OPERATOR ALGEBRAS METHODS
IN
QUANTUM OPEN SYSTEMS

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1. C^*-ALGEBRAS

1.1 First definitions

A *C^*-algebra* is an algebra $\mathcal{A}$ equipped with an involution $A \mapsto A^*$ and a norm $||\cdot||$ satisfying:

i) $A^{**} = A$

ii) $(\lambda A + \mu B)^* = \overline{\lambda} A^* + \overline{\mu} B^*$

iii) $(AB)^* = B^* A^*$

i') $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = 0$

ii') $||\lambda A|| = |\lambda|\, ||A||$

iii') $||A + B|| \leq ||A|| + ||B||$

iv') $||AB|| \leq ||A||\, ||B||$

i") $\mathcal{A}$ is complete for $||\cdot||$

ii") $||AA^*|| = ||A||^2$.

An algebra with an involution as above satisfying i), ii) and iii) is called a *$*$-algebra*.

An algebra satisfying all the conditions above but where ii") is replaced by

ii") $||A^*|| = ||A||$

is called a *Banach algebra*.

The basic examples of C^*-algebras are:

1) $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. The involution is the usual adjoint mapping and the norm is the usual operator norm:

$$||A|| = \sup_{||f||=1} ||Af||$$

.

2) $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the algebra of compact operators on $\mathcal{H}$. It is a sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$.

3) $\mathcal{A} = C_0(X)$, the space of continuous functions vanishing at infinity on a locally compact space X. Recall that a function f is *vanishing at infinity* if for every $\varepsilon > 0$ there exists a compact $K \subset X$ such that $|f| < \varepsilon$ outside of K. The involution on $\mathcal{A}$ is the complex conjugation $\overline{f}$ and the norm is

$$||f|| = \sup_{x \in X} |f(x)|\,.$$

We will see later that examples i) and ii) contain all the cases of C^*-algebras.

Proposition 1.1 – *On a C^*-algebra $\mathcal{A}$ we have $||A^*|| = ||A||$ for all $A \in \mathcal{A}$.*

Proof

We have $||A||^2 = ||A^*A|| \leq ||A^*||\, ||A||$ and thus $||A|| \leq ||A^*||$. Inverting the role of A and A^* gives the result. ■

An element I of a C^*-algebra $\mathcal{A}$ is a *unity* if

$$IA = AI = A$$

for all $A \in \mathcal{A}$.

If a unity exists it is unique and norm 1 (except if $\mathcal{A} = \{0\}$). But it may not always exists. Indeed, in the example $\mathcal{K}(\mathcal{H})$ there is a unity if and only if $\mathcal{H}$ is finite dimensional. In the example $C_0(X)$ there exists a unity if and only if X is compact.

But if a C^*-algebra does not contain a unity one can easily add one as follows. Consider the vector space $\mathcal{A}' = \mathcal{A} \oplus \mathbb{C}$ and provide it with the product

$$(A, \lambda)(B, \mu) = (AB + \lambda B + \mu A, \lambda\mu),$$

the involution

$$(A, \lambda)^* = (A^*, \overline{\lambda})$$

and the norm

$$||(A, \lambda)|| = \sup_{||B||=1} ||AB + \lambda B|| \,.$$

Equipped this way $\mathcal{A}'$ is a C^*-algebra. It admits a unity $(0, 1)$. The algebra $\mathcal{A}$ identifies to the subset of elements of the form $(A, 0)$. The only delicate point is to check that $||(A, \lambda)|| = 0$ if and only if $A = 0$ and $\lambda = 0$. One can assume that $\lambda \neq 0$ for if not we are in $\mathcal{A}$. Thus one can assume that $\lambda = 1$. We have

$$||B - AB|| \leq ||B|| \, ||(-A, 1)|| \,.$$

Thus if $||(-A, 1)|| = 0$ then $B = AB$ for all $B \in \mathcal{A}$. Applying the involution gives $B = BA^*$ for all $B \in \mathcal{A}$. In particular $A^* = AA^* = A$ and thus $B = AB = BA$. This means A is a unity. Contradiction.

Note that the above definition of the norm in $\mathcal{A}'$ comes from the fact that in any C^*-algebra we have

$$||A|| = \sup_{||B||=1} ||AB|| \,.$$

Indeed, there is obviously an inequality $\geq$ between the two terms above. The equality is obtained by considering $B = A^*/\, ||A||$.

1.2 Spectral analysis

Let $\mathcal{A}$ be a C^*-algebra with unity I. An element A of $\mathcal{A}$ is *invertible* if there exists an element A^{-1} of $\mathcal{A}$ such that

$$A^{-1}A = AA^{-1} = I.$$

One calls *resolvant set* of A the set

$$\rho(A) = \{\lambda \in \mathbb{C}; \lambda I - A \text{ is invertible}\}.$$

We put

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

and call it the *spectrum* of A.

If $|\lambda| > ||A||$ then the series

$$\frac{1}{\lambda}\sum_n \left(\frac{A}{\lambda}\right)^n$$

is normally convergent and equals $(\lambda I - A)^{-1}$. This implies that $\sigma(A)$ is included in $B(0, ||A||)$.

Furthermore, if λ_0 belongs to $\rho(A)$ and if $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| < ||\lambda_0 I - A||$, then the series

$$(\lambda_0 I - A)^{-1}\sum_n \left(\frac{\lambda_0 - \lambda}{\lambda_0 I - A}\right)^n$$

normally converges to $(\lambda I - A)^{-1}$. In particular we have proved that:

1) the set $\rho(A)$ is open

2) the mapping $\lambda \mapsto (\lambda I - A)^{-1}$ is analytic on $\rho(A)$

3) the set $\sigma(A)$ is compact.

We define

$$r(A) = \sup\{|\lambda|\, ;\, \lambda \in \sigma(A)\}$$

the *spectral radius* of A.

Theorem 1.2 – *We have for all $A \in \mathcal{A}$*

$$r(A) = \lim_n ||A^n||^{1/n} = \inf_n ||A^n||^{1/n} \leq ||A||\,.$$

In particular the above limit always exists and $\sigma(A)$ is never empty.

Proof

Let n be fixed and let $|\lambda| > ||A^n||^{1/n}$. Every integer m can be written $m = pn + q$ with p, q integers and $q < n$. Thus we have

$$\begin{aligned}\sum_m \left|\left|\left(\frac{A}{\lambda}\right)^m\right|\right| = \sum_m \left|\left|\left(\frac{A}{\lambda}\right)^{pn+q}\right|\right| &\leq \sum_m \left(\frac{||A^n||}{|\lambda|^n}\right)^p \left(\frac{||A||}{|\lambda|}\right)^q \\ &\leq \left(1 + \frac{||A||}{|\lambda|} + \ldots + \left(\frac{||A||}{|\lambda|}\right)^{n-1}\right)\sum_p \left(\frac{||A^n||}{|\lambda|^n}\right)^p < \infty.\end{aligned}$$

Thus the series

$$\frac{1}{\lambda}\sum_m \left(\frac{A}{\lambda}\right)^m$$

converges and is equal to $(\lambda I - A)^{-1}$. This proves that $r(A) \leq ||A^n||^{1/n}$ and thus $r(A) \leq \liminf_n ||A^n||^{1/n}$.

Let us prove that $r(A) \geq \limsup_n ||A^n||^{1/n}$. If we have

$$r(A) < \limsup_n ||A^n||^{1/n}$$

then consider the open set

$$\mathcal{O} = \{\lambda \in \mathbb{C}; r(A) < |\lambda| < \limsup_n ||A^n||^{1/n}\}.$$

On $\mathcal{O}$ all the operators $\lambda I - A$ are invertible, thus so are the operators $I - \frac{1}{\lambda}A$. The mapping $\lambda \mapsto (I - \frac{1}{\lambda}A)^{-1}$ is analytic on $\mathcal{O}$ and its Taylor series $\sum_n (\frac{A}{\lambda})^n$ converges. But the convergence radius of the series $\sum_n z^n A^n$ is exactly $(\limsup_n ||A^n||^{1/n})^{-1}$. This would mean

$$\frac{1}{|\lambda|} < (\limsup_n ||A^n||^{1/n})^{-1}$$

which contradicts the fact that $\lambda \in \mathcal{O}$. We have proved the first part of the theorem.

If $r(A) > 0$ then it is clear that $\sigma(A)$ is not empty. It remains to consider the case $r(A) = 0$. But note that if 0 belongs to $\rho(A)$ this means that A is invertible and $1 = ||A^n A^{-n}|| \leq ||A^n|| \, ||A^{-n}||$. In particular, passing to the limit, we get $r(A) > 0$. Thus if $r(A) = 0$ we must have $0 \in \sigma(A)$. In any case $\sigma(A)$ is non empty. ■

Corollary 1.3 – *A C^*-algebra $\mathcal{A}$ with unity and all of which elements, except 0, are invertible is isomorphic to $\mathbb{C}$.*

Proof

If $A \in \mathcal{A}$ its spectrum $\sigma(A)$ is non empty. Thus there exists a $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not invertible. This means $\lambda I - A = 0$ and $A = \lambda I$. ■

An element A of a C^*-algebra $\mathcal{A}$ with unity is
normal if $A^*A = AA^*$,
self-adjoint if $A = A^*$,
isometric if $A^*A = I$,
unitary if $A^*A = AA^* = I$.

Theorem 1.4 –

a) If A is normal then $r(A) = ||A||$.

b) If A is self-adjoint then $\sigma(A) \subset [-\,||A||\,,\,||A||]$.

c) If A is isometric then $r(A) = 1$.

d) If A is unitary then $\sigma(A) \subset \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.

e) For all $A \in \mathcal{A}$ we have $\sigma(A^*) = \overline{\sigma(A)}$ and $\sigma(A^{-1}) = \sigma(A)^{-1}$.

f) For every polynomial function P we have

$$\sigma(P(A)) = P(\sigma(A)).$$

g) For any two $A, B \in \mathcal{A}$ then

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

Proof

a) If A is normal than

$$\left|\left|A^{2^n}\right|\right|^2 = \left|\left|A^{2^n}A^{*2^n}\right|\right| = \left|\left|(AA^*)^{2^n}\right|\right| = \left|\left|(AA^*)^{2^{n-1}}(AA^*)^{2^{n-1}}\right|\right|$$
$$= \left|\left|(AA^*)^{2^{n-1}}\right|\right|^2 = \ldots = ||AA^*||^{2^n} = ||A||^{2^{n+1}}.$$

One now concludes easily with Theorem 1.2.

b) We only have to prove that the spectrum of any self-adjoint element of $\mathcal{A}$ is a subset of $\mathbb{R}$. Let $\lambda = x + iy$ be an element of $\sigma(A)$, with x, y real. We have $x + i(y + t) \in \sigma(A + itI)$. But

$$||A + itI||^2 = ||(A + itI)(A - itI)|| = \left|\left|A^2 + t^2I\right|\right| \leq ||A||^2 + t^2.$$

This implies

$$|x + i(y + t)|^2 = x^2 + (y + t)^2 \leq ||A||^2 + t^2$$

or else

$$2yt \leq ||A||^2 - x^2 - y^2$$

for all t. This means $y = 0$.

c) If A is isometric then

$$||A^n||^2 = ||A^{*n}A^n|| = \left|\left|A^{*n-1}A^{n-1}\right|\right| = \ldots = ||A^*A|| = ||I|| = 1.$$

d) Assume e) is proved. Then if A is unitary we have

$$\sigma(A) = \overline{\sigma(A^*)} = \overline{\sigma(A^{-1})} = \overline{\sigma(A)}^{-1}.$$

This and c) imply that $\sigma(A)$ is included in the unit circle.

e) The property $\sigma(A^*) = \overline{\sigma(A)}$ is obvious. For the other identity we write $\lambda I - A = \lambda A(A^{-1} - \lambda^{-1}I)$ and $\lambda^{-1}I - A^{-1} = \lambda^{-1}A^{-1}(A - \lambda I)$.

f) Note that if $B = A_1 \ldots A_n$ in $\mathcal{A}$, where all the A_i are two by two commuting, we have that B is invertible if and only if each A_i is invertible. Now choose α and $\alpha_1, \ldots \alpha_n$ in $\mathbb{C}$ such that

$$P(x) - \lambda = \sigma \prod_i (x - \alpha_i).$$

In particular we have

$$P(A) - \lambda I = \alpha \prod_i (A - \alpha_i I).$$

As a consequence $\lambda \in \sigma(P(A))$ if and only if $\alpha_i \in \sigma(A)$ for a i. But as $P(\alpha_i) = \lambda$ this exactly means that λ belongs to $\sigma(P(A)$ if and only if λ belongs to $P(\sigma(A))$.

g) If λ belong to $\rho(BA)$ then

$$(\lambda I - AB)(I + A(\lambda I - BA)^{-1}B) = \lambda I.$$

This proves that $\lambda I - AB$ is invertible, with possible exception of $\lambda = 0$. This proves one inclusion. The converse inclusion is obtained exchanging the role of A and B. ■

Theorem 1.5 – *The norm which makes a $*$-algebra being a C^*-algebra, when it exists, is unique.*

Proof

By the above results we have

$$||A||^2 = ||AA^*|| = r(AA^*)$$

for AA^* is always normal. But $r(AA^*)$ depends only on the algebraic structure of $\mathcal{A}$. ■

Proposition 1.6 – *The set of invertible elements of a C^*-algebra $\mathcal{A}$ is open and the mapping $A \mapsto A^{-1}$ is continuous on this set.*

Proof

If A is invertible and if B is such that $||B - A|| < \left|\left|A^{-1}\right|\right|^{-1}$ then $B = A(I - A^{-1}(A - B))$ is invertible for

$$r(A^{-1}(A - B)) \leq \left|\left|A^{-1}(A - B)\right|\right| < 1$$

and thus $I - A^{-1}(A - B)$ is invertible. The open character is proved. Let us now show the continuity. If $||B - A|| < 1/2 \left|\left|A^{-1}\right|\right|^{-1}$ then

$$\begin{aligned} \left|\left|B^{-1} - A^{-1}\right|\right| &= \left|\left|\sum_{n=0}^{\infty} \left(A^{-1}(A - B)\right)^n A^{-1} - A^{-1}\right|\right| \\ &\leq \sum_{n=1}^{\infty} \left|\left|A^{-1}(A - B)\right|\right|^n \left|\left|A^{-1}\right|\right| \\ &\leq \frac{\left|\left|A^{-1}\right|\right| \, ||A - B||}{1 - ||A^{-1}(A - B)||} \\ &\leq 2\left|\left|A^{-1}\right|\right|^2 ||A - B|| \, . \end{aligned}$$

This proves the continuity. ■

Theorem 1.7 [Functional calculus] – *Let $\mathcal{A}$ be a C^*-algebra with unity. Let A be a self-adjoint element in $\mathcal{A}$. Let $C(\sigma(A))$ be the C^*-algebra of continuous functions on $\sigma(A)$. Then there is a unique morphism of C^*-algebra*

$$\begin{array}{ccc} C(\sigma(A)) & \longrightarrow & \mathcal{A} \\ f & \longmapsto & f(A) \end{array}$$

which sends the function $\mathbb{1}$ on I and the function $id_{\sigma(A)}$ on A.

Furthermore we have

$$\sigma(f(A)) = f(\sigma(A)) \tag{3}$$

for all $f \in C(\sigma(A))$.

Proof

When f is a polynomial function the application $f \mapsto f(A)$ is well-defined and isometric for

$$||f(A)|| = \sup\{|\lambda| \, ; \lambda \in \sigma(f(A))\} = \sup\{|\lambda| \, ; \lambda \in f(\sigma(A))\} = ||f|| \, .$$

Thus it extends to an isometry on $C(\sigma(A))$ by Weierstrass theorem. The extension is easily seen to be a morphism also. The only delicate point to check is the identity (3). Let $\mu \in f(\sigma(A))$, with $\mu = f(\lambda)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of polynomial functions converging to f. The sequence $(f_n(\lambda)I - f_n(A))_{n\in\mathbb{N}}$ converges to $\mu I - f(A)$. As none of the $f_n(\lambda)I - f_n(A)$ is invertible then $\mu I - f(A)$is not either (Proposition 1.6). Thus $f(\sigma(A)) \subset \sigma(f(A))$. Finally, if $\mu \in \mathbb{C} \setminus f(\sigma(A))$ then let $g(t) = (\mu - f(t))^{-1}$. Then g belongs to $C(\sigma(A))$ and $g(A) = (\mu I - f(A))^{-1}$. Thus μ belongs to $\mathbb{C} \setminus \sigma(f(A))$. ■

An element A of a C^*-algebra $\mathcal{A}$ is *positive* if it is self-adjoint and its spectrum is included in $\mathbb{R}^+$.

Theorem 1.8 – *Let A be an element of $\mathcal{A}$. The following assertions are equivalent.*

i) A is positive.

ii) A is self-adjoint and $||tI - A|| \leq t$ for some $t \geq ||A||$.

iii) A is self-adjoint and $||tI - A|| \leq t$ for all $t \geq ||A||$.

iv) $A = B^*B$ for a $B \in \mathcal{A}$.

v) $A = C^2$ for a self-adjoint $C \in \mathcal{A}$.

Proof

Let us first prove that i) implies iii). If i) is assumed then $tI - A$ is a normal operator and

$$||tI - A|| = \sup\{|\lambda|\,;\, \lambda \in \sigma(tI - A)\} = \sup\{|\lambda - t|\,;\, \lambda \in \sigma(A)\} \leq t.$$

This gives iii).

Obviously iii) implies ii). Let us prove that ii) implies i). If ii) is satisfied and if $\lambda \in \sigma(A)$ then $t - \lambda \in \sigma(tI - A)$ and with the same computation as above $|t - \lambda| \leq ||tI - A|| \leq t$. But as $\lambda \leq t$ we must have $\lambda \geq 0$. This proves i).

We have proved the equivalence of the first 3 assertions.

We have that v) implies iv) obviously. In order to show that i) implies v) it suffices to consider $C = \sqrt{A}$ (using the functional calculus of Theorem 1.7 and identity (3)). It remains to prove that iv) implies i). Let $f_+(t) = t \vee 0$ and $f_-(t) = (-t) \vee 0$. Let $A_+ = f_+(A)$ and $A_- = f_-(A)$ (note that when ii) holds then A is automatically self-adjoint and thus accepts the functional calculus of Theorem 1.7). We have $A = A_+ - A_-$ and the elements A_+ and A_- are positive (by (3)). Furthermore the identity $f_+f_- = 0$ implies $A_+A_- = 0$. We have

$$(BA_-)^*(BA_-) = A_-(A_+ - A_-)A_- = -A_-^3.$$

In particular $-(BA_-)^*(BA_-)$ is positive.

Writing $BA_- = S + iT$ with S and T self-adjoint gives

$$(BA_-)(BA_-)^* = -(BA_-)^*(BA_-) + 2(S^2 + T^2).$$

In particular, as the equivalence established between i), ii) and iii) proves it easily (exercise), the set of positive elements of $\mathcal{A}$ is a cone, thus the element $(BA_-)(BA_-)^*$ is positive. As a consequence $\sigma((BA_-)(BA_-)^*) \subset [0, ||B||\,||A_-||]$.

But by Theorem 1.4 g) we must also have $\sigma((BA_-)^*(BA_-)) \subset [0, ||B||\,||A_-||]$. In particular $\sigma(-A_-^3) \subset [0, ||B||^2||A_-||^2]$. This implies $\sigma(A_-^3) = \{0\}$ and $\left|\left|A_-^3\right|\right| = 0 = ||A_-||^3$. That is $A_- = 0$. ■

This notion of positivity defines an order on elements of $\mathcal{A}$, by saying that $U \geq V$ in $\mathcal{A}$ if $U - V$ is a positive element of $\mathcal{A}$.

Proposition 1.9 – *Let U, V be self-adjoint elements of $\mathcal{A}$ such that $U \geq V \geq 0$. Then*

i) $W^*UW \geq W^*VW \geq 0$ for all $W \in \mathcal{A}$;

ii) $(V + \lambda I)^{-1} \geq (U + \lambda I)^{-1}$ for all $\lambda \geq 0$.

Proof

i) is obvious from Theorem 1.8.

ii) As we have $U + \lambda I \geq V + \lambda I$, then by i) we have

$$(V + \lambda I)^{-1/2}(U + \lambda I)V + \lambda I)^{-1/2} \geq I.$$

Now, note that if W is self-adjoint and $W \geq I$ then $\sigma(W) \subset [1, +\infty[$ and $\sigma(W^{-1}) \subset [0, 1]$. In particular $W^{-1} \leq I$. This argument applied to the above inequality shows that

$$(V + \lambda I)^{1/2}(U + \lambda I)^{-1}V + \lambda I)^{1/2} \leq I.$$

Multiplying both sides by $(V + \lambda I)^{-1/2}$ gives the result. ■

1.3 Representations and states

A $*$-*algebra morphism* is a linear mapping $\Pi : \mathcal{A} \to \mathcal{B}$, between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, such that $\Pi(A^*B) = \Pi(A)^*\Pi(B)$ for all $A, B \in \mathcal{A}$.

Such a morphism is always positive, that is it maps positive elements of $\mathcal{A}$ on positive elements of $\mathcal{B}$. Indeed we have $\Pi(A^*A) = \Pi(A)^*\Pi(A)$.

Theorem 1.10 – *If Π is a morphism between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ then Π is continuous, with norm smaller than 1. Furthermore the range of Π is a sub-C^*-algebra of $\mathcal{B}$.*

Proof

If A is self-adjoint then so is $\Pi(A)$ and thus

$$||\Pi(A)|| = \sup\{|\lambda|\,;\lambda \in \sigma(\Pi(A))\}.$$

But it is easy to see that $\sigma(\Pi(A))$ is included in $\sigma(A)$ and consequently

$$||\Pi(A)|| \leq \sup\{|\lambda|\,;\lambda \in \sigma(A)\} = ||A||\,.$$

For a general A now, we have

$$||\Pi(A)||^2 = ||\Pi(A^*A)|| \leq ||A^*A|| = ||A||^2.$$

We have proved the first part of the theorem.

For proving the second part we reduce the problem to the case where $\ker\Pi = \{0\}$. If this is not the case, following Annexe 1.1, we consider the quotient of $\mathcal{A}$ by the two-sided closed ideal $\ker\Pi$: $\mathcal{A}_\Pi = \mathcal{A}/\ker\Pi$ which is a C^*-algebra. We can thus assume $\ker\Pi = \{0\}$. Let $\mathcal{B}_\Pi$ be the image of Π, it is sufficient to prove that it is closed. Consider the inverse morphism Π^{-1} from $\mathcal{B}_\Pi$ onto $\mathcal{A}$. As previously, for A self-adjoint in $\mathcal{A}$ we have

$$||A|| = \left|\left|\Pi^{-1}(\Pi(A))\right|\right| \leq ||\Pi(A)|| \leq ||A|| \,.$$

Thus Π^{-1} and Π are isometric and one concludes easily. ■

A *representation* of a C^*-algebra $\mathcal{A}$ is a couple $(\mathcal{H}, \Pi)$ made of a Hilbert space $\mathcal{H}$ and a morphism Π from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. The representation is *faithful* if $\ker\Pi = \{0\}$.

Proposition 1.11 – *Let $(\mathcal{H}, \Pi)$ be a representation of a C^*-algebra $\mathcal{A}$. Then the following assertions are equivalent.*

i) Π *is faithful.*

ii) $||\Pi(A)|| = ||A||$ *for all* $A \in \mathcal{A}$.

iii) $\Pi(A) > 0$ *if* $A > 0$.

Proof

We have already seen that i) implies ii), in the proof above. Let us prove that ii) implies iii). If $A > 0$ then $||A|| > 0$ and thus $||\Pi(A)|| > 0$ and $\Pi(A) \neq 0$. As we already know that $\Pi(A) \geq 0$, we conclude that $\Pi(A) > 0$. Finally, assume iii) is satisfied. If B belongs to $\ker\Pi$ and $B \neq 0$ then $\Pi(B^*B) = 0$. But $||B^*B|| = ||B||^2 > 0$ and thus $B^*B > 0$. Which is contradictory and ends the proof. ■

Clearly we have not yet discussed the existence of representations for C^*-algebras. The key tool for this existence theorem is the notion of *state*.

A linear form ω on $\mathcal{A}$ is *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$.

Note that for such positive linear form one can easily prove a Cauchy-Schwarz inequality:

$$|\omega(B^*A)|^2 \leq \omega(B^*B)\omega(A^*A).$$

Proposition 1.12 – *Let ω be a linear form on $\mathcal{A}$. Then the following assertions are equivalent.*

i) ω is positive.

ii) ω is continuous with $||\omega|| = \omega(I)$.

Proof

By Theorem 1.8 ii), recall that a self-adjoint element A of $\mathcal{A}$, with $||A|| = 1$ is positive if and only if $||(I - A)|| \leq 1$. In particular, for any $A \in \mathcal{A}$, we have that $||A^*A||\, I - A^*A$ is positive.

If i) is satisfied then $\omega(A^*A) \leq ||A^*A||\, \omega(I)$. By Cauchy-Schwarz we have

$$|\omega(A)| \leq \omega(I)^{1/2} \, |\omega(A^*A)|^{1/2} \leq ||A^*A||^{1/2} \, \omega(I) = ||A|| \, \omega(I). \tag{1.2}$$

This proves ii).

Conversely, if ii) is satisfied. One can assume $\omega(I) = 1$. Let A be a self-adjoint element of $\mathcal{A}$. Write $\omega(A) = \alpha + i\beta$ for some α, β real. For every $\lambda \in I\!R$ we have

$$||A + i\lambda I||^2 = \left|\left|A^2 + \lambda^2 I\right|\right| = ||A||^2 + \lambda^2.$$

Thus we have

$$\beta^2 + 2\lambda\beta + \lambda^2 \leq \left|\alpha^2 + i(\beta + \lambda)\right|^2 = |\omega(A + i\lambda I)|^2 \leq ||A||^2 + \lambda^2.$$

This implies that $\beta = 0$ and $\omega(A)$ is real. Consider now A positive, with $||A|| = 1$. We have

$$|1 - \omega(A)| = |\omega(I - A)| \leq ||I - A|| \leq I.$$

Thus $\omega(A)$ is positive. ■

We call *state* any positive linear form on $\mathcal{A}$ such that $\omega(I) = 1$.

We need an existence theorem for states.

Theorem 1.13 – *Let A be any element of $\mathcal{A}$. Then there exists a state ω on $\mathcal{A}$ such that $\omega(A^*A) = ||A||^2$.*

Proof

On the space $\mathcal{B} = \{\alpha I + \beta A^*A; \alpha, \beta \in \mathbb{C}\}$ we define the linear form

$$f(\alpha I + \beta A^*A) = \alpha + \beta||A||^2.$$

One easily checks that $||f|| = 1$. By Hahn-Banach we extend f to the whole of $\mathcal{A}$ into a norm 1 continuous linear form ω. By the previous proposition ω is a state. ■

We now turn to the construction of a representation which is going to be fundamental for us, the so called Gelfand-Naimark-Segal construction (G.N.S. construction). Indeed, note that if $(\mathcal{H}, \Pi)$ is a representation of a C^*-algebra $\mathcal{A}$ and if Ω is any norm 1 vector of $\mathcal{H}$, then the mapping

$$\omega(A) = <\Omega\,,\, \Pi(A)\Omega>$$

clearly defines a state on $\mathcal{A}$. The G.N.S. construction proves that any C^*-algebra with a state can be represented this way.

Theorem 1.14 (G.N.S. representation) – *Let $\mathcal{A}$ be a C^*-algebra with unit and ω be a state on $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation Π_ω of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\omega)$ and a unit vector Ω_ω of $\mathcal{H}_\omega$ such that*

$$\omega(A) = <\Omega_\omega\,,\, \Pi_\omega(A)\Omega_\omega>$$

for all A. Furthermore the space $\{\Pi_\omega(A)\Omega_\omega; A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.

Such a representation is unique up to unitary isomorphism.

Proof

Let $V_\omega = \{A \in \mathcal{A}; \omega(A^*A = 0)\}$. By Cauchy-Schwarz inequality one easily sees that V_ω is a closed two-sided ideal. We consider the quotient C^*-algebra $\mathcal{A}/V_\omega$. On $\mathcal{A}/V_\omega$ we define

$$<[A]\,,\,[B]> = \omega(B^*A).$$

It is a positive sesquilinear form which makes $\mathcal{A}/V_\omega$ a pre-Hilbert space. Let $\mathcal{H}_\omega$ be the its closure. We put

$$\begin{array}{rccc} L_A : & \mathcal{A}/V_\omega & \to & \mathcal{A}/V_\omega \\ & [B] & \mapsto & [AB]. \end{array}$$

We have

$$< L_A[B]\,,\, L_A[B]> = \omega(B^*A^*AB) \leq ||A||^2\omega(B^*B)$$

for $C \mapsto \omega(B^*CB)$ is a positive linear form equal to $\omega(B^*B)$ on $C = I$. In particular $< L_A[B]\,,\, L_A[B]> \leq ||A||^2 < [B]\,,\, [B]>$. One can extend L_A into a bounded operator $\Pi_\omega(A)$ on $\mathcal{H}_\omega$. If we put $\Omega_\omega = [I]$ then the construction is achieved.

Let us check uniqueness. If $(\mathcal{H}', \Pi', \Omega')$ is another such triple, we have

$$\begin{aligned} < \Pi_\omega(B)\Omega_\omega\,,\, \Pi_\omega(A)\Omega_\omega > &= < \Omega_\omega\,,\, \Pi_\omega(B^*A)\Omega_\omega > = \omega(B^*A) \\ &= < \Omega'\,,\, \Pi'(B^*A)\Omega' > = < \Pi'(B)\Omega'\,,\, \Pi'(A)\Omega' >. \end{aligned}$$

The unitary isomorphism is thus defined by $U : \Pi_\omega(A)\Omega_\omega \mapsto \Pi'(A)\Omega'$. ■

Theorem 1.15 – *Let $\mathcal{A}$ be a C^*-algebra . Then $\mathcal{A}$ is isomorphic to a sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proof

For every state ω we have the G.N.S. representation $(\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega)$. Put $\mathcal{H} = \oplus_\omega \mathcal{H}_\omega$ and $\Pi = \oplus_\omega \Pi_\omega$ where the direct sums run over the set of all states on $\mathcal{A}$.

For every $A \in \mathcal{A}$ there exists a state ω_A such that $||\Pi_{\omega_A}(A)|| = ||A||$ (Theorem 1.12). But we have $||\Pi(A)|| \geq ||\Pi_{\omega_A}(A)|| = ||A||$. Thus we get $||\Pi(A)|| = ||A||$. This means that Π is faithful. ■

1.4 Commutative C^*-algebras

We have shown the very important characterization of C^*-algebras, namely they are exactly the closed $*$-sub-algebras of bounded operators on Hilbert space. We dedicate this last section to prove the (not very useful for us but) interesting characterization of commutative C^*-algebras .

Let $\mathcal{A}$ be a commutative C^*-algebra . A *character* on $\mathcal{A}$ is a linear form χ on $\mathcal{A}$ satisfying

$$\chi(AB) = \chi(A)\chi(B)$$

for all $A, B \in \mathcal{A}$. On then calls *spectrum* of $\mathcal{A}$ the set $\sigma(\mathcal{A})$ of all characters on $\mathcal{A}$.

Proposition 1.16 – *Every character is positive.*

Proof

Let $A \in \mathcal{A}$ and $\lambda \notin \sigma(A)$. Then there exists $B \in \mathcal{A}$ such that $(\lambda I - A)B = I$. Thus $\chi(\lambda I - A)\chi(B) = (\lambda\chi(I) - \chi(A))\chi(B) = \chi(I) = 1$. This implies in particular that $\lambda \neq \chi(A)$. We have proved that $\chi(A)$ always belong to $\sigma(A)$. In particular $\chi(A^*A)$ is always positive. ■

As a corollary every character is a state and thus is continuous. The set $\sigma(\mathcal{A})$ is a subset of $\mathcal{A}^*$, the dual of $\mathcal{A}$.

Theorem 1.17 – *Let $\mathcal{A}$ be a commutative C^*-algebra and X be the spectrum of $\mathcal{A}$ endowed with the $*$-weak topology of $\mathcal{A}^*$. Then X is a Haussdorf locally compact set; it is compact if and only if $\mathcal{A}$ admits a unit.*

Furthermore $\mathcal{A}$ is isomorphic to the C^*-algebra $C_0(X)$ of continuous functions on X which vanish at infinity.

Proof

Let $\omega_0 \in X$. Let A positive be such that $\omega_0(A) > 0$. One can assume $\omega_0(A) > 1$. Let $K = \{\omega \in X; \omega(A) > 1\}$. It is an open neighborhood of ω_0. Its closure $\overline{K}$ is included into $\{\omega \in X; \omega(A) \geq 1\}$. The latest set is closed and included in the unit ball of $\mathcal{A}^*$ which is compact. Thus X is locally compact.

If $\mathcal{A}$ contains a unit I, then the same argument applied to $A = 2I$ shows that X is compact.

Now, for all $A \in \mathcal{A}$ we put $\widehat{A}(\omega) = \omega(A)$. Then $\widehat{A}$ is a continuous complex function and $A \mapsto \widehat{A}$ is a morphism. Furthermore

$$\left|\left|\widehat{A}\right|\right|^2 = \sup_{\omega \in X} \left|\widehat{A}(\omega)\right|^2 = \sup_{\omega \in X} \left|\widehat{A^*A}(\omega)\right| = ||A||^2$$

for it exists an ω such that $|\omega(A^*A)| = ||A||$. Thus $A \mapsto \widehat{A}$ is an isomorphism.

The set $K_\varepsilon = \{\omega \in X; \omega(A) > \varepsilon\}$ is $*$-weakly compact and thus $\widehat{A}$ belong to $C_0(X)$. Finally $\widehat{A}$ separates the points of X, thus by Stone-Weierstrass theorem, the mapping $\widehat{A}$ gives the whole of $C_0(X)$. ■

Appendix: Quotient algebras and approximate identities

A subspace $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$ is a *left ideal* if for all $J \in \mathcal{J}$ and all $A \in \mathcal{A}$ then JA belongs to $\mathcal{J}$. In the same way one obviously defines *right ideals* and *two-sided ideals*.

If $\mathcal{J}$ is a two-sided, self-adjoint ideal of $\mathcal{A}$, one can easily define the quotient algebra $\mathcal{A}/\mathcal{J}$ by the usual rules:

$$\begin{aligned}
\text{i)}&\lambda[X] + \mu[Y] = [\lambda X + \mu Y],\\
\text{ii)}&[X]\,[Y] = [XY],\\
\text{iii)}&[X]^* = [X^*],
\end{aligned}$$

where $[X] = \{X + J; J \in \mathcal{J}\}$ is the equivalence class of $X \in \mathcal{A}$ modulo $\mathcal{J}$. We leave to the reader to check the consistency of the above definitions.

We now define a norm on $\mathcal{A}/\mathcal{J}$ by

$$||[X]|| = \inf\{||X + J||\,; J \in \mathcal{J}\}.$$

The true difficulty is to check that the above norm is a C^*-algebra norm. For this aim we need the notion of approximate identity.

If $\mathcal{J}$ is a left ideal of $\mathcal{A}$ then an *approximate identity* in $\mathcal{J}$ is a generalized sequence $(e_\alpha)_a$ of positive elements of $\mathcal{J}$ satisfying

i) $||e_\alpha|| \leq 1$,

ii) $\alpha \leq \beta$ implies $e_\alpha \leq e_\beta$,

iii) $\lim_\alpha ||Xe_\alpha - X|| = 0$ for all $X \in \mathcal{J}$.

Proposition 1.18 – *Every left ideal $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$ possesses an approximate unit.*

Proof

Let $\mathcal{J}_+$ be the set of positive elements of $\mathcal{J}$. For each $J \in \mathcal{J}_+$ put

$$e_J = J(I+J)^{-1} = I - (I+J)^{-1}.$$

It is a generalized sequence, it is increasing by Proposition 1.9 and $||e_J|| \leq 1$. Let us now fix $X \in \mathcal{J}$. For every $n \in \mathbb{N}$ there exists a $J \in \mathcal{J}_+$ such that $J \geq nX^*X$. Thus

$$(X - Xe_J)^*(X - Xe_J) = (I - e_J)X^*X(I - e_J) \leq \frac{1}{n}(I - e_J)J(I - e_J)$$

by Proposition 1.9. It suffices to prove that

$$\sup_{J \in \mathcal{J}_+} ||J(I - e_J)^2|| < \infty.$$

But note that $J(I - e_J)^2 = J(I+J)^{-2}$ and using the functional calculus this reduces to the obvious remark that $\lambda/(1+\lambda^2)$ is bounded on $\mathbb{R}^+$. ■

We can now prove the main result of the appendix.

Theorem 1.19 – *If $\mathcal{J}$ is a closed, self-adjoint, two-sided ideal of a C^*-algebra $\mathcal{A}$, then the quotient algebra $\mathcal{A}/\mathcal{J}$, equiped with the quotient norm, is a C^*-algebra.*

Proof

Let us first show that

$$||[X]|| = \lim_\alpha ||e_\alpha X - X||$$

for all $X \in \mathcal{J}$. By definition of the quotient we obviously have

$$||[X]|| \leq \lim_\alpha ||e_\alpha X - X||.$$

As $\sigma(e_\alpha) \subset [0,1]$ we have $\sigma(I - e_\alpha) \subset [0,1]$ and $||I - e_\alpha|| \leq 1$. This implies

$$||(X + e_\alpha X) + (Y + e_\alpha Y)|| = ||(I - e_\alpha)(X + Y)|| \leq ||X + Y||.$$

In particular $\limsup_\alpha ||(X + e_\alpha X)|| \leq ||X + Y||$ for every $Y \in \mathcal{J}$. This proves our claim.

Now we have

$$\begin{aligned}||[X]||^2 &= \lim_\alpha ||X - e_\alpha X||^2 = \lim_\alpha ||(X^* - X^* e_\alpha)(X - e_\alpha X)|| \\ &= \lim_\alpha ||(I - e_\alpha)(X^*X + Y^*)(I - e_\alpha)|| \\ &\leq ||X^*X + \Psi||\end{aligned}$$

for every $Y \in \mathcal{J}$. This implies

$$|||[X]|||^2 \leq |||[X]^*[X]|||$$

and thus the result. ∎

2. VON NEUMANN ALGEBRAS

2.1 Topologies on $\mathcal{B}(\mathcal{H})$

As every C^*-algebra is a sub-$*$-algebra of some $\mathcal{B}(\mathcal{H})$, closed for the operator norm topology (or *uniform topology*), then it inherits new topologies, which are weaker.

On $\mathcal{B}(\mathcal{H})$ we define the *strong topology* to be the locally convex topology defined by the semi-norms $P_x(A) = ||Ax||$, $x \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$. This is to say that a base of neighborhood is formed by the sets

$$V(A; x_1, \ldots, x_n; \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}); ||(A - B)x_i|| < \varepsilon, i = 1, \ldots, n\}.$$

On $\mathcal{B}(\mathcal{H})$ we define the *weak topology* to be the locally convex topology defined by the semi-norms $P_{x,y}(A) = |< x\,,\, Ay >|$, $x, y \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$. This is to say that a base of neighborhood is formed by the sets

$$V(A; x_1, \ldots, x_n; y_1, \ldots, y_n; \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}); |< x_i\,,\, Ay_j >| < \varepsilon, i, j = 1, \ldots, n\}.$$

Proposition 2.1 –

i) The weak topology is weaker than the strong topology which is itself weaker than the uniform topology. Once $\mathcal{H}$ is infinite dimensional then these comparisons are strict.

ii) A linear form on $\mathcal{B}(\mathcal{H})$ is strongly continuous if and only if it is weakly continuous.

iii) The strong and the weak closure of any convex subset of $\mathcal{B}(\mathcal{H})$ coincide.

Proof

i) All the comparisons are obvious in the large sense. To make the difference in infinite dimension assume that $\mathcal{H}$ is separable with orthonormal basis $(e_n)_{n \in \mathbb{N}}$. The sequence $(P_n)_{n \in \mathbb{N}}$ of orthogonal projections onto the space generated by $e_1, \ldots, e_n$ converges strongly to I but not uniformly. Furthermore, consider the unilateral shift $S : e_i \mapsto e_{i+1}$. Then S^k converges weakly to 0 when k tends to $+\infty$ but not strongly.

ii) Let $\Psi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a strongly continuous linear form. Then there exists $x_1, \ldots, x_n \in \mathcal{H}$ such that

$$|\Psi(B)| \leq \sum_{i=1}^{n} ||Bx_i||$$

for all $B \in \mathcal{B}(\mathcal{H})$ (classical result on locally convex topologies, not proved here). On $\mathcal{B}(\mathcal{H})^n$ let P be the semi-norm defined by

$$P(A_1, \ldots, A_n) = \sum_{i=1}^{n} ||A_i x_i||\,.$$

On the diagonal of $\mathcal{B}(\mathcal{H})^n$ we define the linear form $\widetilde{\Psi}$ by $\widetilde{\Psi}(A,\dots,A) = \Psi(A)$. We then have $\left|\widetilde{\Psi}(A,\dots,A)\right| \leq P(A,\dots,A)$. By Hahn-Banach, there exists a linear form Ψ on $\mathcal{B}(\mathcal{H})^n$ which extends $\widetilde{\Psi}$ and such that

$$|\Psi(A_1,\dots,A_n)| \leq P(A_1,\dots,A_n).$$

Let Ψ_k be the linear form on $\mathcal{B}(\mathcal{H})$ defined by

$$\Psi_k(A) = \Psi(0,\dots,0,A,0,\dots,0). \qquad (A \text{ is at the } k\text{-th place})$$

Then $|\Psi_k(A)| \leq ||Ax_k||$ for every A. Every vector $y \in \mathcal{H}$ can be written as Ax_k for some $A \in \mathcal{B}(\mathcal{H})$. The linear form $Ax_k \mapsto \Psi_k(A)$ is thus well-defined and continuous on $\mathcal{H}$. By Riesz theorem there exists a $y_k \in \mathcal{H}$ such that $\Psi_k(A) = <y_k\,,\, Ax_k>$. We have proved that

$$\Psi(A) = \sum_{i=1}^{n} <y_k\,,\, Ax_k>.$$

Thus Ψ is weakly continuous.

iii) is an easy consequence of ii) and of the geometric form of Hahn-Banach theorem. ■

Another topology is of importance for us, the *σ-weak topology.* It is the one determined by the semi-norms

$$p_{(x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}}(A) = \sum_{n=0}^{\infty} |<x_n\,,\, Ay_n>|$$

where $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ run over all sequences in $\mathcal{H}$ such that $\sum_n ||x_n||^2 < \infty$ and $\sum_n ||y_n||^2 < \infty$.

Let $\mathcal{T}(\mathcal{H})$ denote the Banach space of trace class operators on $\mathcal{H}$, equiped with the trace norme $||H||_1 = \mathrm{tr}\,|H|$, where $|H| = \sqrt{H^*H}$.

Theorem 2.2 – *The Banach space $\mathcal{B}(\mathcal{H})$ is the topological dual of $\mathcal{T}(\mathcal{H})$ thanks to the duality*

$$(A,T) \mapsto tr(AT),$$

$A \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{T}(\mathcal{H})$. Furthermore the $*$-weak topology on $\mathcal{B}(\mathcal{H})$ associated to this duality is the σ-weak topology.

Proof

The inequality $|\mathrm{tr}(AT)| \leq ||A||\,||T||_1$ proves that $\mathcal{B}(\mathcal{H})$ is included in the topological dual of $\mathcal{T}(\mathcal{H})$. Conversely, let ω be an element of the dual of $\mathcal{T}(\mathcal{H})$. Consider the rank one operators $E_{\xi,\nu} = |\,\xi\,><\,\nu\,|$. One easily checks that $||E_{\xi,\nu}||_1 = ||\xi||\,||\nu||$. Thus $|\omega(E_{\xi,\nu})| \leq ||\omega||\,||\xi||\,||\nu||$. By Riesz theorem there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\omega(E_{\xi,\nu} = <\nu\,,\, A\xi>$. The linear form $\mathrm{tr}(A\,\cdot)$ then coincides with ω on rank one projectors. One concludes that they coincide on $\mathcal{T}(\mathcal{H})$ by density of finite rank operators. This proves the announced duality.

The $*$-weak topology associated to this duality is defined by the seminorms

$$P_T(A) = \mathrm{tr}(AT)$$

where T runs over $\mathcal{T}(\mathcal{H})$. But every trace class operator T writes

$$T = \sum_{n=0}^{\infty} \lambda_n \,|\, \xi_n >< \nu_n|$$

for some orthonormed systems $(\nu_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ and some absolutely summable sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$. Thus

$$\mathrm{tr}(AT) = \sum_{n=0}^{\infty} \lambda_n < \nu_n \, , \, A\Xi_n >$$

and the seminorms P_T are equivalent to those defining the σ-weak topology. ■

Corollary 2.3 – *Every σ-weakly continuous linear form on $\mathcal{B}(\mathcal{H})$ is of the form*

$$A \mapsto tr(AT)$$

for some $T \in \mathcal{T}(\mathcal{H})$. ■

We can now put the first definition of a von Neumann algebra.

A *von Neumann algebra* is a C^*-algebra acting on $\mathcal{H}$ which contains a unit I and which is weakly (strongly) closed.

Of course the whole of $\mathcal{B}(\mathcal{H})$ is the first example of a von Neumann algebra.

Another example, which is actually the archetype of commutative von Neumann algebra, is obtained when considering a measured space (X, μ), with a σ-finite measure μ. The $*$-algebra $L^\infty(X, \mu)$ acts on $\mathcal{H} = L^2(X, \mu)$ by multiplication. One can assume that X is locally compact. The C^*-algebra $C_0(X)$ also acts on $\mathcal{H}$. But every function $f \in L^\infty(X, \mu)$ is almost sure limit of a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0(X)$. By dominated convergence, the space $L^\infty(X, \mu)$ is included in the weak closure of $C_0(X)$. But as $L^\infty(X, \mu)$ is also equal to its weak closure, we have that $L^\infty(X, \mu)$ is the weak closure of $C_0(X)$. We have proved that $L^\infty(X, \mu)$ is a von Neumann algebra and we have obtained it as the weak closure of some C^*-algebra.

2.2 Commutant

Let $\mathcal{M}$ be a subset of $\mathcal{B}(\mathcal{H})$. We put

$$\mathcal{M}' = \{B \in \mathcal{B}(\mathcal{H}); BM = MB \text{ for all } M \in \mathcal{M}\}.$$

The space $\mathcal{M}'$ is called the *commutant* of $\mathcal{M}$. We also define

$$\mathcal{M}'' = (\mathcal{M}')', \ldots, \mathcal{M}^{(n)} = (\mathcal{M}^{(n-1)})', \ldots$$

Proposition 2.4 – *For every subset $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ we have*

i) $\mathcal{M}'$ is weakly closed;

ii) $\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(5)} = \dots$
and $\mathcal{M} \subset \mathcal{M}'' = \mathcal{M}^{(4)} = \dots$

Proof

i) If $(A_n)_{n\in I\!N}$ is a sequence in $\mathcal{M}'$ which converges weakly to A in $\mathcal{B}(\mathcal{H})$ then for all $B \in \mathcal{M}$ and all $x, y \in \mathcal{H}$ we have

$$|{<}\,x\,,\,(AB - BA)y\,{>}| \leq |{<}\,x\,,\,(A - A_n)By\,{>}| + |{<}\,x\,,\,B(A - A_n)y\,{>}| \to_{n\to\infty} 0.$$

Thus A belongs to $\mathcal{M}'$.

ii) If B belongs to $\mathcal{M}'$ and A belongs to $\mathcal{M}$ then $AB = BA$, thus A belongs to $(\mathcal{M}')' = \mathcal{M}''$. This proves the inclusion $\mathcal{M} \subset \mathcal{M}''$. But note that if $\mathcal{M}_1 \subset \mathcal{M}_2$ then clearly $\mathcal{M}_2' \subset \mathcal{M}_1'$. Applying this to the previous inclusion gives $\mathcal{M}''' \subset \mathcal{M}'$. But as $\mathcal{M}'''$ is also equal to $(\mathcal{M}')''$ we should also have the converse inclusion to hold true. This means $\mathcal{M}' = \mathcal{M}'''$. We now conclude easily. ■

Proposition 2.5 – *Let $\mathcal{M}$ be a self-adjoint subset of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{E}$ be a closed subspace of $\mathcal{H}$ and P be the orthogonal projector onto $\mathcal{E}$. Then $\mathcal{E}$ is invariant under $\mathcal{M}$ (in the sense $M\mathcal{E} \subset \mathcal{E}$ for all $M \in \mathcal{M}$) if and only if $P \in \mathcal{M}'$.*

Proof

The space $\mathcal{E}$ is invariant under $M \in \mathcal{M}$ if and only $MP = PMP$. Thus if $\mathcal{E}$ is invariant under $\mathcal{M}$ we have $MP = PMP$ for all $M \in \mathcal{M}$. Applying the involution on this equality and using the fact that $\mathcal{M}$ is self-adjoint, gives $PM = PMP$ for all $M \in \mathcal{M}$. Finally $PM = MP$ for all $M \in \mathcal{M}$ and P belongs to $\mathcal{M}'$. The converse is obvious. ■

Theorem 2.6 [Von Neumann density theorem] – *Let $\mathcal{M}$ be a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ which contains the identity I. Then $\mathcal{M}$ is weakly (strongly) dense in $\mathcal{M}''$.*

Proof

Let $B \in \mathcal{M}''$. Let $x_1, \dots, x_n \in \mathcal{H}$. Let

$$V = \{A \in \mathcal{B}(\mathcal{H});\, ||(A - B)x_i|| < \varepsilon,\, i = 1, \dots, n\}$$

be a strong neighborhood of B. It is sufficient to show that V intersects $\mathcal{M}$. One can assume B to be self-adjoint as it can always be decomposed as a linear combination of two self-adjoint operators which also belong to $\mathcal{M}''$.

Let $\widetilde{\mathcal{H}} = \oplus_{i=1}^n \mathcal{H}$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ be given by $\pi(A) = \oplus_{i=1}^n A$. Let $x = \{x_1, \dots, x_n\} \in \widetilde{\mathcal{H}}$. Let P be the orthogonal projection from $\widetilde{\mathcal{H}}$ onto the closure of $\pi(\mathcal{M})x = \{\pi(A)x; A \in \mathcal{M}\} \subset \widetilde{\mathcal{H}}$. By Proposition 2.5 we have that P belongs to $\pi(\mathcal{M})'$.

If one identifies $\mathcal{B}(\widetilde{\mathcal{H}})$ to $M_n(\mathcal{B}(\mathcal{H}))$ it is easy to see that $\pi(\mathcal{M})' = M_n(\mathcal{M}')$ and $\pi(\mathcal{M}'') \subset M_n(\mathcal{M}')'$ (be aware that the prime symbols above are relative to different operator spaces!).

This means that $\pi(B)$ belong to $\pi(\mathcal{M}'') \subset M_n(\mathcal{M}')' = \underline{\pi(\mathcal{M})''}.$ In particular B commutes with $P \in \pi(\mathcal{M})'$. This means that the space $\overline{\pi(\mathcal{M})x}$ is invariant under $\pi(B)$. In particular

$$\pi(B)\,(\pi(I)x) = \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix}$$

belongs to $\overline{\pi(\mathcal{M})x}$. This means that there exists a $A \in \mathcal{M}$ such that $||(B-A)x_i||$ is small for all $i = 1, \dots n$. Thus A belongs to $\mathcal{M} \cap V$. ■

As immediate corollary we have a characterization of von Neumann algebras.

Corollary 2.7 [Bicommutant theorem] – *Let $\mathcal{M}$ be a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ which contains I. Then the following assertions are equivalent.*

i) $\mathcal{M}$ is weakly (strongly) closed.

ii) $\mathcal{M} = \mathcal{M}''$.

As I always belong to $\mathcal{M}''$, we have that a C^*-algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$.

■

2.3 Predual, normal states

Let $\mathcal{M}$ be a von Neumann algebra. Put $\mathcal{M}_1 = \{M \in \mathcal{M}; ||M|| \leq 1\}$. It is a weakly closed subset of the unit ball of $\mathcal{B}(\mathcal{H})$ which is weakly compact. Thus $\mathcal{M}_1$ is weakly compact. Note that the weak topology and the σ-weak topology coincide on $\mathcal{M}_1$ (exercise).

We denote by $\mathcal{M}_*$ the space of weakly (σ-weakly) continuous linear forms on $\mathcal{M}_1$. The space $\mathcal{M}_*$ is called the *predual* of $\mathcal{M}$, for a reason that will appear clear in next proposition. If Ψ belongs to $\mathcal{M}_*$ then $\Psi(\mathcal{M}_1)$ is compact in $\mathbb{C}$, thus Ψ is norm continuous. Thus $\mathcal{M}_*$ is a subspace of $\mathcal{M}^*$ the topological dual of $\mathcal{M}$.

Proposition 2.6 –

i) $\mathcal{M}_*$ is closed in $\mathcal{M}^*$, it is thus a Banach space.

ii) $\mathcal{M}$ is the dual of $\mathcal{M}_*$.

Proof

i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_*$ which converges to a f in $\mathcal{M}^*$, that is

$$\sup_{||A||=1} |f_n(A) - f(A)| \longrightarrow_{n\to\infty} 0.$$

We want to show that f belongs to $\mathcal{M}_*$, that is f is weakly continuous on $\mathcal{M}_1$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_1$ which converges weakly to $A \in \mathcal{M}_1$. Then

$$\begin{aligned} |f(A_n) - f(A)| &\leq |f(A_n) - f_m(A_n)| + |f_m(A) - f(A)| + |f_m(A_n) - f_m(A)| \\ &\leq 2 \sup_{||B||=1} |f_m(B) - f(B)| + |f_m(A_n) - f_m(A)| \end{aligned}$$

$$\begin{aligned}&\to_{n\to\infty} 2 \sup_{||B||=1} |f_m(B) - f(B)| \\ &\to_{m\to\infty} 0.\end{aligned}$$

This proves i).

ii) For a $A \in \mathcal{M}$ we put

$$||A||_{du} = \sup_{\omega\in\mathcal{M}_*;||\omega||=1} |\omega(A)|$$

the norm of A for the duality announced in the statement of ii). Clearly we have $||A||_{du} \leq ||A||$.

For $x, y \in \mathcal{H}$ we denote by $\omega_{x,y}$ the linear form $A \mapsto <y\,,\, Ax>$ on $\mathcal{B}(\mathcal{H})$ and $\omega_{x,y}{}_{|\mathcal{M}}$ the restriction of $\omega_{x,y}$ to $\mathcal{M}$. We have

$$||A|| = \sup_{||x||=||y||=1} |<y\,,\, Ax>| \leq \sup_{\omega=\omega_{x,y};||\omega||=1} |\omega(A)| \leq ||A||_{du}\,.$$

Thus $\mathcal{M}$ is indeed identified linearly and isometrically to a subspace of $(\mathcal{M}_*)^*$. We just have to prove that this identification is onto.

Let ϕ be a continuous linear form on $\mathcal{M}_*$. Let $\phi'(x,y) = \phi(\omega_{x,y}{}_{|\mathcal{M}})$. Then ϕ' is a continuous sesquilinear form on $\mathcal{H}$, it is thus of the form $\phi'(x,y) = <y\,,\, Ax>$ for some $A \in \mathcal{B}(\mathcal{H})$.

If T' is a self-adjoint element of $\mathcal{M}'$ then $\omega_{T'x,y}{}_{|\mathcal{M}} = \omega_{x,T'y}{}_{|\mathcal{M}}$ and

$$<AT'x\,,\, y> = <T'Ax\,,\, y>$$

for all $x, y \in \mathcal{H}$. Thus A belong to $\mathcal{M}'' = \mathcal{M}$.

As $\omega_{x,y}(A) = <y\,,\, Ax> = \phi'(x,y) = \phi(\omega_{x,y}{}_{|\mathcal{M}})$ then the image of A in $(\mathcal{M}_*)^*$ coincides with ϕ at least on the $\omega_{x,y}$. Now, it remains to show that this is sufficient for A and ϕ to coincide everywhere. That is, we have to prove that an element a of $(\mathcal{M}_*)^*$ which vanishes on all the $\omega_{x,y}$ is null. But all the elements of $\mathcal{M}_*$ are linear forms ω of the form $\omega(A) = \mathrm{tr}(\rho A)$ for some trace class operator ρ. As every trace class operator ρ writes as

$$\rho = \sum_n \lambda_n\, |x_n\rangle\langle x_n|$$

for some orthonormal basis $(x_n)_{n\in\mathbb{N}}$ and some summable sequence $(\lambda_n)_{n\in\mathbb{N}}$, we have that

$$\omega = \sum_n \lambda_n\, \omega_{x_n,x_n}$$

where the series above is convergent in $\mathcal{M}_*$. One concludes easily. ■

The two main examples of von Neumann algebra have well-known preduals. Indeed, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then $\mathcal{M}_* = \mathcal{T}(\mathcal{H})$ the space of trace class operators.

If $\mathcal{M} = L^\infty(X,\mu)$ then $\mathcal{M}_* = L^1(X,\mu)$.

Theorem 2.7 [Sakai theorem] – *A C^*-algebra is a von Neumann algebra if and only if it is the dual of some Banach space.*

Admitted. ■

A state on a von Neumann algebra $\mathcal{M}$ is called *normal* if it is σ-weakly continuous. The following characterization is now straightforward.

Theorem 2.8 – *On a von Neumann algebra $\mathcal{M}$, for a state ω on $\mathcal{M}$, the following assertions are equivalent.*

i) The state ω is normal

ii) There exists a positive, trace class operator ρ on $\mathcal{H}$ such that $\mathrm{tr}\rho = 1$ *and*

$$\omega(A) = \mathrm{tr}(\rho A)$$

for all $A \in \mathcal{M}$.

■

3.THE ALGEBRA OF CANONICAL COMMUTATION RELATIONS

3.1 Fock spaces

In classical mechanics a point system is characterized by its coordinates $Q_i(t)$ and impulsion $P_i(t)$, $i = 1 \ldots n$. In the Hamiltonian description of motion equations there exists a fundamental function $H(P, Q)$ of motion, which describes the system and satisfies Euler-Lagrange equations:

$$\frac{\partial H}{\partial P_i} = \dot{Q}_i \ , \quad \frac{\partial H}{\partial Q_i} = -\dot{P}_i \ .$$

If $f(P, Q)$ is a functional of the trajectory, we then have the evolution equation

$$\frac{df}{dt} = \frac{\partial f}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial f}{\partial Q}\frac{\partial Q}{\partial t}$$

or else

$$\frac{df}{dt} = \{f, H\}$$

where $\{g, f\}$ denote the *Poisson bracket* of f by h:

$$\{g, h\} = \frac{\partial g}{\partial P}\frac{\partial h}{\partial Q} - \frac{\partial g}{\partial Q}\frac{\partial h}{\partial P} \ .$$

In particular we have

$$\begin{aligned} &\{P_i, P_j\} = \{Q_i, Q_j\} = 0 \\ &\{P_i, Q_j\} = \delta_{ij} \ . \end{aligned}$$

It happens that it is not exactly the definitions of the P_i and Q_i which is important, but the relations above. Indeed, a change of coordinates $P'(P, Q)$, $Q'(P, Q)$ will give rise to the same motion equations if and only if P' and Q' satisfy the relations above.

In quantum mechanics it is essentially the same situation. We have a self-adjoint operator H (the Hamiltonian) which describes all the evolution of the system via the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = H \ \psi(t) \ .$$

There are also self-adjoint operators Q_i, P_i which represent the position and the impulsion of the system and which evolve following

$$\begin{aligned} &Q_i(t) = e^{itH}Q_i e^{itH} \\ &P_i(t) = e^{-itH}P_i e^{itH} \ . \end{aligned}$$

Thus any observable A defined from P and Q satisfies the evolution equation

$$\frac{d}{dt}A(t) = -\frac{i}{\hbar}[A(t), H]$$

where $[\cdot, \cdot]$ denotes the commutator.

But the operators P_i, Q_i satisfy the relation

$$[P_i, P_j] = [Q_i, Q_j] = 0$$
$$[Q_i, P_j] = i\hbar\delta_{ij} I \ .$$

Once again, it is not the choice of the representations of P_i and Q_i which is important, it is the relations above. It is called *commutation relation.*

In quantum field theory we have an infinite number of degrees of freedom. The operators position and impulsion are indexed by $\mathbb{R}^3$ (for example): we have a field of operators and the relations

$$[P(x), P(y)] = [Q(x), Q(y)] = 0$$
$$[Q(x), P(y)] = i\hbar\delta(x-y)I \ .$$

If one puts $a(x) = \frac{1}{\sqrt{2}}(Q(x) + iP(x))$ and $a^*(x) = \frac{1}{\sqrt{2}}(Q(x) - iP(x))$ then $a(x)$ and $a^*(x)$ are mutually adjoint and satisfy the *canonical commutation relations* (CCR)

$$[a(x), a(y)] = [a^*(x), a^*(y)] = 0$$
$$[a(x), a^*(y)] = \hbar\delta(x-y)I \ .$$

Actually it happens that these equations are valid only for a particular family of particles: the bosons (photons, mesons, gravitons,...). There is another family of particles: the fermions (electrons, muons, neutrinos, protons, neutrons, baryons,...) for which the correct relations are the *canonical anticommutation relations* (CAR)

$$\{b(x), b(y)\} = \{b^*(x), b^*(y)\} = 0$$
$$\{b(x), b^*(y)\} = \hbar\delta(x-y)I$$

where $\{A, B\} = AB + BC$ is the *anticommutator* of operators.

A natural problem, which has given rise to a huge literature, is to find concrete realisations of these relations. Let us see the simplest example: find two self-adjoint operators P and Q such that

$$QP - PQ = i\hbar I \ .$$

In a certain sense there is only one solution. This solution is realized on $L^2(\mathbb{R})$ by $Q = x$ (multiplication by x) and $P = i\hbar\frac{d}{dx}$. It is the *Schrödinger representation* of the CCR. But in full generality this problem is not well-posed. We need to be able to define the operators PQ and QP on good common domains. One can construct pathological counter-examples (Reed-Simon).

The problem is well-posed if we transform it in terms of bounded operators. Let $W_{x,y} = e^{-i(xP-yQ)}$ and $W_z = W_{x,y}$ when $z = x + iy \in \mathbb{C}$. We then have the *Weyl commutation relations*

$$W_z W_{z'} = e^{-i\Im\langle z, z'\rangle} W_{z+z'} \ .$$

Posed in these terms the problem has only one solution: the symmetric Fock space (Stone-von Neumann theorem).

The anticommutation relations as they are written with $b(x)$ and $b^*(x)$ have a more direct solution for $b(x)$ and $b^*(x)$ have to be bounded. We will come back to that later.

The importance of Fock space comes from the fact they give an easy realization of the CCR and CAR. They are also a natural tool for quantum field theory, second quantization... (all sorts of physical important notions that we will not develop here). The physical ideal around Fock spaces is the following. If $\mathcal{H}$ is the Hilbert space describing a system of one particle, then $\mathcal{H}\otimes\mathcal{H}$ describes a system consisting of two particles of the same type. The space $\mathcal{H}^{\otimes n}=\mathcal{H}\otimes\cdots\otimes\mathcal{H}$, n-fold, describes n such particles. Finally the space $\oplus_{n\in\mathbb{N}}\mathcal{H}^{\otimes n}$ describes a system where there can be any number of such particles which can disappear (annihilate) or be created. But depending on the type of particles (bosons or fermions) we deal with, there are some symmetries which force to look at certain subspaces of $\oplus_n\mathcal{H}^{\otimes n}$. We did not aim to describe the physics behind Fock spaces (we are not able to), but we just wanted to motivate them. Let us come back to mathematics.

Let $\mathcal{H}$ be a complex Hilbert space. For any integer $n\geq 1$ put

$$\mathcal{H}^{\otimes n}=\mathcal{H}\otimes\cdots\otimes\mathcal{H}$$

the *n-fold tensor product* of $\mathcal{H}$. That is, the Hilbert space obtained after completion of the pre-Hilbert space of finite linear combinations of elements of the form $u_1\otimes\cdots\otimes u_n$, with the scalar product

$$\langle u_1\otimes\cdots\otimes u_n, v_1\otimes\cdots\otimes v_n\rangle=\langle u_1,v_1\rangle\cdots\langle u_n,v_n\rangle\ .$$

For $u_1,\ldots,u_n\in\mathcal{H}$ we define the *symmetric tensor product*

$$u_1\circ\cdots\circ u_n=\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}\ ,$$

where S_n is the group of permutations of $\{1,2\ldots n\}$, and the *antisymmetric tensor product*

$$u_1\wedge\cdots\wedge u_n=\frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon_\sigma u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}\ ,$$

where ε_σ is the signature of the permutation σ.

The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the $u_1\circ\cdots\circ u_n$ (*resp.* $u_1\wedge\cdots\wedge u_n$) is denoted $\mathcal{H}^{\circ n}$ (*resp.* $\mathcal{H}^{\wedge n}$). It is called the n-fold symmetric (*resp.* antisymmetric) tensor product of $\mathcal{H}$.

Sometimes, when the notation is clear, one denotes by $\mathcal{H}_n$ the space $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$, and one calls it the *n-th chaos* of $\mathcal{H}$. In any of the three cases we put

$$\mathcal{H}_0=\mathbb{C}\ .$$

The element $1\in\mathbb{C}=\mathcal{H}_0$ plays an important role. One denotes it by $\mathbb{1}$ (usually by Ω in the literature) and one calls it the *vacuum vector*.

If one computes

$$\langle u_1\wedge\cdots\wedge u_n, v_1\wedge\cdots\wedge v_n\rangle=\frac{1}{(n!)^2}\sum_{\sigma,\tau\in S_n}\varepsilon_\sigma\varepsilon_\tau\langle u_{\sigma(1)},v_{\tau(1)}\rangle\cdots\langle u_{\sigma(n)},v_{\tau(n)}\rangle$$

one finds

$$\frac{1}{n!}\det(\langle u_i,v_j\rangle)_{ij}.$$

In order to remove the $n!$ factor we put a scalar product on $\mathcal{H}^{\wedge n}$ which is different from the one induced by $\mathcal{H}^{\otimes n}$, namely:

$$\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n \rangle_\wedge = \det(\langle u_i, v_j \rangle)_{ij} \ .$$

This way, we have

$$\|u_1 \wedge \cdots \wedge u_n\|_\wedge^2 = n! \|u_1 \wedge \cdots \wedge u_n\|_\otimes^2 \ .$$

In the same way, on $\mathcal{H}^{\circ n}$ we put

$$\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle_\circ = \operatorname{per}(\langle u_i, v_j \rangle)_{ij} \ ,$$

where per denotes the *permanent* of the matrix (that is, the determinant without the minus signs). This way we get

$$\|u_1 \circ \cdots \circ u_n\|_\circ^2 = n! \|u_1 \circ \cdots \circ u_n\|_\otimes^2 \ .$$

We call *free (or full) Fock space* over $\mathcal{H}$ the space

$$\Gamma_f(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \ .$$

We call *symmetric (or bosonic) Fock space* over $\mathcal{H}$ the space

$$\Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circ n} \ .$$

We call *antisymmetric (or fermionic) Fock space* over $\mathcal{H}$ the space

$$\Gamma_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n}.$$

It is understood that in the definition of $\Gamma_f(\mathcal{H})$, $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ each of the spaces $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$ is equipped with its own scalar product $\langle \cdot, \cdot \rangle_\otimes$, $\langle \cdot, \cdot \rangle_\circ$ or $\langle \cdot, \cdot \rangle_\wedge$. In other words, the elements of $\Gamma_f(\mathcal{H})$ (*resp.* $\Gamma_s(\mathcal{H})$, $\Gamma_a(\mathcal{H})$) are those series $f = \sum_{n \in \mathbb{N}} f_n$ such that $f_n \in \mathcal{H}^{\otimes n}$ (*resp.* $\mathcal{H}^{\circ n}$, $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} \|f_n\|_\varepsilon^2 < \infty$$

for $\varepsilon = \otimes$ (*resp.* $\circ$, $\wedge$).

If one want to write everything in terms of the usual tensor norm, we simply have that an element $f = \sum_{n \in \mathbb{N}} f_n$ is in $\Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) if $f_n \in \mathcal{H}^{\circ n}$ (*resp.* $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} n! \|f_n\|_\otimes^2 < \infty \ .$$

The simplest case is obtained by taking $\mathcal{H} = \mathbb{C}$, this gives $\Gamma_s(\mathbb{C}) = \ell^2(\mathbb{N})$. If $\mathcal{H}$ is of finite dimension n then $\mathcal{H}^{\wedge m} = 0$ for $m > n$ and thus $\Gamma_a(\mathcal{H})$ is of finite dimension 2^n; this is never the case for $\Gamma_s(\mathcal{H})$.

In physics, one usually consider bosonic or fermionic Fock spaces over $\mathcal{H} = L^2(\mathbb{R}^3)$.

In quantum probability it is the space $\Gamma_s(L^2(\mathbb{R}^+))$ which is important for quantum stochastic calculus (we will meet this space during the second semester).

We now only consider symmetric Fock spaces $\Gamma_s(\mathcal{H})$.

For a $u \in \mathcal{H}$ one notes that $u \circ \cdots \circ u = u \otimes \cdots \otimes u$. The *coherent vector* (or *exponential vector*) associated to u is

$$\varepsilon(u) = \sum_{n \in I\!N} \frac{u^{\otimes n}}{n!}$$

so that

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u,v \rangle}$$

in $\Gamma_s(\mathcal{H})$.

Proposition 3.1 – *The vector space $\mathcal{E}$ of finite linear combinations of coherent vectors, is dense in $\Gamma_s(\mathcal{H})$.*

Every finite family of coherent vectors is linearly independent.

Proof

Let us prove the independence. Let $u_1 \ldots u_n \in \mathcal{H}$. The set

$$E_{i,j} = \big\{u \in \mathcal{H}; \langle u, u_i \rangle \neq \langle u, u_j \rangle \big\},$$

for $i \neq j$, is open and dense in $\mathcal{H}$. Thus the set $\bigcap_{i,j} E_{i,j}$ is non empty. Thus there exists a $v \in \mathcal{H}$ such that the $\theta_j = \langle v, u_j \rangle$ are two by two different. Now, if $\sum_{i=1}^n \alpha_i \, \varepsilon(u_i) = 0$ this implies that

$$0 = \Big\langle \varepsilon(zv), \sum_{i=1}^n \alpha_i \varepsilon(u_i) \Big\rangle = \sum_{i=1}^n \alpha_i e^{z\theta_i}$$

for all $z \in \mathbb{C}$. Thus the α_i all vanish and the family $\big\{\varepsilon(u_1) \ldots \varepsilon(u_n)\big\}$ is free.

In order to show the density, we first notice that the set $\{u \circ \cdots \circ u, u \in \mathcal{H}\}$ is total in $\Gamma_s(\mathcal{H})$ for

$$u_1 \circ \cdots \circ u_n = \sum_{\varepsilon_i = \pm 1} (\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)^{\circ n} \ .$$

But $u^{\circ n} = \frac{d^n}{dt^n} \varepsilon(tu)_{|t=0}$. This gives the result. ■

Corollary 3.2 – *If $S \subset \mathcal{H}$ is dense subset, then the space $\mathcal{E}(S)$ generated by the $\varepsilon(u)$, $u \in S$, is dense in $\Gamma_s(\mathcal{H})$.*

Proof

We have

$$\|\varepsilon(u) - \varepsilon(v)\|^2 = e^{\|u\|^2} + e^{e\|v\|^2} - 2\Re e^{\langle u,v \rangle} \ .$$

Thus the mapping $u \mapsto \varepsilon(u)$ is continuous. We now conclude easily from Proposition 3.1. ■

Theorem 3.3 – *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces. Then there exists a unique unitary isomorphism*

$$\begin{aligned} U : \Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) &\longrightarrow \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2) \\ \varepsilon(u \oplus v) &\longmapsto \varepsilon(u) \otimes \varepsilon(v). \end{aligned}$$

Proof

The space $\mathcal{E}(\mathcal{H}_i)$ is dense in $\Gamma_s(\mathcal{H}_i)$, $i = 1, 2$, and $\{\varepsilon(u) \otimes \varepsilon(v)\ ;\ u \in \mathcal{H}_1,\ v \in \mathcal{H}_2\}$ is total in $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$. Furthermore, we have

$$\begin{aligned}
\langle \varepsilon(u \oplus v), \varepsilon(u' \oplus v') \rangle &= e^{\langle u \oplus v, u' \oplus v' \rangle} \\
&= e^{\langle u, u' \rangle + \langle v, v' \rangle} \\
&= e^{\langle u, u' \rangle} e^{\langle v, v' \rangle} \\
&= \langle \varepsilon(u), \varepsilon(u') \rangle \langle \varepsilon(v), \varepsilon(v') \rangle \\
&= \langle \varepsilon(u) \otimes \varepsilon(v), \varepsilon(u') \otimes \varepsilon(v') \rangle\ .
\end{aligned}$$

Thus the mapping U is isometric. One concludes easily. ■

An example we will follow all along this chapter: the space $\Gamma_s(\mathbb{C})$. It is equal to $\ell^2(\mathbb{N})$ but it can be advantageously interpreted as $L^2(\mathbb{R})$. Indeed, let U be the mapping from $\Gamma_s(\mathbb{C})$ to $L^2(\mathbb{R})$ which maps $\varepsilon(z)$ to the function $f_z(x) = (2\pi)^{-1/4} e^{zx - s^2/2 - x^2/4}$. It is easy to see that U extends to a unitary isomorphism. We will come back to this example later.

There is an interesting characterization of the space $\Gamma_s(\mathcal{H})$ which says roughly that $\Gamma_s(\mathcal{H})$ is the exponential of $\mathcal{H}$. Idea which is already confirmed by Theorem 3.3.

Theorem 3.4 – *Let $\mathcal{H}$ be a separable Hilbert space. If K is a Hilbert space such that there exists a mapping*

$$\begin{aligned}
\lambda : \mathcal{H} &\longrightarrow K \\
u &\longmapsto \lambda(u)
\end{aligned}$$

satisfying

i) $\langle \lambda(u), \lambda(v) \rangle = e^{\langle u, v \rangle}$ *for all* $u, v \in \mathcal{H}$

ii) $\{\lambda(u)\ ;\ u \in \mathcal{H}\}$ *is total in* K.

Then there exists a unique unitary isomorphism

$$\begin{aligned}
U : K &\longrightarrow \Gamma_s(\mathcal{H}) \\
\lambda(u) &\longmapsto \varepsilon(u).
\end{aligned}$$

Proof

Clearly U is isometric and maps a dense subspace onto a dense subspace. ■

It is useful to stop a moment in order to describe $\Gamma_s(\mathcal{H})$ when $\mathcal{H}$ is of the form $L^2(E, \mathcal{E}, m)$. We are going to see that if $(E, \mathcal{E}, m)$ is a measured, non atomic, σ-finite, separable measured space then $\Gamma(L^2(E, \mathcal{E}, m))$ can be written as $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ for an explicit measured space $(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$.

If $\mathcal{H} = L^2(E, \mathcal{E}, m)$, then $\mathcal{H}^{\otimes n}$ interprets naturally as $L^2(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ and $\mathcal{H}^{\circ n}$ interprets as $L^2_{sym}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ the space of symmetric, square integrable functions on E^n.

If $f(x_1 \dots x_n)$ is a n-variable *symmetric* function on E then

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $\sigma \in S_n$. If the x_i are two by two different we thus can see f as a function on the *set* $\{x_1, \dots, x_n\}$. But as m is non atomic, almost all the $(x_1 \dots x_n) \in E^n$ satisfy $x_i \neq x_j$ once $i \neq j$. An element of $\Gamma_s(\mathcal{H})$ is of the form $f = \sum_{n \in I\!N} f_n$ where each f_n is a function on the n-element subsets of E. Thus f can be seen as a function on the finite subsets of E.

More rigorously, let $\mathcal{P}$ be the set of finite subsets of E. Then $\mathcal{P} = \cup_{n \in I\!N} \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n-elements subsets of E. Let f_n be an element of $L^2_{sym}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$, we define f on $\mathcal{P}$ by

$$\begin{cases} f(\sigma) = 0 & \text{if } \sigma \in \mathcal{P} \text{ and } |\sigma| = n \ , \\ f(\{x_1, \dots, x_n\}) = f_n(x_1, \dots, x_n) & \text{otherwise.} \end{cases}$$

Let $\mathcal{E}_{\mathcal{P}}$ be the smallest σ-field on $\mathcal{P}$ which makes all these functions measurable on $\mathcal{P}$.

Let $\Delta_n \subset E^n$ be the set of $(x_1 \dots x_n)$ such that $x_i \neq x_j$ once $i \neq j$. By the non-atomicity of m, we have $m(E^n \backslash \Delta^n) = 0$. For $F \in \mathcal{E}_{\mathcal{P}}$ we put

$$\mu(F) = 1\!\!1_{\emptyset}(F) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Delta_n} 1\!\!1_{F \cap \mathcal{P}_n}(x_1, \dots, x_n) dm(x_1) \cdots dm(x_n) \ .$$

For example, if $E = I\!R$ with the Lebesgue structure, then $\mathcal{P}_n$ can be identified to the increasing simplex $\Sigma_n = \{x_1 < \cdots < x_n \in I\!R\} \subset I\!R^n$. Thus $\mathcal{P}^n$ inherits the Lebesgue measure from $I\!R^n$.

The measure μ we have defined is σ-finite, it possesses only one atom: $\{\emptyset\}$ which has mass 1. We call $(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ the *symmetric measure space* over $(E, \mathcal{E}, m)$. This construction is due to Guichardet.

For all $u \in L^2(E, \mathcal{E}, m)$ one defines by π_u the element of $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ which satisfies

$$\pi_u(\sigma) = \begin{cases} 1 & \text{if } \ \sigma = \emptyset \\ \prod_{s \in \sigma} u(s) & \text{otherwise} \end{cases}$$

for all $\sigma \in \mathcal{P}$.

Theorem 3.4 – *The mapping $\pi_u \longmapsto \varepsilon(u)$ extends to a unitary isomorphism from $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ onto $\Gamma_s(L^2(E, \mathcal{E}, m))$.*

Proof

Clearly $\langle \pi_u, \pi_v \rangle = e^{\langle u, v \rangle} = \langle \varepsilon(u), \varepsilon(v) \rangle$. The set of functions π_u is total in $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$. One concludes easily. ■

3.2 Creation and annihilation operators

We now come back to general symmetric and antisymmetric Fock spaces $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$.

For $u \in \mathcal{H}$ we define the following operators:

$$\begin{array}{rccc}
a^*(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n+1)} \\
 & u_1 \circ \cdots \circ u_n & \longmapsto & u \circ u_1 \circ \cdots \circ u_n \\
 & & & \\
b^*(u): & \mathcal{H}^{\wedge n} & \longrightarrow & \mathcal{H}^{\wedge(n+1)} \\
 & u_1 \wedge \cdots \wedge u_n & \longmapsto & u \wedge u_1 \wedge \cdots \wedge u_n \\
 & & & \\
a(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n-1)} \\
 & u_1 \circ \cdots \circ u_n & \longmapsto & \sum_{i=1}^n \langle u, u_i \rangle u_1 \circ \cdots \circ \widehat{u_i} \circ \cdots \circ u_n \\
 & & & \\
b(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n-1)} \\
 & u_1 \wedge \cdots \wedge u_n & \longmapsto & \sum_{i=1}^n (-1)^i \langle u, u_i \rangle u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_n \ .
\end{array}$$

These operators are respectively called *bosonic creation operator*, *fermionic creation operator*, *bosonic annihilation operator* and *fermionic annihilation operator*.

Notice that $a^*(u)$ and $b^*(u)$ depend linearly on u, whereas $a(u)$ and $b(u)$ depend antilinearly on u. Actually, one often finds in the literature notations with “bras” and kets”: $a^*_{|u\rangle}$, $b^*_{|u\rangle}$, $a_{\langle u|}$, $b_{\langle u|}$.

Note that

$$\begin{aligned}
a^*(u)\mathbb{1} = b^*(u)\mathbb{1} &= u \\
a(u)\mathbb{1} = b(u)\mathbb{1} &= 0 \ .
\end{aligned}$$

All the operators above extend to the space $\Gamma^f_s(\mathcal{H})$(*resp.* $\Gamma^f_a(\mathcal{H})$) of finite sums of chaos that is those $f = \sum_{n \in \mathbb{N}} f_n \in \Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) such that only a finite number of f_n do not vanish. This subspace is dense in the corresponding Fock space. It is included in the domain of the operators $a^*(u)$, $b^*(u)$, $a(u)$, $b(u)$ (defined as operators on $\Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$)), and it is stable under their action. On this space we have the following relations:

$$\begin{aligned}
\langle a^*(u)f, g \rangle &= \langle f, a(u)g \rangle \\
[a(u), a(v)] &= [a^*(u), a^*(v)] = 0 \\
[a(u), a^*(v)] &= \langle u, v \rangle I \\
\langle b^*(u)f, g \rangle &= \langle f, b(u)g \rangle \\
\{b(u), b(v)\} &= \{b^*(u), b^*(v)\} = 0 \\
\{b(u), b^*(v)\} &= \langle u, v \rangle I
\end{aligned}$$

In other words, when restricted to $\Gamma^f_s(\mathcal{H})$ (*resp.* $\Gamma^f_a(\mathcal{H})$) the operators $a(u)$ and $a^*(u)$ (*resp.* $b(u)$ and $b^*(u)$) are mutually adjoint and they satisfy the *CCR* (*resp.* *CAR*).

Proposition 3.6 – *For all $u \in \mathcal{H}$ we have*

i) $b^*(u)^2 = 0$,

ii) $\|b(u)\| = \|b^*(u)\| = \|u\|$.

Proof

The anticommutation relation $\{b^*(u), b^*(u)\} = 0$ means $2b^*(u)b^*(u) = 0$, this gives *i)*.

We have

$$\begin{aligned} b^*(u)b(u)b^*(u)b(u) &= b^*(u)\{b(u), b^*(u)\}b(u) \\ &= \|u\|^2 b^*(u)b(u) \ . \end{aligned}$$

Thus

$$\begin{aligned} \|b(u)\|^4 = \|b^*(u)b(u)b^*(u)b(u)\| &= \|u\|^2 \|b^*(u)b(u)\| \\ &= \|u\|^2 \|b(u)\|^2 \ . \end{aligned}$$

As the operator $b(u)$ is null if and only if $u = 0$ we easily deduce that $\|b(u)\| = \|u\|$. ■

The identity *i)* expresses the so-called *Pauli exclusion principle*: "One cannot have together two fermionic particles in the same state".

The bosonic case is less simple for the operators $a^*(u)$ and $a(u)$ are never bounded. Indeed, we have $a(u)v^{\circ n} = n\langle u, v\rangle v^{\circ(n-1)}$, thus the coherent vectors are in the domain of $a(u)$ and

$$a(u)\varepsilon(v) = \langle u, v\rangle \varepsilon(v) \ .$$

In particular

$$\begin{aligned} \sup_{\|h\|=1} \|a(u)h\| &\geq \sup_{v\in\mathcal{H}} \|a(u)e^{-\|v\|^2/2}\varepsilon(v)\| \\ &= \sup_{v\in\mathcal{H}} |\langle u, v\rangle| = +\infty \ . \end{aligned}$$

Thus $a(u)$ is not bounded.

The action of $a^*(u)$ can be also be made explicit. Indeed, we have

$$a^*(u)v^{\circ n} = u \circ v \circ \cdots \circ v = \frac{d}{d\varepsilon}(u + \varepsilon v)^{\circ n}_{|\varepsilon=0} \ .$$

Thus $\varepsilon(v)$ is in the domain of $a^*(u)$ and

$$a^*(u)\varepsilon(v) = \frac{d}{d\varepsilon}\varepsilon(u + \varepsilon v)_{|\varepsilon=0} \ .$$

The operators $a(u)$ and $a^*(u)$ are thus closable (they have a densely defined adjoint). We extend them by closure, while keeping the same notations $a(u)$, $a^*(u)$.

Proposition 3.7 – *We have $a^*(u) = a(u)^*$.*

Proof

On $\Gamma_s^f(\mathcal{H})$ we have $\langle f, a(u)g\rangle = \langle a^*(u)f, g\rangle$. We extend this relation to $f \in \operatorname{Dom} a^*(u)$. The mapping $g \mapsto \langle f, a(u)g\rangle$ is thus continuous and $f \in \operatorname{Dom} a(u)^*$. We have proved that $a^*(u) \subset a(u)^*$.

Conversely, if $f \in \operatorname{Dom} a(u)^*$ and if $h = a(u)^* f$. We decompose f and h in chaoses: $f = \sum_n f_n$ and $h = \sum_n h_n$. We have $\langle f, a(u)g\rangle = \langle h, g\rangle$ for all $g \in \Gamma_s^f(\mathcal{H})$.

Thus, taking $g \in \mathcal{H}^{\circ n}$ we get $\langle f_{n-1}, a(u)g\rangle = \langle h_n, g\rangle$ that is, $\langle a^*(u)f_{n-1}, g\rangle = \langle h_n, g\rangle$. This shows that $h_n = a^*(u)f_{n-1}$ This way $\sum_n \|a^*(u)f_n\|^2$ is finite, f belongs to $\operatorname{Dom} a^*(u)$ and $a^*(u)f = a(u)^*f$. ■

In physics, the space $\mathcal{H}$ is often $L^2(\mathbb{R}^3)$. An element h_n of $\mathcal{H}^{\circ n}$ is thus a symmetric function of n variables on $\mathbb{R}^3$. With our definitions we have

$$\big(a(f)h_n\big)(x_1 \dots x_{n-1}) = \int h_n(x_1 \dots x_{n-1}, x)\overline{f}(x)\; dx$$

and

$$\big(a^*(f)h_n\big)(x_1 \dots x_{n+1}) = \sum_{i=1}^{n+1} h_n(x_1 \dots \hat{x}_i \dots x_{n+1})f(x_i)\ .$$

But in the physic literature one often use creation and annihilation operators indexed by the points of $\mathbb{R}^3$, instead of the elements of $L^2(\mathbb{R}^3)$. One can find there $a(x)$ and $a^*(x)$ formally defined by

$$a(f) = \int \overline{f}(x)a(x)\; dx$$

$$a^*(f) = \int f(x)a^*(x)\; dx$$

with

$$\big(a(x)h_n\big)(x_1 \dots x_{n-1}) = h_n(x_1 \dots x_{n-1}, x)$$

$$\big(a^*(x)h_n\big)(x_1 \dots x_{n+1}) = \sum_{i=1}^{n+1} \delta(x - x_i)h_n(x_1 \dots \hat{x}_i \dots x_{n+1})\ .$$

If one comes back to our example $\Gamma_s(\mathbb{C}) \simeq L^2(\mathbb{R})$, we have the creation and annihilation operators $a^*(z)$, $a(z)$, $z \in \mathbb{C}$. They are actually determined by two operators $a^* = a^*(1)$ and $a = a(1)$. They operate on coherent vectors by

$$a \in \varepsilon(z) = z(\varepsilon(z), \quad a^*\varepsilon(z) = \frac{d}{d\varepsilon}\varepsilon(z+\varepsilon)_{|\varepsilon=0}\ .$$

On $L^2(\mathbb{R})$ this gives

$$af_z(x) = zf_z(x) = \Big(\frac{d}{dx} + \frac{x}{2}\Big)f_z(x)$$

$$a^*f_s(x) = \frac{d}{d\varepsilon}f_{z+\varepsilon}(x)_{|\varepsilon=0} = (x-z)f_z(x) = \Big(\frac{x}{2} - \frac{d}{dx}\Big)f_z(x)\ .$$

The operators $Q = a + a^*$ and $P = i(a - a^*)$ are thus respectively represented by the operator x and $2i\frac{d}{dx}$ on $L^2(\mathbb{R})$. That is, the Schrödinger representation of the CCR (with $\hbar = 2$).

To conclude in this section, note that the operator $Q = a + a^*$ is an observable, in the physical sens. If we are given a state on $L^2(\mathbb{R})$, for example the vaccum state $\mathbb{1}$,then the observable Q has a natural probability law. This law isthe one which describes the probabilistic behaviour of the observable Q if one tries to measure it in the state $1\!\!1$. One can also see this law in the following way: the mapping $t \mapsto <1\!\!1\,,\, e^{itQ}1\!\!1>$ satisfies the Bochner criterion and is thus the Fourier transform of some probability measure μ.

From the postulats of quantm mechanics, the n-th moment of this law are given by

$$<1\!\!1\,,\, Q^n 1\!\!1>.$$

Passing by the $L^2(I\!\!R)$ interpretation, this quantity equals

$$< f_0\,,\, x^n f_0 > = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2}\, dx$$

that is the n-th moment of the standard normal law $\mathcal{N}(0,1)$. The observable Q, in the vaccum state, follows the $\mathcal{N}(0,1)$ law.

3.3 Second quantization

If one is given an operator A from an Hilbert space $\mathcal{H}$ to another $\mathcal{K}$, it is possible to rise, in a natural way, this operator into an operator $\Gamma(A)$ from $\Gamma_s(\mathcal{H})$ to $\Gamma_s(K)$ (and in a similar way from $\Gamma_a(\mathcal{H})$ to $\Gamma_a(K)$) by putting

$$\Gamma(A)(u_1 \circ \cdots \circ u_n) = Au_1 \circ \cdots \circ Au_n \ .$$

One easily sees that

$$\Gamma(A)\varepsilon(u) = \varepsilon(Au) \ .$$

This operator $\Gamma(A)$ is called the *second quantization* of A.

One must be careful that even if A is bounded operator, $\Gamma(A)$ is not bounded in general. Indeed, if $\|A\| > 1$ then $\Gamma(A)$ is not bounded. But on easily sees that

$$\Gamma(AB) = \Gamma(A)\Gamma(B)$$

and

$$\Gamma(A^*) = \Gamma(A)^*.$$

In particular if A is unitary, then so is $\Gamma(A)$. Even more, if $(U_t)_{t\in I\!\!R}$ is a strongly continuous one parameter group of unitary operators then so is $(\Gamma(U_t)_{t\in I\!\!R})$.█

In other words, if $U_t = e^{itH}$ for some self-adjoint operator H, then $\Gamma(U_t) = e^{itH'}$ for some self-adjoint operator H'. The operator H' is denoted $\Lambda(H)$ (or sometimes $d\Gamma(H)$ in the litterature) and called the *differential second quantization of* H.

One easily checks that

$$\Lambda(H)\, u_1 \circ \cdots \circ u_n = \sum_{i=1}^{n} u_1 \circ \cdots \circ Hu_i \circ \cdots \circ u_n$$

and $\Lambda(H)1\!\!1 = 0$.

In particular, if $H = I$ we have

$$\Lambda(I)\, u_1 \circ \cdots \circ u_n = n\, u_1 \circ \cdots \circ u_n \ .$$

This operator is called *number operator.*

Proposition 3.7 – *We have*

$$\Lambda(H)\varepsilon(u) = a^*(Hu)\varepsilon(u) \ .$$

Proof

We have $\Lambda(H)u^{\circ n} = n(Hu) \circ u \circ \cdots \circ u$. Thus

$$\Lambda(H)\frac{u^{\circ n}}{n!} = (Hu) \circ \frac{u^{\circ(n-1)}}{(n-1)!} = a^*(Hu)\frac{u^{\circ(n-1)}}{(n-1)!} \ .$$

■

Proposition 3.8 – *For all $u \in \mathcal{H}$, we have*

$$\Lambda(|u\rangle\langle u|) = a^*_{|u\rangle} a_{\langle u|} \ .$$

Proof

Indeed, we have

$$\begin{aligned}\Lambda(|u\rangle\langle u|)\varepsilon(v) &= a^*(\langle u, v\rangle u)\varepsilon(v) \\ &= \langle u, v\rangle a^*(u)\varepsilon(v) \\ &= a^*_{|u\rangle} a_{\langle u|}\varepsilon(v) \ .\end{aligned}$$

■

Coming back to $L^2(\mathbb{R})$, there is only one differential second quantization:

$$\Lambda(I) = \Lambda = a^*a \ .$$

We obtain

$$\begin{aligned}\Lambda &= \Big(\frac{x}{2} - \frac{d}{dx}\Big)\Big(\frac{x}{2} + \frac{d}{dx}\Big) \\ &= \frac{x^2}{4} - \frac{d^2}{dx^2} - \frac{1}{2}\end{aligned}$$

that is

$$\Lambda + \frac{1}{2} = \frac{x^2}{4} - \frac{d^2}{dx^2}$$

the Hamiltonian of the one dimensional harmonic oscillator.

Note that Λ is self-adjoint and its law in the vacuum state is just the Dirac mass in 0, for $\Lambda \mathbb{1} = 0$.

3.4 Weyl operators

Let $\mathcal{H}$ be a Hilbert space. Let G be the group of displacements of $\mathcal{H}$ that is,

$$G = \big\{(U, u) \ ; \quad U \in \mathcal{U}(\mathcal{H}), \quad u \in \mathcal{H}\big\} \ ,$$

where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$. This group acts on $\mathcal{H}$ by

$$(U, u)h = Uh + u \ .$$

The composition law of G is thus

$$(U,u)(V,v)=(UV,Uv+u)$$

and in particular

$$(U,u)^{-1}=(U^*,-U^*u)\ .$$

For every $\alpha=(U,u)\in G$ one defines the *Weyl operator* W_α on $\Gamma_s(\mathcal{H})$ by

$$W_\alpha\,\varepsilon(v)=e^{-\|u\|^2/2-\langle u,Uv\rangle}\varepsilon(Uv+u)\ .$$

In particular

$$W_\alpha W_\beta=e^{-i\Im\langle u,Uv\rangle}W_{\alpha\beta}$$

for all $\alpha=(U,u)$, $\beta=(V,v)$ in G. These are called the *Weyl commutation relations.*

Proposition 3.8 – *The Weyl operators W_α are unitary.*

Proof

We have

$$\begin{aligned}\langle W_\alpha\varepsilon(k),W_\alpha\varepsilon(\ell)\rangle&=e^{-\|u\|^2-\langle Uk,u\rangle-\langle u,U\ell\rangle}\langle\varepsilon(Uk+u),\varepsilon(U\ell+u)\rangle\\&=e^{-\|u\|^2-\langle Uk,u\rangle-\langle u,U\ell\rangle}e^{\langle Uk+u,U\ell+u\rangle}\\&=e^{\langle Uk,U\ell\rangle}=e^{\langle k,\ell\rangle}=\langle\varepsilon(k),\varepsilon(\ell)\rangle\ .\end{aligned}$$

Thus W_α extends to an isometry. But we furthermore have

$$\begin{aligned}W_\alpha W_{\alpha^{-1}}&=e^{-i\Im\langle u,-UU^*u\rangle}W_{\alpha\alpha^{-1}}\\&=e^{-i\Im(-\|u\|^2)}W_{(I,0)}\\&=I\ .\end{aligned}$$

Thus W_α is invertible. ■

The mapping :

$$\begin{aligned}G&\longrightarrow\mathcal{U}(\Gamma(\mathcal{H}))\\\alpha&\longmapsto W_\alpha\end{aligned}$$

is a unitary projective representation of G.

If one considers the group $\widetilde{G}$ of (U,u,t), $U\in\mathcal{U}(\mathcal{H})$, $u\in\mathcal{H}$ and $t\in\mathbb{R}$ with

$$(U,u,t)(V,v,s)=(UV,Uv+u,t+s+\Im\langle u,Uv\rangle)\ ;$$

we obtain the so-called *Heisenberg group* of $\mathcal{H}$. The mapping $(U,u,t)\longmapsto W_{(U,u)}e^{it}$ is thus a unitary representation of $\widetilde{G}$.

Conversely, if $W_{(U,u,t)}$ is a unitary representation of the Heisenberg group of $\mathcal{H}$ we then have

$$W_{(U,u,t)}=W_{(U,u,0)}W_{(I,0,t)}$$

and

$$W_{(I,0,t)}=W_{(I,0,s)}W_{(I,0,t+s)}\ .$$

This means that

$$W_{(U,u,t)} = W_{(U,u,0)} e^{itH}$$

for some self-adjoint operator H, and the $W_{(U,u,0)}$ satisfy the Weyl commutation relations.

If we come back to our Weyl operators $W_{(U,u)}$ one easily sees that

$$W_{(U,u)} = W_{(I,u)} W_{(U,0)} \ .$$

By definition $W_{(U,0)}\varepsilon(k) = \varepsilon(Uk)$ and thus

$$W_{(U,0)} = \Gamma(U) \ .$$

Finally, write W_u for $W_{(I,u)}$. Then

$$W_u W_v = e^{-i\Im\langle u,v\rangle} W_{u+v} \ .$$

These relations are often also called Weyl commutation relations. As a consequence $(W_{(I+tu)})_{t\in\mathbb{R}}$ is a unitary group; it is strongly continuous (*exercise*).

Proposition 3.9 – *We have*

$$W_{(I,tu)} = e^{it\frac{1}{i}(a(u)-a^*(u))} \ .$$

Proof

$$\begin{aligned}
\frac{1}{i}\frac{d}{dt}\Big|_{t=0} W_{(I,tu)}\varepsilon(k) &= \frac{1}{i}\frac{d}{dt}\Big|_{t=0} e^{-\frac{t^2}{2}\|u\|^2 - t\langle u,k\rangle}\varepsilon(k+tu) \\
&= -\frac{1}{i}\langle u,k\rangle\varepsilon(k) + \frac{1}{i}\frac{d}{dt}\Big|_{t=0}\varepsilon(k+tu) \\
&= \frac{1}{i}(-a(u) + a^*(u))\varepsilon(k) \ .
\end{aligned}$$

■

Coming back to our example on $L^2(\mathbb{R})$, the Weyl operators are defined by

$$W_z\varepsilon(z') = e^{-\frac{|z|^2}{2} - \overline{z}z'}\varepsilon(z+z') \ .$$

These operators are very helpfull for computing the law of some observables.

Proposition 3.10 – *The observable* $1/i\,(za^* - \overline{z}a)$ *follows a law* $\mathcal{N}(0,|z|^2)$ *in the vaccum state.*

Proof

We have

$$\begin{aligned}
< 1\!\!1\,,\, e^{t(za^*-\overline{z}a)} 1\!\!1 > &= < 1\!\!1\,,\, W_{itz} 1\!\!1 > \\
&= < \mathcal{E}(0)\,,\, W_{itz}\mathcal{E}(0) > \\
&= < \mathcal{E}(0)\,,\, \mathcal{E}(itz) > e^{-t^2|z|^2/2} \\
&= e^{-t^2|z|^2/2}.
\end{aligned}$$

■

Proposition 3.11 – *The observable $\Lambda+\alpha I$ follows the law δ_α in the vaccum state.*

Proof

Indeed,

$$<\mathbb{1}\,,\,(\Lambda+\alpha I)^n\mathbb{1}> \,=\alpha^n<\mathbb{1}\,,\,\mathbb{1}> \,=\alpha^n.$$

■

Let us now compute the law, in the vaccum state, of the observable $\Lambda+za^*-\overline{z}a+|z|^2\,I$, sum of the two previous observables.

Lemma 3.12 –

$$W_{-z}e^{it\Lambda}W_z=e^{it(\Lambda+za^*-\overline{z}a+|z|^2I)}.$$

Proof

It suffices to show that $W_{-z}\Lambda W_z=\Lambda+za^*-\overline{z}a+|z|^2\,I$. We have

$$\begin{aligned}<\mathcal{E}(z_1)\,,\,W_{-z}\Lambda W_z\,\mathcal{E}(z_2)> &= \,<aW_z\mathcal{E}(z_1)\,,\,aW_z\mathcal{E}(z_2)>\\ &= \,<(z_1+z)\mathcal{E}(z_1+z)\,,\,(z_2+z)\mathcal{E}(z_2+z)>\times\\ &\qquad\times\, e^{-|z|^2-\overline{z}z_2-\overline{z_1}z}\\ &= (\overline{z_1}z_2+\overline{z_1}z+z_2\overline{z}+|z|^2)e^{\overline{z_1}z_2}.\end{aligned}$$

■

Proposition 3.13 – *The law of the observable $\Lambda+za^*-\overline{z}a+|z|^2\,I$ in the vaccum state is the Poisson law $\mathcal{P}(|z|^2)$.*

Proof

We have

$$\begin{aligned}<\mathbb{1}\,,\,e^{it(\Lambda+za^*-\overline{z}a+|z|^2I)}\mathbb{1}> &= \,<W_z\mathbb{1}\,,\,e^{it\Lambda}W_z\mathbb{1}>\\ &= e^{-|z|^2}<\mathcal{E}(z)\,,\,e^{it\Lambda}\mathcal{E}(z)>\\ &= e^{-|z|^2}e^{|z|^2e^{it}}\\ &= e^{|z|^2(e^{it}-1)}.\end{aligned}$$

■

Comments: This result may seem very surprising: the sum of a Gaussian variable and a deterministic one, gives a Poisson distribution! Of course such a phenomena cannot be realised with usual random variables. What does this mean?

Actually it is one of the manifestation of the fact that **the random phenomena attached to quantum mechanics cannot be modelized by usual probability theory**.

There has been many attemps to give a probabilistic model to the stochastic phenomena of quantum mechanics, such as the "hidden variable theory" which tried to give a model of the randomness in measurement by saying that many parameters of the systems are unknown to us (hidden variables) since the begining and that their effects appear at the measurement and give this uncertainty, this randomness.

But it has been proved by Bell, that if one tries to attach random variables behind each observable and try to find (complicated) rules that explain the principles of quantum mechanics, then this reaches an impossibility. Indeed, taking the spin of a particle in three well-choosen directions, one obtains three Bernoulli variables, but their correlations cannot be obtained by any triplet of classical Bernoulli variables.

What then can we do to express the probabilistic effects of measurement in a probabilistic language? Actually, one does need to look very far away. Quantum mechanics in itself, in its axioms, contains the germ of a new probability theory. Indeed, now accept to consider a probability space to be a couple $(\mathcal{H}, \Psi)$, where $\mathcal{H}$ is a Hilbert space and Ψ is a normalized vector of $\mathcal{H}$; instead of the usual $(\Omega, \mathcal{F}, P)$. Accept to consider a random variable to be a self-adjoint operator A on $\mathcal{H}$, instead of a measurable function $X : \Omega \to \mathbb{R}$. Accept that the probability distribution of A under the state Ψ is the one described above:

$$E \longmapsto < \Psi \,,\, \mathbb{1}_E(A)\Psi >.$$

Then what do we obtain ?

Actually this probability theory, as stated here, is equivalent to the usual one when considering a single random variable. Indeed, a classical probabilistic situation $(\Omega, \mathcal{F}, P, X)$ is easily seen to be also a quantum one $(\mathcal{H}, \Psi, A)$ by putting $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$, $\Psi = \mathbb{1}$ and $A = \mathcal{M}_X$ the operator of multiplication by X. The (quantum) distribution of A is then the same as the (classical) distribution of X.

Conversely, given a quantum triplet $(\mathcal{H}, \Psi, A)$, then by the spectral theorem the operator A can be represented as a multiplication operator on some measured space (by diagonalization).

Where does the difference lie? When considering two non commuting observables on $\mathcal{H}$ (for example P, Q on $L^2(\mathbb{R})$), then each of them is a classical random variable, but on its own probability space (they cannot be diagonalized simultaneously). We have put together two classical random variables which have nothing to do together, in a same context. We have not stick the together by declaring them independant, there is a dependency ($[Q, P] = iI$) which has some consequences (uncertainty principle for example) which cannot be expressed in classical terms.

This is to say that what quantum mechanics teaches us is that the observables, under measurement, behave as true random variables, but each one with its own random, and that their interdependency cannot be expressed in a simpler way than the axioms of quantum mechanics.

It is then not a surprise that adding two observables which do not commute we obtain a distribution which has nothing to do with the convolution of their respective distributions. This is the example above. One also obtains a surprising one with $P^2 + Q^2$ (exercise).

3.5 The CCR algebra

We now denote by $W(f)$ the Weyl operator $W_{(I,f)}$, $f \in \mathcal{H}$.

Theorem 3.14 – *Let $\mathcal{K}$ be any (algebraic) subspace of a Hilbert space $\mathcal{H}$. There exists a C^*-algebra, denoted $CCR(\mathcal{K})$ of operators on $\Gamma(\mathcal{H})$, unique up to isomorphism, generated by nonzero elements $W(f)$, $f \in \mathcal{K}$, such that*

$$W(f)^* = W(-f) \text{ for all } f \in \mathcal{K}$$
$$W(f)W(g) = W(f+g)e^{-i\Im < f\,,g >} \text{ for all } f, g \in \mathcal{K}.$$

Proof

The existence of a C^*-algebra satisfying the two conditions is obvious. It suffices to consider the C^*-algebra generated by the Weyl operators $W(f), f \in \mathcal{K}$ of $\Gamma(\mathcal{H})$.

We now give the proof of uniqueness but it can omited by the reader as it makes use of tools that are not pertinent for us.

Put $b(f,g) = \exp(-i\Im < f\,,\,g >/2)$ for all $f, g \in \mathcal{K}$. For every $F \in \ell^2(\mathcal{K})$ put

$$(R_b(g)F)(f) = b(f,g)F(f+g)$$
$$(R(g)F)(f) = F(f+g).$$

then R is a unitary representation of the additive abelian group $\mathcal{K}$ in $\ell^2(\mathcal{K})$ and R_b is also a unitary representation, but up to a multiplier b.

Assume we have $\mathcal{U}_1$ and $\mathcal{U}_2$, two CCR algebras on $\mathcal{K}$, with associated Weyl elements W_i, $i = 1, 2$. Assume they are faithfully represented in $\mathcal{H}_1$ and $\mathcal{H}_2$. On the space $\ell^2(\mathcal{K}; \mathcal{H}_i) = \ell^2(\mathcal{K}) \otimes \mathcal{H}_i$ we put

$$((W_i \times R)(g)\Psi)(f) = W_i(g)\Psi(f+g).$$

Finally, define U_i, unitary operator on $\ell^2(\mathcal{K}; \mathcal{H}_i)$ by

$$(U_i\Psi)(f) = W_i(f)\Psi(f).$$

Then a simple computation proves that

$$U_i(W_i \times R)(g)U_i^* = I_i \otimes R_b(g).$$

If $\mathcal{B}_i$ denotes the C^*-algebra generated by $\{(W_i \times R)(g); g \in \mathcal{K}\}$, then there exists a $*$-isomorphism τ from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that

$$\tau((W_1 \times R)(g)) = (W_2 \times R)(g).$$

Thus it would be sufficient now to find $*$-isomorphisms τ_i from $\mathcal{U}_i$ to $\mathcal{B}_i$ such that

$$\tau_i(W_i(g)) = (W_i \times R)(g).$$

Now set $W = W_i$. It suffices to show that

$$\left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| = \left|\left|\sum_{i=1}^{n} \lambda_i W(f_i)\right|\right|$$

for all $\lambda_i \in \mathbb{C}$, $f_i \in \mathcal{K}$.

The representation $W \times R$ is, via Fourier transform on $\ell^2(\mathcal{K})$, unitary equivalent to the representation $W \times \widehat{R}$ on $\ell^2(\widehat{\mathcal{K}}, \mathcal{H})$ defined by

$$((W \times \widehat{R})(g)\Psi)(\chi) = W(g)\chi(g)\Psi(\chi), \chi \in \widehat{\mathcal{K}}$$

and hence

$$\left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| = \sup_{\chi \in \widehat{\mathcal{K}}} \left|\left|\sum_{i=1}^{n} \lambda_i \chi(f_i) W(f_i)\right|\right| .$$

The set of characters $\{\chi_g\}$ on $\mathcal{K}$ of the form $\chi_g(f) = b(f,g)^2$ is dense in $\widehat{\mathcal{K}}$ for it is a subgroup with annihilator zero.

Note that $\chi_g(f)W(f) = W(g)W(f)W(g)^*$, thus

$$\begin{aligned} \left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| &= \sup_{g \in \mathcal{K}} \left|\left|\sum_{i=1}^{n} \lambda_i \chi_g(f_i) W(f_i)\right|\right| \\ &= \sup_{g \in \mathcal{K}} \left|\left| W(g) \sum_{i=1}^{n} \lambda_i W(f_i) W(g)^* \right|\right| \\ &= \left|\left|\sum_{i=1}^{n} \lambda_i W(f_i)\right|\right| . \end{aligned}$$

■

Proposition 3.15 – *Let $\mathcal{K} \subset \mathcal{H}$ be a subspace of $\mathcal{H}$. It follows that $CCR(\mathcal{K}) = CCR(\mathcal{H})$ if and only if $\mathcal{K} = \mathcal{H}$.*

Proof

If $\mathcal{K} \neq \mathcal{H}$, then consider the representation of $CCR(\mathcal{H})$ on $\ell^2(\mathcal{H})$ defined by

$$(W(g)F)(f) = b(f,g)F(f+g)$$

with the same notation as in the proof of uniqueness in the above theorem. If $g \in \mathcal{H} \setminus \mathcal{K}$ then

$$\left(\left(W(g) - \sum_{i=1}^{n} \lambda_i W(g_i)\right) F\right)(f) = b(f,g)\left(F(f+g) + \\ - \sum_{i=1}^{n} \lambda_i b(f, g_i - g) F(f + g_i)\right) .$$

If F is supported by $\mathcal{K}$ then

$$\left|\left|\left(W(g) - \sum_{i=1}^{n} \lambda_i W(g_i)\right) F\right|\right| \geq ||F||$$

for the vector $f \mapsto F(f+g)$ is orthogonal to each of the vectors $f \mapsto b(f, g_i - g)F(f + g - i)$.

Therefore

$$\inf_{A \in CCR(\mathcal{K})} ||W(g) - A|| \geq 1$$

and hence $W(g) \notin CCR(\mathcal{K})$. ■

4. THE FREE BOSON GAS

The aim of this chapter is to use the Fock space structure in order to give a mathematical description of a free boson gas. In particular we shall construct the state describing such a gas a temperature equilibrium.

4.1 Quantification of the wave equation

Consider the usual wave equation

$$\phi'' = \Delta\phi$$

where $\phi(x,t)$ is a function on $\mathbb{R}^3 \times \mathbb{R}$. This evolution equation can be described in an Hamiltonian structure

$$\begin{cases} \phi' = \pi \\ \pi' = \Delta\phi \end{cases}$$

which are the Hamilton equation associated to the Hamiltonian

$$H(\phi,\pi) = \frac{1}{2}\int_{\mathbb{R}^3} \left\{ |\nabla\phi|^2 + |\pi|^2 \right\} d^3x.$$

The Poisson bracket which describes the dynamic of any observable F via

$$F' = \{F, H\}$$

is the following

$$\{F,G\} = \int \left[\frac{\partial F}{\partial\phi(x)} \frac{\partial G}{\partial\pi(x)} - \frac{\partial F}{\partial\pi(x)} \frac{\partial G}{\partial\phi(x)} \right] dx.$$

In particular

$$\{\phi(x), \pi(y)\} = \delta(x-y).$$

Considering the Fourier transform

$$\widehat{f}(k) = \int \frac{1}{(2\pi)^{3/2}} e^{-ik.x} f(x)\, d^3x$$

and putting

$$a(k) = \frac{1}{\sqrt{2\,|k|}} \left(|k|\, \widehat{\phi}(k) + i\widehat{\pi}(k) \right)$$

we find

$$H = \int \omega(k)\, |a(k)|^2\, d^3x, \qquad \text{with } \omega(k) = |k|$$

and the relations

$$i\{a(k), \overline{a}(k')\} = \delta(k-k')$$

$$\{a(k), H\} = -i\omega(k)a(k).$$

The field operators are then recovered by

$$\phi(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \left(e^{i(k.x-\omega(k)t)} a(k) + e^{-i(k.x-\omega(k)t)} \overline{a}(k) \right) d^3x$$

$$\pi(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{\sqrt{\omega(k)}}{2} \frac{1}{i} \left(e^{i(k.x-\omega(k)t)} a(k) + e^{-i(k.x-\omega(k)t)} \overline{a}(k) \right) d^3x.$$

We have a complete description of the evolution of the dynamic.

In order to describe the quantum system equivalent to the above system, we *quantify*. In practice this means that functions are replaced by operators and the Poisson bracket $\{\cdot,\cdot\}$ is replaced by the commutator $-i[\cdot,\cdot]$.

Thus we should have a field of operators satisfying

$$[a(k), a^*(k')] = \delta(k-k'),$$

an Hamiltonian given by

$$H = \int \omega(k) a^*(k) a(k)\, d^3k$$

so that

$$e^{itH} a(k) e^{-itH} = e^{-i\omega(k)t} a(k).$$

At time $t=0$ we have the canonical field operators:

$$\phi(x) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \left(e^{ik.x} a(k) + e^{-ik.x} \overline{a}(k)\right) d^3x$$

$$\pi(x) = \int \frac{1}{(2\pi)^{3/2}} \frac{\sqrt{\omega(k)}}{2} \frac{1}{i} \left(e^{ik.x} a(k) + e^{-ik.x} \overline{a}(k)\right) d^3x$$

so that

$$[\phi(x), \pi(y)] = i\delta(x-y).$$

It is also useful to consider the *Segal field* operators:

$$\Phi(f) = \int \frac{1}{\sqrt{2}} \left(a^*(k) f(k) + a(k) \overline{f(k)}\right) d^3k, \qquad f \in L^2(\mathbb{R}^3; \mathbb{C}).$$

They are self-adjoint operators which satisfy

$$[\Phi(f), \Phi(g)] = i\Im < f\,, g >$$

and

$$e^{itH} \Phi(f) e^{-itH} = \Phi(e^{i\omega(k)t} f).$$

Tha reader has of course understood that the operators $a(k), a^*(k), \Phi(f), ...etc$ can be realized in the boson Fock space $\Gamma_s(L^2(\mathbb{R}^3; \mathbb{C}))$. That is, more precisely

$$\Phi(f) = \frac{1}{\sqrt{2}} (a^*(f) + a(f))$$

and

$$W(f) = e^{i\Phi(f)}.$$

In particular, the above Hamiltonian H is actually given by

$$H = \Lambda(h)$$

where h is the multiplication operator by $\omega(k) = |k|$ on $L^2(\mathbb{R}^3; \mathbb{C})$.

Actually, as the field we have quantified is the *massless Klein-Gordon field*, that is we have quantified the wave equation for photons, our Hamiltonian describes a quantum gas of **photons**.

If one wants to describe a quantum gas of massive bosons one has to quantify the*massive Klein-Gordon equation*

$$\phi'' = \Delta\phi + m_0^2\phi$$

where m_0 is the mass of the particle. In the Schrödinger approximation (i.e. non-relativistic approximation)of this equation, the Hamiltonian is then

$$H = \Lambda(h)$$

where h is the multiplication operator by $\omega(k) = k^2$.

For what we are going to study in the sequel, the difference between the two Hamiltonian will not be of much importance.

4.2 Thermal equilibrium states

Let us give a hint to justify the equilibrium state we are going to study in this course. We shall explain the so-called *Gibbs-Boltzmann distribution* in statistical mechanics. The following discussion is very incomplete and shall only motivate the reader for reading text books on the subject.

Consider a classical gas made of a very large number of particles. This gas may be in any state (position and velocity of each particle) which leads to some possible energies for the whole system. Put $\Omega(E)$ to be the number of different states which give the same energy E. Put

$$S = k \ln \Omega(E)$$

the so-called *entropy of the system* (k is Boltzmann constant). Finally put

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

Let us describe what happens when two systems are juxtaposed and communicate. The number of configuration $\Omega(E, E_1)$ for the total system, with total energy E and energy E_1 in the subsystem 1, is given by

$$\Omega(E, E_1) = \Omega_1(E_1)\Omega_2(E - E_1)$$

with obvious notations. Thus the total entropy of this state $S(E, E_1)$ satisfies

$$S(E, E_1) = S_1(E_1) + S_2(E_2)$$

where $E_2 = E - E_1$.

The most probable energy for the system 1 is the one, denoted E_1, which maximizes $\Omega(E, E_1)$. That is which maximizes the total entropy $S(E, E_1)$. This gives

$$\frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2}.$$

This in particular implies $T_1 = T_2$. So for two systems in contact and in equilibrium, the above quantity T is constant. Actually, this constant T_i is the temperature of the system i (this can be taken as a definition of the temperature). The above identity only says that two independant systems, when they are put in contact, have same temperature (after equilibrium).

Now in this setup we want to define the state of a system which is put at equilibrium at temperature T. In order to obtain this, one has to put our system 1 in contact with a *thermostat*, that is an infinite source, which is itself at temperature T. The probability to find the system 1 at energy E_1 and the total system at energy E is

$$p = \frac{\Omega_2(E - E_1)}{\Omega(E)}.$$

As $S_2 = k \ln \Omega_2(E_2)$ we have $\Omega_2(E_2) = \exp(S_2/k)$ and the above probability is $p = C \exp(S_2(E - E_1)/k)$. But

$$S_2(E - E_1) \simeq S_2(E) - E_1 \frac{\partial S_2}{\partial E}$$

for the total energy E is far arger than E_1. But as the system 2 is at temperature T we have

$$\frac{\partial S_2}{\partial E} = T$$

and finally

$$p = Ce^{-E_1/kT}.$$

Thus the probability distribution of the energy of a system, when put at equilibrium at temperature T is

$$Ce^{-\beta H}$$

where $\beta = 1/(kT)$ and H is the Hamiltonian of the system (the energy). This is the so-called *Gibbs-Boltzmann* equilibrium state.

In the quantum situation, if a system is decribed by a Hamiltonian (operator) H, then the state representing the system at equilibrium at temperature T is the state

$$A \longmapsto \mathrm{Tr}(e^{-\beta H} A)$$

on the C^*-algebra of observables.

Consider on the Fock space $\Phi = \Gamma_s(L^2(\mathbb{R}^3))$ an Hamiltonian of the form $K = \Lambda(h)$ for some self-adjoint operator h on $L^2(\mathbb{R}^3)$. In order to be able to define the equilibrium state at temperature $T = 1/\beta$ we need to have the operator $\exp(-\beta K)$ to be trace-class.

Proposition 4.1 – *Let h be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $K = \Lambda(h)$ be defined on $\Gamma_s(\mathcal{H})$. Then the following conditions are equivalent.*

i) $\exp(-\beta K)$ *is trace-class on* $\Phi_s(\mathcal{H})$.

ii) $\exp(-\beta h)$ *is trace-class on* h *and* $\beta h > 0$.

Proof

ii) implies i): Let (ε_n) denote the eigenvalues of H in increasing (*resp.* decreasing) order if $\beta > 0$ (*resp.* $\beta < 0$). We have

$$\mathrm{Tr}_{\mathcal{H}^{\circ m}}(e^{-\beta K}) \leq \sum_{n_1,\ldots,n_m \geq 0} \exp\left(-\beta \sum_{p=1}^{m} \varepsilon_{n_p}\right).$$

Hence, using $\beta\varepsilon_m > 0$ we get

$$\begin{aligned}
0 \leq \mathrm{Tr}(e^{-\beta K}) &= \sum_m \mathrm{Tr}_{\mathcal{H}^{\circ m}}(e^{-\beta K}) \\
&\leq \prod_m (1 - e^{-\beta\varepsilon_m})^{-1} \\
&= \prod_m (1 + e^{-\beta\varepsilon_m}(1 - e^{-\beta\varepsilon_m})^{-1}) \\
&\leq \exp\left(\sum_m e^{-\beta\varepsilon_m}(1 - e^{-\beta\varepsilon_m})\right) \\
&\leq \exp\left((1 - e^{-\beta\varepsilon_0})^{-1}\mathrm{Tr}(e^{-\beta h})\right).
\end{aligned}$$

This proves the result in one direction.

i) implies ii): The restriction of K to the one-particle space is h and hence $\exp -\beta h$ must be trace-class. But from the above inequality it follows $\beta\varepsilon_m > 0$ for all m, i.e. $\beta h > 0$. ■

In the case of an Hamiltonian such that the above trace-class condition is satisfied one can compute the associated state explicitely.

Theorem 4.2 – *Let h be a self-adjoint operator on $\mathcal{H}$. Let $\beta \in \mathbb{R}$. Assume $\exp(-\beta h)$ is trace-class and $\beta h > 0$. Let*

$$\omega(A) = \frac{\mathrm{Tr}(e^{-\beta K}A)}{\mathrm{Tr}(e^{-\beta K})}$$

be the Gibbs grand canonical equilibrium state over $CCR(\mathcal{H})$, where $K = \Lambda(h)$. Then ω is given by

$$\omega(a^*(f)a(g)) = \langle g\,,\, e^{-\beta h}(I - e^{-\beta h})^{-1} f\rangle$$

or else

$$\omega(W(f)) = \exp\left(-\frac{1}{4}\langle f\,,\,(I + e^{-\beta h})(I - e^{-\beta h})^{-1} f\rangle\right).$$

Proof

First note that if $\Psi \in \mathcal{H}^{\circ m}$ then

$$\|a(f_1)\ldots a(f_n)\Psi\| \leq m^{n/2}\,\|\Psi\|\,\|f_1\|\ldots\|f_n\|\,.$$

The space $\mathcal{H}^{\circ m}$ is stable under $\exp(-\beta K)$ and thus

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}^{\circ m}}\left(e^{-\beta K/2}a^*(f_n)\ldots a^*(f_1)a(f_1)\ldots a(f_n)e^{-\beta K/2}\right) \\
&\qquad\qquad \leq m^n \mathrm{Tr}_{\mathcal{H}^{\circ m}}(e^{-\beta K/2})\|f_1\|^2\ldots\|f_n\|^2. \qquad (4.1)
\end{aligned}$$

A simple adaptation of the estimates of the above Proposition 4.1 shows that the operators

$$A_f = a(f_1)\ldots a(f_n)e^{-\beta K/2}$$

have a bounded closure $\overline{A_f}$ and both $\overline{A_f^* A_f}$ and $\overline{A_f A_f^*}$ are trace-class. Thus one can extend the state ω to any polynomial in the $a(f)$ and $a^*(f)$. Moreover this extension is continuous in the sense that

$$|\omega(a^*(f_1)\dots a^*(f_n)a(g_1)\dots a(g_n))| \leq C \prod_i ||f_i||^2 \prod_i ||g_i||^2.$$

Now note that

$$e^{-\beta K/2} a^*(f) = a^*(e^{-\beta h/2} f) e^{-\beta K/2}.$$

Thus

$$\begin{aligned} \omega(a^*(f)a(g)) &= \frac{\operatorname{Tr}(a^*(e^{-\beta h/2}f)e^{-\beta K} a(e^{-\beta h/2}g))}{\operatorname{Tr}(e^{-\beta K})} \\ &= \omega(a(e^{-\beta h/2}f)a^*(e^{-\beta h/2}g)) \\ &= \omega(a^*(e^{-\beta h/2}f)a(e^{-\beta h/2}g)) + <g\,,\, e^{-\beta h} f>. \end{aligned}$$

Iterating this identity gives

$$\omega(a^*(f)a(g)) = \omega(a^*(e^{-\beta h/2}f)a(e^{-\beta h/2}g)) + \sum_{m=1}^{n} <g\,,\, e^{-m\beta h} f>.$$

By hypothesis $\beta h > 0$ and thus

$$\lim_{n \to +\infty} \left|\left| e^{-n\beta h/2} f \right|\right| = 0.$$

But the mappings $f, g \mapsto \omega(a^*(f)a(g))$ are continuous by the previous observation. Therefore, in the limit $n \to +\infty$ we get the result for $\omega(a^*(f)a(g))$.

The same method applied to polynomials of a^* and a show that ω is determined by the above two-point function (*quasi-free state* in the literature) and we get the announced result for $\omega(W(f))$ this way (after some computations!). ■

The problem is that in our case, with $h = \omega(k) = |k|$ (or even $\omega(k) = k^2$) we do not have h to be trace-class, $e^{-\beta h}$ neither. So we cannot construct the above Gibbs state this way. The method which is then used in physics (and in mathematics) is to restrict the Fock space to a compact box, to restrict the Hamiltonian with boundary conditions which ensure that we have discrete spectrum, tracability...etc and to pass to the limit on the size of the box (*thermodynamical limit*).

Indeed, let $\Lambda = [-L, L]^3 \subset \mathbb{R}^3$. Put $\mathcal{H}_\Lambda = L^2(\Lambda)$. Put h_Λ to be the square root of the Laplacian restricted to Λ with Dirichlet boundary conditions. Put $\mathcal{C}_\Lambda = CCR(\mathcal{H}_\Lambda)$ and $\mathcal{C} = \overline{\bigvee_\Lambda \mathcal{C}_\Lambda}$. The state

$$\omega_\Lambda^\beta(W(f)) = \exp\left(-\frac{1}{4} < f\,,\, \coth(\beta h_\Lambda/2) f >\right)$$

is the Gibbs state (associated to the temperature $T = 1/k\beta$) on $\mathcal{C}_\Lambda$. One can show (admitted here) that these states converge, when L tends to $+\infty$, to a state ω_β on $\mathcal{C}$ given by

$$\omega_\beta(W(f)) = \exp(-\frac{1}{4} < f\,,\, \coth(\beta\omega/2) f >)$$

for any f in some $L^2(\Lambda)$.

The above state is the one we will consider as being the equilibrium state of the photon gas at temperature $T = 1/k\beta$. This state can be justified as above: it is the thermodynamical limit of the usual Gibbs states on compact restrictions of our system. But we will also see later on that it can be obtained as the only β-K.M.S. state on this system, where the notion of K.M.S. state, as we will see, is the natural extension of the notion of Gibbs state.

This prevents us from giving a rigorous proof of the above facts.

4.3 The cyclic representation of the Gibbs state

It will be of great use for us later to consider the G.N.S. representation (or cyclic representation) of $(\mathcal{C}, \omega_\beta)$.

Consider a Hilbert space $\mathcal{H}$. The conjugated space $\overline{\mathcal{H}}$ is a Hilbert space together with an anti-unitary isomorphism $\Psi \mapsto \overline{\Psi}$ from $\mathcal{H}$ to $\overline{\mathcal{H}}$.

For any operator $A \in \mathcal{B}(\mathcal{H})$ one defines the operator $\overline{A} \in \mathcal{B}(\overline{\mathcal{H}})$ by

$$\overline{A}\overline{f} = \overline{Af}.$$

Now consider a positive operator ρ on $\mathcal{H}$ and let $Q(\rho)$ denote the domain of $\rho^{1/2}$. On the space $\Gamma_s(\mathcal{H} \oplus \overline{\mathcal{H}})$ one puts, for every $f \in Q(\rho)$

$$\begin{aligned} W_{\rho,\ell}(f) &= W\left((1+\rho)^{1/2} f \oplus \overline{\rho}^{1/2}\overline{f}\right) \\ W_{\rho,r}(\overline{f}) &= W\left(\rho^{1/2} f \oplus (1+\overline{\rho})^{1/2}\overline{f}\right). \end{aligned}$$

A simple computation shows that

$$\begin{aligned} W_{\rho,\ell}(f)W_{\rho,\ell}(g) &= W_{\rho,\ell}(f+g)e^{-i\Im\left\{< f\,,\,(1+\rho)g > + < \overline{f}\,,\,\overline{\rho g} >\right\}/2} \\ &= W_{\rho,\ell}(f+g)e^{-i\Im< f\,,\,g >/2} \\ W_{\rho,r}(\overline{f})W_{\rho,r}(\overline{g}) &= W_{\rho,\ell}(\overline{f}+\overline{g})e^{-i\Im< \overline{f}\,,\,\overline{g}/2 >}. \end{aligned}$$

They thus give two representations of $CCR(Q(\rho))$.

The associated field operators are

$$\begin{aligned} a_{\rho,\ell}(f) &= a\left((1+\rho)^{1/2} f \oplus 0\right) + a^*\left(0 \oplus \overline{\rho}^{1/2}\overline{f}\right) \\ a^*_{\rho,\ell}(f) &= a\left(0 \oplus \overline{\rho}^{1/2}\overline{f}\right) + a^*\left((1+\rho)^{1/2} f \oplus 0\right) \\ a_{\rho,r}(\overline{f}) &= a^*\left(\rho^{1/2} f \oplus 0\right) + a\left(0 \oplus (1+\overline{\rho})^{1/2}\overline{f}\right) \\ a^*_{\rho,r}(\overline{f}) &= a^*\left(0 \oplus (1+\overline{\rho})^{1/2}\overline{f}\right) + a\left(\rho^{1/2} f \oplus 0\right). \end{aligned}$$

Consider the state ω on $CCR(Q(\rho))$ defined by

$$\begin{aligned} \omega(W(f)) &= <\mathbb{1}, W_{\rho,\ell}(f)\mathbb{1}> \\ &= e^{-\frac{1}{4}\left|\left|(1+\rho)^{1/2} f \oplus \overline{\rho}^{1/2}\overline{f}\right|\right|^2} \\ &= e^{-\frac{1}{4}\left\{< f\,,\,(1+\rho)f > + < \overline{f}\,,\,\overline{\rho f} >\right\}} \\ &= e^{-\frac{1}{4}||f||^2} e^{-\frac{1}{2}< f\,,\,\rho f >}. \end{aligned}$$

We would have obtained the same with

$$<\mathbb{1},\, W_{\rho,r}(\overline{f})\mathbb{1}>.$$

Thus by putting

$$\rho = \frac{1}{e^{\beta\omega}-1}$$

we obtain

$$\omega = \omega_\beta.$$

We have found a representation of $(\mathcal{C},\omega_\beta)$ which has all the properties of the G.N.S. representation, except that we have not yet proved the cyclicity. Let us prove it.

By the exponential property of the Fock space, we identify $\Gamma_s(\mathcal{H}\oplus\overline{\mathcal{H}})$ to $\Gamma_s(\mathcal{H})\otimes\Gamma_s(\overline{\mathcal{H}})$. In this identification we have in particular

$$\begin{aligned} a_{\rho,\ell}(f) &\simeq a\left((1+\rho)^{1/2}f\right)\otimes I + I\otimes a^*\left(\overline{\rho}^{1/2}\overline{f}\right) \\ a^*_{\rho,\ell}(f) &\simeq a^*\left((1+\rho)^{1/2}f\right)\otimes I + I\otimes a\left(\overline{\rho}^{1/2}\overline{f}\right). \end{aligned}$$

Put

$$D_{n,m} = \mathrm{Ran}\left((1+\rho)^{1/2}\right)^{\circ n}\otimes \mathrm{Ran}\left(\overline{\rho}^{1/2}\right)^{\circ m}.$$

Clearly, the algebraic sum $\oplus_{n,m}D_{n,m}$ is dense in $\Gamma_s(\mathcal{H})\otimes\Gamma_s(\overline{\mathcal{H}})$.

We prove by induction that $\oplus_{n+m\leq N}D_{n,m}$ is obtained by action of polynomials of $a_{\rho,\ell}$ and $a^*_{\rho,\ell}$ acting on $\mathbb{1}\otimes\mathbb{1}$. This is quite clear for $N=1$. Assume this is true for N. Let $n+m=N+1$ and $\Psi\in D_{n,m}$. Then either

$$\Psi = a^*\left((1+\rho)^{1/2}f\right)\otimes I\,\Phi$$

for some $\Phi\in D_{n-1,m}$ or

$$\Psi = I\otimes a^*\left(\overline{\rho}^{1/2}\overline{f}\right)\,\Phi$$

for some $\Phi\in D_{n,m-1}$. Both cases are treated similarly, let us consider the first one. We then have

$$\Psi = a^*_{\rho,\ell}(f)\Phi - I\otimes a(\overline{\rho}^{1/2})\Phi.$$

The last term belongs to $D_{n,m-1}$. We have proved our induction.

We have proved the cyclicity of $\mathbb{1}\otimes\mathbb{1}$ under the polynomials of $a_{\rho,\ell}$ and $a^*_{\rho,\ell}$. We thus have the cyclicity of the representation.

Let us resume the situation. For each positive operator ρ we have constructed

– a state ω_ρ on $\mathcal{C}_\rho = CCR(Q(\rho))$ by

$$\omega_\rho(W(f)) = e^{-\frac{1}{4}||f||^2-\frac{1}{2}<f\,,\rho f>}.$$

– two G.N.S. representations of $(\mathcal{C}_\rho,\omega_\rho)$

$$(\mathcal{H}_{\omega_\rho},\Pi_{\omega_\rho},\Omega_{\omega_\rho}) = \begin{cases} (\Gamma_s(\mathcal{H}\oplus\overline{\mathcal{H}}),\Pi_{\rho,\ell},\Omega) & \text{a representation} \\ (\Gamma_s(\mathcal{H}\oplus\overline{\mathcal{H}}),\Pi_{\rho,r},\Omega) & \text{an anti-representation} \end{cases}$$

with

$$\Pi_{\rho,\ell}(W(f)) = W_{\rho,\ell}(W(f)) \qquad \Pi_{\rho,r}(W(f)) = W_{\rho,r}(W(f)).$$

The uniqueness of the G.N.S. representation proves the existence of an anti-unitary operator

$$J \,:\, \Phi_s(\mathcal{H}\oplus\overline{\mathcal{H}}) \longrightarrow \Phi_s(\mathcal{H}\oplus\overline{\mathcal{H}})$$

such that

$$W_{\rho,r}(\overline{f}) = JW_{\rho,\ell}(f)J^*$$
$$\text{and } J\mathbb{1} = \mathbb{1}.$$

One easily finds the following explicit form:

$$J = \Gamma(c)$$

where

$$\begin{array}{rccc} c\,: & \mathcal{H}\oplus\overline{\mathcal{H}} & \longrightarrow & \mathcal{H}\oplus\overline{\mathcal{H}} \\ & f\oplus\overline{g} & \longmapsto & g\oplus\overline{f}. \end{array}$$

In particular $J = J^*$ and $J^2 = I$.

Another important property of this Araki-Woods representation is that

$$[W_{\rho,\ell}(f), W_{\rho,r}(\overline{g})] = 0$$

for all $f, g \in Q(\rho)$. This implies

$$\Pi_{\rho,\ell}(\mathcal{C}_\rho) \subset \Pi_{\rho,r}(\mathcal{C}_\rho)'$$
$$\Pi_{\rho,r}(\mathcal{C}_\rho) \subset \Pi_{\rho,\ell}(\mathcal{C}_\rho)'.$$

Put

$$\mathcal{M}_\rho = \Pi_{\rho,\ell}(\mathcal{C}_\rho)''$$

to be the *algebra of observables of* $CCR(Q(\rho))$ *in the state* ω_ρ. We have

$$\mathcal{M}_\rho' = \Pi_{\rho,\ell}(\mathcal{C}_\rho)' \supset \Pi_{\rho,r}(\mathcal{C}_\rho) = J\Pi_{\rho,\ell}(\mathcal{C}_\rho)J$$

and

$$\mathcal{M}_\rho' \supset (J\Pi_{\rho,\ell}(\mathcal{C}_\rho)J)'' = J\Pi_{\rho,\ell}(\mathcal{C}_\rho)''J = J\mathcal{M}_\rho J.$$

Actually, we will see later with the modular theory that

$$J\mathcal{M}_\rho J = \mathcal{M}_\rho'.$$

Finally, note that $\mathbb{1}$ is cyclic for $\mathcal{M}_\rho$ and $J\mathcal{M}_\rho J$. Thus it is cyclic for $\mathcal{M}_\rho'$. Thus it is separating for $\mathcal{M}_\rho$. The state ω_ρ is thus faithfull and its normal extension

$$\omega_\rho(A) = <\mathbb{1}\,,\, A\mathbb{1}>$$

on $\mathcal{M}_\rho$ is also faithfull.

5. MODULAR THEORY OF VON NEUMANN ALGEBRAS

5.1 The modular operators

The starting point here is a couple $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a von Neumann algebra acting on some Hilbert space, ω is a normal faithful state on $\mathcal{M}$. Recall that ω is then of the form

$$\omega(A) = \mathrm{Tr}(\rho A)$$

for a strictly positive ρ, with $\mathrm{Tr}\rho = 1$.

Let us consider the G.N.S. representation of $(\mathcal{M}, \omega)$. That is, a triple $(\mathcal{H}, \Pi, \Omega)$ such that

i) Π is a morphism from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$.

ii) $\omega(A) = <\Omega\,,\,\Pi(A)\Omega>$

iii) $\Pi(\mathcal{M})\Omega$ is dense in $\mathcal{H}$.

From now on, we omit to mention the representation Π and identify $\mathcal{M}$ and $\mathcal{M}'$ with $\Pi(\mathcal{M})$ and $\Pi(\mathcal{M}')$. We thus write $\omega(A) = <\Omega\,,\,A\Omega>$.

Proposition 5.1 – *The vector Ω is cyclic and separating for $\mathcal{M}$ and $\mathcal{M}'$.*

Proof

Ω is cyclic for $\mathcal{M}$ by iii) above. Let us see that it is separating for $\mathcal{M}$. If $A \in \mathcal{M}$ is such that $A\Omega = 0$ then $\omega(A) = 0$, but as ω is faithful this implies $A = 0$.

Let us now see that these properties of Ω on $\mathcal{M}$ imply the same ones on $\mathcal{M}'$. If A' belongs to $\mathcal{M}'$ and $A'\Omega = 0$ then $A'B\Omega = BA'\Omega = 0$ for all $B \in \mathcal{M}$. Thus A' vanishes on a dense subspace of $\mathcal{H}$, it is thus the null operator. This proves that Ω is separating for $\mathcal{M}'$.

Finally, let P' be the orthogonal projector onto the space $\mathcal{M}'\Omega$. As it is the projection onto a $\mathcal{M}'$-invariant space, it belongs to $(\mathcal{M}')' = \mathcal{M}$. But $P\Omega = \Omega$ and thus $(I - P)\Omega = 0$. As Ω is separating for $\mathcal{M}$ this implies $I - P = 0$ and Ω is cyclic for $\mathcal{M}'$. ■

As a consequence the (**anti-linear**) operators

$$\begin{array}{cccc} S_0 : & \mathcal{M}\Omega & \longrightarrow & \mathcal{M}\Omega \\ & A\Omega & \longmapsto & A^*\Omega \end{array}$$

$$\begin{array}{cccc} F_0 : & \mathcal{M}'\Omega & \longrightarrow & \mathcal{M}'\Omega \\ & B\Omega & \longmapsto & B^*\Omega \end{array}$$

are well-defined (by the separability of Ω) on dense domains.

Proposition 5.2 – *The operators S_0 and F_0 are closable and $\overline{F}_0 = S_0^*$, $\overline{S}_0 = F_0^*$.*

Proof

For all $A \in \mathcal{M}$, $B \in \mathcal{M}'$ we have

$$<B\Omega\,,\,S_0A\Omega> = <B\Omega\,,\,A^*\Omega> = <A\Omega\,,\,B^*\Omega> = <A\Omega\,,\,F_0B\Omega>.$$

This proves that $F_0 \subset S_0^*$ and $S_0 \subset F_0^*$. The operators S_0 and F_0 are thus closable.

Let us show that $\overline{F}_0 = S_0^*$. Actually it is sufficient to show that $S_0^* \subset \overline{F}_0$. Let $x \in \operatorname{Dom} S_0^*$ and $y = S_0^* x$. For any $A \in \mathcal{M}$ we have

$$<A\Omega\,,\,y> = <A\Omega\,,\,S_0^*x> = <x\,,\,S_0A\Omega> = <x\,,\,A^*\Omega>.$$

If we define the operators Q_0 and Q_0^+ by

$$\begin{array}{ccccc} Q_0 & : & A\Omega & \longmapsto & Ax \\ Q_0^+ & : & A\Omega & \longmapsto & Ay \end{array}$$

we then have

$$\begin{aligned} <B\Omega\,,\,Q_0A\Omega> &= <B\Omega\,,\,Ax> = <A^*B\Omega\,,\,x> \\ &= <y\,,\,B^*A\Omega> = <By\,,\,A\Omega> \\ &= <Q_0^+B\Omega\,,\,A\Omega>. \end{aligned}$$

This proves that $Q_0^+ \subset Q_0^*$ and Q_0 is closable. Let $Q = \overline{Q}_0$. Note that we have

$$Q_0AB\Omega = ABx = AQ_0B\Omega.$$

This proves that $Q_0A = AQ_0$ on $\operatorname{Dom} Q_0$ and thus $AQ \subset QA$ for all $A \in \mathcal{M}$. This means that Q is *affiliated* to $\mathcal{M}'$, that is, it fails from belonging to $\mathcal{M}'$ only by the fact it is an unbounded operator; but every bounded function of Q is thus in $\mathcal{M}'$. In particular, if $Q = U\,|Q|$ is the polar decomposition of Q then U belongs to $\mathcal{M}'$ and the spectral projections of $|Q|$ also belong to $\mathcal{M}'$.

Let $E_n = \mathbb{1}_{[0,n]}(|Q|)$. The operator $Q_n = UE_n\,|Q|$ thus belongs to $\mathcal{M}'$ and

$$\begin{aligned} Q_n\Omega &= UE_n\,|Q|\,\Omega = UE_nU^*U\,|Q|\,\Omega \\ &= UE_nU^*Q_0\Omega = UE_nU^*x. \end{aligned}$$

Furthermore we have

$$Q_n^*\Omega = E_n\,|Q|\,U^*\Omega = E_nQ_0^+\Omega = E_ny.$$

This way UE_nU^*x belongs to $\operatorname{Dom} F_0$ and $F_0(UE_nU^*x) = E_ny$. But E_n tends to I and UU^* is the orthogonal projector onto $\operatorname{Ran} Q$, which contains x.

Finally, we have proved that $x \in \operatorname{Dom} \overline{F}_0$ and $\overline{F}_0x = y = S_0^*x$. That is, $S_0^* \subset \overline{F}_0$.

The other case is treated similarly. ■

We now put $S = \overline{S}_0$ anf $F = \overline{F}_0$.

Lemma 5.3 – *We have*

$$S = S^{-1}.$$

Proof

Let $z \in \operatorname{Dom} S^*$. We have

$$<S_0A\Omega\,,\,S^*z> = <A^*\Omega\,,\,S_0^*z> = <z\,,\,S_0A^*\Omega> = <z\,,\,A\Omega>.$$

Thus $S^* z$ belongs to $\operatorname{Dom} S_0^* = S^*$ and $(S^*)^2 z = z$.

Let $y \in \operatorname{Dom} S$ and $z \in \operatorname{Dom} S^*$, we have $S^* z \in \operatorname{Dom} S^*$ and

$$< S^* z \,,\, Sy > = < y \,,\, (S^*)^2 z > = < y \,,\, z >.$$

This means that Sy belongs to $\operatorname{Dom} S^{**} = \operatorname{Dom} S$ and $S^2 y = S^{**} S y = y$.

We have proved that $\operatorname{Dom} S^2 = \operatorname{Dom} S$ and $S^2 = I$ on $\operatorname{Dom} S$. ■

We had proved in Proposition 5.2 that $F = S^*$. Thus the operators FS and SF are (self-adjoint) positive. The operators F and S have their range equal to their domain, they are invertible and equal to their inverse.

Let $\Delta = FS = S^* S$. Then Δ is invertible, with inverse $\Delta^{-1} = SF = SS^*$.

As S, Δ and thus $\Delta^{1/2}$ have a dense range then the partial anti-isometry J such that

$$S = J(S^* S)^{1/2}$$

(modular decomposition of S) is an anti-isometry from $\mathcal{H}$ to $\mathcal{H}$.

Furthermore

$$S = J\Delta^{1/2} = (SS^*)^{1/2} J = \Delta^{-1/2} J.$$

Let x belong to $\operatorname{Dom} S$. Then

$$x = S^2 x = J\Delta^{1/2}\Delta^{-1/2} J x = J^2 x$$

and thus $J^2 = I$.

Note the following relations

$$\begin{aligned} S &= J\Delta^{1/2} \\ F = S^* &= \Delta^{1/2} J \\ \Delta^{-1} &= J\Delta J. \end{aligned}$$

The operator Δ has a spectral measure (E_λ). Thus the operator $\Delta^{-1} = J\Delta J$ has the spectral measure $(JE_\lambda J)$. Let f be a bounded Borel function, we have

$$\begin{aligned} < f(\Delta^{-1}) x \,,\, x > &= \int \overline{f}(\lambda)\, d\!< JE_\lambda J x \,,\, x > \\ &= \int \overline{f}(\lambda)\, d\!< Jx \,,\, E_\lambda J x > \\ &= \int \overline{f}(\lambda)\, d\!< E_\lambda J x \,,\, Jx > \\ &= < f(\Delta) J x \,,\, Jx > \\ &= < Jx \,,\, \overline{f}(\Delta) J x > \\ &= < J\overline{f}(\Delta) J x \,,\, x >. \end{aligned}$$

This proves

$$f(\Delta^{-1}) = J\overline{f}(\Delta) J.$$

In particular

$$\Delta^{it} = J\Delta^{it}J$$
$$\Delta^{it}J = J\Delta^{it}.$$

Finally note that $S\Omega = F\Omega = \Omega$ and thus $\Delta\Omega = FS\Omega = \Omega$ which finally gives

$$\Delta^{1/2}\Omega = \Omega.$$

Let us now resume the situation we have already described.

Theorem 5.4 – *There exists an anti-unitary operator J from $\mathcal{H}$ to $\mathcal{H}$ and an (unbounded) invertible, positive operator Δ such that*

$$\Delta = FS,\ \Delta^{-1} = SF,\ J^2 = I$$
$$S = J\Delta^{1/2} = \Delta^{-1/2}J$$
$$F = J\Delta^{-1/2} = \Delta^{1/2}J$$
$$J\Delta^{it} = \Delta^{-it}J$$
$$J\Omega = \Delta\Omega = \Omega.$$

■

The operator Δ is called the *modular operator* and J is the *modular conjugation.*

It is interesting to note the following. If the state ω were *tracial*, that is, $\omega(AB) = \omega(BA)$ for all A, B, we would have

$$||S_0A\Omega||^2 = ||A^*\Omega||^2 = < A^*\Omega\,,\, A^*\Omega > = \omega(AA^*) = \omega(A^*A) = ||A\Omega||^2.$$

Thus S_0 would be an isometry and thus

$$S = J = F$$
$$\Delta = I.$$

5.2 The modular group

Let $A, B, C \in \mathcal{M}$. We have

$$SASBC\Omega = SAC^*B^*\Omega = BCA^*\Omega = BSAC^*\Omega = BSASC\Omega.$$

This proves that B and SAS commute. Thus SAS is affiliated to $\mathcal{M}'$.

Let us assume for a moment that Δ **is bounded**. In that case the operators $\Delta^{-1} = J\Delta J$, S and F are also bounded.

We have seen that

$$S\mathcal{M}S \subset \mathcal{M}'$$
$$F\mathcal{M}'F \subset \mathcal{M}.$$

This way we have

$$\begin{aligned}\Delta\mathcal{M}\Delta^{-1} &= \Delta^{1/2}JJ\Delta^{1/2}\mathcal{M}\Delta^{-1/2}JJ\Delta^{-1/2}\\ &= FS\mathcal{M}SF \subset F\mathcal{M}'F \subset \mathcal{M}.\end{aligned}$$

We also have

$$\Delta^n \mathcal{M} \Delta^{-n} \subset \mathcal{M}$$

for all $n \in I\!\!N$.

For any $A \in \mathcal{M}$, $A' \in \mathcal{M}'$, the function

$$f(z) = ||\Delta||^{-2z} < \phi, [\Delta^z A \Delta^{-z}, A']\Psi >$$

is analytic on $\mathbb{C}$. It vanishes for $z = 0, 1, 2, ...$

As $\left|\left|\Delta^{-1}\right|\right| \, ||J\Delta J|| = ||\Delta||$ we have

$$|f(z)| = O\left(||\Delta||^{-2\Re z} \left(||\Delta||^{|\Re z|}\right)^2\right) = O(1)$$

when $\Re z > 0$.

By Carlson's theorem we have $f(z) = 0$ for all $z \in \mathbb{C}$. Thus

$$\Delta^z \mathcal{M} \Delta^{-z} \subset \mathcal{M}'' = \mathcal{M}$$

for all $z \in \mathbb{C}$. But

$$\mathcal{M} = \Delta^z(\Delta^{-z} \mathcal{M} \Delta^z)\Delta^{-z} \subset \Delta^z \mathcal{M} \Delta^{-z}$$

and finally

$$\Delta^z \mathcal{M} \Delta^{-z} = \mathcal{M}.$$

Furthermore

$$\begin{aligned} J\mathcal{M}J &= J\Delta^{1/2}\mathcal{M}\Delta^{-1/2}J = S\mathcal{M}S \subset \mathcal{M}' \\ J\mathcal{M}'J &= J\Delta^{-1/2}\mathcal{M}\Delta^{1/2}J = F\mathcal{M}F \subset \mathcal{M}. \end{aligned}$$

We have proved

$$J\mathcal{M}J = \mathcal{M}'.$$

The results we have obtained here are fundamental and extend to the case when Δ is unbounded. This is what the following theorem says. We do not prove it as it implies pages of difficult analytic considerations. We hope that the above computations make it credible.

Theorem 5.5 [Tomita-Takesaki's theorem] – *In any case we have*

$$\begin{aligned} J\mathcal{M}J &= \mathcal{M}' \\ \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M}. \end{aligned}$$

■

Put

$$\sigma_t(A) = \Delta^{it} A \Delta^{-it}.$$

This defines a one parameter group of automorphisms of $\mathcal{M}$.

Proposition 5.6 – *We have, for all $A, B \in \mathcal{M}$*

$$\omega(A\sigma_t(B)) = \omega(\sigma_{t+i}(B)A). \tag{1}$$

Proof

$$\begin{aligned}
<\Omega\,,\,A\Delta^{it}B\Delta^{-it}\Omega> &= <\Delta^{-it}A^*\Omega\,,\,B\Omega> \\
&= <\Delta^{-it-1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <\Delta^{-it-1}\Delta^{1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <J\Delta^{-it+1}J\Delta^{1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <J\Delta^{1/2}B\Omega\,,\,\Delta^{-it+1}J\Delta^{1/2}A^*\Omega> \\
&= <B^*\Omega\,,\,\Delta^{-it+1}A\Omega> \\
&= <\Omega\,,\,B\Delta^{-i(t+i)}A\Omega> \\
&= <\Omega\,,\,\Delta^{i(t+i)}B\Delta^{-i(t+i)}A\Omega> \\
&= \omega(\sigma_{t+i}(B)A).
\end{aligned}$$

■

It is interesting to relate the above equality with the following result.

Proposition 5.7 – *Let ω be a state of the form*

$$\omega(A) = \mathrm{Tr}(\rho A)$$

on $\mathcal{B}(\mathcal{K})$ for some trace-class positive ρ with $\mathrm{Tr}\rho = 1$. Let (σ_t) be the following group of automorphisms of $\mathcal{B}(\mathcal{K})$:

$$\sigma_t(A) = e^{itH}Ae^{-itH}$$

for some self-adjoint operator H on $\mathcal{K}$. Then the following assertions are equivalent.

i) For all $A, B \in \mathcal{B}(\mathcal{K})$, all $t \in \mathbb{R}$ and a fixed $\beta \in \mathbb{R}$ we have

$$\omega(A\sigma_t(B)) = \omega(\sigma_{t-\beta i}(B)A).$$

ii) ρ is given by

$$\rho = \frac{1}{Z}e^{-\beta H},$$

where $Z = \mathrm{Tr}(\exp(-\beta H))$.

Proof

ii) implies i): We compute directly

$$\begin{aligned}
\omega(A\sigma_t(B)) &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}Ae^{itH}Be^{-itH}) \\
&= \frac{1}{Z}\mathrm{Tr}(Ae^{itH}Be^{(-it-\beta)H}) \\
&= \frac{1}{Z}\mathrm{Tr}(Ae^{-\beta H}e^{(it+\beta)H}Be^{(-it-\beta)H}) \\
&= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}e^{(it+\beta)H}Be^{(-it-\beta)H}A) \\
&= \omega(\sigma_{t-\beta i}(B)A).
\end{aligned}$$

i) implies ii): We have

$$\begin{aligned}\mathrm{Tr}(AB\rho) &= \mathrm{Tr}(\rho AB) = \omega(AB)\\ &= \omega(\sigma_{-\beta i}(B)A) = \mathrm{Tr}(\rho e^{\beta H}Be^{-\beta H}A) = \mathrm{Tr}(A\rho e^{\beta H}Be^{-\beta H}).\end{aligned}$$

As this is valid for any A we conclude that

$$B\rho = \rho e^{\beta H}Be^{-\beta H}$$

for all B. This means

$$B\left(\rho e^{\beta H}\right) = \left(\rho e^{\beta H}\right)B.$$

As this is valid for all B we conclude that $\rho\exp(\beta H)$ is a multiple of the identity. This gives ii). ■

Another very interesting result to add to Proposition 5.6 is that the modular group is the only one to perform the relation (1).

Theorem 5.8 – *$\sigma_.$ is the only automorphism group to satisfy (1) on $\mathcal{M}$ for the given state ω.*

Proof

Let $\tau_.$ be another automorphism group on $\mathcal{M}$ which satisfies (1). Define the operators U_t by

$$U_tA\Omega = \tau_t(A)\Omega.$$

Then U_t is unitary for

$$\begin{aligned}||U_tA\Omega||^2 &= <\tau_t(A)\Omega\,,\,\tau_t(A)\Omega> = <\Omega\,,\,\tau_t(A^*A)\Omega>\\ &= \omega(\tau_t(A^*A)) = \omega(\tau_{t+i}(I)A^*A)\\ &= \omega(A^*A) = ||A\Omega||^2.\end{aligned}$$

The family $U_.$ is clearly a group, it is thus of the form $U_t = \exp itM$ for a self-adjoint operator M.

Note that $U_t\Omega = \Omega$ and thus $M\Omega = 0$.

Let A, B be entire elements for $\tau_.$, then the relation $\omega(\tau_i(B)A) = \omega(AB)$ implies

$$\begin{aligned}<B^*\Omega\,,\,\Delta A\Omega> &= <\Delta^{1/2}B^*\Omega\,,\,JJ\Delta^{1/2}A\Omega>\\ &= <A^*\Omega\,,\,B\Omega>\\ &= \omega(AB)\\ &= \omega(\tau_i(B)A)\\ &= <\Omega\,,\,e^{-M}Be^{M}A\Omega>\\ &= <B^*\Omega\,,\,e^{M}A\Omega>.\end{aligned}$$

This means

$$\Delta = e^M$$

and $\tau = \sigma$. ■

5.3 Self-dual cone and standard form

We put

$$\mathcal{P} = \overline{\{AJAJ\Omega; A \in \mathcal{M}\}}.$$

Proposition 5.9 –

i) $\mathcal{P} = \overline{\Delta^{1/4}\mathcal{M}_+\Omega} = \overline{\Delta^{-1/4}\mathcal{M}'_+\Omega}$ *and thus* $\mathcal{P}$ *is a convex cone.*

ii) $\Delta^{it}\mathcal{P} = \mathcal{P}$ *for all* t*.*

iii) If f *is of positive type then* $f(\log \Delta)\mathcal{P} \subset \mathcal{P}$*.*

iv) If $\xi \in \mathcal{P}$ *then* $J\xi = \xi$*.*

v) If $A \in \mathcal{M}$ *then* $AJAJ\mathcal{P} \subset \mathcal{P}$*.*

Proof

i) Let $\mathcal{M}_0$ be the $*$-algebra of elements of $\mathcal{M}$ which are *entire* for the modular group σ. (that is, $t \mapsto \sigma_t(A)$ admits an analytic extension). We shall admit here that $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$.

For every $A \in \mathcal{M}_0$ we have

$$\begin{aligned}
\Delta^{1/4}AA^*\Omega &= \sigma_{-i/4}(A)\sigma_{i/4}(A)^*\Omega \\
&= \sigma_{-i/4}(A)J\Delta^{1/2}\sigma_{i/4}(A)\Omega \\
&= \sigma_{-i/4}(A)J\sigma_{-i/4}(A)J\Omega \\
&= BJBJ\Omega
\end{aligned}$$

where $B = \sigma_{-i/4}(A)$. By $\sigma_{-i/4}(\mathcal{M}_0) = \mathcal{M}_0$ and by the density of $\mathcal{M}_0$ in $\mathcal{M}$ we have

$$BJBJ\Omega \in \overline{\Delta^{1/4}\mathcal{M}_+\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}}$$

for all $B \in \mathcal{M}$. Thus

$$\mathcal{P} \subset \overline{\Delta^{1/4}\mathcal{M}_+\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}}.$$

Conversely, $\mathcal{M}_0^+\Omega$ is dense in $\overline{\mathcal{M}_+\Omega}$. Let $\Psi \in \overline{\mathcal{M}_+\Omega}$. There exists a sequence $(A_n) \subset \mathcal{M}_0^+$ such that $A_n\Omega \to \Psi$. We know by the above that $\Delta^{1/4}A_n\Omega$ belongs to $\mathcal{P}$. But

$$J\Delta^{1/2}A_n\Omega = A_n\Omega \to \Psi = J\Delta^{1/2}\Psi$$

and thus

$$\left|\left|\Delta^{1/4}(\Psi - A_n\Omega)\right|\right|^2 = <\Psi - A_n\Omega\,,\, \Delta^{1/2}(\Psi - A_n\Omega)> \to 0.$$

Thus $\Delta^{1/4}\Psi$ belongs to $\mathcal{P}$ and $\overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}} \subset \mathcal{P}$.

This proves the first equality of i). The second one is treated exactly in the same way.

ii) We have

$$\Delta^{it}\Delta^{1/4}\mathcal{M}_+\Omega = \Delta^{1/4}\Delta^{it}\mathcal{M}_+\Omega = \Delta^{1/4}\sigma_t(\mathcal{M}_+)\Omega = \Delta^{1/4}\mathcal{M}_+\Omega.$$

iii) If f is of positive type then f is the Fourier transform of some positive, finite, Borel measure μ on $\mathbb{R}$. In particular

$$f(\log \Delta) = \int \Delta^{it}\, d\mu(t).$$

One concludes with ii) now.

iv) $JAJAJ\Omega = JAJA\Omega = AJAJ\Omega$.

v) $AJAJBJBJ\Omega = ABJAJJBJ\Omega = ABJABJ\Omega$. ■

Theorem 5.10 –

i) P is self-dual*, that is $\mathcal{P} = \mathcal{P}^\vee$ where*

$$\mathcal{P}^\vee = \{x \in \mathcal{H}; <y\,,\, x> \geq 0, \forall y \in \mathcal{P}\}.$$

ii) $\mathcal{P}$ is pointed*, that is,*

$$\mathcal{P} \cap (-\mathcal{P}) = \{0\}.$$

iii) If $J\xi = \xi$ then ξ admits a unique decomposition as $\xi = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \in \mathcal{P}$ and ξ_1 orthogonal to ξ_2.

iv) The linear span of $\mathcal{P}$ is the whole of $\mathcal{H}$.

Proof

i) If $A \in \mathcal{M}_+$ and $A' \in \mathcal{M}'_+$ then

$$<\Delta^{1/4}A\Omega\,,\, \Delta^{-1/4}A'\Omega> = <A\Omega\,,\, A'\Omega> = <\Omega\,,\, A^{1/2}A'A^{1/2}\Omega> \geq 0.$$

Thus $\mathcal{P}$ is included in $\mathcal{P}^\vee$.

Conversely, if $\xi \in \mathcal{P}^\vee$, that is $<\xi\,,\,\nu> \geq 0$ for all $\nu \in \mathcal{P}$, we put

$$\xi_n = f_n(\log\Delta)\xi$$

where $f_n(x) = \exp(-x^2/2n^2)$. Then ξ_n belongs to $\cap_{\alpha\in\mathbb{C}} \operatorname{Dom}\Delta^\alpha$ and ξ_n converges to ξ. We know that $f_n(\log\Delta)\nu$ belongs to $\mathcal{P}$ and thus

$$<\xi_n\,,\,\nu> = <\xi\,,\, f_n(\log\Delta)\nu> \geq 0.$$

Let $A \in \mathcal{M}_+$ then $\Delta^{1/4}A\Omega$ belongs to $\mathcal{P}$ and

$$<\Delta^{1/4}\xi_n\,,\, A\Omega> = <\xi_n\,,\, \Delta^{1/4}A\Omega> \geq 0.$$

Thus $\Delta^{1/4}\xi_n$ belongs to $\overline{\mathcal{M}_+\Omega}^\vee$ which coincides with $\overline{\mathcal{M}'_+\Omega}$ (admitted). This finally gives that ξ_n belongs to $\Delta^{-1/4}\overline{\mathcal{M}'_+\Omega} \subset \mathcal{P}$. This proves i).

ii) If $\xi \in \mathcal{P} \cap (-\mathcal{P}) = \mathcal{P} \cap (-\mathcal{P}^\vee)$ then $<\xi\,,\,-\xi> \geq 0$ and $\xi = 0$.

iii) If $J\xi = \xi$ then, as $\mathcal{P}$ is convex and closed, there exists a unique $\xi_1 \in \mathcal{P}$ such that

$$||\xi - \xi_1|| = \inf\{||\xi - \nu||\,; \nu \in \mathcal{P}\}.$$

We put $\xi_2 = \xi_1 - \xi$. Let $\nu \in \mathcal{P}$ and $\lambda > 0$. Then $\xi_1 + \lambda\nu$ belongs to $\mathcal{P}$ and

$$||\xi - \xi_1||^2 \leq ||\xi_1 + \lambda\nu - \xi||^2.$$

That is $||\xi_2||^2 \leq ||\xi_2 + \lambda\nu||^2$, or else $\lambda^2||\nu||^2 + 2\lambda\Re<\xi_2\,,\,\nu> \geq 0$. This implies that $\Re<\xi_2\,,\,\nu>$ is positive. But as $J\xi_2 = \xi_2$ and $J\nu = \nu$ then

$$<\xi_2\,,\,\nu> = <J\xi_2\,,\, J\nu> = \overline{<\xi_2\,,\,\nu>}.$$

That is $<\xi_2\,,\,\nu> \geq 0$ and $\xi_2 \in \mathcal{P}^\vee = \mathcal{P}$.

iv) If ξ is orthogonal to the linear span of $\mathcal{P}$ then ξ belongs to $\mathcal{P}^\vee = \mathcal{P}$. thus $<\xi\,,\,\xi> = 0$ and $\xi = 0$. ■

Theorem 5.11 [Universality] –

1) If $\xi \in \mathcal{P}$ then ξ is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}$.

2) If $\xi \in \mathcal{P}$ is cyclic for $\mathcal{M}$ then $J_\xi, \mathcal{P}_\xi$ associated to $(\mathcal{M}, \xi)$ satisfy

$$J_\xi = J \quad \textit{and} \quad \mathcal{P}_\xi = \mathcal{P}.$$

Proof

1) If ξ is cyclic for $\mathcal{M}$ then $J\xi$ is cyclic for $\mathcal{M}' = J\mathcal{M}J$ and thus $\xi = J\xi$ is separating for $\mathcal{M}$. And conversely.

2) Define as before (the closed version of)

$$\begin{array}{rcccc} S_\xi & : & A\xi & \longmapsto & A^*\xi \\ F_\xi & : & A'\xi & \longmapsto & A'^*\xi. \end{array}$$

We have

$$\begin{aligned} JF_\xi JA\xi &= JF_\xi JAJ\xi \\ &= J(JAJ)^*\xi \\ &= A^*\xi \\ &= S_\xi A\xi. \end{aligned}$$

This proves that $S_\xi \subset JF_\xi J$. By a symmetric argument $F_\xi \subset JS_\xi J$ and thus $JS_\xi = F_\xi J$.

Note that

$$(JS_\xi)^* = S_\xi^* J = F_\xi J = JS_\xi.$$

This means that JS_ξ is self-adjoint. Let us prove that it is positive. We have

$$< A\xi\,,\, JS_\xi A\xi > = < A\xi\,,\, JA^*\xi > = < \xi\,,\, A^*JA^*\xi >$$

which is a positive quantity for ξ and A^*JA^*J belong to $\mathcal{P}$. This proves the positivity of JS_ξ.

We have

$$S_\xi = J_\xi \Delta_\xi^{1/2} = J(JS_\xi).$$

By uniqueness of the polar decomposition we must have $J = J_\xi$.

Finally, we have that $\mathcal{P}_\xi$ is generated by the $AJ_\xi AJ_\xi \xi = AJAJ\xi$. But as ξ belongs to $\mathcal{P}$ we have that $AJAJ\xi$ belongs to $\mathcal{P}$ and thus $\mathcal{P}_\xi \subset \mathcal{P}$. Finally, $\mathcal{P} = \mathcal{P}^\vee \subset \mathcal{P}_\xi^\vee = \mathcal{P}_\xi$ and $\mathcal{P} = \mathcal{P}_\xi$. ■

The following theorem is very usefull and powerfull, but its proof is very long, tedious and cannot be resumed, thus we prefer not enter into it and give the result as it is (cf [B-R], p. 108-117).

For every $\xi \in \mathcal{P}$ one can define a particular normal positive form

$$\omega_\xi(A) = <\xi\,,\, A\xi>$$

on $\mathcal{M}$. That is, $\xi \in \mathcal{M}_{*+}$.

Theorem 5.12 –

1) For every $\omega \in \mathcal{M}_{*+}$ *there exists a unique* $\xi \in \mathcal{P}$ *such that*

$$\omega = \omega_\xi .$$

2) The mapping $\xi \longmapsto \omega_\xi$ *is an homeomorphism and*

$$||\xi - \nu||^2 \leq ||\omega_\xi - \omega_\nu||^2 \leq ||\xi - \nu|| \; ||\xi + \nu|| \,.$$

■

We denote by $\omega \longmapsto \xi(\omega)$ the inverse mapping of $\xi \longmapsto \omega_\xi$.

Corollary 5.13 – *There exists a unique unitary representation*

$$\alpha \in Aut(\mathcal{M}) \longmapsto U_\alpha$$

of the group of $*$*-automorphisms of* $\mathcal{M}$ *on* $\mathcal{H}$*, such that*

i) $U_\alpha A U_\alpha^* = \alpha(A)$*, for all* $A \in \mathcal{M}$*,*

ii) $U_\alpha \mathcal{P} \subset \mathcal{P}$ *and, moreover,*

$$U_\alpha \xi(\omega) = \xi({\alpha^{-1}}^*(\omega))$$

for all $\omega \in \mathcal{M}_{*+}$ *and where* $(\alpha^*\omega)(A) = \omega(\alpha(A))$*.*

iii) $[U_\alpha, J] = 0$*.*

Proof

Let $\alpha \in \mathrm{Aut}(\mathcal{M})$. Let $\xi \in \mathcal{P}$ be the representant of the state

$$A \longmapsto <\Omega\,,\, \alpha^{-1}(A)\Omega>.$$

That is,

$$<\xi\,,\, A\xi> = <\Omega\,,\, \alpha^{-1}(A)\Omega>.$$

In particular ξ is separating for $\mathcal{M}$ and hence cyclic. Define the operator

$$UA\Omega = \alpha(A)\xi.$$

We have

$$||UA\Omega||^2 = <\xi\,,\, \alpha(A^*A)\xi> = <\Omega\,,\, A^*A\Omega> = ||A\Omega||^2.$$

Thus U is unitary. In particular

$$U^*A\xi = \alpha^{-1}(A)\Omega.$$

Now, for $A, B \in \mathcal{M}$ we have

$$UAU^*B\xi = UA\alpha^{-1}(B)\Omega = \alpha(A\alpha^{-1}(B))\xi = \alpha(A)B\xi$$

and

$$\alpha(A) = UAU^*.$$

We have proved the existence of the unitary representation.

Note that

$$\begin{aligned} SU^*A\xi &= S\alpha^{-1}(A)\Omega \\ &= \alpha^{-1}(A)^*\Omega \\ &= \alpha^{-1}(A^*)\Omega \\ &= U^*A^*\xi \\ &= U^*S_\xi A\xi. \end{aligned}$$

Hence by closure

$$J\Delta^{1/2}U^* = U^*J_\xi\Delta_\xi^{1/2} = U^*J\Delta_\xi^{1/2}.$$

That is

$$UJU^*U\Delta^{1/2}U^* = J\Delta_\xi^{1/2}.$$

By uniqueness of the polar decomposition we must have $UJU^* = J$. This gives iii).

For $A \in \mathcal{M}$ we have

$$UAJAJ\Omega = \alpha(A)J\alpha(A)J\xi.$$

Since ξ belongs to $\mathcal{P}$ we deduce

$$U\mathcal{P} = \mathcal{P}.$$

If $\phi \in \mathcal{M}_{*+}$ we have

$$\begin{aligned} < U\xi(\phi)\,,\, AU\xi(\phi) > &= < \xi(\phi)\,,\, U^*AU\xi(\phi) > \\ &= < \xi(\phi)\,,\, \alpha^{-1}(A)\xi(\phi) > \\ &= \phi(\alpha^{-1}(A)) \\ &= (\alpha^{-1^*}(\phi))(A) \\ &= < \xi(\alpha^{-1^*}(\phi))\,,\, A\xi(\alpha^{-1^*}(\phi)) >. \end{aligned}$$

By uniqueness of the representing vector in $\mathcal{P}$

$$U(\alpha)\xi(\phi) = \xi(\alpha^{-1^*}(\phi)).$$

This gives ii) and also the uniqueness of the unitary representation. ■

6. W^*-DYNAMICAL SYSTEMS

6.1 Cyclic representation of finite systems

We shall now concentrate on simple systems in order to illustrate the tools we have developed before.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} = \mathcal{B}(\mathcal{H})$ the associated von Neumann algebra. Let ρ be a (trace-class) operator, which is positive and $\mathrm{Tr}\rho = 1$. Then ρ can be written as

$$\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$$

for some orthonormal family $\{\Psi_i\}$ and some $0 \leq p_i \leq 1$.

On $\mathcal{M}$ consider the state

$$\omega(A) = \mathrm{Tr}(\rho A) = \sum_i p_i < \Psi_i\, ,\, A\Psi_i >.$$

Thus we have

$$\omega(A^*A) = \sum_i p_i \left|\left|A\Psi_i\right|\right|^2.$$

The state ω is faithfull if and only if the above quantity vanishes in the only case $A = 0$. One easily checks (exercise) that ω is faithfull if and only if $p_i > 0$ for all i and

$$\sum_i |\Psi_i\rangle\langle\Psi_i| = I$$

That is, if and only if $\rho > 0$. We now on assume that ω is faithfull.

As ρ is trace-class then $\rho^{1/2}$ is Hilbert-Schmidt. Let us call $L^2(\mathcal{H})$ the space of Hilbert-Schmidt operators on $\mathcal{H}$. This space is a Hilbert space when equiped with the scalar product

$$< A\, ,\, B > = \mathrm{Tr}(A^*B).$$

We put $\mathcal{H}_\omega = L^2(\mathcal{H})$. We now have a representation of $\mathcal{M}$ in $\mathcal{H}_\omega$ by

$$\Pi_l(A)a = Aa.$$

Put $\Omega_\omega = \rho^{1/2}$. Then clearly $(\mathcal{H}_\omega, \Pi_l, \Omega_\omega)$ is the cyclic representation of $(\mathcal{M}, \omega)$.

Note that we also have an anti-representation

$$\Pi_r(A)a = aA^*.$$

The vector Ω_ω is also cyclic for this anti-representation.

The two representations Π_l and Π_r are linked by the relation

$$\Pi_r(A) = J\Pi_l(A)J$$

where

$$\begin{array}{cccc} J\ : & L^2(\mathcal{H}) & \longrightarrow & L^2(\mathcal{H}) \\ & a & \longmapsto & a^* \end{array}$$

is a anti-unitary involution.

There is a canonical isomorphism

$$\begin{array}{ccc} L^2(\mathcal{H}) & \longrightarrow & \mathcal{H}\otimes\overline{\mathcal{H}} \\ |\Psi\rangle\langle\phi| & \longmapsto & \Psi\otimes\overline{\phi} \end{array}$$

reading the above via this isomorphism we get

$$\Pi_l(A) = A\otimes I$$
$$\Pi_r(A) = I\otimes\overline{A}$$
$$J\ :\ \Psi\otimes\overline{\phi}\longmapsto\phi\otimes\overline{\Psi}.$$

The von Neumann algebra

$$\mathcal{N} = \Pi_l(\mathcal{M})$$

clearly satisfies

$$\mathcal{N}' = J\mathcal{N}J = \Pi_r(\mathcal{M}).$$

Finally, put

$$\mathcal{P} = L^2_+(\mathcal{H}) = \{a\in L^2(\mathcal{H}); a\geq 0\}.$$

This is a cone, it coincides with

$$\{AJAJ\Omega_\omega; A\in\mathcal{M}\}$$

(exercise).

Its dual is given by

$$\mathcal{P}^\vee = \{b\in L^2(\mathcal{H}); <b\,,\,a>\,\geq 0, \forall a\in\mathcal{P}\}.$$

But note that

$$\mathrm{Tr}(b^*a) = \mathrm{Tr}(a^{1/2}b^*a^{1/2})$$

is positive for all a if and only if $b\geq 0$ (exercise). This means

$$\mathcal{P}^\vee = \mathcal{P}.$$

For every density matrix h (i.e., trace-class, positive, with $\mathrm{Tr}h = 1$) there exists a unique element $h^{1/2}$ in $\mathcal{P}$ such that

$$\omega_h(A) = \,<h^{1/2}\,,\,\Pi_l(A)h^{1/2}>.$$

Note that if $A\in\mathcal{N}$ and $b\in\mathcal{P}$ then

$$AJAJb = AbA^*\geq 0$$

and thus $AJAJ\mathcal{P}\subset\mathcal{P}$. Finally

$$Jb = b^* = b.$$

Let us look at the dynamics under this point of view.

Let H be a self-adjoint operator on $\mathcal{H}$ and consider the automorphism group

$$\alpha_t(A) = e^{itH}Ae^{-itH}$$

on $\mathcal{M}$. This group is σ-weakly-continuous (exercise).

We have

$$\begin{aligned}\Pi_l(\alpha_t(A))b &= e^{itH}Ae^{-itH}b\\ &= e^{itH}A(e^{-itH}be^{itH})e^{-itH}\\ &= U_t^*\Pi_l(A)U_tb\end{aligned}$$

where

$$\begin{array}{cccccc} U_t & : & L^2(\mathcal{H}) & \longrightarrow & L^2(\mathcal{H}) & \\ & & b & \longmapsto & e^{-itH}be^{itH} & \simeq (e^{-itH}\otimes e^{it\overline{H}})b \end{array}$$

that is,

$$U_t = e^{-itL}$$

where

$$L \,:\, b \longmapsto Hb - bH$$

and thus

$$L \simeq H\otimes I - I\otimes \overline{H}.$$

By construction

$$U_t^*\mathcal{N}U_t = \mathcal{N}.$$

Note that

$$U_tJb = U_tb^* = e^{-itH}b^*e^{itH} = (e^{-itH}be^{itH})^* = JU_tb.$$

In particular

$$U_t^*\mathcal{N}'U_t = \mathcal{N}'$$

and $J\,\mathrm{Dom}\,L\subset \mathrm{Dom}\,L$ and

$$JL + LJ = 0.$$

Finally, it is clear that

$$U_t\mathcal{P}\subset\mathcal{P}.$$

The operator L is called the *standard Liouvillian.*

Let

$$h = \frac{1}{Z_\beta}e^{-\beta H}.$$

Consider the state

$$\omega_h(A) = \mathrm{Tr}(hA)$$

on $\mathcal{M}$. The function

$$\begin{aligned}F_{A,B}(t) = \omega_h(A\alpha_t(B)) &= \frac{1}{Z_\beta}\mathrm{Tr}\left(e^{-\beta H}Ae^{itH}Be^{-itH}\right)\\ &= \frac{1}{Z_\beta}\mathrm{Tr}\left(e^{-(\beta+it)H}Ae^{itH}B\right)\end{aligned}$$

is analytic for $0<\Im t<\beta$, it is continuous on the boundary of this domain and as seen before

$$F_{A,B}(t+i\beta) = \omega_h(\alpha_t(B)A).$$

A state ω_h satisfying the above is called (α,β)-K.M.S.

Note that a (α,β)-K.M.S state ω has to be invariant under α for the function

$$f_z(A) = \omega(\alpha_z(A))$$

is analytic on some dense set of A's, bounded on the strip $0 \leq \Im t \leq \beta$ and $i\beta$-periodic by hypothesis. Thus it has to be constant.

Conversely, if h is a density matrix, then

$$\sigma_t(A) = h^{it} A h^{-it}$$

is an automorphism group. The state ω_h satisfies

$$\omega_h(A\sigma_{t-i}(B)) = \omega_h(\sigma_t(B)A),$$

that is, ω_h is $(\sigma, -1)$-K.M.S.

The group σ_t is the modular group of ω_h on $\mathcal{N}$. Thus in the standard representation

$$\Pi_l(\sigma_t(A)) = e^{it\mathcal{L}} \Pi_l(A) e^{-it\mathcal{L}}.$$

Note that $\mathcal{L}h^{1/2} = 0$ and $\mathcal{L} = -\beta L$.

Conversly, given a state h, let us prove that ω_h is (α, β)-K.M.S. if and only if

$$\mathcal{L} = -\beta L.$$

Indeed,

$$\begin{gathered} \omega_h(A\alpha_{-i\beta}(B)) = \omega_h(BA) \\ < h^{1/2}\,,\, Ae^{-\beta L} B e^{\beta L} h^{1/2} > = < h^{1/2}\,,\, BAh^{1/2} > \\ < A^* h^{1/2}\,,\, e^{-\beta L} B h^{1/2} > = < B^* h^{1/2}\,,\, Ah^{1/2} > \end{gathered}$$

for $Lh^{1/2} = 0$ by the α-invariance of ω_h

$$< A^* h^{1/2}\,,\, e^{-\beta L} B h^{1/2} > = < \Delta_h^{1/2} A^* h^{1/2}\,,\, \Delta_h^{1/2} B h^{1/2} >$$

thus

$$e^{-\beta L} B h^{1/2} = \Delta_h^{1/2} B h^{1/2}.$$

But by the same method, we would get

$$e^{\mathcal{L}} B h^{1/2} = \Delta_h^{1/2} B h^{1/2}$$

for ω_h is always $(\sigma, -1)$-K.M.S.

6.2 W^*-Dynamical systems

A W^*-dynamical system is a couple $(\mathcal{M}, \alpha)$, where $\mathcal{M}$ is a von Neumann algebra and α is a σ-weakly continuous automorphism group of $\mathcal{M}$.

If we are furthermore given a faithfull normal state ω on $\mathcal{M}$, we consider the G.N.S. representation $(\mathcal{H}, \Pi, \Omega)$ of $(\mathcal{M}, \omega)$, we transport the group α into $\alpha = \Pi \circ \alpha$ and we put ourselves into the standard representation $(\mathcal{M}, \mathcal{H}, \Omega, J, \mathcal{P})$.

Theorem 6.1 – *There exists a unique self-adjoint operator L on $\mathcal{H}$ such that*

1) $\alpha(A) = e^{itL} A e^{-itL}$

for all $A \in \mathcal{M}$ and

2) $L\mathcal{P} \subset \mathcal{P}$.

Proof

There exists $\Psi_t \in \mathcal{P}$ such that

$$<\Psi_t\,,\, A\Psi_t> \,=\, <\Omega\,,\, \alpha_t(A)\Omega> \,=\omega\circ\alpha_t(A).$$

As the above state is faithfull then Ψ_t is separating for $\mathcal{M}$, thus cyclic for $\mathcal{M}$. Put

$$U_tA\Omega = \alpha_{-t}(A)\Psi_t$$

for all $A \in \mathcal{M}$. As before U_t is unitary and

$$U_{-t}AU_t = \alpha_t(A).$$

Clearly $(U_t)_{t\in\mathbb{R}}$ is a group and we write it as

$$U_t = e^{itL}.$$

We have

$$U_tA^*\Omega = U_tJ\Delta_\Omega^{1/2}A\Omega$$

but also

$$\begin{aligned} U_tA^*\Omega &= \alpha_{-t}(A)^*\Psi_t \\ &= J\Delta_{\Psi_t}^{1/2}\alpha_{-t}(A)\Psi_t \\ &= J\Delta_{\Psi_t}^{1/2}U_tA\Omega. \end{aligned}$$

This gives

$$J\Delta_\Omega^{1/2} = U_{-t}JU_tU_{-t}\Delta_{\Psi_t}^{1/2}U_t.$$

By uniqueness of the polar decomposition we get

$$[U_t, J] = 0$$

that is,

$$JL + LJ = 0$$

and also

$$\Delta_{U_t\Omega} = U_t\Delta_\Omega U_{-t}.$$

Finally, note that

$$U_tAJAJ\Omega = U_tAU_{-t}JU_tAU_{-t}JU_t\Omega = \alpha_{-t}(A)J\alpha_{-t}(A)J\Psi_t$$

and thus belongs to $\mathcal{P}$. By closure we get $U_t\mathcal{P} \subset \mathcal{P}$. This gives the uniqueness of U and of L. ■

Theorem 6.2 – *Let μ be a normal state on $\mathcal{M}$ with associated vector Ψ_μ in $\mathcal{P}$. Then μ is α-invariant if and only if*

$$L\Psi_\mu = 0.$$

Proof

Indeed, if U is given by the above theorem, then

$$\mu(\alpha_t(A)) = \,<U_t\Psi_\mu\,,\, AU_t\Psi_\mu>$$

but it should also equal

$$\omega(A) = < \Psi_\mu \, , \, A\Psi_\mu >.$$

As $U_t \Psi_\mu$ belongs to $\mathcal{P}$, then by uniqueness of the representing element in $\mathcal{P}$ we have $U_t\Psi_\mu = \Psi_\mu$ and $L\Psi_\mu = \Psi_\mu$.

Conversly, if $L\Psi_\mu = 0$ and $V_t = e^{itL}$ has the property 1), we have in particular

$$V_t A \Psi_\mu = V_t A V_{-t} \Psi_\mu = \alpha_{-t}(A)\Psi_\mu = U_t A \Psi_\mu.$$

Thus $V_t = U_t$. ■

The operator L is called the *standard Liouvillian* of the dynamical system.

Note the following. If one add to L any operator L' affiliated to $\mathcal{M}'$ and such that $L + L'$ is essentially self-adjoint then the operator $\widetilde{L} = L + L'$ still satisfies

$$\alpha(A) = e^{it\widetilde{L}} A e^{-it\widetilde{L}}.$$

The property 1) of the above theorem leaves some indetermination on L (L is known up to the sum with some element of $\mathcal{M}'_{a.a.}$). The property 2) then fixes the good choice for L within this class.

6.3 K.M.S. states

Let $(\mathcal{M}, \alpha)$ be as above. A normal state ω on $\mathcal{M}$ is called (α, β)-K.M.S., for a $\beta \in \mathbb{R}^*$, if for all $A, B \in \mathcal{M}$ there exists a function $F_{AB}(z)$ analytic on the strip $\Im z \in]0, \beta[$ (or $]\beta, 0[$), continuous on the closure of the strip, such that

$$F_{AB}(t) = \omega(A\alpha_t(B))$$
$$F_{AB}(t + i\beta) = \omega(\alpha_t(B)A)$$

for all $t \in \mathbb{R}$.

If $\beta = -1$ one says α-K.M.S. state only.

Note the following properties.

1) The above definition of a (α, β)-K.M.S. state is equivalent (admitted) to the following one:

$$\omega(BA) = \omega(A\alpha_{i\beta}(B))$$

for all A, B in a σ-weakly dense, α-invariant, subalgebra of $\mathcal{M}$.

2) If ω is (α, β)-K.M.S. then it is α-invariant

Indeed, for A entire element of $\mathcal{M}$ for α, we have that $f(z) = \omega(\alpha_z(A))$ is entire. Furthermore

$$\sup_{o \leq \Im z \leq \beta} |f(z)| \leq \sup_{0 \leq y \leq \beta} ||\alpha_{iy}(A)|| < \infty.$$

Thus f is bounded on the strip. Furthermore

$$f(z + i\beta) = \omega(I\alpha_{z+i\beta}(A)) = \omega(\alpha_z(A)I) = f(z).$$

Thus f is $i\beta$-periodic, thus bounded, thus constant.

3) If ω is (α_t, β)-K.M.S. then ω is $(\alpha_{\lambda t}, \lambda\beta)$-K.M.S. and in particular ω is $\alpha_{-t/\beta}$-K.M.S.

Note that there no relation in general between (α, β)-K.M.S. states and (α, β')-K.M.S. states.

4) If ω is (α, β)-K.M.S., then Ω_ω is separating for $\Pi_\omega(\mathcal{M})$. In particular ω is faithfull and Ω_ω is cyclic for $\Pi(\mathcal{M})'$.

Indeed, if $A\Omega_\omega = 0$ then $\omega(A^*A) = 0$ and $\omega(A^*\alpha_t(A)) = 0$ for

$$\omega(A^*\alpha_t(A)) = < \Omega_\omega \, , \, A^* e^{itL} A e^{-itL} \Omega_\omega > = < \Omega_\omega \, , \, A^* e^{itL} A \Omega_\omega > = 0.$$

This means $F_{A^*A}(t) = 0$ for all $t \in \mathbb{R}$ and $F_{A^*A}(z) = 0$ on the strip (edge of the wedge theorem). In particular $F_{A^*A}(i\beta) = \omega(AA^*) = 0$ and $A^*\Omega_\omega = 0$. Now if $B^*A\Omega_\omega = 0$ then $A^*B\Omega_\omega = 0$ for all $B \in \mathcal{M}$. Thus $A^* = 0$ and $A = 0$.

5) In the same way as above, if L denotes the standard Liouvillian of α, for the K.M.S. state ω, if Δ is the modular operator associated to $(\mathcal{M}, \omega)$ then

$$L = -\beta \log \Delta.$$

6.4 Pertubation of W^*-dynamical systems

Let $(\mathcal{M}, \alpha, \omega)$ be a W^*-dynamical system. Consider $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ its standard form. Let L be the associated standard Liouvillian.

We now consider an essentially self-adjoint operator V on $\mathcal{H}$ which is affiliated to $\mathcal{M}$ (every bounded function of V belongs to $\mathcal{M}$).

Proposition 6.3 – *If* $H_V = L + V$ *is essentially self-adjoint on* $\mathrm{Dom}\, L \cap \mathrm{Dom}\, V$ *then*

$$\alpha_t^V(A) = e^{itH_V} A e^{-itH_V}$$

defines a new W^**-dynamical system on* $\mathcal{M}$.

Proof

By Trotter's formula

$$e^{itH_V} = s - \lim_{n \to +\infty} \left(e^{iLt/n} e^{iVt/n} \right)^n .$$

Thus

$$\alpha_t^V(A) = w - \lim_{n \to +\infty} \left(e^{iLt/n} e^{iVt/n} \right)^n A \left(e^{-iVt/n} e^{-iLt/n} \right)^n .$$

As $e^{iVt/n}$ belongs to $\mathcal{M}$ we have that $\alpha_t^V(\mathcal{M}) \subset \mathcal{M}$ for $\mathcal{M}$ is weakly closed. ■

Proposition 6.4 – *If*

$$L_V = L + V - JVJ$$

is essentially self-adjoint on $\mathrm{Dom}\, L \cap \mathrm{Dom}\, V \cap \mathrm{Dom}\, JVJ$ *then* L_V *is the standard Liouvillian of* α^V.

Proof

Trotter's formula gives

$$\begin{aligned} e^{itL_V} A e^{-itL_V} &= w - \lim_{n\to+\infty} \left(e^{iH_V t/n} e^{-iJVJt/n}\right)^n A \left(e^{iJVJt/n} e^{-iH_V t/n}\right)^n \\ &= w - \lim_{n\to+\infty} \left(e^{iH_V t/n}\right)^n A \left(e^{-iH_V t/n}\right)^n = \alpha_t^V(A) \end{aligned}$$

for $e^{\pm iJVJt/n}$ belongs to $\mathcal{M}'$.

Furthermore, as V and JVJ commute, the operator $V - JVJ$ is essentially self-adjoint on $\mathrm{Dom}\, V \cap \mathrm{Dom}\, JVJ$ and thus

$$e^{i(V-JVJ)t} = e^{itV} e^{-itJVJ} = e^{itV} J e^{itV} J$$

(recall that J is anti-linear!) and

$$e^{itL_V} = s - \lim_{n\to+\infty} \left(e^{iLt/n} e^{iVt/n} J e^{iVt/n} J\right)^n .$$

As $e^{iLt/n}\mathcal{P} \subset \mathcal{P}$ and $e^{iVt/n} J e^{iVt/n} J\mathcal{P} \subset \mathcal{P}$, this shows that $\mathcal{P}\,(=\overline{\mathcal{P}})$ is invariant under e^{itL_V}. ■

6.5 Ergodic theory

Let $(\mathcal{M}, \alpha, \omega)$ be a W^*-dynamical system and L be its standard Liouvillian. We assume that ω is faithfull.

Theorem 6.5 – *The following assertions are equivalent*

a) (Ergodicity) For all $A \in \mathcal{M}$, all normal state μ

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \mu(\alpha_t(A))\, dt = \omega(A).$$

b) For all $A, B \in \mathcal{M}$

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \omega(A\alpha_t(B))\, dt = \omega(A)\omega(B).$$

c) $1\!\!1_{\{0\}}(L) = |\Omega\rangle\langle\Omega|$, i.e., 0 is simple eigenvalue of L.

Proof

Let us prove the so-called von Neumann's ergodic theorem, that is,

$$1\!\!1_{\{0\}}(L) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} e^{itL}\, dt.$$

Indeed, for any $f \in \mathrm{Dom}\, L$, let μ_f be the spectral measure

$$\mu_f(A) = ||1\!\!1_A(L) f||^2.$$

We have

$$\begin{aligned} < f\,,\, \frac{1}{2T} \int_{-T}^{T} e^{itL}\, dt f > &= \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R}} e^{it\lambda}\, d\mu_f(\lambda)\, dt \\ &= \int_{\mathbb{R}} \frac{1}{\lambda T} \frac{e^{i\lambda T} - e^{-i\lambda T}}{2i}\, d\mu_f(\lambda) \\ &= \int_{\mathbb{R}} \frac{\sin \lambda T}{\lambda T}\, d\mu_f(\lambda) \end{aligned}$$

which converges to $\mu_f(\{0\})$ when T tends to $+\infty$. This gives the above formula.

This shows the equivalence between b) and c) for

$$\omega(A\alpha_t(B)) = < A^*\Omega\,,\, e^{itL} B\Omega >.$$

Assume a) is satisfied. Let $D \in \mathcal{M}'$, let $\Psi_\mu = D\Omega$ to which we associate the normal state μ. Then

$$\mu(\alpha_t(A)) = < D\Omega\,,\, \alpha_t(A)D\Omega > = < \Omega\,,\, (D^*D)\alpha_t(A)\Omega >$$

which converges, in ergodic limit, to

$$< \Omega\,,\, (D^*D)1\!\!1_{\{0\}}(L)A\Omega > = < \Omega\,,\, A\Omega >$$

by hypothesis a). This gives c).

If ω is faithfull (it is actually the only place where we use this assumption) then Ω is cyclic for $\mathcal{M}'$ and the μ's as above are dense in $\mathcal{M}_*$. This gives the converse direction. ■

Theorem 6.6 [Return to equilibrium] – *Consider the following assertions.*

a) (Return to equilibrium) For all $A \in \mathcal{M}$, *all normal state* μ

$$\lim_{t\to\pm\infty} \mu(\alpha_t(A)) = \omega(A).$$

b) (Mixing) For all $A, B \in \mathcal{M}$

$$\lim_{t\to\pm\infty} \mu(A\alpha_t(B)) = \omega(A)\omega(B).$$

c) $w - \lim_{t\to\pm\infty} e^{itL} = |\Omega\rangle\langle\Omega|$.

d) The spectrum of L is purely absolutely continuous except for the only simple eigenvalue 0.

Then a), b) and c) are equivalent, and d) implies the 3 first ones.

Proof

d) implies c) is a classical result of spectral theory that we do not develop here.

The identity

$$\omega(A\alpha_t(B)) = < A^*\Omega\,,\, e^{itL} B\Omega >$$

shows the equivalence between b) and c).

a) implies b) is proved as in the previous theorem.

b) implies a) also, using here the faithfullness. ■

6.6 Pauli-Fierz models

Pauli-Fierz models are representing simple systems in interaction with some large environment. For example a N-level atom in contact with a heat bath. Let us describe this model in its standard representation.

The small system is described by a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^n$, an Hamiltonian H and the thermal equilibrium state

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}.$$

In standard representation, we have

$$\begin{aligned} \mathcal{H}_S &= \mathcal{H} \otimes \overline{\mathcal{H}} \simeq L^2(\mathcal{H}) \\ \Omega_S &= \frac{e^{-\beta H/2}}{\mathrm{Tr}(e^{-\beta H})^{1/2}} \\ J_S &: \phi \otimes \overline{\Psi} \longmapsto \Psi \otimes \overline{\phi} \\ \mathcal{P}_S &= L^2_+(\mathcal{H}) \\ L_S &= H \otimes I - I \otimes \overline{H} \\ \mathcal{M}_S &= \mathcal{B}(\mathcal{H}) \otimes I. \end{aligned}$$

The reservoir is modelised by the free boson gaz: $\mathcal{K} = L^2(\mathbb{R}^3)$, $h = \omega(k) = |k|$, $\rho = (e^{\beta h} - 1)^{-1}$, that is, in standard representation:

$$\begin{aligned} \mathcal{H}_R &= \Gamma_s(\mathcal{K} \oplus \overline{\mathcal{K}}) \simeq \Gamma_s(\mathcal{K}) \otimes \Gamma_s(\overline{\mathcal{K}}) \\ \Omega_R &= \text{Fock vacuum} = \mathbb{1} \\ J_R &= \Gamma(C), \text{ with } C \ : \ (f, \overline{g}) \longmapsto (g, \overline{f}) \\ L_R &= d\Gamma(p), \text{ with } p = h \oplus (-h) \\ \mathcal{M}_R &= \{W_{\rho,\ell}(f); f \in \mathrm{Dom}\, \rho^{1/2}\}. \end{aligned}$$

Thus the combined system is represented by

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_S \otimes \mathcal{H}_R \\ \mathcal{M} &= \mathcal{M}_S \otimes \mathcal{M}_R \\ \Omega &= \Omega_S \otimes \Omega_R \\ \mathcal{P} &= \mathcal{P}_S \otimes \mathcal{P}_R \\ J &= J_R \otimes J_R \\ L &= L_S \otimes I + I \otimes L_S = L_S + L_R. \end{aligned}$$

Now we add a coupling V between the two systems, that is an operator V on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$, which is affiliated to $\mathcal{M}$ and which represents the way the two systems now interact. We put typically

$$V = Q^* \otimes a(g) + Q \otimes a^*(g).$$

The Liouvillian of the total system is now

$$L_V = L + V - JVJ$$

whose spectrum has to be studied in order to conclude on the ergodic properties, or return to equilibrium properties of our system.

SPECTRUM OF SELF-ADJOINT OPERATORS

Let T be a self-adjoint operator on $\mathcal{H}$. Recall that the resolvent set $\rho(T)$ is the set

$$\begin{aligned}\rho(T) &= \{z \in \mathbb{C}; (z-T) \text{ is injective and } (z-T)^{-1} \text{ is bounded}\} \\ &= \{z \in \mathbb{C}; (z-T) \text{ is bijective}\}\end{aligned}$$

The spectrum $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is then a non-empty, closed subset of $\mathbb{R}$.

We call *point spectrum* of T the set $\sigma_p(T)$ of eigenvalues of T.

Theorem 1 – *The following statements are equivalent for a self-adjoint operator T.*

i) $z \in \rho(T)$.

ii) There exists a $c > 0$ *such that* $||(z-T)f|| \geq c\,||f||$ *for all* $f \in \operatorname{Dom} T$.

iii) $\operatorname{Ran}(z-T) = \mathcal{H}$.

Proof

i) implies iii) is obvious.

iii) implies i): If $\operatorname{Ran} z - T = \mathcal{H}$ then $\operatorname{Ker} \overline{z} - T = \{0\}$. If z is real then $\operatorname{Ker} z - T = \{0\}$ and $z - T$ is bijective; if z is not real then one knows that z belongs to $\rho(T)$.

i) implies ii): This is just the boudedness of $(z-T)^{-1}$.

ii) implies i): By the inequality we have $\operatorname{Ker} z - T = \{0\}$ and $z - T$ is injective. But furthermore

$$||(z-T)^{-1}f|| \leq \frac{1}{c}\,||(z-T)(z-T)^{-1}f|| = \frac{1}{c}\,||f||\,.$$

■

Theorem 2 – *Let T be self-adjoint on $\mathcal{H}$ and E be its spectral measure. The following statements are equivalent.*

i) $s \in \sigma(T)$.

ii) There exists a sequence $(f_n)_{n\in\mathbb{N}}$ *in* $\operatorname{Dom} T$ *such that* $\liminf ||f_n|| > 0$ *and* $(s-T)f_n \to 0$.

iii) $E(s-\varepsilon, s+\varepsilon) \neq 0$ *for every* $\varepsilon > 0$.

Proof

ii) implies i): If $s \in \rho(T)$ then $(s-T)$ is invertible with continuous inverse and thus there exists no sequence $(f_n)_{n\in\mathbb{N}}$ such as in ii).

i) implies ii): If for every $c > 0$ there exists a $f \in \operatorname{Dom} T$ such that

$$||(s-T)f|| < c\,||f||\,,$$

then taking f_n to be one of these f, with norme 1, associated to $c = 1/n$, gives the requiered sequence.

ii) implies iii): If there exists a ε such that $E(s-\varepsilon, s+\varepsilon)=0$ then, for all $f \in \operatorname{Dom} T$

$$||(s-T)f||^2 = \int |s-t|^2 \, d||E(t)f||^2 \geq \varepsilon^2 \int d||E(t)f||^2 = \varepsilon^2 ||f||^2.$$

By Theorem 1 this means that $s \in \rho(T)$.

iii) implies ii): If $E(s-1/n, s+1/n) \neq 0$ for all n, then let $f_n \in \operatorname{Ran} E([s-1/n, s+1/n])$ with $||f_n|| = 1$. We have

$$||(s-T)f_n||^2 = \int |s-t|^2 \, d||E(t)f_n||^2 \leq \frac{1}{n^2} \int d||E(t)f_n||^2 = \frac{1}{n^2} ||f_n||^2.$$

This gives the requiered sequence. ■

Proposition 3 – *Every isolated point of $\sigma(T)$ is an eigenvalue of T.*

Proof

If $(\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(T) = \{\lambda\}$ then $E(\lambda-\varepsilon, \lambda) = E(\lambda, \lambda+\varepsilon) = 0$. By Theorem 2, we know that as $\lambda \in \sigma(T)$ then $E(\{\lambda\}) = E(\lambda-\varepsilon, \lambda+\varepsilon) \neq 0$.

One then easily checks that this implies that λ is an eigenvalue (take a v is $\operatorname{Ran} E(\{\lambda\})$ and compute Tv). ■

Denote by $\sigma_e(T)$ the set of $s \in \sigma(T)$ such that s is either an isolated eigenvalue with infinite multiplicity, or an accumulation point. This set is called the *essential spectrum* of T.

Denote by $\sigma_d(T)$ the set $\sigma(T) \setminus \sigma_e(T)$, that is, the set of eigenvalues with finite multiplicity. This is called the *discrete spectrum* of T.

Theorem 4 – *With the same notation as above, the following assertions are equivalent.*

i) $s \in \sigma_e(T)$.

ii) There exists a sequence $(f_n)_{n\in \mathbb{N}}$ *such that* $f_n \underset{w}{\longrightarrow} 0$, $\liminf ||f_n|| > 0$ *and* $||(s-T)f_n|| \longrightarrow 0$.

iii) For every $\varepsilon > 0$, $\dim E(s-\varepsilon, s+\varepsilon) = +\infty$.

Proof

i) implies ii): If s is an eigenvalue with infinite multiplicity then there exists an infinite orthonormal family $(f_n)_{n\in \mathbb{N}}$ in $\operatorname{Ker} s-T$. This sequence satisfies ii).

If s is an accumulation point then take $(s_n)_{n\in \mathbb{N}}$ to be a sequence in $\sigma(T)$ of two by two different real numbers, converging to s. Choose ε_n such that the intervals $(s_n-\varepsilon_n, s_n+\varepsilon_n)$ are disjoint. As s_n belongs to $\sigma(T)$ then $E(s_n-\varepsilon_n, s_n+\varepsilon_n) \neq 0$. Choose a f_n with norme 1 and in the range of the above projector. Then the sequence $(f_n)_{n\in \mathbb{N}}$ is orthonormal and $(s-T)f_n$ converges to 0.

ii) implies iii): If $\dim E(s-\varepsilon, s+\varepsilon) < \infty$ for some $\varepsilon > 0$, then this projector is compact. If $(g_n)_{n\in \mathbb{N}}$ is a sequence such as in ii) then $E(s-\varepsilon, s+\varepsilon)g_n \to 0$.

Furthermore

$$\begin{aligned}
\left|\left|(s-T)g_n\right|\right|^2 &= \int |s-t|^2 \; d\left|\left|E(t)g_n\right|\right|^2 \\
&\geq \varepsilon^2 \left[\int d\left|\left|E(t)g_n\right|\right|^2 - \int 1\!\!1_{(s-\varepsilon,s+\varepsilon)}(t)\, d\left|\left|E(t)g_n\right|\right|^2\right] \\
&= \varepsilon^2 \left(\left|\left|g_n\right|\right|^2 - \left|\left|E(s-\varepsilon,s+\varepsilon)g_n\right|\right|^2\right).
\end{aligned}$$

In particular $\liminf \left|\left|(s-T)g_n\right|\right|^2 \geq \varepsilon^2 \liminf \left|\left|g_n\right|\right|^2 > 0$, which leads to a contradiction.

iii) implies i): If $\dim E(\{s\}) = \infty$ then s is an eigenvalue with infinite multiplicity. If $\dim E(\{s\}) < \infty$ and, by hypothesis, $\dim E(s-\varepsilon, s+\varepsilon) = \infty$ for all $\varepsilon > 0$, then $(s-\varepsilon, s) \cup (s, s+\varepsilon)$ contains at least one spectral point for every $\varepsilon > 0$. Thus s is an accumulation point of $\sigma(T)$. ■

We shall now distinguish several subspaces of $\mathcal{H}$ depending on the spectral properties of T. For a $f \in \operatorname{Dom} T$ we denote by μ_f the measure $d\left|\left|E(t)f\right|\right|^2$.

We put $\mathcal{H}_p$ to be the closed linear subspace generated by the eigenvectors of T, it is the *discontinuous subspace* of $\mathcal{H}$ relative to T.

We denote by $\mathcal{H}_c$ the orthogonal of $\mathcal{H}_p$, the *continuous subspace.*

Let $\mathcal{H}_{sc}$ be the subspace of $f \in \mathcal{H}_c$ such that μ_f is singular with respect to the Lebesgue measure λ, that is, μ_f is supported by a null set for λ. This is the *singular continuous subspace.*

Let $\mathcal{H}_{ac} = \mathcal{H}_c \ominus \mathcal{H}_{sc}$, the *absolutely continuous* subspace.

Let $\mathcal{H}_s = \mathcal{H}_p \oplus \mathcal{H}_{sc}$, the *singular* subspace.

Theorem 5 –

a) $\mathcal{H}_p$ *is the space of* $f \in \mathcal{H}$ *such that* μ_f *is atomic.*

b) $\mathcal{H}_c$ *is the space of* $f \in \mathcal{H}$ *such that* $\mu_f(\{t\}) = 0$ *for every* t*, that is, such that* $t \longmapsto \mu_f([0,t])$ *is continuous.*

c) $\mathcal{H}_s$ *is the space of* $f \in \mathcal{H}$ *such that* μ_f *is singular.*

d) $\mathcal{H}_{ac}$ *is the space of* $f \in \mathcal{H}$ *such that* μ_f *is absolutely continuous.*

Proof

a) If f belongs to $\mathcal{H}_p$ then $f = \sum_i c_i f_i$ with $Tf_i = \lambda_i f_i$. thus $f = \sum_i E(\{\lambda_i\})f$ and μ_f is carried by this set of λ_i's.

Conversely, if μ_f is atomic, carried by some λ_i's then $E(\{\lambda_i\}) \neq 0$, thus λ_i is an eigenvalue of T ...etc.

b) Any $E(\{t\})f$ belongs to $\mathcal{H}_p$. Thus if $f \in \mathcal{H}_c = \mathcal{H}_p^{\perp}$ then $\mu_f(\{t\}) = 0$ for all t.

Conversely, if $\mu_f(\{t\}) = 0$ for all t, then $\mu_f(A) = 0$ for all countable A. If g belongs to $\mathcal{H}_p$ then there exists a countable A such that $g = E(A)g$ and thus f is orthogonal to g and f belongs to $\mathcal{H}_c$.

c) We have $\mathcal{H}_s = \mathcal{H}_p \oplus \mathcal{H}_{sc}$. Let $f = f_p + f_{sc} \in \mathcal{H}_s$. Then $E(A)f_p = f_p$ for a countable A and $E(N)f_{sc} = f_{sc}$ for a null N. Finally, $E(A \cup N)f = f$ and $A \cup N$ is null.

Conversely, if $E(N)f = f$ for a null N, let A be the set of t in N which are jumping times for the function $||E([0,t])f||^2$. As this function is increasing, the set A is countable. Thus $E(A)f$ belongs to $\mathcal{H}_p$. Note that

$$E(\{t\})(f - E(A)f) = E(\{t\})f - E(\{t\} \cap A)f = 0$$

for all t. Thus if g belongs to $\mathcal{H}_p$ clearly $f - E(A)f$ is orthogonal to g. This shows $f - E(A)f$ belongs to $\mathcal{H}_c$. But because of the equality $E(N)(f - E(A)f) = f - E(A)f$ we know that $f - E(A)f$ belongs to $\mathcal{H}_{sc}$. This gives the decomposition of f.

d) Let $f \in \mathcal{H}_{ac} = \mathcal{H}_s^{\perp}$. Then $||E(N)f||^2 = \, < f \, , \, E(N)f > \, = 0$ for any null set N. This shows that μ_f is absolutely continuous.

Conversely, if $E(N)f = 0$ for any null set N then f is orthogonal to $\mathcal{H}_s$ and $f \in \mathcal{H}_{ac}$. ■

Let M be a closed subspace of $\mathcal{H}$ and P the orthogonal projection onto M. We say that M *reduces* T if $PT \subset TP$ (in particular $P \,\mathrm{Dom}\, T \subset \mathrm{Dom}\, T$ and also $(I - P)T \subset T(I - P)$).

Theorem 6 – *The subspace M reduces the self-adjoint operator T if and only if $PE([0,t]) = E([0,t])P$ for all t.*

Proof

If M reduces T then

$$\begin{aligned}(z - T)^{-1}P &= (z - T)^{-1}P(z - T)(z - T)^{-1} \\ &\subset (z - T)^{-1}(z - T)P(z - T)^{-1} = P(z - T)^{-1}.\end{aligned}$$

Therefore $(z - T)^{-1}P = P(z - T)^{-1}$ for $\mathrm{Dom}(z - T)^{-1}P = \mathcal{H}$. It follows from the formula

$$< g \, , \, E(a,b)f > \, = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2i\pi} \int_{a+\delta}^{b+\delta} < g \, , \, ((t - i\varepsilon - T)^{-1} - (t + i\varepsilon - T)^{-1})f > dt$$

that $PE([0,t]) = E([0,t])P$ for all t.

Conversely, if $PE([0,t]) = E([0,t])P$ for all t holds true then

$$\int |t|^2 \, d||E(t)Pf||^2 = \int |t|^2 \, d||PE(t)f||^2 \leq \int |t|^2 \, d||E(t)f||^2$$

and thus $Pf \in \mathrm{Dom}\, T$ for all $f \in \mathrm{Dom}\, T$. The identity $PTf = TPf$ is then easily obtained by the spectral theorem, approximating the identity function by a sequence of step functions. ■

Define $\mathrm{Dom}\, T_M = M \cap \mathrm{Dom}\, T$ and $T_M f = Tf$ on $\mathrm{Dom}\, T_M$. This defines an operator T_M on M.

Note that $\mathrm{Dom}\, T_{M^\perp}$ and $T_{M^\perp}$ are defined in the same way and that $\mathrm{Dom}\, T = \mathrm{Dom}\, T_M \oplus \mathrm{Dom}\, T_{M^\perp}$.

Note that $\mathrm{Dom}\, T_M$ is dense in M for the orthogonal of $\overline{\mathrm{Dom}\, T_M}$ in M is $\mathrm{Dom}\, T^{\perp} \cap M = \{0\}$ by the equality $\mathrm{Dom}\, T = \mathrm{Dom}\, T_M \oplus \mathrm{Dom}\, T_{M^\perp}$.

Theorem 7 – *If T is self-adjoint on M with spectral family E and M reduces T, then T_M (and $T_{M^\perp}$) is self-adjoint on M (on $M^\perp$). Furthermore $\sigma(T) = \sigma(T_M) \cup \sigma(T_{M^\perp})$.*

Proof

Clearly T_M is symmetric. It remains to prove $\mathrm{Dom}\, T_M^* \subset \mathrm{Dom}\, T_M$. Let $g \in \mathrm{Dom}\, T_M^*$ then

$$\begin{aligned} < T_M^* g \,,\, f > &= < T_M^* g \,,\, f_1 > = < g \,,\, T_M f_1 > \\ &= < g \,,\, T_M f_1 + T_{M^\perp} f_2 > = < g \,,\, T f > \end{aligned}$$

for all $f = f_1 + f_2 \in \mathrm{Dom}\, T = \mathrm{Dom}\, T_M \oplus \mathrm{Dom}\, T_{M^\perp}$. This proves that $g \in \mathrm{Dom}\, T^* \cap M = \mathrm{Dom}\, T \cap M = \mathrm{Dom}\, T_M$. We have proved the self-adjointness.

Note that

$$||(z - T)f||^2 = ||(z - T_M)f_1||^2 + ||(z - T_{M^\perp})f_2||^2.$$

By Theorem 1 we thus get $\rho(T) = \rho(T_M) \cap \rho(T_{M^\perp})$. ■

Theorem 8 – *The subspaces $\mathcal{H}_p$, $\mathcal{H}_c$, $\mathcal{H}_{sc}$, $\mathcal{H}_{ac}$ and $\mathcal{H}_s$ reduce the operator T.*

Proof

It is sufficient to prove it for $\mathcal{H}_p$ and $\mathcal{H}_s$, as the other ones are deduced by orthogonality.

For any $f \in \mathcal{H}_p$ there exists a countable A such that $E(A)f = f$. Thus $E([0,t])f = E([0,t])E(A)f = E(A)E([0,t])f$ and $E([0,t])f$ belongs to $\mathcal{H}_p$. This proves $E([0,t])P_p = P_p E([0,t])P_p$. Passing to the adjoint we get $E([0,t])P_p = E([0,t])P_p$.

The other one is treated in the same way. ■

We denote by T_p, T_c, T_{sc}, T_{ac} and T_s the restrictions of T to the corresponding subspaces. These operators are called *discontinuous, continuous, singular continuous, absolutely continuous, singular parts* of T.

Define $\sigma_c(T)$, $\sigma_{sc}(T)$, $\sigma_{ac}(T)$, $\sigma_s(T)$ to be the spectrum associated to the respective operator. Be careful that $\sigma(T_p)$ is not in general $\sigma_p(T)$ for $\sigma_p(T)$ is not a spectrum and thus not closed in general; only one has for sure $\sigma(T_p) = \overline{\sigma_p(T)}$. We have

$$\sigma(T) = \overline{\sigma_p(T)} \cup \sigma_{sc}(T) \cup \sigma_{ac}(T) = \sigma_s(T) \cup \sigma_{ac}(T) = \overline{\sigma_p(T)} \cup \sigma_c(T).$$

These subsets of $\sigma(T)$ are called *point-spectrum, continuous spectrum, singular continuous spectrum, absolutely continuous spectrum, singular spectrum* of T.

We can now prove the missing part of Theorem 6.6

If the spectrum of L is purely absolutely continuous except for the only simple eigenvalue 0 then

$$w - \lim_{t \to \pm\infty} e^{itL} = |\Omega\rangle\langle\Omega|.$$

Indeed, the space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \mathbb{C}\Omega \oplus \mathcal{H}_{ac}.$$

Let $f \in \mathcal{H}$ and let us decompose it as $f = <\Omega\,,\, f>\Omega + f_{ac}$, with obvious notations. Then clearly

$$< f\,,\, e^{itL} f > = \left|< \Omega\,,\, f >\right|^2 + < f_{ac}\,,\, e^{itL} f_{ac} >.$$

As f_{ac} belongs to the absolutely continuous part of $\mathcal{H}$ relative to T then $d\mu_{f_{ac}}(x) = g(x)\,dx$ for some integrable function g. Consequently, by the spectral theorem

$$\begin{aligned} < f_{ac}\,,\, e^{itL} f_{ac} > &= \int_{\mathbb{R}} e^{itx}\, d\mu_{f_{ac}}(x) \\ &= \int_{\mathbb{R}} e^{itx} g(x)\, dx \\ &= \widehat{g}(t) \end{aligned}$$

which converges to 0 when $|t| \to \infty$, by Riemann-Lebesgue's Lemma. ■