{(n)} - X_{t_1}^{(n)}$, $Y_{t_2}^{(n)} - Y_{t_1}^{(n)}$, $Z_{t_2}^{(n)} - Z_{t_1}^{(n)}, \dots,$ $X_{t_m}^{(n)} - X_{t_{m-1}}^{(n)}$, $Y_{t_m}^{(n)} - Y_{t_{m-1}}^{(n)}$, $Z_{t_m}^{(n)} - Z_{t_{m-1}}^{(n)}$. A monomial of Q is a product of the form $A_{i_1} \dots A_{i_k}$ for some distinct $i_1, \dots, i_k$ in $\{1, \dots, m\}$ where A_i is a product depending only on the increments $X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}$, $Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}$, $Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}$. Since the A_i's are commuting variables, the totally symmetrized polynomial of the monomial $A_{i_1} \dots A_{i_k}$ is equal to the product $\widehat{A_{i_1}} \dots \widehat{A_{i_k}}$. Consequently, it is enough to prove the theorem for any polynomial A_i.
Let $i \geq 1$ fixed, for every $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$, we begin by determining the asymptotic distribution of the linear combination

$$(\nu_1^2 + \nu_2^2 + \nu_3^2)^{-1/2} \Big(\nu_1 (X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}) + \nu_2 (Y_{t_i}^{(n)} - Y_{t_{i-1}}^{(n)}) + \nu_3 (Z_{t_i}^{(n)} - Z_{t_{i-1}}^{(n)}) \Big) \quad (1)$$

which can be rewritten as

$$\frac{1}{\sqrt{n}} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \left(\frac{\nu_1 x_k + \nu_2 y_k + \nu_3 z_k}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}} \right).$$

Consider the matrix

$$\begin{aligned} A &= \frac{1}{\sqrt{\nu_1^2+\nu_2^2+\nu_3^2}} \Big(\nu_1(\sigma_x - v_1 I) + \nu_2(\sigma_y - v_2 I) + \nu_3(\sigma_z - v_3 I) \Big) \\ &= \frac{1}{\sqrt{\nu_1^2+\nu_2^2+\nu_3^2}} \begin{pmatrix} -\nu_1 v_1 - \nu_2 v_2 + \nu_3(1 - v_3) & \nu_1 - i\nu_2 \\ \nu_1 + i\,\nu_2 & -\nu_1 v_1 - \nu_2 v_2 - \nu_3(1 + v_3) \end{pmatrix} \end{aligned}$$

which we denote by

$$\begin{pmatrix} a_1 & a_3 \\ \overline{a}_3 & a_2 \end{pmatrix},$$

$a_1, a_2 \in \mathbb{R}, a_3 \in \mathbb{C}$.

From assumption (**A**) we can write, for every $n \geq 0$

$$\Phi^n(\rho) = \begin{pmatrix} \alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\ \overline{\beta}_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4) \end{pmatrix}$$

where each sequence $(\phi_n(i))_n$ satisfies: $\phi_n(i) = o(1/\sqrt{n})$.

Let $k \geq 1$, the expectation and the variance of A in the state $\Phi^k(\rho)$ are respectively equal to

$$\mathrm{Trace}(A\Phi^k(\rho))$$

and

$$\mathrm{Trace}(A^2\Phi^k(\rho)) - \mathrm{Trace}(A\Phi^k(\rho))^2.$$

If both following conditions are satisfied:

$$\sum_{k=[nt_{i-1}]+1}^{[nt_i]} \mathrm{Trace}(A\Phi^k(\rho)) = o(\sqrt{n}) \tag{2}$$

and

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} [\mathrm{Trace}(A^2\Phi^k(\rho)) - \mathrm{Trace}(A\Phi^k(\rho))^2] = a(t_i - t_{i-1}), \tag{3}$$

then (see Theorem 2.8.42 in [3]) the asymptotic distribution of (1) is the Normal distribution $\mathcal{N}(0, a(t_i - t_{i-1})), a > 0$.

Let us first prove (2). For every $k \geq 1$, a simple computation gives

$$\mathrm{Trace}(A\Phi^k(\rho)) = [a_1\alpha_\infty + a_3\bar{\beta}_\infty + \overline{a}_3\beta_\infty + a_2(1-\alpha_\infty) + o(1/\sqrt{n})] = o(1/\sqrt{n}),$$

hence

$$\sum_{k=[nt_{i-1}]+1}^{[nt_i]} \mathrm{Trace}(A\Phi^k(\rho)) = \sum_{k=[nt_{i-1}]+1}^{[nt_i]} o(1/\sqrt{n}) = o(\sqrt{n}).$$

This gives (2).

Let us prove (3). Note that the sequence $(\mathrm{Trace}(A\Phi^n(\rho)))_n$ converges to 0 , as n tends to infinity. As a consequence, it is enough to prove that

$$\frac{1}{n} \sum_{k=[nt_{i-1}]+1}^{[nt_i]} \mathrm{Trace}(A^2\Phi^k(\rho))$$

converges to a strictly positive constant. A straightforward computation gives

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\sum_{k=[nt_{i-1}]+1}^{[nt_i]}\mathrm{Trace}(A^2\Phi^k(\rho))\\
=\;& a_1^2\alpha_\infty + a_2^2\,(1-\alpha_\infty) + |a_3|^2 + (a_1+a_2)(a_3\bar\beta_\infty + \bar a_3\beta_\infty)\\
=\;& \frac{(t_i-t_{i-1})}{\nu_1^2+\nu_2^2+\nu_3^2}\Big[\nu_1^2(1-v_1^2)+\nu_2^2(1-v_2^2)+\nu_3^2(1-v_3^2)\\
&-2\nu_1\nu_2v_1v_2 - 2\nu_1\nu_3v_1v_3 - 2\nu_2\nu_3v_2v_3\Big].
\end{aligned}$$

This means that, for every $\nu_1,\nu_2,\nu_3\in\mathbb{R}$, for any $p\geq 1$, the expectation

$$w\Big[\Big(\nu_1(X^{(n)}_{t_i}-X^{(n)}_{t_{i-1}})+\nu_2(Y^{(n)}_{t_i}-Y^{(n)}_{t_{i-1}})+\nu_3(Z^{(n)}_{t_i}-Z^{(n)}_{t_{i-1}})\Big)^p\Big]$$

converges to

$$\mathbb{E}\Big[\Big(\nu_1(B^{(1)}_{t_i}-B^{(1)}_{t_{i-1}})+\nu_2(B^{(2)}_{t_i}-B^{(2)}_{t_{i-1}})+\nu_3(B^{(3)}_{t_i}-B^{(3)}_{t_{i-1}})\Big)^p\Big],$$

where $(B^{(1)}_t,B^{(2)}_t,B^{(3)}_t)$ is a 3-dimensional Brownian motion with the announced covariance matrix.

The polynomial

$$\Big(\nu_1(X^{(n)}_{t_i}-X^{(n)}_{t_{i-1}})+\nu_2(Y^{(n)}_{t_i}-Y^{(n)}_{t_{i-1}})+\nu_3(Z^{(n)}_{t_i}-Z^{(n)}_{t_{i-1}})\Big)^p$$

can be developed as the sum

$$\sum_{0\leq p_1+p_2\leq p}\nu_1^{p_1}\nu_2^{p_2}\nu_3^{p-p_1-p_2}\sum_{\mathcal{P}}S_1S_2\dots S_p$$

where the summation in the last sum runs over all partitions $\mathcal{P}=\{A,B,C\}$ of $\{1,\dots,p\}$ such that $|A|=p_1,|B|=p_2,|C|=p-p_1-p_2$, with the convention:

$$S_j=\begin{cases} X^{(n)}_{t_i}-X^{(n)}_{t_{i-1}} & \text{if}\quad j\in A\\ Y^{(n)}_{t_i}-Y^{(n)}_{t_{i-1}} & \text{if}\quad j\in B\\ Z^{(n)}_{t_i}-Z^{(n)}_{t_{i-1}} & \text{if}\quad j\in C\,.\end{cases}$$

The expectation under w of the above expression converges to the corresponding expression involving the expectation ($\mathbb{E}[\,\cdot\,]$) of the Brownian motion $(B^{(1)}_t,B^{(2)}_t,B^{(3)}_t)$. As this holds for any $\nu_1,\nu_2,\nu_3\in\mathbb{R}$, we deduce that

$w[\sum_{\mathcal{P}} S_1 S_2 \ldots S_p]$ converges to the corresponding expectation for the Brownian motion.

We can conclude the proof by noticing that $\widehat{A_i}$ can be written, modulo multiplication by a constant, as $\sum_{\mathcal{P}} S_1 S_2 \ldots S_p$ for some p. △

Let us discuss the class of polynomials for which Theorem 3.1 holds. In the particular case when the map Φ is the identity map and $\rho = 1/2I$ (in that case $v_i = 0$ for $i = 1, 2, 3$ and $C = I$), Biane [1] proved the convergence of the expectations in Theorem 3.1 for any polynomial in $3m$ non-commuting variables. It is a natural question to ask whether our result holds for any polynomial P instead of $\widehat{P}$, or at least for a larger class.

Let us give an example of a polynomial for which the convergence in our setting does not hold. Take $P(X, Y) = XY$. From Theorem 3.1, the expectation under the state ω of

$$X_t^{(n)} Y_t^{(n)} + Y_t^{(n)} X_t^{(n)}$$

converges as $n \to +\infty$ to $2\ \mathbb{E}[B_t^{(1)} B_t^{(2)}]$.
Since we have the following commutation relations

$$[(\sigma_x - v_1\ I), (\sigma_y - v_2\ I)] = 2i\sigma_z, \quad [(\sigma_y - v_2\ I), (\sigma_z - v_3\ I)] = 2i\sigma_x$$

and

$$[(\sigma_z - v_3\ I), (\sigma_x - v_1\ I)] = 2i\sigma_y,$$

we deduce that

$$[X_t^{(n)}, Y_t^{(n)}] = 2in^{-1/2}\ Z_t^{(n)} + 2itv_3\ I, \quad [Y_t^{(n)}, Z_t^{(n)}] = 2in^{-1/2}\ X_t^{(n)} + 2itv_1\ I$$

and

$$[Z_t^{(n)}, X_t^{(n)}] = 2in^{-1/2}\ Y_t^{(n)} + 2itv_2\ I. \tag{4}$$

Then the expectation under the state ω of

$$X_t^{(n)} Y_t^{(n)} = \frac{1}{2}\Big[\widehat{P}(X, Y) + [X_t^{(n)}, Y_t^{(n)}]\Big]$$

converges to $\mathbb{E}[B_t^{(1)} B_t^{(2)}] + itv_3 \neq \mathbb{E}[B_t^{(1)} B_t^{(2)}]$, if v_3 is non zero.

Furthermore, by considering the polynomial $P(X, Y) = XY^3 + Y^3X$, it is possible to show that the convergence in Theorem 3.1 can not be enlarged to the class of symmetric polynomials. Straightforward computations gives that $P(X,Y)$ can be rewritten as

$$\widehat{XY^3}+\widehat{YX^3}+\frac{3}{4}[X,Y](Y^2-X^2)+\frac{1}{2}(Y[X,Y]Y-X[X,Y]X)+\frac{1}{4}(Y^2-X^2)[X,Y]$$

so the expectation $w[P(X_t^{(n)},Y_t^{(n)})]$ converges as n tends to $+\infty$ to

$$\mathbb{E}[P(B_t^{(1)},B_t^{(2)})]+3iv_3t(v_1^2-v_2^2)$$

which is not equal to $\mathbb{E}[P(B_t^{(1)},B_t^{(2)})]$ if $v_3\neq 0$ and $|v_1|\neq|v_2|$.

In the following corollary we give a condition under which the convergence in Theorem 3.1 holds for any polynomial in $3m$ non-commuting variables.

Corollaire 3.1. *In the case when ρ_∞ is equal to $\frac{1}{2}I$, the convergence holds for any polynomial P in $3m$ non-commuting variables, i.e. for every $t_1 < t_2 < \ldots < t_m$, the following convergence holds:*

$$\begin{aligned}\lim_{n\to+\infty} w&\left[P(X_{t_1}^{(n)},Y_{t_1}^{(n)},Z_{t_1}^{(n)},\ldots,X_{t_m}^{(n)},Y_{t_m}^{(n)},Z_{t_m}^{(n)})\right]\\&=\mathbb{E}\left[P(B_{t_1}^{(1)},B_{t_1}^{(2)},B_{t_1}^{(3)},\ldots,B_{t_m}^{(1)},B_{t_m}^{(2)},B_{t_m}^{(3)})\right]\end{aligned}$$

where $(B_t^{(1)},B_t^{(2)},B_t^{(3)})_{t\geq 0}$ is a three-dimensional centered Brownian motion with covariance matrix tI_3.

Proof:
We consider the polynomials of the form $S=\frac{1}{N}\sum_{\mathcal{P}} S_1S_2\ldots S_{p_1+p_2+p_3}$ where the summation is done over all partitions $\mathcal{P}=\{A,B,C\}$ of the set $\{1,\ldots,p_1+p_2+p_3\}$ such that $|A|=p_1,|B|=p_2,|C|=p_3$, with the convention:

$$S_j=\begin{cases}X_{t_i}^{(n)}-X_{t_{i-1}}^{(n)} & \text{if} \quad j\in A\\ Y_{t_i}^{(n)}-Y_{t_{i-1}}^{(n)} & \text{if} \quad j\in B\\ Z_{t_i}^{(n)}-Z_{t_{i-1}}^{(n)} & \text{if} \quad j\in C\end{cases}$$

and N is the number of terms in the sum.

From Theorem 3.1 the expectation under the state w of S converges to

$$\mathbb{E}\Big[\prod_{j=1}^{3}(B_{t_i}^{(j)}-B_{t_{i-1}}^{(j)})^{p_j}\Big].$$

Using the commutation relations (4) with the v_i's being all equal to zero, monomials of S differ of each other by $n^{-1/2}$ times a polynomial of total degree less or equal to $(p_1+p_2+p_3)-1$. It is easy to conclude by induction. $\triangle$

4 Examples

4.1 King-Ruskai-Szarek-Werner's representation

The set of 2×2 self-adjoint matrices forms a four dimensional real vector subspace of $M_2(\mathbb{C})$. A convenient basis of this space is given by $\mathcal{B} = \{I, \sigma_x, \sigma_y, \sigma_z\}$. Each state ρ on $M_2(\mathbb{C})$ can then be written as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-i\,y \\ x+i\,y & 1-z \end{pmatrix}$$

where x, y, z are reals such that $x^2 + y^2 + z^2 \leq 1$. Equivalently, in the basis $\mathcal{B}$,

$$\rho = \frac{1}{2}(I + x\ \sigma_x + y\ \sigma_y + z\ \sigma_z)$$

with x, y, z defined above. Thus, the set of density matrices can be identified with the unit ball in $\mathbb{R}^3$. The pure states, that is, the ones for which $x^2 + y^2 + z^2 = 1$, constitutes the Bloch sphere.

The noise coming from interactions between the qubit states and the environment is represented by the action of a completely positive and trace-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$. Kraus and Choi [2, 7, 8] gave an abstract representation of these particular maps in terms of Kraus operators: There exists at most four matrices L_i such that for any density matrix ρ,

$$\Phi(\rho) = \sum_{1 \leq i \leq 4} L_i^* \rho L_i$$

with $\sum_i L_i L_i^* = I$. The matrices L_i are usually called the *Kraus operators* of Φ. This representation is unique up to a unitary transformation. Recently, King, Ruskai et al [10, 6] obtained a precise characterization of completely positive and trace-preserving maps from $M_2(\mathbb{C})$ as follows. The map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ being linear and preserving the trace, it can be represented as a unique 4×4-matrix in the basis $\mathcal{B}$ given by

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathbf{T} \end{pmatrix}$$

with $\mathbf{0} = (0, 0, 0)$, $\mathbf{t} \in \mathbb{R}^3$ and $\mathbf{T}$ a real 3×3-matrix. King, Ruskai et al [10, 6] proved that via changes of basis, this matrix can be reduced to

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix} \tag{5}$$

Necessary and sufficient conditions under which the map Φ with reduced matrix T for which $|t_3| + |\lambda_3| \leq 1$ is completely positive are (see [6])

$$(\lambda_1 + \lambda_2)^2 \leq (1+\lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2)\left(\frac{1+\lambda_3 \pm t_3}{1-\lambda_3 \pm t_3}\right) \leq (1+\lambda_3)^2 - t_3^2 \tag{6}$$

$$(\lambda_1 - \lambda_2)^2 \leq (1-\lambda_3)^2 - t_3^2 - (t_1^2 + t_2^2)\left(\frac{1-\lambda_3 \pm t_3}{1+\lambda_3 \pm t_3}\right) \leq (1-\lambda_3)^2 - t_3^2 \tag{7}$$

$$\begin{aligned} &\left[1 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - (t_1^2 + t_2^2 + t_3^2)\right]^2 \\ &\quad \geq 4\left[\lambda_1^2(t_1^2 + \lambda_2^2) + \lambda_2^2(t_2^2 + \lambda_3^2) + \lambda_3^2(t_3^2 + \lambda_1^2) - 2\lambda_1\lambda_2\lambda_3\right]. \end{aligned} \tag{8}$$

We now apply Theorem 3.1 in this setting. Let Φ be a completely positive and trace preserving map with matrix T given in (5), with coefficients $t_i, \lambda_i, i = 1,2,3$ satisfying conditions (6), (7) and (8). Moreover, we assume that $|\lambda_i| < 1, i = 1,2,3$. For every $n \geq 0$,

$$\Phi^n(\rho) = \frac{1}{2}\begin{pmatrix} 1 + \phi_n(3) & \phi_n(1) - i\ \phi_n(2) \\ \phi_n(1) + i\ \phi_n(2) & 1 - \phi_n(3) \end{pmatrix}$$

where the sequences $(\phi_n(i))_{n\geq 0}$, $i = 1,2,3$ satisfy the induction relations:

$$\phi_n(i) = \lambda_i \phi_{n-1}(i) + t_i.$$

with initial conditions $\phi_0(1) = x, \phi_0(2) = y$ and $\phi_0(3) = z$. Explicit formulae can easily be obtained. We get, for every $n \geq 0$,

$$\phi_n(1) = \left(x - \frac{t_1}{1-\lambda_1}\right)\lambda_1^n + \frac{t_1}{1-\lambda_1}$$

$$\phi_n(2) = \left(y - \frac{t_2}{1-\lambda_2}\right)\lambda_2^n + \frac{t_2}{1-\lambda_2}$$

$$\phi_n(3) = \left(z - \frac{t_3}{1-\lambda_3}\right)\lambda_3^n + \frac{t_3}{1-\lambda_3}.$$

Hence, for any state ρ, for any $n \geq 1$,

$$\Phi_n(\rho) = \rho_\infty + o(|\lambda|_{max}^n)$$

where $|\lambda|_{max} = \max_{i=1,2,3} |\lambda_i|$ and

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \overline{\beta}_\infty & 1-\alpha_\infty \end{pmatrix}$$

with $\alpha_\infty = \frac{1}{2}\left(1 + \frac{t_3}{1-\lambda_3}\right)$ and $\beta_\infty = \frac{1}{2}\left(\frac{t_1}{1-\lambda_1} - i\,\frac{t_2}{1-\lambda_2}\right)$. Theorem 3.1 applies with $v_i = \frac{t_i}{1-\lambda_i}$, $i = 1, 2, 3$.

We now give some examples of well-known quantum channels. For each of them we give their Kraus operators, their corresponding matrix T in the King-Ruskai-Szarek-Werner's representation, as well as the vector $v = (v_1, v_2, v_3)$ and the covariance matrix C obtained in Theorem 3.1. It is worth noticing that if Φ is a *unital* map, i.e. such that $\Phi(I) = I$, then the covariance matrix C is equal to the identity matrix I_3.

1. *The depolarizing channel:*

   Kraus operators: for some $0 \leq p \leq 1$,

   $$L_1 = \sqrt{1-p}I, L_2 = \sqrt{\frac{p}{3}}\sigma_x, L_3 = \sqrt{\frac{p}{3}}\sigma_y, L_4 = \sqrt{\frac{p}{3}}\sigma_z.$$

   King-Ruskai-Szarek-Werner's representation:

   $$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{4p}{3} & 0 & 0 \\ 0 & 0 & 1-\frac{4p}{3} & 0 \\ 0 & 0 & 0 & 1-\frac{4p}{3} \end{pmatrix}$$

   The vector v is the null vector and the covariance matrix C in this case is given by the identity matrix I_3.

2. *Phase-damping channel:*

Kraus operators: for some $0 \leq p \leq 1$,

$$L_1 = \sqrt{1-p}\, I,\ L_2 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, L_3 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The vector v is the null vector and the covariance matrix C in this case is given by I_3.

3. *Amplitude-damping channel:*

   Kraus operators: for some $0 \leq p \leq 1$,

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, L_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

   King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ t & 0 & 0 & 1-p \end{pmatrix}$$

   The vector v is equal to $(0, 0, 1)$. The covariance matrix in this case is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. *Trigonometric parameterization:*

   Consider the particular Kraus operators

$$L_1 = \left[\cos(\frac{v}{2})\cos(\frac{u}{2})\right] I + \left[\sin(\frac{v}{2})\sin(\frac{u}{2})\right]\sigma_z$$

and

$$L_2 = \left[\sin(\frac{v}{2})\cos(\frac{u}{2})\right]\sigma_x - i\left[\cos(\frac{v}{2})\sin(\frac{u}{2})\right]\sigma_y.$$

King-Ruskai-Szarek-Werner's representation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix}$$

The vector v is equal to $(0, 0, \dfrac{\sin u \sin v}{1 - \cos u \cos v})$. The covariance matrix in this case is given by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - v_3^2 \end{pmatrix}$$

with $v_3 = \dfrac{\sin u \sin v}{1 - \cos u \cos v}$.

4.2 CP map associated to a Markov chain

With every Markov chain with two states and transition matrix given by

$$P = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}, \; p, q \in (0,1)$$

is associated a completely positive and trace preserving map, denoted by Φ, with the Kraus operators:

$$L_1 = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} = \frac{\sqrt{p}}{2}(I + \sigma_z) + \frac{\sqrt{1-p}}{2}(\sigma_x + i\sigma_y)$$

and

$$L_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{q} & \sqrt{1-q} \end{pmatrix} = \frac{\sqrt{1-q}}{2}(I - \sigma_z) + \frac{\sqrt{q}}{2}(\sigma_x - i\sigma_y).$$

Let ρ be the density matrix

$$\frac{1}{2}\begin{pmatrix} 1+z & x - i\,y \\ x + i\,y & 1-z \end{pmatrix}$$

where x, y, z are reals such that $x^2 + y^2 + z^2 \leq 1$. The map Φ transforms the density matrix ρ into a new one given by

$$\Phi(\rho) = L_1^* \rho L_1 + L_2^* \rho L_2.$$

By induction, for every $n \geq 0$,

$$\Phi^n(\rho) = \begin{pmatrix} p_n & r_n \\ r_n & 1 - p_n \end{pmatrix}$$

where the sequences $(p_n)_{n \geq 0}$, and $(r_n)_{n \geq 0}$ satisfy the recurrence relations: for every $n \geq 1$,

$$p_n = p_{n-1}(p - q) + q$$

and

$$r_n = \sqrt{q(1-q)} + p_{n-1}(\sqrt{p(1-p)} - \sqrt{q(1-q)})$$

with the initial condition $p_0 = (1 + z)/2$. Assumption (A) is then clearly satisfied with

$$\rho_\infty = \frac{1}{1 + q - p} \begin{pmatrix} q & \beta \\ \beta & 1 - p \end{pmatrix}$$

where $\beta = \left[q\sqrt{p(1-p)} + (1-p)\sqrt{q(1-q)} \right]$. Then, applying Theorem 3.1, if P is a polynomial of $3m$ non-commuting variables, for every $0 < t_1 < t_2 < \ldots < t_m$, the following convergence holds

$$\lim_{n \to +\infty} w \left[\widehat{P}(X_{t_1}^{(n)}, Y_{t_1}^{(n)}, Z_{t_1}^{(n)}, \ldots, X_{t_m}^{(n)}, Y_{t_m}^{(n)}, Z_{t_m}^{(n)}) \right]$$
$$= \mathbb{E} \left[P(B_{t_1}^{(1)}, B_{t_1}^{(2)}, B_{t_1}^{(3)}, \ldots, B_{t_m}^{(1)}, B_{t_m}^{(2)}, B_{t_m}^{(3)}) \right]$$

where $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})_{t \geq 0}$ is a three-dimensional centered Brownian motion with Covariance matrix Ct where

$$C = \begin{pmatrix} 1 - v_1^2 & 0 & -v_1 v_2 \\ 0 & 1 & 0 \\ -v_1 v_2 & 0 & 1 - v_2^2 \end{pmatrix}$$

with

$$v_1 = \frac{2}{1 + q - p} [q\sqrt{p(1-p)} + (1-p)\sqrt{q(1-q)}]$$

and

$$v_2 = \frac{p + q - 1}{1 + q - p}.$$

5 Large deviation principle

Let Γ be a Polish space endowed with the Borel σ-field $\mathcal{B}(\Gamma)$. A good *rate function* is a lower semi-continuous function $\Lambda^* : \Gamma \rightarrow [0,\infty]$ with compact level sets $\{x; \Lambda^*(x) \leq \alpha\}, \alpha \in [0,\infty[$. Let $v = (v_n)_n \uparrow \infty$ be an increasing sequence of positive reals. A sequence of random variables $(Y_n)_n$ with values in Γ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy *a Large Deviation Principle* (LDP) with speed $v = (v_n)_n$ and the good rate function Λ^* if for every Borel set $B \in \mathcal{B}(\Gamma)$,

$$\begin{aligned} -\inf_{x \in B^o} \Lambda^*(x) &\leq \liminf_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \\ &\leq \limsup_n \frac{1}{v_n} \log \mathbb{P}(Y_n \in B) \leq -\inf_{x \in \bar{B}} \Lambda^*(x). \end{aligned}$$

For every $k \geq 1$, we define

$$\bar{x}_k = I \otimes \ldots \otimes I \otimes \sigma_x \otimes I \otimes \ldots$$

$$\bar{y}_k = I \otimes \ldots \otimes I \otimes \sigma_y \otimes I \otimes \ldots$$

$$\bar{z}_k = I \otimes \ldots \otimes I \otimes \sigma_z \otimes I \otimes \ldots$$

where each $\sigma_.$ appears on the k^{th} place.
For every $n \geq 1$, we consider the processes

$$\bar{X}_n = \sum_{k=1}^n \bar{x}_k, \ \bar{Y}_n = \sum_{k=1}^n \bar{y}_k, \ \bar{Z}_n = \sum_{k=1}^n \bar{z}_k$$

with initial conditions

$$\bar{X}_0 = \bar{Y}_0 = \bar{Z}_0 = 0.$$

To each vector $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$, we associate the euclidean norm $||\nu|| = \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}$ and $\langle .,. \rangle$ the corresponding inner product.

Theorem 5.1. *Let Φ be a completely positive and trace-preserving map for which there exists a state*

$$\rho_\infty = \begin{pmatrix} \alpha_\infty & \beta_\infty \\ \overline{\beta}_\infty & 1 - \alpha_\infty \end{pmatrix}$$

such that for any given state ρ,

$$\Phi^n(\rho) = \rho_\infty + o(1).$$

For every $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^{3,*}$*, the sequence*

$$\Big(\frac{\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n}{n}\Big)_{n \geq 1}$$

satisfies a LDP with speed n *and the good rate function*

$$I(x) = \begin{cases} \frac{1}{2}\Big[\Big(1 + \frac{x}{||\nu||}\Big) \log\Big(\frac{||\nu||+x}{||\nu||+\langle\, \nu, v\,\rangle}\Big) \\ \quad + \Big(1 - \frac{x}{||\nu||}\Big) \log\Big(\frac{||\nu||-x}{||\nu||-\langle\, \nu, v\,\rangle}\Big)\Big] & \text{if } \ |x| < ||\nu||. \\ \\ +\infty & \text{otherwise.} \end{cases}$$

where $v_1 = 2\,\mathrm{Re}(\beta_\infty), v_2 = -2\,\mathrm{Im}(\beta_\infty), v_3 = 2\alpha_\infty - 1$.

Proof:
The matrix

$$B := \nu_1 \sigma_x + \nu_2 \sigma_y + \nu_3 \sigma_z \ \ = \ \begin{pmatrix} \nu_3 & \nu_1 - i\nu_2 \\ \nu_1 + i\ \nu_2 & -\nu_3 \end{pmatrix}$$

has two distinct eigenvalues $\pm ||\nu||$.
For every $n \geq 0$, we can write

$$\Phi^n(\rho) = \begin{pmatrix} \alpha_\infty + \phi_n(1) & \beta_\infty + \phi_n(2) \\ \overline{\beta}_\infty + \phi_n(3) & 1 - \alpha_\infty + \phi_n(4) \end{pmatrix}$$

where the four sequences $(\phi_n(i))_{n \geq 0}$ satisfy $\phi_n(i) = o(1)$.

For any $k \geq 1$, the expectation of B in the state $\Phi^k(\rho)$ is equal to

$$\mathrm{Trace}(B\ \Phi^k(\rho)) = \langle\, \nu, v\,\rangle + \varepsilon_k,$$

with $\varepsilon_n = o(1)$. As a consequence, the distribution of B is

$$p_k(||\nu||) = \frac{1}{2}\left[1 + \frac{1}{|\nu|}(\langle\, \nu, v\,\rangle + \varepsilon_k)\right] = 1 - p_k(-||\nu||).$$

Using the fact that $\nu_1 \bar{X}_n + \nu_2 \bar{Y}_n + \nu_3 \bar{Z}_n$ is the sum of n commuting matrices, we get that

$$\begin{aligned} &\frac{1}{n} \log w\ (\exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n)) \\ &\quad = \frac{1}{n} \sum_{k=1}^{n} \log\Big(e^{||\nu|| t} p_k(||\nu||) + e^{-||\nu|| t}(1 - p_k(||\nu||))\Big) \end{aligned}$$

Since $\varepsilon_n = o(1)$, we obtain that

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{1}{n} \log w \; (\exp t(\nu_1 X_n + \nu_2 Y_n + \nu_3 Z_n)) \\
&\quad = \log \Big(\cosh(||\nu|| t) + \frac{\langle \nu, v \rangle}{||\nu||} \sinh(||\nu|| t) \Big) \\
&\quad = \log \Big(\cosh(||\nu|| t) \Big) + \log \Big(1 + \frac{\langle \nu, v \rangle}{||\nu||} \tanh(||\nu|| t) \Big).
\end{aligned}$$

We denote by $\Lambda(t)$ this function of t.

For every $t \in \mathbb{R}$, the function Λ is finite and differentiable on $\mathbb{R}$, then, by Gärtner-Ellis Theorem (see [4]), the LDP holds with the good rate function

$$I(x) = \sup_{t\in\mathbb{R}} \{tx - \Lambda(t)\}.$$

A simple computation leads to the rate function given in the theorem.
$\triangle$

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