Lecture 1

OPERATOR AND SPECTRAL THEORY

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Abstract This lecture is a complete introduction to the general theory of operators on Hilbert spaces. We particularly focus on those tools that are essentials in Quantum Mechanics: unbounded operators, multiplication operators, self-adjointness, spectrum, functional calculus, spectral measures and von Neumann's Spectral Theorem.

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1.1 Basic Definitions

1.1.1 Operators, Domains, Graphs

Definition 1.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. An *operator* T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear mapping from a subspace $\mathrm{Dom}\,\mathsf{T}$ of $\mathcal{H}_1$ to the space $\mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2$ we shall simply say that T is an *operator on* $\mathcal{H}_1$.

The space $\mathrm{Dom}\,\mathsf{T}$ is called the *domain* of T. It is very important in many situations to consider only operators whose domain is dense in $\mathcal{H}_1$, but be aware that this may not always be the case.

One denotes by $\mathrm{Ran}\,\mathsf{T}$ the *range* of T, that is, the set $\{\mathsf{T}f\,;\ f \in \mathrm{Dom}\,\mathsf{T}\}$. It is a subspace of $\mathcal{H}_2$. One denotes by $\mathrm{Ker}\ \mathsf{T}$ the *kernel* of T, that is the set $\{f \in \mathrm{Dom}\,\mathsf{T}\,; \mathsf{T}f = 0\}$. It is a subspace of $\mathcal{H}_1$.

If T is such that $\mathrm{Ker}\ \mathsf{T} = \{0\}$ then T is injective and admits an *inverse* T^{-1} which is an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$, with domain $\mathrm{Dom}\,\mathsf{T}^{-1} = \mathrm{Ran}\,\mathsf{T}$ (if it is dense) and defined by

$$\mathsf{T}^{-1}g = f \quad \text{if} \quad g = \mathsf{T}f\,.$$

Definition 1.2. If T is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\lambda \in \mathbb{C}$, then the operator $\lambda\mathsf{T}$ has domain $\mathrm{Dom}\,\lambda\mathsf{T} = \mathrm{Dom}\,\mathsf{T}$ and action

$$(\lambda\mathsf{T})f = \lambda(\mathsf{T}f)\,.$$

If S and T are operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ then the operator $\mathsf{S}+\mathsf{T}$ has domain

$$\mathrm{Dom}(\mathsf{S}+\mathsf{T}) = (\mathrm{Dom}\,\mathsf{S}) \cap (\mathrm{Dom}\,\mathsf{T})$$

(if it is dense) and action

$$(\mathsf{S}+\mathsf{T})f = \mathsf{S}f + \mathsf{T}f\,.$$

If T is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and S is an operator from $\mathcal{H}_2$ to $\mathcal{H}_3$, then the operator ST is an operator from $\mathcal{H}_1$ to $\mathcal{H}_3$, with domain

$$\mathrm{Dom}(\mathsf{ST}) = \{f \in \mathrm{Dom}\,\mathsf{T}\,;\ \mathsf{T}f \in \mathrm{Dom}\,\mathsf{S}\}$$

(if it is a dense domain) and action

$$(\mathsf{ST})f = \mathsf{S}(\mathsf{T}f)\,.$$

Definition 1.3. Important examples of operators for us are the *multiplication operators*. Let $(\Omega, \mathcal{F}, \mu)$ be a measured space and $f : \Omega \to \mathbb{C}$ be a measurable function. Then the operator M_f of multiplication by f is defined as an operator on $L^2(\Omega, \mathcal{F}, \mu)$, with domain

$$\mathrm{Dom}\,\mathsf{M}_f = \big\{g \in L^2(\Omega, \mathcal{F}, \mu)\,;\ fg \in L^2(\Omega, \mathcal{F}, \mu)\big\}$$

and action

$$\mathsf{M}_f\, g = fg\,.$$

Lemma 1.4. *The domain of M_f is always dense in $L^2(\Omega, \mathcal{F}, \mu)$.*

Proof. Let f be fixed in $L^2(\Omega, \mathcal{F}, \mu)$. For any $g \in L^2(\Omega, \mathcal{F}, \mu)$ and any $n \in \mathbb{N}$ put $E_n = \{\omega \in \Omega\,;\ |f(\omega)| \leq n\}$ and $g_n = g\,\mathbb{1}_{E_n}$. Then clearly g_n belongs to $\mathrm{Dom}\,\mathsf{M}_f$ and (g_n) converges to g in $L^2(\Omega, \mathcal{F}, \mu)$ by Lebesgue's Theorem. □

Definition 1.5. The *graph* of an operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is the set

$$\Gamma(\mathsf{T}) = \big\{(f, \mathsf{T}f)\,;\ f \in \mathrm{Dom}\,\mathsf{T}\big\} \subset \mathcal{H}_1 \times \mathcal{H}_2\,.$$

Recall that $\mathcal{H}_1 \times \mathcal{H}_2$ is a Hilbert space for the scalar product

$$\langle (x_1, x_2), (x_1', x_2') \rangle = \langle x_1, x_1' \rangle + \langle x_2, x_2' \rangle\,.$$

Proposition 1.6. *A subset G of $\mathcal{H}_1 \times \mathcal{H}_2$ is the graph of an operator if and only if it is a subspace of $\mathcal{H}_1 \times \mathcal{H}_2$ and $(0, g) \in G$ implies $g = 0$.*

In particular, every subspace of a graph is a graph.

Proof. If G is the graph of an operator, then the properties above are obviously satisfied. Conversely, if G satisfies the properties above, let

$$\mathrm{Dom}\,\mathsf{T} = \{f \in \mathcal{H}_1\,;\ \text{there exists } g \in \mathcal{H}_2 \text{ with } (f, g) \in G\big\}\,.$$

By hypothesis the g associated to f is unique and $\mathrm{Dom}\,\mathsf{T}$ is a subspace. If one defines $\mathsf{T}f = g$ on $\mathrm{Dom}\,\mathsf{T}$, then it is easy to check that T is linear. Thus T is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ whose graph is G. □

Definition 1.7. An operator S is an *extension* of an operator T if $\Gamma(\mathsf{T}) \subset \Gamma(\mathsf{S})$. This situation is simply denoted by $\mathsf{T} \subset \mathsf{S}$. This property is clearly equivalent to $\mathrm{Dom}\,\mathsf{T} \subset \mathrm{Dom}\,\mathsf{S}$ and $\mathsf{T}f = \mathsf{S}f$ for all $f \in \mathrm{Dom}\,\mathsf{T}$. In this situation we also say that T is a *restriction* of S.

1.1.2 Bounded Operators

We do not recall here the well-known facts about bounded operators on Hilbert spaces, their continuity and their associated operator norm. We just recall some important theorems and setup a few notations.

Theorem 1.8. *If T is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with a dense domain, then there exists a unique bounded extension S of T which is defined on the whole of $\mathcal{H}_1$. This extension is also a bounded operator and it satisfies $\|\mathsf{S}\| = \|\mathsf{T}\|$.*

From now on, every bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is considered to be defined on the whole of $\mathcal{H}_1$.

The following theorem is of much use.

Theorem 1.9 (Riesz Theorem). *Every continuous operator T from a Hilbert space $\mathcal{H}$ to $\mathbb{C}$ (i.e. every continuous linear form on $\mathcal{H}$) is of the form*

$$\mathsf{T}\varphi = \langle \psi, \varphi \rangle$$

for a $\psi \in \mathcal{H}$. This ψ associated to T is unique.

Definition 1.10. The space of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the space of bounded operators on $\mathcal{H}_1$ is denoted by $\mathcal{B}(\mathcal{H}_1)$. On these spaces co-exist many different topologies which are all very useful. Let us here recall the three main ones.

– The *uniform topology* or *operator-norm topology* is the topology induced by the operator norm.

– The *strong topology* is the one induced by the seminorms

$$n_x(\mathsf{T}) = \|\mathsf{T}x\| \ , \quad x \in \mathcal{H}_1 \, .$$

– The *weak topology* is the one induced by the seminorms

$$n_{x,y}(\mathsf{T}) = |\langle \mathsf{T}x \, , \, y \rangle| \ , \quad x \in \mathcal{H}_1 \, , \ y \in \mathcal{H}_2 \, .$$

The space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ then enjoys nice properties with respect to these topologies.

Theorem 1.11.
1) The space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space when equipped with the operator norm.
2) The space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is complete for the strong topology.
3) The space $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is complete for the weak topology.
4) If (T_n) converges strongly (or weakly) to T in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ then

$$\|\mathsf{T}\| \leq \liminf_n \|\mathsf{T}_n\| .$$

1.1.3 Closed and Closable Operators

Definition 1.12. An operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is *closed* if $\Gamma(\mathsf{T})$ is a closed set in $\mathcal{H}_1 \times \mathcal{H}_2$. An operator T is *closable* if it admits a closed extension.

It is easy to check that if S_1 and S_2 are two closed extension of T, then their intersection (the operator S_1 defined on $\mathrm{Dom}\,\mathsf{S}_1 \cap \mathrm{Dom}\,\mathsf{S}_2$) is again a closed extension of T. The smallest (in the sense of the restriction comparison) closed extension of a closable operator T is called the *closure* of T and is denoted by $\overline{\mathsf{T}}$.

In order to close any operator one may be tempted to take the closure of its graph. But the problem is that the closure of a graph is not a graph in general. As a counter-example, consider a Hilbert space $\mathcal{H}$, with orthonormal basis (e_n). Consider an element $\varphi = \sum_n \alpha_n e_n$ of $\mathcal{H}$, with only a finite number of α_n being null. Let D be the vector space generated by the finite linear combinations of e_n's and φ. Define an operator T on D by

$$\mathsf{T}\left(\lambda\varphi + \sum_{i=1}^{n} \lambda_i e_i\right) = \lambda\varphi .$$

Then $\overline{\Gamma(\mathsf{T})}$ contains (φ, φ) obviously, but also $(\varphi, 0)$ as a limit of $(\sum_{i=1}^N \alpha_i e_i, 0)$. The closure of the graph of T is not an operator graph (by Proposition 1.6).

Proposition 1.13. *If T is closable then $\Gamma(\overline{\mathsf{T}}) = \overline{\Gamma(\mathsf{T})}$.*

Proof. If S is any closed extension of T then $\overline{\Gamma(\mathsf{T})} \subset \Gamma(\mathsf{S})$. As a consequence, the subspace $\overline{\Gamma(\mathsf{T})}$ is the graph of some operator R (Proposition 1.6). We have that $\mathsf{R} \subset \mathsf{S}$ and also that R is closed. We have proved that R is the smallest closed extension of T, that is, $\mathsf{R} = \overline{\mathsf{T}}$. □

We then have the following easy characterizations.

Theorem 1.14. *Let T be an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

1) The operator T *is closed if and only if, for every sequence* (f_n) *in* $\mathrm{Dom}\,\mathsf{T}$ *such that* (f_n) *converges to some* $f \in \mathcal{H}_1$ *and* $(\mathsf{T}f_n)$ *converges to some* $g \in \mathcal{H}_2$ *, we have* $f \in \mathrm{Dom}\,\mathsf{T}$ *and* $\mathsf{T}f = g$ *.*

2) The operator T *is closable if and only, if for every sequence* (f_n) *in* $\mathrm{Dom}\,\mathsf{T}$ *such that* (f_n) *converges to* 0 *and* $(\mathsf{T}f_n)$ *converges to some* $g \in \mathcal{H}_2$ *, we have* $g = 0$ *.*

3) If T *is closable then* $\mathrm{Dom}\,\overline{\mathsf{T}}$ *coincides with the set of* $f \in \mathcal{H}_1$ *such that there exists a sequence* (f_n) *in* $\mathrm{Dom}\,\mathsf{T}$ *converging to* f *and such that* $(\mathsf{T}f_n)$ *converges in* $\mathcal{H}_2$*.*

On this domain, the operator $\overline{\mathsf{T}}$ *is given by* $\overline{\mathsf{T}}f = \lim\, \mathsf{T}f_n$ *.*

Proof. 1) This is just the definition of "$\Gamma(\mathsf{T})$ is closed".

2) If T is closable, then $\mathsf{T} \subset \mathsf{S}$ for some closed operator S. If $(f_n) \subset \mathrm{Dom}\,\mathsf{T}$ tends to 0 and $(\mathsf{T}f_n)$ tends to g, then $(0, g)$ belongs to $\Gamma(\mathsf{S})$ and thus $g = 0$. This proves one direction.

Conversely, if the property above holds true, then consider the closure $\overline{\Gamma(\mathsf{T})}$. It is the graph of some operator S by our hypothesis and by Proposition 1.6. Hence S is closed and T is closable.

3) This is just the definition of the operator whose graph is $\overline{\Gamma(\mathsf{T})}$, that is, the operator $\overline{\mathsf{T}}$. □

Note the following easy property which is often useful.

Proposition 1.15. *If* T *is a closed operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *then* $\mathrm{Ker}\,\mathsf{T}$ *is a closed subspace of* $\mathcal{H}_1$*.*

Proof. If (f_n) is a sequence in $\mathrm{Ker}\,\mathsf{T}$ which converges to a $f \in \mathcal{H}_1$, then the sequence $(\mathsf{T}f_n)$ is constant equal to 0 and hence convergent. By Theorem 1.14, as T is closed, this implies that f belongs to $\mathrm{Dom}\,\mathsf{T}$ and that $\mathsf{T}f = \lim \mathsf{T}f_n = 0$. Hence f belongs to $\mathrm{Ker}\,\mathsf{T}$. □

Now we show that the inverse of a closed operator is closed too.

Definition 1.16. Define the mapping

$$\begin{aligned} \mathsf{V} : \mathcal{H}_1 \times \mathcal{H}_2 &\longrightarrow \mathcal{H}_2 \times \mathcal{H}_1 \\ (\varphi, \psi) &\longmapsto (\psi, \varphi)\,. \end{aligned}$$

Proposition 1.17. *Let* T *be a densely defined and injective operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$*, then the following holds.*

1) $\Gamma(\mathsf{T}^{-1}) = \mathsf{V}\,(\Gamma(\mathsf{T}))$.

2) If T *is a closed and injective operator then* T^{-1} *is closed.*

Proof. 1) is obvious, let us prove 2). If T is closed and injective we have

$$\Gamma(\mathsf{T}^{-1}) = \mathsf{V}\left(\Gamma(\mathsf{T})\right) = \mathsf{V}\left(\overline{\Gamma(\mathsf{T})}\right) = \mathsf{V}(\Gamma(\mathsf{T})^{\perp\perp})\,.$$

But it is easy to check that we always have $\mathsf{V}(E^\perp) = \mathsf{V}(E)^\perp$ for all subspace E of $\mathcal{H}_1 \times \mathcal{H}_2$. Hence we have

$$\Gamma(\mathsf{T}^{-1}) = \mathsf{V}(\Gamma(\mathsf{T}))^{\perp\perp} = \overline{\Gamma(\mathsf{T}^{-1})}\,.$$

That is, T^{-1} is a closed operator. □

1.2 Adjoint

1.2.1 Definitions, Basic Properties

Definition 1.18. Let T be an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, with a dense domain $\operatorname{Dom}\mathsf{T}$ (here the density hypothesis is fundamental). We define the space

$$\operatorname{Dom}\mathsf{T}^* = \{\varphi \in \mathcal{H}_2\,;\ \psi \mapsto \langle \varphi, \mathsf{T}\psi\rangle \text{ is continuous on } \operatorname{Dom}\mathsf{T}\}\ .$$

For every $\varphi \in \operatorname{Dom}\mathsf{T}^*$, the continuous mapping $\psi \mapsto \langle \varphi, \mathsf{T}\psi\rangle$, defined on $\operatorname{Dom}\mathsf{T}$, can be extended to a continuous linear form on $\mathcal{H}_1$ (by Theorem 1.8). By Riesz Theorem (Theorem 1.9) this form can be written as $\psi \mapsto \langle \mathsf{A}(\varphi), \psi\rangle$ for some unique element $\mathsf{A}(\varphi) \in \mathcal{H}_2$. One can easily check that $\mathsf{A}(\varphi)$ is linear in φ, on $\operatorname{Dom}\mathsf{T}^*$. Thus the mapping $\varphi \mapsto \mathsf{A}(\varphi)$ defines an operator from $\operatorname{Dom}\mathsf{T}^* \subset \mathcal{H}_2$ to $\mathcal{H}_1$. This operator is denoted by T^* and called the *adjoint* of T. Be aware that $\operatorname{Dom}\mathsf{T}^*$ has no reason to be dense in $\mathcal{H}_2$, in general!

The fundamental relation which defines the adjoint operator is thus

$$\boxed{\langle \varphi, \mathsf{T}\psi\rangle = \langle \mathsf{T}^*\varphi, \psi\rangle}$$

for all $\psi \in \operatorname{Dom}\mathsf{T}$, all $\varphi \in \operatorname{Dom}\mathsf{T}^*$.

We now collect some basic properties associated to adjoint operators.

Proposition 1.19. *Let T be a (densely defined) operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.*
1) We have $\operatorname{Ker}\mathsf{T}^* = (\operatorname{Ran}\mathsf{T})^\perp$.
2) If $\operatorname{Dom}\mathsf{T}^*$ *is dense, then* $\mathsf{T} \subset \mathsf{T}^{**}$.
3) The operator T is bounded if and only if T^* is bounded. In that case we have

$$\|\mathsf{T}^*\| = \|\mathsf{T}\|\,, \tag{1.1}$$

but also

$$\|\mathsf{T}^*\mathsf{T}\| = \|\mathsf{T}\|^2\,. \tag{1.2}$$

Proof. 1) A vector $\varphi \in \mathcal{H}_1$ belongs to $(\mathrm{Ran}\,\mathsf{T})^\perp$ if and only if

$$\langle \varphi, \mathsf{T}\psi \rangle = 0 \quad \text{for all} \quad \psi \in \mathrm{Dom}\,\mathsf{T}\,.$$

This is clearly equivalent to saying that $\varphi \in \mathrm{Dom}\,\mathsf{T}^*$ and $\mathsf{T}^*\varphi = 0$. In other words this is equivalent to $\varphi \in \mathrm{Ker}\ \mathsf{T}^*$.

2) Let $g \in \mathrm{Dom}\,\mathsf{T}^*$ and $f \in \mathrm{Dom}\,\mathsf{T}$, we then have

$$\langle f, \mathsf{T}^* g \rangle = \langle \mathsf{T}f, g \rangle\,.$$

The expression above is thus continuous in g on $\mathrm{Dom}\,\mathsf{T}^*$, this proves that f belongs to $\mathrm{Dom}\,\mathsf{T}^{**}$ and that $\mathsf{T}^{**}f = \mathsf{T}f$. We have proved the inclusion $\mathsf{T} \subset \mathsf{T}^{**}$.

3) For every $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ we have

$$|\langle g, \mathsf{T}f \rangle| \leq \|f\|\ \|\mathsf{T}\|\ \|g\|\,.$$

The expression above is thus continuous in f on $\mathcal{H}_1$. This proves that $\mathrm{Dom}\,\mathsf{T}^* = \mathcal{H}_2$ and that

$$|\langle \mathsf{T}^* g, f \rangle| \leq \|f\|\ \|\mathsf{T}\|\ \|g\|\,.$$

This inequality shows that T^* is a bounded operator with $\|\mathsf{T}^*\| \leq \|\mathsf{T}\|$.

Applying this to T^* shows that T^{**} is a bounded operator with norm dominated by $\|\mathsf{T}^*\|$. But 2) shows that $\mathsf{T} = \mathsf{T}^{**}$. This means that $\|T\| \leq \|T^*\|$, hence we have proved the first equality of norms.

Let us prove the second equality of norms. As $\|\mathsf{T}^*\mathsf{T}\| \leq \|\mathsf{T}^*\|\,\|\mathsf{T}\| = \|\mathsf{T}\|^2$, it is sufficient to prove that $\|\mathsf{T}^*\mathsf{T}\| \geq \|\mathsf{T}\|^2$. We have

$$\|\mathsf{T}\mathsf{T}^*\| \geq \sup_{\|f\|=1} |\langle f, \mathsf{T}^*\mathsf{T}f \rangle| = \sup_{\|f\|=1} |\langle \mathsf{T}f, \mathsf{T}f \rangle| = \sup_{\|f\|=1} \|\mathsf{T}f\|^2 = \|\mathsf{T}\|^2\,.$$

This gives the result. □

Proposition 1.20. *Let* S *and* T *be two densely defined operators from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *. Let* R *be a densely defined operator from* $\mathcal{H}_0$ *to* $\mathcal{H}_1$*.*

1) If $\lambda \in \mathbb{C}^*$ *then* $(\lambda\mathsf{T})^* = \overline{\lambda}\,\mathsf{T}^*$*.*

2) If $\mathsf{S}+\mathsf{T}$ *is densely defined then* $\mathsf{S}^* + \mathsf{T}^* \subset (\mathsf{S}+\mathsf{T})^*$*.*

3) If $\mathsf{T}\,\mathsf{R}$ *is densely defined then* $\mathsf{R}^*\mathsf{T}^* \subset (\mathsf{T}\,\mathsf{R})^*$*.*

4) If T *is bounded then we have equalities in 2) and 3) above.*

Proof. 1), 2) and 3) are easy from the definitions and left to the reader. Let us prove 4). Assume that T is bounded. If $g \in \mathrm{Dom}(\mathsf{S}+\mathsf{T})^*$ then for all $f \in \mathrm{Dom}(\mathsf{S}+\mathsf{T}) = \mathrm{Dom}\,\mathsf{S}$ we have

$$\langle [(\mathsf{S}+\mathsf{T})^* - \mathsf{T}^*]\ g\,,\ f \rangle = \langle g\,,\ (\mathsf{S}+\mathsf{T})f \rangle - \langle g\,,\ \mathsf{T}f \rangle = \langle g\,,\ \mathsf{S}f \rangle\ .$$

This means that g belongs to $\mathrm{Dom}\,\mathsf{S}^* = \mathrm{Dom}(\mathsf{S}^* + \mathsf{T}^*)$. Together with 2), this proves the equality $\mathsf{S}^* + \mathsf{T}^* = (\mathsf{S} + \mathsf{T})^*$.

In the same way, if T is bounded then take $g \in \mathrm{Dom}(\mathsf{T}\,\mathsf{R})^*$ and $f \in \mathrm{Dom}(\mathsf{T}\,\mathsf{R}) = \mathrm{Dom}\,\mathsf{R}$. We have

$$\langle(\mathsf{T}\,\mathsf{R})^* g\,,\, f\rangle = \langle g\,,\, \mathsf{T}\,\mathsf{R}\, f\rangle = \langle \mathsf{T}^* g\,,\, \mathsf{R} f\rangle\ .$$

This shows that $\mathsf{T}^* g$ belongs to $\mathrm{Dom}\,\mathsf{R}^*$. Hence g belongs to $\mathrm{Dom}\,\mathsf{R}^*\mathsf{T}^*$ and the equality $\mathsf{R}^*\mathsf{T}^* = (\mathsf{T}\,\mathsf{R})^*$ is now proved, with the help of 3). □

1.2.2 Adjoint and Closability

We are now interested in a set of results which relate the properties of T^* and the closability of T.

Theorem 1.21. *Let T be a (densely defined) operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

1) The operator T^* is always a closed operator.

2) The operator T is closable if and only if $\mathrm{Dom}\,\mathsf{T}^*$ is dense. In that case we have $\overline{\mathsf{T}} = \mathsf{T}^{**}$.

3) If T is closable then $\left(\overline{\mathsf{T}}\right)^* = \mathsf{T}^*$.

Proof. Let U be the mapping from $\mathcal{H}_1 \times \mathcal{H}_2$ to $\mathcal{H}_2 \times \mathcal{H}_1$ defined by

$$\mathsf{U}(\varphi, \psi) = (-\psi, \varphi)\,.$$

We have $\mathsf{U}^*(\psi, \varphi) = (\varphi, -\psi)$ and hence $\mathsf{U}^* = \mathsf{U}^{-1}$. This implies in particular that $\mathsf{U}\mathsf{U}^* = \mathsf{U}^*\mathsf{U} = \mathrm{I}$. As a consequence, if E is a subspace of $\mathcal{H}_1 \times \mathcal{H}_2$ it is easy to check that

$$\mathsf{U}(E^\perp) = \mathsf{U}(E)^\perp$$

and also that

$$\mathsf{U}^{-1}(E^\perp) = \mathsf{U}^{-1}(E)^\perp\,.$$

1) Note that a pair (φ, ψ) belongs to $\mathsf{U}(\Gamma(\mathsf{T}))^\perp$ if and only if

$$\langle(\varphi, \psi), (-\mathsf{T}\eta, \eta)\rangle = 0$$

for all $\eta \in \mathrm{Dom}\,\mathsf{T}$, that is,

$$\langle\psi, \eta\rangle = \langle\varphi, \mathsf{T}\eta\rangle$$

for all $\eta \in \mathrm{Dom}\,\mathsf{T}$, that is, $(\psi, \varphi) \in \Gamma(\mathsf{T}^*)$. We have proved that

$$\Gamma(\mathsf{T}^*) = \mathsf{U}(\Gamma(\mathsf{T}))^\perp\,.$$

In particular $\Gamma(\mathsf{T}^*)$ is closed and the operator T^* is closed.

2) If $\operatorname{Dom} \mathsf{T}^*$ is dense then the operator $(\mathsf{T}^*)^*$ is well-defined, it is closed by 1) above, it is an extension of T by Proposition 1.19. Hence T admits a closed extension, that is, T is closable.

Conversely, if $\operatorname{Dom} \mathsf{T}^*$ is not dense, there exists a $\varphi \neq 0$ which belongs to $(\operatorname{Dom} \mathsf{T}^*)^\perp$. This implies that $(\varphi, 0)$ belongs to $\Gamma(\mathsf{T}^*)^\perp$ and consequently $(0, \varphi) = \mathsf{U}^{-1}(-\varphi, 0)$ belongs to $\mathsf{U}^{-1}(\Gamma(\mathsf{T}^*)^\perp) = \mathsf{U}^{-1}\big(\Gamma(\mathsf{T}^*)\big)^\perp$. On the other hand we have

$$\begin{aligned}
\mathsf{U}^{-1}\left(\Gamma(\mathsf{T}^*)\right)^\perp &= \mathsf{U}^{-1}\left(\mathsf{U}\left(\Gamma(\mathsf{T})\right)^\perp\right)^\perp \\
&= \mathsf{U}^{-1}\left(\mathsf{U}\left(\Gamma(\mathsf{T})^{\perp\perp}\right)\right) \\
&= \Gamma(\mathsf{T})^{\perp\perp} \\
&= \overline{\Gamma(\mathsf{T})}\,.
\end{aligned}$$

We have proved that $\overline{\Gamma(\mathsf{T})}$ contains $(0, \varphi)$ with some $\varphi \neq 0$. Hence $\overline{\Gamma(\mathsf{T})}$ is not the graph of an operator (Proposition 1.6). Hence T is not closable.

We now have to prove that $\overline{\mathsf{T}} = \mathsf{T}^{**}$. In exactly the same way as for U, it is easy to check that, for all operator S from $\mathcal{H}_2$ to $\mathcal{H}_1$ we have

$$\mathsf{U}^{-1}\left(\Gamma(\mathsf{S})\right)^\perp = \Gamma(\mathsf{S}^*)\,.$$

In particular

$$\overline{\Gamma(\mathsf{T})} = \mathsf{U}^{-1}\left(\Gamma(\mathsf{T}^*)\right)^\perp = \Gamma(\mathsf{T}^{**})\,.$$

This proves that $\mathsf{T}^{**} = \overline{\mathsf{T}}$.

3) is now immediate, for

$$\mathsf{T}^* = \overline{\mathsf{T}^*} = \mathsf{T}^{***} = \left(\overline{\mathsf{T}}\right)^*\,. \qquad \square$$

We now easily get a very strong commutation relation between inverse and adjoint mappings.

Proposition 1.22. *Let* T *be a densely defined and injective operator. If* $\operatorname{Ran} \mathsf{T}$ *is dense then* T^* *is injective and*

$$(\mathsf{T}^*)^{-1} = (\mathsf{T}^{-1})^*\,,$$

including the equality of domains.

Proof. We have $\operatorname{Ker}\, \mathsf{T}^* = (\operatorname{Ran} \mathsf{T})^\perp = \{0\}$, thus T^* is injective. Now, recall Def. 1.16, as well as the notations and results developed in the proof of Theorem 1.21, we have

$$\Gamma((\mathsf{T}^{-1})^*) = \mathsf{U}^{-1}(\Gamma(\mathsf{T}^{-1})^\perp) = \mathsf{U}^{-1}\big(\mathsf{V}(\Gamma(\mathsf{T}))^\perp\big) = \mathsf{U}^{-1}\big(\mathsf{V}(\Gamma(\mathsf{T})^\perp)\big)\,.$$

But obviously we have $\mathsf{U}^{-1}\mathsf{V} = \mathsf{V}\mathsf{U}$, hence we get

$$\Gamma((\mathsf{T}^{-1})^*) = \mathsf{V}\mathsf{U}(\Gamma(\mathsf{T})^{\perp}) = \mathsf{V}(\Gamma(\mathsf{T}^*)) = \Gamma((\mathsf{T}^*)^{-1})\,.$$

This proves the claim. □

1.2.3 The Case of Multiplication Operators

Let us come back to our examples of multiplication operators.

Definition 1.23. Let f be a measurable function from $(\Omega, \mathcal{F}, \mu)$ to $\mathbb{C}$. We put

$$\mathrm{essup}_\mu |f| = \inf\{\sup_{x\in E} |f(x)|\ ;\ E \in \mathcal{F}\,,\ \mu(\Omega \setminus E) = 0\}\,.$$

This quantity is the *essential supremum* of $|f|$ *with respect to* μ. The function f is *essentially bounded* if $\mathrm{essup}_\mu |f| < \infty$.

Theorem 1.24. *The multiplication operator* M_f *is bounded if and only if the function* f *is essentially bounded. In that case we have*

$$\|\mathsf{M}_f\| = \mathrm{essup}_\mu\ |f|\ .$$

Proof. If $C = \mathrm{essup}_\mu |f| < \infty$ then, for all $g \in L^2(\Omega, \mathcal{F}, \mu)$

$$\begin{aligned} \|\mathsf{M}_f\, g\|^2 &= \int |f(\omega)g(\omega)|^2 \,\mathrm{d}\mu(\omega) \\ &\leq C^2 \int |g(\omega)|^2 \,\mathrm{d}\mu(\omega) \qquad\qquad \text{(left as an exercise)} \\ &\leq C^2 \|g\|^2\,. \end{aligned}$$

Thus M_f is a bounded operator and $\|\mathsf{M}_f\| \leq \mathrm{essup}_\mu |f|$.

In particular, if $C = 0$ we have $\|\mathsf{M}_f\| = 0$. Assume that $C > 0$ and let $\varepsilon \in [0, C[$; define $E_\varepsilon = \{\omega\,;\ |f(\omega)| \geq C - \varepsilon\}$. The set A_ε has a strictly positive measure for μ (for otherwise C would not be the essential supremum of f). Let g be an element of $L^2(\varOmega, \mathcal{F}, \mu)$ which vanishes outside of A_ε. We have

$$\begin{aligned} \|\mathsf{M}_f\, g\|^2 &= \int_\varOmega |f(\omega)\, g(\omega)|^2 \,\mathrm{d}\mu(\omega) \\ &\geq (C-\varepsilon)^2 \int_{A_\varepsilon} |g(\omega)|^2 \,\mathrm{d}\mu(\omega) \\ &= (C-\varepsilon)^2 \|g\|^2. \end{aligned}$$

This proves that $\|\mathsf{M}_f\| \geq C - \varepsilon$. Hence, we have proved that $\|\mathsf{M}_f\| = C$.

Conversely, if f is not essentially bounded, then for all $n \in \mathbb{N}$ the set $A_n = \{\omega\,;\ |f(\omega)| \geq n\}$ has a strictly positive measure for μ. For any $g \in \mathrm{Dom}\,\mathsf{M}_f$

which vanishes outside of A_n, we have

$$\begin{aligned}\|\mathsf{M}_f\, g\| &= \int |f(\omega)g(\omega)|^2 \,\mathrm{d}\mu(\omega)\\ &\geq n\int_{A_n} |g(\omega)|^2\,\mathrm{d}\mu(\omega)\\ &= n\|g\|^2.\end{aligned}$$

Thus M_f is not bounded. □

Theorem 1.25. *The adjoint of the operator* M_f *is* $\mathsf{M}_{\overline{f}}$, *with same domain.*

Proof. It is clear that $\mathrm{Dom}\,\mathsf{M}_{\overline{f}} = \mathrm{Dom}\,\mathsf{M}_f$. It is also very easy to check that $\mathsf{M}_{\overline{f}} \subset \mathsf{M}_f^*$. We have to prove that $\mathsf{M}_f^* \subset \mathsf{M}_{\overline{f}}$. If g belongs to $\mathrm{Dom}\,\mathsf{M}_f^*$ then

$$\left\langle \mathsf{M}_f^*\, g\,,\, h\right\rangle = \langle g\,,\, \mathsf{M}_f\, h\rangle = \int_\Omega \overline{g(\omega)}f(\omega)h(\omega)\,\mathrm{d}\mu(\omega)$$

for all $h \in \mathrm{Dom}\,\mathsf{M}_f$. As a consequence

$$\int_\Omega \overline{(\mathsf{M}_f^*\, g - g\overline{f})(\omega)}\, h(\omega)\,\mathrm{d}\mu(\omega) = 0$$

for all $h \in \mathrm{Dom}\,\mathsf{M}_f$. Let $A_n = \{\omega\,;\ |f(\omega)| \leq n\}$. Then $1\!\!1_{A_n}h$ belongs to $\mathrm{Dom}\,\mathsf{M}_f$ for all $h \in L^2(\Omega, \mathcal{F}, \mu)$ and

$$\int_\Omega \overline{(\mathsf{M}_f^*\, g - g\overline{f})(\omega)}\, 1\!\!1_{A_n}(\omega)h(\omega)\,\mathrm{d}\mu(\omega) = 0$$

for all $h \in L^2(\Omega, \mathcal{F}, \mu)$. This means that $(\mathsf{M}_f^*\, g - g\overline{f})1\!\!1_{A_n} = 0$ as an element of $L^2(\Omega, \mathcal{F}, \mu)$. As this holds for all n, we have $\mathsf{M}_f^*\, g - g\overline{f} = 0$ as an element of Ω (left as an exercise). In particular $g\overline{f}$ belongs to $L^2(\Omega, \mathcal{F}, \mu)$ and hence g belongs to $\mathrm{Dom}\,\mathsf{M}_{\overline{f}}$. We have also proved that $\mathsf{M}_{\overline{f}}\, g = \mathsf{M}_f^*\, g$. □

Corollary 1.26. *For all* f *in* $L^2(\Omega, \mathcal{F}, \mu)$, *the multiplication operator* M_f *is a closed operator.*

Proof. By Theorem 1.25 we have $\mathsf{M}_f = (\mathsf{M}_{\overline{f}})^*$. By Theorem 1.21 the operator M_f is thus closed. □

1.3 Orthogonal Projectors, Unitaries, Isometries

In this section we collect the main definitions and properties concerning orthogonal projectors, unitary operators, isometries and partial isometries. As these are very common bounded operators we make a rather quick presentation.

1.3.1 Orthogonal Projectors

Let us first recall very basic facts concerning orthogonal projectors.

Definition 1.27. If E is a closed subspace of a Hilbert space $\mathcal{H}$, recall that $\mathcal{H}$ can be decomposed into $\mathcal{H} = E \oplus E^\perp$. Every $f \in \mathcal{H}$ can be decomposed in a unique way into $f = g + h$ with $g \in E$ and $h \in E^\perp$. The operator P_E from $\mathcal{H}$ to E defined by

$$\mathsf{P}_E\, f = g$$

is called the *orthogonal projector* from $\mathcal{H}$ onto E.

In the following theorem we recall the main facts that characterize orthogonal projectors. These are very well-known facts, we do not develop their proof.

Theorem 1.28. *A densely defined operator* P *on a Hilbert space* $\mathcal{H}$ *is an orthogonal projector if and only if*

$$\mathsf{P}^2 = \mathsf{P}^* = \mathsf{P}\,,$$

including the equalities of domains.

It is then automatically a bounded operator with $\|\mathsf{P}\| = 1$ *and as a consequence it is defined on the whole of* $\mathcal{H}$.

In that case, the operator P *is the orthogonal projector onto the closed subspace* $E = \operatorname{Ran} \mathsf{P}$ *and the operator* $\mathrm{I} - \mathsf{P}$ *is the orthogonal projector onto* $E^\perp = \operatorname{Ker} \mathsf{P}$. *In particular, for every* $f \in \mathcal{H}$ *we have the Pythagoras relation*

$$\|f\|^2 = \|\mathsf{P}f\|^2 + \|(\mathrm{I} - \mathsf{P})f\|^2\,.$$

Here is a list of basic properties shared by orthogonal projectors. All these proofs are easy and left to the reader, they constitute good exercises for whom is not used to these objects.

Proposition 1.29. *Let* P *and* Q *be two orthogonal projectors on some Hilbert space* $\mathcal{H}$.

1) The operator PQ *is an orthogonal projector if and only if* $\mathsf{PQ} = \mathsf{QP}$. *In this case* $\operatorname{Ran}(\mathsf{PQ}) = (\operatorname{Ran} \mathsf{P}) \cap (\operatorname{Ran} \mathsf{Q})$.

2) The operator $\mathsf{P} + \mathsf{Q}$ *is an orthogonal projector if and only if* $\mathsf{PQ} = 0$, *that is, if and only if* $\operatorname{Ran} \mathsf{P}$ *is orthogonal to* $\operatorname{Ran} \mathsf{Q}$. *In this case* $\operatorname{Ran}(\mathsf{P} + \mathsf{Q}) = \operatorname{Ran} \mathsf{P} \oplus \operatorname{Ran} \mathsf{Q}$.

3) We have $\mathsf{PQ} = \mathsf{P}$ *if and only if* $\mathsf{QP} = \mathsf{P}$. *This is also equivalent to* $\operatorname{Ran} \mathsf{P} \subset \operatorname{Ran} \mathsf{Q}$.

1.3.2 Unitaries and Isometries

We now turn to isometries and unitary operators.

Definition 1.30. An operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ is an *isometry* if it is densely defined on $\mathcal{H}_1$ and if $\|\mathsf{U}f\| = \|f\|$ for all f. In particular an isometry is a bounded operator with norm 1 and it gets extended to the whole of $\mathcal{H}_1$ (Theorem 1.8).

An operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ is *unitary* if it is an isometry and onto.

The following characterizations are well-known and immediate.

Theorem 1.31. *Let* U *be a densely defined operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$. *The following assertions are equivalent.*

i) U *is an isometry.*

ii) $\mathsf{U}^*\mathsf{U} = \mathrm{I}$.

The following assertions are equivalent.

*i') * U *is unitary.*

ii') $\mathsf{U}^*\mathsf{U} = \mathsf{U}\mathsf{U}^* = \mathrm{I}$.

iii') U^* *is unitary.*

Note that identity ii') could be more precisely written if one wants to make the underlying space appearing in the identity operator: $\mathsf{U}^*\mathsf{U} = \mathrm{I}_{\mathcal{H}_1}$ and $\mathsf{U}\mathsf{U}^* = \mathrm{I}_{\mathcal{H}_2}$.

Isometries are naturally related to orthogonal projectors in the following way.

Proposition 1.32. *If* U *is an isometry then* $\mathsf{U}\mathsf{U}^*$ *is the orthogonal projector onto* $\operatorname{Ran}\mathsf{U}$.

Proof. We have $(\mathsf{U}\mathsf{U}^*)^* = \mathsf{U}\mathsf{U}^*$ and $(\mathsf{U}\mathsf{U}^*)^2 = \mathsf{U}\mathsf{U}^*\mathsf{U}\mathsf{U}^* = \mathsf{U}\mathsf{U}^*$. Thus $\mathsf{U}\mathsf{U}^*$ is an orthogonal projector (Theorem 1.28).

If $f = \mathsf{U}g$ is an element of $\operatorname{Ran}\mathsf{U}$ then $\mathsf{U}\mathsf{U}^*f = \mathsf{U}\mathsf{U}^*\mathsf{U}g = \mathsf{U}g = f$. Thus $\operatorname{Ran}\mathsf{U}$ is included in $\operatorname{Ran}(\mathsf{U}\mathsf{U}^*)$. But $\operatorname{Ran}(\mathsf{U}\mathsf{U}^*)$ is included in $\operatorname{Ran}\mathsf{U}$ obviously, hence we have proved that $\operatorname{Ran}(\mathsf{U}\mathsf{U}^*) = \operatorname{Ran}\mathsf{U}$. □

1.3.3 Partial Isometries

Definition 1.33. An operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a *partial isometry* if there exists a closed subspace E of $\mathcal{H}_1$ such that

$$\begin{cases} \|\mathsf{U}f\| = \|f\|, & \text{for all } f \in E, \\ \mathsf{U}f = 0, & \text{for all } f \in E^\perp. \end{cases}$$

It is in particular a bounded operator.

In this case, the space $F = \operatorname{Ran} \mathsf{U}$ is closed (exercise). The space E is called the *initial space* of U, the space F is called the *final space* of U.

Theorem 1.34. *A bounded operator* U *is a partial isometry if and only if* $\mathsf{U}^*\mathsf{U}$ *is an orthogonal projector. In that case*

1) $\mathsf{U}^*\mathsf{U}$ *is the orthogonal projector onto the initial space of* U

and

2) $\mathsf{U}\mathsf{U}^*$ *is also an orthogonal projector, it is the orthogonal projector onto the final space of* U*.*

Proof. If U is a partial isometry then let P_E be the orthogonal projector onto E, the initial space of U. We have, for all $f \in \mathcal{H}_1$

$$\begin{aligned}\langle \mathsf{U}f \,,\, \mathsf{U}f\rangle &= \langle \mathsf{U}\mathsf{P}_E\, f \,,\, \mathsf{U}\mathsf{P}_E\, f\rangle \\ &= \langle \mathsf{P}_E\, f \,,\, \mathsf{P}_E\, f\rangle \ .\end{aligned}$$

By polarization we get, for all $f, g \in \mathcal{H}_1$

$$\langle \mathsf{U}f \,,\, \mathsf{U}g\rangle = \langle \mathsf{P}_E\, f \,,\, \mathsf{P}_E\, g\rangle \ ,$$

that is,

$$\langle \mathsf{U}^*\mathsf{U}f \,,\, g\rangle = \langle \mathsf{P}_E\, f \,,\, g\rangle \ .$$

This proves that $\mathsf{U}^*\mathsf{U} = \mathsf{P}_E$.

Conversely, assume that $\mathsf{U}^*\mathsf{U}$ is the orthogonal projector onto a closed subspace E. Then, for all $f \in E$, we have

$$\|f\|^2 = \langle f \,,\, f\rangle = \langle \mathsf{U}^*\mathsf{U}f \,,\, f\rangle = \langle \mathsf{U}f \,,\, \mathsf{U}f\rangle = \|\mathsf{U}f\|^2$$

and U is an isometry on E. Furthermore, for all $f \in E^\perp$, we have

$$0 = \langle \mathsf{U}^*\mathsf{U}f \,,\, f\rangle = \langle \mathsf{U}f \,,\, \mathsf{U}f\rangle = \|\mathsf{U}f\|^2$$

and U vanishes on $E^\perp$. We have proved that U is a partial isometry with initial space E.

Now consider a partial isometry U with in initial space E and final space F. Let P_E be the orthogonal projector onto E. We have $\mathsf{U} = \mathsf{U}\mathsf{P}_E = \mathsf{U}\mathsf{U}^*\mathsf{U}$ and consequently $\mathsf{U}\mathsf{U}^* = \mathsf{U}\mathsf{U}^*\mathsf{U}\mathsf{U}^*$. This proves that $\mathsf{U}\mathsf{U}^*$ is an orthogonal projection. Let $G = \text{Ran } \mathsf{U}\mathsf{U}^*$ be its range. The relation $\mathsf{U}\mathsf{U}^*\mathsf{U} = \mathsf{U}$ implies that Ran U is included in G. But as $G = \text{Ran } \mathsf{U}\mathsf{U}^* \subset \text{Ran } \mathsf{U}$, we finally conclude that $G = \text{Ran } \mathsf{U} = F$. $\square$

1.4 Symmetric and Self-Adjoint Operators

In the case of finite dimensional Hilbert spaces, it is well-known that the hermitian matrices (if we want the spectrum to be real), or more generally the set of normal matrices, are the set of matrices that can be diagonalized in some orthonormal basis. They admit in particular an easy functional calculus. Recall that these two set of matrices are those such that $\mathsf{T} = \mathsf{T}^*$, *resp.* $\mathsf{T}\mathsf{T}^* = \mathsf{T}^*\mathsf{T}$.

In the infinite dimensional case the two set of operators that admit a kind of diagonalization in some orthonormal basis are the self-adjoint, *resp.* the normal ones. They are basically defined by the same relations $\mathsf{T} = \mathsf{T}^*$, *resp.* $\mathsf{T}\mathsf{T}^* = \mathsf{T}^*\mathsf{T}$, but with all the subtleties involved by unbounded operators and their domain constraints. We enter here in a very touchy business; for instance proving that a given operator is self-adjoint or not may be a very difficult problem.

1.4.1 Basic Definitions

A densely defined operator T from $\mathcal{H}$ to $\mathcal{H}$ is *symmetric* if

$$\langle f\,,\,\mathsf{T}g\rangle = \langle \mathsf{T}f\,,\,g\rangle$$

for all $f, g \in \mathrm{Dom}\,\mathsf{T}$. This is clearly equivalent to saying that

$$\mathsf{T} \subset \mathsf{T}^*\,.$$

Here recall Theorem 1.21. Note that symmetric operators are always closable for $\mathrm{Dom}\,\mathsf{T}^*$ contains $\mathrm{Dom}\,\mathsf{T}$ which is dense. This means that T^* is a closed extension of T and consequently we have

$$\overline{\mathsf{T}} = \mathsf{T}^{**} \subset \mathsf{T}^*\,.$$

But also, by Proposition 1.19, we have

$$\mathsf{T} \subset \mathsf{T}^{**}\,.$$

We thus distinguish four different cases among symmetric operators:

a) $\mathsf{T} \subset \mathsf{T}^{**} \subset \mathsf{T}^*$, for which we say that T is *symmetric*,

b) $\mathsf{T} = \mathsf{T}^{**} \subset \mathsf{T}^*$, for which we say that T is *closed symmetric*,

c) $\mathsf{T} \subset \mathsf{T}^{**} = \mathsf{T}^*$, for which we say that T is *essentially self-adjoint*,

d) $\mathsf{T} = \mathsf{T}^{**} = \mathsf{T}^*$, for which we say that T is *self-adjoint*.

The last two cases are the ones which are really of interest. Actually self-adjoint operators are really the good objects, but the essentially self-adjoint operators are not so far from self-adjointness, as prove the next two results.

Proposition 1.35. *Let* T *be a symmetric operator on* $\mathcal{H}$. *Then* T *is essentially self-adjoint if and only if* $\overline{\mathsf{T}}$ *is self-adjoint.*

Proof. If T is essentially self-adjoint we have $\mathsf{T}^{**} = \overline{\mathsf{T}}$ and $\overline{\mathsf{T}}^* = \mathsf{T}^{***} = \mathsf{T}^{**} = \overline{\mathsf{T}}$, thus $\overline{\mathsf{T}}$ is self-adjoint.

Conversely, if $\overline{\mathsf{T}}$ is self-adjoint, $\overline{\mathsf{T}} = \overline{\mathsf{T}}^* = \mathsf{T}^*$, by Theorem 1.21, Property 3). Hence $\mathsf{T}^{**} = \mathsf{T}^*$ and T is essentially self-adjoint.

Proposition 1.36. *If* T *is essentially self-adjoint then it admits a unique self-adjoint extension.*

Proof. If S is a self-adjoint extension of T then S is closed (Theorem 1.21) and thus $\mathsf{T}^{**} = \overline{\mathsf{T}} \subset \mathsf{S}$. But, it is easy to check from the definition that if $\mathsf{T} \subset \mathsf{S}$ then $\mathsf{S}^* \subset \mathsf{T}^*$. Hence we have $\mathsf{S} = \mathsf{S}^* \subset \mathsf{T}^* = \mathsf{T}^{***} = \mathsf{T}^{**}$. We have proved that $\mathsf{S} = \mathsf{T}^{**} = \overline{\mathsf{T}}$. □

In the case of bounded symmetric operators things get much simpler.

Proposition 1.37. *If* T *is a bounded symmetric operator on* $\mathcal{H}$ *then* T *is self-adjoint. Furthermore we have*

$$\|\mathsf{T}\| = \sup\{|\langle x\,,\,\mathsf{T}x\rangle|\ ;\ x \in \mathcal{H}\,,\ \|x\| = 1\}\,. \tag{1.3}$$

Proof. If T is symmetric and bounded, then it is defined everywhere and hence self-adjoint.

We have $|\langle x\,,\,\mathsf{T}x\rangle| \leq \|x\|\ \|\mathsf{T}x\| \leq \|\mathsf{T}\|$ if $\|x\| = 1$. Hence

$$\|\mathsf{T}\| \geq \sup\{|\langle x\,,\,\mathsf{T}x\rangle|\ ;\ x \in \mathcal{H}\,,\ \|x\| = 1\}\,.$$

In the opposite direction, let us put

$$C = \sup\{|\langle x\,,\,\mathsf{T}x\rangle|\ ;\ x \in \mathcal{H}\,,\ \|x\| = 1\}\,.$$

We have, by the polarization identity, for all $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$

$$\begin{aligned}
|\mathrm{Re}\,\langle y\,,\,\mathsf{T}x\rangle| &\leq \frac{1}{4}\big[\,|\langle x+y\,,\,\mathsf{T}(x+y)\rangle| + |\langle x-y\,,\,\mathsf{T}(x-y)\rangle|\,\big] \\
&\leq \frac{1}{4}\,C\,\big[\|x+y\|^2 + \|x-y\|^2\big] \\
&= \frac{1}{4}\,C\,\big[2\|x\|^2 + 2\|y\|^2\big] \\
&= C\,.
\end{aligned}$$

Choose $a \in \mathbb{C}$ such that $|a| = 1$ and $a\,\langle y\,,\,\mathsf{T}x\rangle = |\langle y\,,\,\mathsf{T}x\rangle|$. Then

$$|\langle y\,,\,\mathsf{T}x\rangle| = \langle \overline{a}\,y\,,\,\mathsf{T}x\rangle = |\mathrm{Re}\,\langle \overline{a}\,y\,,\,\mathsf{T}x\rangle|\ ,$$

for $\langle \overline{a}\,y\,,\,\mathsf{T}x\rangle$ is a positive real number. Hence, we have $|\langle y\,,\,\mathsf{T}x\rangle| \leq C$. This proves that $\|\mathsf{T}\| \leq C$. □

We end up this subsection with the very nice and important property that $\mathsf{T}^*\mathsf{T}$ is always a self-adjoint operator, under the only condition that T is a closed operator.

Definition 1.38. For every operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ one defines

$$\langle f\,,\,g\rangle_{\mathsf{T}} = \langle f\,,\,g\rangle_{\mathcal{H}_1} + \langle \mathsf{T}f\,,\,\mathsf{T}g\rangle_{\mathcal{H}_2}\ ,$$

for all $f, g \in \mathrm{Dom}\,\mathsf{T}$. It is easy to check that this defines a scalar product on $\mathrm{Dom}\,\mathsf{T}$. We denote by $\|\cdot\|_{\mathsf{T}}$ the norm associated to this scalar product.

Note that if T is a closed operator then $\mathrm{Dom}\,\mathsf{T}$ is a Hilbert space for this scalar product.

Theorem 1.39. *If T is a densely defined and closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, then $\mathsf{T}^*\mathsf{T}$ is a self-adjoint operator on $\mathcal{H}_1$.*

Furthermore, the domain of $\mathsf{T}^*\mathsf{T}$ is dense in $\mathrm{Dom}\,\mathsf{T}$ for $\|\cdot\|_{\mathsf{T}}$.

Proof. In the proof of Theorem 1.21 we have proved the relation $\mathsf{U}(\Gamma(\mathsf{T}))^\perp = \Gamma(\mathsf{T}^*)$. In other words the space $\mathcal{H}_1 \times \mathcal{H}_2$ can be decomposed as

$$\mathcal{H}_1 \times \mathcal{H}_2 = \mathsf{U}(\Gamma(\mathsf{T})) \oplus \Gamma(\mathsf{T}^*)\,.$$

This identity has many consequences; in particular, for every $f \in \mathcal{H}_2$ and every $g \in \mathcal{H}_1$ there exist a unique $x \in \mathrm{Dom}\,\mathsf{T}$ and a unique $y \in \mathrm{Dom}\,\mathsf{T}^*$ such that

$$(f, g) = (-\mathsf{T}x\,,\,x) + (y\,,\,\mathsf{T}^*y)\,, \tag{1.4}$$

that is,

$$\begin{cases} f = y - \mathsf{T}x \\ g = x + \mathsf{T}^*y\,. \end{cases} \tag{1.5}$$

Furthermore, taking the $\mathcal{H}_1 \times \mathcal{H}_2$-norm in (1.4) gives

$$\|f\|^2 + \|g\|^2 = \|x\|^2 + \|\mathsf{T}x\|^2 + \|y\|^2 + \|\mathsf{T}^*y\|^2\,. \tag{1.6}$$

Applying (1.5) to the case $f = 0$ shows that for every $g \in \mathcal{H}_1$ there exists a $x \in \mathrm{Dom}\,\mathsf{T}$ and a $y \in \mathrm{Dom}\,\mathsf{T}^*$ such that

$$\begin{cases} y = \mathsf{T}x \\ g = x + \mathsf{T}^*y\,. \end{cases}$$

This is to say that $x \in \mathrm{Dom}(\mathsf{T}^*\mathsf{T})$ and

$$g = (\mathrm{I} + \mathsf{T}^*\mathsf{T})x\,.$$

The relation (1.6) gives $\|x\| \le \|g\|$ in this case. Hence the operator $\mathrm{I} + \mathsf{T}^*\mathsf{T}$ is injective and $\mathsf{S} = (\mathrm{I} + \mathsf{T}^*\mathsf{T})^{-1}$ is a bounded operator. For all $g, g' \in \mathcal{H}_1$ we have, for some $x, x' \in \mathrm{Dom}(\mathsf{T}^*\mathsf{T})$

$$\langle g' \,,\, \mathsf{S}g\rangle = \langle x' \,,\, x + \mathsf{T}^*\mathsf{T}x\rangle = \langle x' \,,\, x\rangle + \langle \mathsf{T}x' \,,\, \mathsf{T}x\rangle\ .$$

In particular, we obviously have $\langle g' \,,\, \mathsf{S}g\rangle = \langle \mathsf{S}g' \,,\, g\rangle$ and S is a self-adjoint operator. By Proposition 1.22 the operator $\mathrm{I} + \mathsf{T}^*\mathsf{T} = \mathsf{S}^{-1}$ is self-adjoint, hence $\mathsf{T}^*\mathsf{T}$ is self-adjoint.

Let us prove the density property. If $h \in \mathrm{Dom}\,\mathsf{T}$ is orthogonal to every $x \in \mathrm{Dom}(\mathsf{T}^*\mathsf{T})$ for the scalar product $\langle\cdot\,,\,\cdot\rangle_{\mathsf{T}}$ then, for all $x \in \mathrm{Dom}\,\mathsf{T}^*\mathsf{T}$ we have

$$0 = \langle x\,,\, h\rangle_{\mathsf{T}} = \langle x\,,\, h\rangle + \langle \mathsf{T}x\,,\, \mathsf{T}h\rangle = \langle x\,,\, h\rangle + \langle \mathsf{T}^*\mathsf{T}x\,,\, h\rangle\ .$$

That is, h belongs to $\mathrm{Ran}(\mathrm{I} + \mathsf{T}^*\mathsf{T})^{\perp}$, which is $\mathcal{H}_1^{\perp}$ as was noticed above. Hence $h = 0$ and the density is proved. □

1.4.2 Basic Criterions

Self-adjoint operators are key operators in quantum physics, they are used over and ever, but the main difficulty is to prove that a given (unbounded) operator is indeed self-adjoint. Recognizing that an operator is symmetric is very easy, but proving that it is self-adjoint or essentially self-adjoint often needs to enter into very fine technicalities concerning domains. We now present the main criterions for essential self-adjointness and self-adjointness.

Theorem 1.40. *Let T be a symmetric operator on $\mathcal{H}$. The following assertions are equivalent.*

i) T *is self-adjoint.*

ii) T *is closed and* $\mathrm{Ker}\,(\mathsf{T}^* \pm \mathrm{i}\,\mathrm{I}) = \{0\}$ *.*

iii) $\mathrm{Ran}\,(\mathsf{T} \pm \mathrm{i}\,\mathrm{I}) = \mathcal{H}$*.*

Proof.

i) ⇒ ii): If T is self-adjoint then it is closed (Theorem 1.21). Furthermore, if $(\mathsf{T}^* + \mathrm{i}\,\mathrm{I})\varphi = 0$ then $\mathsf{T}\varphi = -\mathrm{i}\varphi$ and $\langle \mathsf{T}\varphi, \varphi\rangle = \mathrm{i}\langle\varphi, \varphi\rangle$ together with $\langle \varphi, \mathsf{T}\varphi\rangle = -\mathrm{i}\langle\varphi, \varphi\rangle$. As $\langle \mathsf{T}\varphi, \varphi\rangle = \langle \varphi, \mathsf{T}\varphi\rangle$ this implies $\varphi = 0$. The same holds in the same way for $\mathsf{T}^* - \mathrm{i}\,\mathrm{I}$.

ii) ⇒ iii): We have $\mathrm{Ran}\,(\mathsf{T} \pm \mathrm{i}\,\mathrm{I})^{\perp} = \mathrm{Ker}\,(\mathsf{T}^* \mp \mathrm{i}\,\mathrm{I}) = \{0\}$. Thus $\mathrm{Ran}\,(\mathsf{T} \pm \mathrm{i}\,\mathrm{I})$ is dense. We just have to prove that it is closed. Let (φ_n) be a sequence in $\mathrm{Dom}\,\mathsf{T}$ such that $((\mathsf{T} + \mathrm{i}\,\mathrm{I})\varphi_n)$ converges to a $\psi \in \mathcal{H}$. As T is symmetric we have

$$\|(\mathsf{T}+\mathrm{i}\,\mathrm{I})\varphi_n\|^2=\|\mathsf{T}\varphi_n\|^2+\|\varphi_n\|^2.$$

This shows that the sequence (φ_n) converges to a limit φ and $(\mathsf{T}\varphi_n)$ also converges (replacing φ_n by $\varphi_n-\varphi_m$ in the above identity shows that both sequences are Cauchy). As T is closed we must have that φ belongs to $\mathrm{Dom}\,\mathsf{T}$ and $\mathsf{T}\varphi=\lim\mathsf{T}\varphi_n$ (Theorem 1.14). Altogether, the sequence $((\mathsf{T}+\mathrm{i}\,\mathrm{I})\varphi_n)$ converges to $\psi=(\mathsf{T}+\mathrm{i}\,\mathrm{I})\varphi$ and hence ψ belongs to $\mathrm{Ran}(\mathsf{T}+\mathrm{i}\,\mathrm{I})$. The proof is the same for $\mathrm{Ran}(\mathsf{T}-\mathrm{i}\,\mathrm{I})$.

iii) ⇒ i): If φ belongs to $\mathrm{Dom}\,\mathsf{T}^*$ then by hypothesis there exists $\eta\in\mathrm{Dom}\,\mathsf{T}$ such that $(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})\varphi=(\mathsf{T}-\mathrm{i}\,\mathrm{I})\eta$. In particular, as T^* coincides with T on $\mathrm{Dom}\,\mathsf{T}$, we have $(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})(\eta-\varphi)=0$ and $\eta-\varphi$ belongs to $\mathrm{Ker}\,(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})$. But, by Proposition 1.19, we have $\mathrm{Ker}\,(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})=\mathrm{Ran}(\mathsf{T}+\mathrm{i}\,\mathrm{I})^\perp=\{0\}$, by hypothesis. This proves that $\eta=\varphi$. We conclude that $\mathrm{Dom}\,\mathsf{T}^*=\mathrm{Dom}\,\mathsf{T}$, hence T is self-adjoint. □

From the above theorem, we rather easily deduce a criterion for essential self-adjointness.

Corollary 1.41. *Let* T *be a symmetric operator on* $\mathcal{H}$*. The following assertions are equivalent.*

i) T *is essentially self-adjoint.*

ii) $\mathrm{Ker}\,(\mathsf{T}^*\pm\mathrm{i}\,\mathrm{I})=\{0\}$.

iii) $\mathrm{Ran}(\mathsf{T}\pm\mathrm{i}\,\mathrm{I})$ *is dense.*

Proof.

i) ⇒ ii): If T is essentially self-adjoint then $\overline{\mathsf{T}}$ is self-adjoint. By the theorem above we have $\mathrm{Ker}(\overline{\mathsf{T}}^*\pm\mathrm{i}\,\mathrm{I})=\{0\}$. But $\overline{\mathsf{T}}^*=\mathsf{T}^*$ (Theorem 1.21), hence the result.

ii) ⇒ iii): As $\mathrm{Ran}(\mathsf{T}\pm\mathrm{i}\,\mathrm{I})=\mathrm{Ker}(\mathsf{T}^*\mp\mathrm{i}\,\mathrm{I})^\perp=\mathcal{H}$, this shows that $\mathrm{Ran}(\mathsf{T}\pm\mathrm{i}\,\mathrm{I})$ is dense.

iii) ⇒ i): Let $\varphi\in\mathrm{Dom}\,\mathsf{T}^*$. Then there exists a sequence (η_n) in $\mathrm{Dom}\,\mathsf{T}$ such that $(\mathsf{T}-\mathrm{i}\,\mathrm{I})\eta_n$ converges to $(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})\varphi$. But, as T is symmetric, we have

$$\left\|(\mathsf{T}-\mathrm{i}\,\mathrm{I})\eta_n\right\|^2=\left\|\mathsf{T}\eta_n\right\|^2+\left\|\eta_n\right\|^2.$$

In particular, both the sequences $(\mathsf{T}\eta_n)$ and (η_n) are convergent in $\mathcal{H}$. Let $\eta=\lim\eta_n$. As T is symmetric then T is closable and, by Theorem 1.14, the vector η belongs to $\mathrm{Dom}\,\overline{\mathsf{T}}$ and $\overline{\mathsf{T}}\eta=\lim\mathsf{T}\eta_n$. This shows that $(\overline{\mathsf{T}}-\mathrm{i}\,\mathrm{I})\eta=(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})\varphi$. In particular, $(\mathsf{T}^*-\mathrm{i}\,\mathrm{I})(\eta-\varphi)=0$ and thus $\varphi=\eta$. This shows that φ belongs to $\mathrm{Dom}\,\overline{\mathsf{T}}$, hence $\mathrm{Dom}\,\mathsf{T}^*=\mathrm{Dom}\,\overline{\mathsf{T}}$ and T is essentially self-adjoint.
□

1.4.3 Normal Operators

Another class of operators shares many properties with the self-adjoint operators, it is the class of *normal operators*.

Definition 1.42. An operator T on $\mathcal{H}$ is *normal* if $\mathrm{Dom}\,\mathsf{T} = \mathrm{Dom}\,\mathsf{T}^*$ and if

$$\|\mathsf{T}f\| = \|\mathsf{T}^* f\|$$

for all $f \in \mathrm{Dom}\,\mathsf{T}$.

Obviously, self-adjoint operators are normal operators. It is also clear that unitary operators (on $\mathcal{H}$) are normal.

We now list the main properties of normal operators. Property 3) is usually the most useful criterion.

Proposition 1.43.

1) Every normal operator is closed.

2) If S and T are normal operators such that $\mathsf{S} \subset \mathsf{T}$ then $\mathsf{S} = \mathsf{T}$.

3) A densely defined and closed operator T is normal if and only if

$$\mathsf{T}^*\,\mathsf{T} = \mathsf{T}\,\mathsf{T}^*\,, \tag{1.7}$$

including the equality of domains.

4) A densely defined and closed operator T is normal if and only if T^* is normal.

Proof.

1) Let T be a normal operator on $\mathcal{H}$. Let (f_n) be sequence in $\mathrm{Dom}\,\mathsf{T}$ such that (f_n) converges to some $f \in \mathcal{H}$ and $(\mathsf{T}f_n)$ is convergent too. Then $(f_n) \subset \mathrm{Dom}\,\mathsf{T}^*$ and as $\|\mathsf{T}^*(f_n - f_m)\| = \|\mathsf{T}(f_n - f_m)\|$, the sequence $(\mathsf{T}^* f_n)$ is convergent too. But T^* is a closed operator (Theorem 1.21) and hence $f \in \mathrm{Dom}\,\mathsf{T}^*$ and $\mathsf{T}^* f = \lim \mathsf{T}^* f_n$ (Theorem 1.14). This means that f belongs to $\mathrm{Dom}\,\mathsf{T}$ and that $\mathsf{T}f = \lim \mathsf{T}f_n$ too. This proves that T is closed.

2) If $\mathrm{Dom}\,\mathsf{S} \subset \mathrm{Dom}\,\mathsf{T}$ then $\mathrm{Dom}\,\mathsf{T}^* \subset \mathrm{Dom}\,\mathsf{S}^*$. But as $\mathrm{Dom}\,\mathsf{T}^* = \mathrm{Dom}\,\mathsf{T}$ and $\mathrm{Dom}\,\mathsf{S}^* = \mathrm{Dom}\,\mathsf{S}$, all four domains are equal.

3) If T is normal then by 1) it is is closed and by Theorem 1.39 the operator $\mathsf{T}^*\mathsf{T}$ is self-adjoint. As T^* is closed (Theorem 1.21) and densely defined, Theorem 1.39 applies too and $\mathsf{T}\mathsf{T}^*$ is a self-adjoint operator. Notice that, as $\mathrm{Dom}\,\mathsf{T} = \mathrm{Dom}\,\mathsf{T}^*$, we obviously have $\mathrm{Dom}(\mathsf{T}^*\mathsf{T}) = \mathrm{Dom}(\mathsf{T}\mathsf{T}^*)$. Now, for all $f \in \mathrm{Dom}(\mathsf{T}^*\mathsf{T}) \subset \mathrm{Dom}\,\mathsf{T} = \mathrm{Dom}\,\mathsf{T}^* \supset \mathrm{Dom}(\mathsf{T}\mathsf{T}^*)$ we have

$$\langle \mathsf{T}^*\mathsf{T}f\,,\,f\rangle = \langle \mathsf{T}f\,,\,\mathsf{T}f\rangle = \|\mathsf{T}f\|^2 = \|\mathsf{T}^*f\|^2 = \langle \mathsf{T}^*f\,,\,\mathsf{T}^*f\rangle = \langle \mathsf{T}\mathsf{T}^*f\,,\,f\rangle\ .$$

By polarization, we deduce that $\mathsf{T}^*\mathsf{T} = \mathsf{T}\mathsf{T}^*$ on $\mathrm{Dom}(\mathsf{T}^*\mathsf{T})$. We have proved the "only if" direction.

Conversely, if $\mathsf{T}^*\mathsf{T} = \mathsf{T}\mathsf{T}^*$, then for all $f \in \mathrm{Dom}(\mathsf{T}^*\mathsf{T}) \subset \mathrm{Dom}\,\mathsf{T}$ we have $f \in \mathrm{Dom}(\mathsf{T}\mathsf{T}^*) \subset \mathrm{Dom}\,\mathsf{T}^*$ and

$$\|\mathsf{T}f\|^2 = \langle \mathsf{T}f\,,\,\mathsf{T}f\rangle = \langle f\,,\,\mathsf{T}^*\mathsf{T}f\rangle = \langle f\,,\,\mathsf{T}\mathsf{T}^*f\rangle = \langle \mathsf{T}^*f\,,\,\mathsf{T}^*f\rangle = \|\mathsf{T}^*f\|^2\,.$$

Hence the equality $\|\mathsf{T}f\| = \|\mathsf{T}^*f\|$ is proved on $\mathrm{Dom}(\mathsf{T}^*\mathsf{T})$.

In Theorem 1.39 it is also proved that $\mathrm{Dom}(\mathsf{T}^*\mathsf{T})$ is dense in $\mathrm{Dom}\,\mathsf{T}$ for the norm $\|\cdot\|_{\mathsf{T}}$ associated to the scalar product $\langle\cdot\,,\,\cdot\rangle_{\mathsf{T}}$. In the same way we have the density of $\mathrm{Dom}(\mathsf{T}\mathsf{T}^*)$ in $\mathrm{Dom}\,\mathsf{T}^*$ but for the scalar product associated to T^*. Let $f \in \mathrm{Dom}\,\mathsf{T}$ and (f_n) be a sequence in $\mathrm{Dom}(\mathsf{T}^*\mathsf{T})$ converging to f for $\|\cdot\|_{\mathsf{T}}$. In particular (f_n) converges to f in $\mathcal{H}$ and the sequence $(\mathsf{T}f_n)$ is Cauchy in $\mathcal{H}$. This implies that the sequence (T^*f_n) is Cauchy too, hence convergent. As T^* is closed, we have that $f \in \mathrm{Dom}\,\mathsf{T}^*$ and $\mathsf{T}^*f = \lim \mathsf{T}^*f_n$. We have proved that $\mathrm{Dom}\,\mathsf{T} \subset \mathrm{Dom}\,\mathsf{T}^*$ and, passing to the limit, that $\|\mathsf{T}f\| = \|\mathsf{T}^*f\|$ on $\mathrm{Dom}\,\mathsf{T}$.

If $f \in \mathrm{Dom}\,\mathsf{T}^*$ we proceed in the same way, choose a sequence (f_n) in $\mathrm{Dom}(\mathsf{T}\mathsf{T}^*)$ which converges to f in $\|\cdot\|_{\mathsf{T}^*}$. Then (f_n) converges to f in $\mathcal{H}$ and $(\mathsf{T}f_n)$ is Cauchy in $\mathcal{H}$. By hypothesis T is closed, hence f belongs to $\mathrm{Dom}\,\mathsf{T}$. We have proved $\mathrm{Dom}\,\mathsf{T}^* \subset \mathrm{Dom}\,\mathsf{T}$.

This finally gives the equality $\mathrm{Dom}\,\mathsf{T} = \mathrm{Dom}\,\mathsf{T}^*$ and the norm equality on that domain. Hence T is a normal operator.

4) is obvious for the role of T and T^* is symmetric in 3) (when having noticed that $\mathsf{T}^{**} = \mathsf{T}$ for T is closed). □

1.5 Spectrum

In finite dimension, one of the most important tool for studying linear applications is certainly the notion of eigenvalues (with the attached notion of eigenvectors), culminating with the well-known diagonalization theorems. For the study of linear operators in infinite dimension, the notion of eigenvalue has to be replaced by a larger notion, the *spectrum* of the operator. This spectrum has a more complicated structure than just being the list of eigenvalues of the operator, but it is nevertheless the essential notion for the study of operators and of their "diagonalization".

1.5.1 Functional Analysis Background

Before entering into the notion of spectrum, we need to make clear a set of fundamental results in Fuctional Analysis. The first result we need is very well-known, its proof can be found in any textbook on Functional Analysis (see the "Notes" section at the end of this chapter).

Theorem 1.44 (Open Mapping Theorem). *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two Banach spaces. Let T be a linear mapping from $\mathcal{B}_1$ to $\mathcal{B}_2$ which is continuous and onto. Then for every open set M in $\mathcal{B}_1$ the set $\mathsf{T}(M)$ is open in $\mathcal{B}_2$.*

This theorem has very important consequences. One of the most useful for us is the following theorem.

Theorem 1.45 (Closed Graph Theorem). *Let T be an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. The following assertions are equivalent.*

i) T *is bounded.*

ii) T *is closed and* $\mathrm{Dom}\,\mathsf{T} = \mathcal{H}_1$.

Proof. One direction is obvious: every bounded operator is closed and has its domain equal to $\mathcal{H}_1$. In the converse direction, if T is closed and $\mathrm{Dom}\,\mathsf{T} = \mathcal{H}_1$ then the graph $\Gamma(\mathsf{T})$ is a Banach space. Consider the following mappings on $\Gamma(\mathsf{T})$:

$$\begin{aligned} \pi_1 &: (x, \mathsf{T}x) \mapsto x \\ \pi_2 &: (x, \mathsf{T}x) \mapsto \mathsf{T}x \,. \end{aligned}$$

The mapping π_1 is linear, continuous and bijective. Thus, by the Open Mapping Theorem above, the mapping π_1^{-1} is continuous. As π_2 is also continuous and $\mathsf{T} = \pi_2 \circ \pi_1^{-1}$, the operator T is continuous (hence bounded). □

The following consequence is also very important.

Corollary 1.46 (Hellinger-Toeplitz Theorem). *Let T be an operator on $\mathcal{H}$. If T is symmetric and everywhere defined then T is bounded.*

Proof. By the Closed Graph Theorem above it is sufficient to prove that $\Gamma(\mathsf{T})$ is closed. If (φ_n) is a sequence in $\mathcal{H}$ which converges to φ and such that $(\mathsf{T}\varphi_n)$ converges to $\psi \in \mathcal{H}$, then for all $\eta \in \mathcal{H}$ we have

$$\begin{aligned} \langle \eta, \psi \rangle &= \lim_{n \to +\infty} \langle \eta, \mathsf{T}\varphi_n \rangle = \lim_{n \to +\infty} \langle \mathsf{T}\eta, \varphi_n \rangle \\ &= \langle \mathsf{T}\eta, \varphi \rangle = \langle \eta, \mathsf{T}\varphi \rangle \ . \end{aligned}$$

This proves that $\psi = \mathsf{T}\varphi$. □

1.5.2 Spectrum

Definition 1.47. Let T be an operator from $\mathcal{H}$ to $\mathcal{H}$. An element $v \neq 0$ in $\mathrm{Dom}\,\mathsf{T}$ is an *eigenvector* for T if there exists $\lambda \in \mathbb{C}$ such that $\mathsf{T}v = \lambda v$. In this case λ is an *eigenvalue* of T.

Clearly λ is an eigenvalue of T if and only if the operator $\mathsf{T}-\lambda\mathrm{I}$ is not injective. If λ is not an eigenvalue for T then $\mathsf{T}-\lambda\mathrm{I}$ is injective, the operator

$$\mathsf{R}_\lambda(\mathsf{T}) = (\lambda\mathrm{I}-\mathsf{T})^{-1}$$

is well defined and called the *resolvent of* T *at* λ. We call *resolvent set* of T the set

$$\boxed{\rho(\mathsf{T}) = \big\{\lambda\in\mathbb{C}\,;\ \mathsf{T}-\lambda\mathrm{I} \text{ is injective and } (\mathsf{T}-\lambda\mathrm{I})^{-1} \text{ is bounded}\big\}\,.}$$

If T is not closed then $\mathsf{T}-\lambda\mathrm{I}$ is not closed and $(\mathsf{T}-\lambda\mathrm{I})^{-1}$ neither (Proposition 1.17). In that case we have $\rho(\mathsf{T})=\emptyset$. On the other hand if we assume T to be closed, we have, by Proposition 1.17 and by the Closed Graph Theorem

$$\rho(\mathsf{T}) = \big\{\lambda\in\mathbb{C}\,;\ \mathsf{T}-\lambda\mathrm{I} \text{ is one to one}\big\}\,.$$

Theorem 1.48. *Let* T *be a closed operator on* $\mathcal{H}$*. The resolvent set* $\rho(\mathsf{T})$ *is open. On* $\rho(\mathsf{T})$ *the mapping* $\lambda\mapsto\mathsf{R}_\lambda(\mathsf{T})$ *is analytic. The operators* $\mathsf{R}_\lambda(\mathsf{T})$*,* $\lambda\in\rho(\mathsf{T})$*, commute with each other. Furthermore we have the* resolvent identity

$$\mathsf{R}_\lambda(\mathsf{T})-\mathsf{R}_\mu(\mathsf{T}) = (\mu-\lambda)\mathsf{R}_\lambda(\mathsf{T})\mathsf{R}_\mu(\mathsf{T})\,. \tag{1.8}$$

Proof. Let $\lambda_0\in\rho(\mathsf{T})$, let λ be such that $|\lambda-\lambda_0| < \|\mathsf{R}_{\lambda_0}(\mathsf{T})\|^{-1}$. Then the series

$$\mathsf{R}_{\lambda_0}(\mathsf{T})\Big(\mathrm{I}+\sum_{n=1}^\infty(\lambda_0-\lambda)^n\mathsf{R}_{\lambda_0}(\mathsf{T})^n\Big)$$

is normally convergent and its sum is equal to $\mathsf{R}_\lambda(\mathsf{T})$ (exercise). Thus $\rho(\mathsf{T})$ is open and $\lambda\mapsto R_\lambda(\mathsf{T})$ is analytic on this set.

We furthermore have

$$\begin{aligned}\mathsf{R}_\lambda(\mathsf{T})-\mathsf{R}_\mu(\mathsf{T}) &= \mathsf{R}_\lambda(\mathsf{T})(\mu\,\mathrm{I}-\mathsf{T})\mathsf{R}_\mu(\mathsf{T})-\mathsf{R}_\lambda(\mathsf{T})(\lambda\mathrm{I}-\mathsf{T})\mathsf{R}_\mu(\mathsf{T})\\ &= (\mu-\lambda)\mathsf{R}_\lambda(\mathsf{T})\mathsf{R}_\mu(\mathsf{T})\,.\end{aligned}$$

This gives the resolvent identity and the commutation property. □

Definition 1.49. We put

$$\sigma(\mathsf{T}) = \mathbb{C}\setminus\rho(\mathsf{T})\,.$$

This set is called the *spectrum* of T.

In the following, for any subset A of $\mathbb{C}$, we denote by $\widehat{A}$ the conjugate set of A, that is,

$$\widehat{A} = \{\overline{a}\,;\ a\in A\}$$

(sorry, we cannot denote it by $\overline{A}$ for it may be very confusing with the closure of A).

Proposition 1.50. *Let* T *be a closed operator. Then*

$$\sigma(\mathsf{T}^*) = \widehat{\sigma(\mathsf{T})}\,,$$
$$\rho(\mathsf{T}^*) = \widehat{\rho(\mathsf{T})}\,.$$

In particular, the spectrum of any self-adjoint operator is real.

Proof. It is sufficient to prove that $\rho(\mathsf{T}) \subset \widehat{\rho(\mathsf{T}^*)}$, for if this is true, applying Theorem 1.21, we get

$$\rho(\mathsf{T}^*) \subset \widehat{\rho(\mathsf{T}^{**})} = \widehat{\rho(\mathsf{T})}$$

and consequently

$$\widehat{\rho(\mathsf{T}^*)} \subset \widehat{\widehat{\rho(\mathsf{T})}} = \rho(\mathsf{T})\,.$$

Let $\lambda \in \rho(\mathsf{T})$, then $\lambda\mathrm{I} - \mathsf{T}$ is one to one and $\overline{\lambda}\mathrm{I} - \mathsf{T}^*$ is injective (Proposition 1.19). By Proposition 1.22 we have $(\overline{\lambda}\mathrm{I} - \mathsf{T}^*)^{-1} = \mathsf{R}_\lambda(\mathsf{T})^*$ and it is a bounded operator (Proposition 1.19). We have proved that $\overline{\lambda}\mathrm{I} - \mathsf{T}^*$ admits a bounded inverse, thus $\overline{\lambda}$ belongs to $\rho(\mathsf{T}^*)$. □

Proposition 1.51. *If* T *is a bounded operator on* $\mathcal{H}$ *then* $\sigma(\mathsf{T})$ *is included in the ball* $B(0, \|\mathsf{T}\|)$. *In particular* $\sigma(\mathsf{T})$ *is compact.*

Proof. If T is a bounded operator, the series

$$\frac{1}{\lambda}\sum_n \left(\frac{1}{\lambda}\mathsf{T}\right)^n$$

is normally convergent when $|\lambda| > \|\mathsf{T}\|$ and its sum is equal to $\mathsf{R}_\lambda(\mathsf{T})$. Hence all these λ belong to $\rho(\mathsf{T})$. This proves that $\sigma(\mathsf{T})$ is included in the ball $B(0, \|\mathsf{T}\|)$. As $\rho(\mathsf{T})$ is open (Theorem 1.48) then $\sigma(\mathsf{T})$ is closed and bounded in $\mathbb{C}$, hence compact. □

1.5.3 Spectral Radius

Definition 1.52. For every bounded operator T on $\mathcal{H}$, we define

$$r(\mathsf{T}) = \sup\{|\lambda|\,;\ \lambda \in \sigma(\mathsf{T})\}\,,$$

the *spectral radius* of T.

In particular, by Proposition 1.50, we have

$$r(\mathsf{T}) = r(\mathsf{T}^*)\,.$$

The main results concerning the spectral radius are gathered in the following theorem.

Theorem 1.53. *For every bounded operator* T *on* $\mathcal{H}$ *the following holds.*
1) The spectral radius of T *is given by*

$$r(\mathsf{T}) = \lim_n \|\mathsf{T}^n\|^{1/n} = \inf_n \|\mathsf{T}^n\|^{1/n}\,.$$

2) The spectrum $\sigma(\mathsf{T})$ *is never empty.*
3) If T *is normal, then* $r(\mathsf{T}) = \|\mathsf{T}\|$.

Proof.

1) Let $m \in \mathbb{N}$ be fixed. For any $n \in \mathbb{N}$, we can write

$$n = pm + q$$

with $p, q \in \mathbb{N}$ and $q < m$. We thus have

$$\|\mathsf{T}^n\| \leq \|\mathsf{T}^m\|^p \, \|\mathsf{T}^q\| \leq (1 \vee \|\mathsf{T}\|^m) \, \|\mathsf{T}^m\|^p$$

and

$$\limsup_{n\to+\infty} \|\mathsf{T}^n\|^{1/n} \leq \limsup_{n\to+\infty} \left(1 \vee \|\mathsf{T}\|^{m/n}\right) \, \|\mathsf{T}^m\|^{p/n}\,.$$

But $\|\mathsf{T}\|^{m/n}$ converges to 1 as n tends to $+\infty$ and p/n converges to $1/m$ as n tends to $+\infty$. This gives $\limsup_{n\to+\infty} \|\mathsf{T}^n\|^{1/n} \leq \|\mathsf{T}^m\|^{1/m}$. As a consequence we get

$$\limsup_{n\to+\infty} \|\mathsf{T}^n\|^{1/n} \leq \inf_m \|\mathsf{T}^m\|^{1/m} \leq \liminf_{m\to+\infty} \|\mathsf{T}^m\|^{1/m}\,.$$

This proves that $\lim_n \|\mathsf{T}^n\|^{1/n}$ exists and is equal to $\inf_n \|\mathsf{T}^n\|^{1/n}$.

The convergence radius of the series $\sum_n z^{n+1}\mathsf{T}^n$ is $(\limsup_n \|\mathsf{T}^n\|^{1/n})^{-1}$. Hence, if we choose $|\lambda| > \limsup_n \|\mathsf{T}^n\|^{1/n}$ the series $\sum_n \mathsf{T}^n/\lambda^{n+1}$ converges to a bounded operator which is $\mathsf{R}_\lambda(\mathsf{T})$, that is, λ belongs to $\rho(\mathsf{T})$. This proves that

$$r(\mathsf{T}) \leq \limsup_{n\to+\infty} \|\mathsf{T}^n\|^{1/n} = \lim_{n\to+\infty} \|\mathsf{T}^n\|^{1/n}\,.$$

We now have to prove that it cannot be strictly smaller. If $\sigma(\mathsf{T})$ were included in $B(0, r)$ for some $r < \lim_n \|\mathsf{T}^n\|^{1/n}$ then

$$\mathsf{R}_\lambda(\mathsf{T}) = (\lambda \mathrm{I} - \mathsf{T})^{-1} = \frac{1}{\lambda}(\mathrm{I} - \frac{1}{\lambda}\mathsf{T})^{-1}$$

would be analytic on $|\lambda| > r$. This means that the mapping $z \mapsto (1 - z\mathsf{T})^{-1}$ would be analytic on $B(0, r^{-1})$ and the convergence radius of $\sum_n z^n \, \mathsf{T}^n$ would be greater than $r^{-1} > (\lim_n \|\mathsf{T}^n\|^{1/n})^{-1}$, which leads to a contradiction.

2) If $\sigma(\mathsf{T}) = \emptyset$, then $\lambda \mapsto \mathsf{R}_\lambda(\mathsf{T})$ is analytic on $\mathbb{C}$. For $|\lambda| \geq \|\mathsf{T}\|$ we have

$$\mathsf{R}_\lambda(\mathsf{T}) = \frac{1}{\lambda} \sum_n \left(\frac{1}{\lambda} \mathsf{T} \right)^n$$

and $\mathsf{R}_\lambda(\mathsf{T})$ tends to 0 when λ tends to $+\infty$. Hence $\lambda \mapsto \mathsf{R}_\lambda(\mathsf{T})$ would be constant equal to 0 by Liouville's Theorem. This is clearly impossible.

3) Assume first that T is self-adjoint. We have, by Equation (1.2)

$$\|\mathsf{T}\|^2 = \|\mathsf{T}^*\mathsf{T}\| = \|\mathsf{T}^2\| \,.$$

In the same way, we get, for all $n \in \mathbb{N}$

$$\|\mathsf{T}^{2^n}\| = \|\mathsf{T}\|^{2^n} \,.$$

This gives

$$r(\mathsf{T}) = \lim_{n \to +\infty} \|\mathsf{T}^{2^n}\|^{2^{-n}} = \|\mathsf{T}\| \,.$$

Now, if T is normal, we have, using that $\mathsf{T}^*\mathsf{T}$ is self-adjoint (Theorem 1.39)

$$\begin{aligned} \|\mathsf{T}\|^2 = \|\mathsf{T}^*\mathsf{T}\| = r(\mathsf{T}^*\mathsf{T}) &= \lim_n \left\| (\mathsf{T}^*\mathsf{T})^{2^n} \right\|^{2^{-n}} \\ &= \lim_n \left\| (\mathsf{T}^*)^{2^n} \mathsf{T}^{2^n} \right\|^{2^{-n}} = \lim_n \left\| \left(\mathsf{T}^{2^n}\right)^* \mathsf{T}^{2^n} \right\|^{2^{-n}} \\ &= \left(\lim_n \left\| \mathsf{T}^{2^n} \right\|^{2^{-n}} \right)^2 = r(\mathsf{T})^2 \,. \end{aligned}$$

This proves the equality for normal operators. □

1.6 Spectral Theorem for Bounded Normal Operators

Before extending it to unbounded normal operator we develop here the Spectral Theorem for bounded normal operators, under its two main forms: the functional calculus form and the multiplication operator form. Not only the Spectral Theorem for bounded normal operators is a key step before the general Spectral Theorem, but also it allows several important applications that we shall see in Sects. 1.7 and 1.8.

Even for bounded operators, the Spectral Theorem is a difficult theorem with a very long proof. In particular it is a rather reasonable theorem if one restricts to self-adjoint operators only, but the extension to normal operators is far from obvious.

1.6.1 Polynomial Functions

For bounded operators the notion of composition of operators is obviously defined, in particular a functional calculus for polynomial functions is easy to define. In this section all polynomial functions are elements of $\mathbb{C}[X]$.

Proposition 1.54. *Let* T *be a bounded operator on* $\mathcal{H}$ *and* P *be a polynomial function. Then*

$$\sigma(P(\mathsf{T})) = P(\sigma(\mathsf{T}))\,.$$

Proof. Notice that if $\mathsf{T}_1, \ldots, \mathsf{T}_n$ are 2 by 2 commuting bounded operators and if $\mathsf{B} = \mathsf{T}_1 \cdots \mathsf{T}_n$ then B is invertible in $\mathcal{B}(\mathcal{H})$ if and only if all the T_i's are invertible in $\mathcal{B}(\mathcal{H})$.

Notice also that, we can always write $P(x) - \lambda$ under the form

$$\alpha \prod_{i=1}^{n} (x - \alpha_i)\,,$$

for some $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$. We then have

$$P(\mathsf{T}) - \lambda \mathrm{I} = \alpha \prod_{i=1}^{n} (\mathsf{T} - \alpha_i \, \mathrm{I})\,.$$

If λ belongs to $\sigma(P(\mathsf{T}))$ then $P(\mathsf{T}) - \lambda\mathrm{I}$ is not invertible in $\mathcal{B}(\mathcal{H})$. Hence one of the $\mathsf{T} - \alpha_i\,\mathrm{I}$ above is not invertible in $\mathcal{B}(\mathcal{H})$, that is, one of the α_i above belongs to $\sigma(\mathsf{T})$. As we have $P(\alpha_i) = \lambda$ this shows that λ belongs to $P(\sigma(\mathsf{T}))$. We have proved the inclusion $\sigma(P(\mathsf{T})) \subset P(\sigma(\mathsf{T}))$.

Conversely, if $\lambda \in P(\sigma(\mathsf{T}))$, then $\lambda = P(\beta)$ for some $\beta \in \sigma(\mathsf{T})$. In particular β is one of the roots of $P(x) - \lambda$, that is, β is one of the α_i above. This means that one of the $\mathsf{T} - \alpha_i\,\mathrm{I}$ is not invertible in $\mathcal{B}(\mathcal{H})$ and hence $P(\mathsf{T}) - \lambda\mathrm{I}$ is not invertible in $\mathcal{B}(\mathcal{H})$. That is $\lambda \in \sigma(P(\mathsf{T}))$. □

This rather simple result has a very powerful consequence.

Theorem 1.55. *If* T *is a normal and bounded operator on* $\mathcal{H}$*, if* P *is a polynomial function then* $P(\mathsf{T})$ *is a bounded normal operator too and*

$$\boxed{\|P(\mathsf{T})\| = \|P\|_{\infty, \sigma(\mathsf{T})}\,.} \tag{1.9}$$

In particular the operator $P(\mathsf{T})$ *depends only on the values of* P *on* $\sigma(\mathsf{T})$*.*

Proof. The fact that $P(\mathsf{T})$ is also a normal operator is rather obvious, using the characterization of normality with Equation (1.7). The norm identity is then a direct consequence of the previous proposition and of the identity $r(T) = \|T\|$ for normal bounded operators (Theorem 1.53).

This norm identity immediately gives that if P and Q are two polynomial functions coinciding on $\sigma(\mathsf{T})$ then the operators $P(\mathsf{T})$ and $Q(\mathsf{T})$ coincide. □

We shall need the following result later on.

Proposition 1.56. *The set of invertible elements in $\mathcal{B}(\mathcal{H})$ is open.*

Proof. Assume that A is invertible in $\mathcal{B}(\mathcal{H})$. For every bounded operator B we have

$$\mathsf{B} = \mathsf{A}(I - \mathsf{A}^{-1}(\mathsf{A} - \mathsf{B}))\,.$$

But if we furthermore assume that $\|\mathsf{B} - \mathsf{A}\| < \|\mathsf{A}^{-1}\|^{-1}$ then we have, using Proposition 1.51

$$r(\mathsf{A}^{-1}(\mathsf{A} - \mathsf{B})) \leq \left\|\mathsf{A}^{-1}(\mathsf{A} - \mathsf{B})\right\| < 1$$

and hence the value 1 is not in the spectrum of $\mathsf{A}^{-1}(\mathsf{A} - \mathsf{B})$, that is, the operator $\mathsf{I} - \mathsf{A}^{-1}(\mathsf{A} - \mathsf{B})$ is invertible in $\mathcal{B}(\mathcal{H})$. Hence B is invertible and we have proved that there exists an open ball around A made of bounded operators invertible in $\mathcal{B}(\mathcal{H})$. □

1.6.2 Continuous Functions

In the following we shall restrict to bounded self-adjoint operators. The first step is to define a continuous functional calculus for them.

Theorem 1.57 (Continuous Functional Calculus for Bounded Self-Adjoint Operators). *Let T be a bounded self-adjoint operator on $\mathcal{H}$. Then there exists a unique homomorphism of unital $*$-algebras*

$$\begin{aligned} \Phi : C_0(\sigma(\mathsf{T})) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ f \quad &\longmapsto \Phi(f) = f(\mathsf{T}) \end{aligned}$$

such that

$$\Phi(\mathrm{id}) = \mathsf{T} \tag{1.10}$$

and

$$\boxed{\|f(\mathsf{T})\| = \|f\|_{\infty,\sigma(\mathsf{T})}\ ,} \tag{1.11}$$

for all $f \in C_0(\sigma(\mathsf{T}))$.

We furthermore have the following properties, for all $f \in C_0(\sigma(\mathsf{T}))$.

1) If $\mathsf{T}\psi = \lambda\,\psi$ then $f(\mathsf{T})\psi = f(\lambda)\,\psi$.

2) The operator $f(\mathsf{T})$ is normal; if f is real then $f(\mathsf{T})$ is self-adjoint.

3) If S is any bounded operator commuting with T, then S also commutes with $f(\mathsf{T})$.

4) We have

$$\boxed{\sigma(f(\mathsf{T})) = f(\sigma(\mathsf{T}))\,.}$$

Definition 1.58. Before giving the proof, let us recall that "*homomorphism of unital $*$-algebras*" means, for all $f, g \in C_0(\sigma(\mathsf{T}))$, all $\lambda, \mu \in \mathbb{C}$

i) $\Phi(\lambda f + \mu\, g) = \lambda\, \Phi(f) + \mu\, \Phi(g)$,

ii) $\Phi(fg) = \Phi(f)\, \Phi(g)$,

iii) $\Phi(f)^* = \Phi(\overline{f})$,

iv) $\Phi(\mathbb{1}) = \mathrm{I}$.

Proof (of the Theorem). Let us first construct such a morphism. Let T be a bounded self-adjoint operator on $\mathcal{H}$. For a polynomial function P we put $\Phi(P) = P(\mathsf{T})$ in the usual sense. In that case all the properties announced above can be easily proved, using Proposition 1.54 and Theorem 1.55.

By Stone-Weierstrass' Theorem the morphism Φ extends by continuity to all continuous functions on $\sigma(T)$. As $\sigma(T)$ is compact (Proposition 1.51), the convergence is uniform and thus all the properties of isometric $*$-algebra homomorphism are transferred easily (exercise).

In order to prove uniqueness of the morphism, first note that the properties $\Phi(\mathbb{1}) = \mathrm{I}$, $\Phi(\mathrm{id}) = \mathsf{T}$ and the $*$-algebra morphism property impose that two such morphisms coincide on polynomials. By the isometry property (1.11) and the Stone-Weierstrass Theorem, the morphisms coincide on all continuous functions. Uniqueness is proved.

Property 1) is easily verified for polynomial functions. One then passes to the limit for continuous functions.

Let us prove Property 2) now. As

$$\Phi(f)\Phi(f)^* = \Phi(f\overline{f}) = \Phi(\overline{f}f) = \Phi(f)^*\Phi(f)\,,$$

we have that $\Phi(f)$ is a normal operator. The self-adjointness property is obvious, for $\Phi(f)^* = \Phi(\overline{f}) = \Phi(f)$ if f is real.

Property 3) is obvious when f is a polynomial function. One easily passes to the limit for continuous functions using the uniform convergence.

Let us finally prove 4). Let $f \in C_0(\sigma(\mathsf{T}))$ and λ be a complex number which does not belong to $\mathrm{Ran}\, f$. In particular λ does not belong to $f(\sigma(\mathsf{T}))$ and the function $g = (f - \lambda\mathbb{1})^{-1}$ is finite, continuous and bounded on $\sigma(\mathsf{T})$. By the functional calculus we have

$$g(\mathsf{T}) = (f(\mathsf{T}) - \lambda \mathrm{I})^{-1}\,.$$

Thus λ does not belong to $\sigma(f(\mathsf{T}))$. We have proved $\sigma(f(\mathsf{T})) \subset f(\sigma(\mathsf{T}))$.

Conversely, if $\lambda \in f(\sigma(\mathsf{T}))$ then $\lambda = f(\mu)$ for some $\mu \in \sigma(\mathsf{T})$. If (f_n) is a sequence of polynomial functions converging uniformly to f on $\sigma(\mathsf{T})$ then $f_n(\mathsf{T}) - f_n(\mu)\mathrm{I}$ converges to $f(\mathsf{T}) - \lambda\mathrm{I}$. We have $f_n(\mu) \in f_n(\sigma(\mathsf{T})) = \sigma(f_n(\mathsf{T}))$ (Proposition 1.54), hence $f_n(\mathsf{T}) - f_n(\mu)\mathrm{I}$ is not invertible in $\mathcal{B}(\mathcal{H})$. Consequently $f(\mathsf{T}) - \lambda\mathrm{I}$ is not either (Proposition 1.56). We have proved that $\lambda \in \sigma(f(\mathsf{T}))$ and the converse inclusion is proved. $\square$

1.6.3 Bounded Borel Functions

We shall now extend the functional calculus to bounded Borel functions.

Definition 1.59. Let T be a bounded self-adjoint operator on $\mathcal{H}$ and let $\psi \in \mathcal{H}$. The linear form

$$\begin{array}{ccc} C_0(\sigma(\mathsf{T})) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \langle \psi, f(\mathsf{T})\,\psi \rangle \end{array}$$

is continuous by Theorem 1.57. Hence, by Riesz Representation Theorem, there exists a measure μ_ψ on $(\sigma(\mathsf{T}), \mathrm{Bor}(\sigma(T)))$ such that $\mu_\psi(\sigma(\mathsf{T})) = \|\psi\|^2$ and

$$\langle \psi, f(\mathsf{T})\,\psi \rangle = \int_{\sigma(\mathsf{T})} f(\lambda)\,\mathrm{d}\mu_\psi(\lambda)$$

for all $f \in C_0(\sigma(\mathsf{T}))$. In the following we shall write $L^2(\sigma(\mathsf{T}), \mu_\psi)$ for short, instead of $L^2(\sigma(\mathsf{T}), \mathrm{Bor}(\sigma(\mathsf{T})), \mu_\psi)$.

Definition 1.60. Let T be a bounded self-adjoint operator on $\mathcal{H}$. For any $f \in \mathcal{B}_b(\sigma(\mathsf{T}))$ we define a bounded operator $f(\mathsf{T})$ by

$$\langle \psi\,,\, f(\mathsf{T})\,\psi \rangle = \int_{\sigma(\mathsf{T})} f(\lambda)\,\mathrm{d}\mu_\psi(\lambda)\,. \tag{1.12}$$

Indeed, it is easy to see that the polarized formula $\langle \phi\,,\, f(\mathsf{T})\,\psi \rangle$ defines a bounded operator on $\mathcal{H}$ (exercise).

Theorem 1.61 (Bounded Borel Functional Calculus for Bounded Self-Adjoint Operators). *Let T be a bounded self-adjoint operator on $\mathcal{H}$. There exists a unique map*

$$\begin{array}{ccc} \Phi : \mathcal{B}_b(\sigma(\mathsf{T})) & \longrightarrow & \mathcal{B}(\mathcal{H}) \\ f & \longmapsto & \Phi(f) = f(\mathsf{T}) \end{array}$$

such that

i) Φ is a unital $*$-algebra homomorphism,

ii) Φ is norm-continuous, more precisely $\|\Phi(f)\| \leq \|f\|_{\infty,\sigma(\mathsf{T})}$,

iii) $\Phi(\mathrm{id}) = \mathsf{T}$,

iv) if (f_n) is a bounded sequence in $\mathcal{B}_b(\sigma(\mathsf{T}))$ converging pointwise to f, then $\Phi(f_n)$ converges strongly to $\Phi(f)$.

We furthermore have the following properties, for all $f \in \mathcal{B}_b(\sigma(\mathsf{T}))$.

1) If $\mathsf{T}\psi = \lambda\,\psi$ then $f(\mathsf{T})\psi = f(\lambda)\,\psi$.

2) The operator $f(\mathsf{T})$ is normal; if f is real then $f(\mathsf{T})$ is self-adjoint.

3) If S is a bounded operator commuting with T, then S commutes with the operator $\Phi(f)$.

Proof. We shall be rather elliptic in this proof, for it uses arguments of same type as those of Theorem 1.57.

Consider the mapping $\Phi(f) = f(\mathsf{T})$ as defined by (1.12). Properties ii) and iii) are immediate from the definition. One shall then show that it satisfies iv), but this is just an easy application of Lebesgue's Theorem. Then, as bounded Borel functions can be approximated by bounded sequences of continuous functions, it is easy to prove i) by passing to the limit from the corresponding property for continuous functions (in the same way as we previously deduced it from polynomial functions). This proves that the bounded Borel functional calculus defined by (1.12) satisfies i) – iv).

Let us prove that it is unique. Well, it is always the same argument: the unital $*$-algebra morphism property (which contains the property $\Phi(\mathbb{1}) = \mathrm{I}$) and the property iii) make the mapping Φ being fixed on polynomial functions. The conditions ii) and iv) easily impose that Φ is fixed on all continuous functions and then on all bounded Borel functions.

All the properties 1), 2) and 3) are easily obtained again by an approximation argument, from similar properties satisfied by the continuous functional calculus. □

The most interesting bounded functions to be applied to a bounded self-adjoint operator T are the indicator functions. They are the cornerstone of spectral integration that we shall develop in Sect. 1.9. For the moment, the following properties are deduced immediately from the theorem above.

Corollary 1.62. *If T is a bounded self-adjoint operator on $\mathcal{H}$ and if E is any Borel subset of $\mathbb{R}$, then the operator $\mathbb{1}_E(\mathsf{T})$ is an orthogonal projector of $\mathcal{H}$. For any Borel subsets E, F of $\mathbb{R}$ we have*

$$\mathbb{1}_E(\mathsf{T})\,\mathbb{1}_F(\mathsf{T}) = \mathbb{1}_{E\cap F}(\mathsf{T})\,.$$

If $E \cap F = \emptyset$ then we have

$$\mathbb{1}_E(\mathsf{T}) + \mathbb{1}_F(\mathsf{T}) = \mathbb{1}_{E\cup F}(\mathsf{T})\,.$$

1.6.4 Multiplication Operator Form

Another extremely important form of the Spectral Theorem is the *multiplication operator form.*

Definition 1.63. We say that a vector $\psi \in \mathcal{H}$ is *cyclic* for a bounded operator T on $\mathcal{H}$ if the set $\{\mathsf{T}^n\psi\,;\ n \in \mathbb{N}\}$ is total in $\mathcal{H}$.

Proposition 1.64. *If T is a bounded self-adjoint operator on $\mathcal{H}$ and if T admits a cyclic vector ψ, then there exists a unitary operator U from $\mathcal{H}$ to $L^2(\sigma(\mathsf{T}), \mu_\psi)$ such that, for all $f \in L^2(\sigma(\mathsf{T}), \mu_\psi)$ we have*

$$[\mathsf{U}\mathsf{T}\mathsf{U}^* f](x) = x\, f(x)\,.$$

Proof. Put $\mathsf{U} f(\mathsf{T})\psi = f$ for all $f \in C_0(\sigma(\mathsf{T}))$. The operator U is well-defined for

$$\begin{aligned}\|f(\mathsf{T})\psi\|^2 &= \langle \psi, f(\mathsf{T})^* f(\mathsf{T})\, \psi\rangle \\ &= \langle \psi, (\overline{f} f)(\mathsf{T})\psi\rangle \\ &= \int |f(\lambda)|^2 \, \mathrm{d}\mu_\psi(\lambda) \\ &= \|f\|^2\,,\end{aligned}$$

hence U is an isometry. We extend U by continuity to $\mathcal{H}$ which is the closure of $\{f(\mathsf{T})\psi\,;\ f \in C_0(\sigma(\mathsf{T}))\}$, by cyclicity of ψ. The extension is an isometry too. But obviously $\mathrm{Ran}\,\mathsf{U}$ contains $C_0(\sigma(\mathsf{T}))$ and $\mathrm{Ran}\,\mathsf{U}$ is closed. Thus $\mathrm{Ran}\,\mathsf{U} = L^2(\sigma(\mathsf{T}), \mu_\psi)$, by classical density arguments, and U is unitary.

Finally, we have

$$[\mathsf{U}\mathsf{T}\mathsf{U}^* f](x) = [\mathsf{U}\mathsf{T} f(\mathsf{T})\psi](x) = x\, f(x)$$

and we have proved the proposition. □

Proposition 1.65. *Let T be a bounded self-adjoint operator on $\mathcal{H}$. The Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of Hilbert spaces*

$$\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$$

where

i) each $\mathcal{H}_n$ is invariant under T,

ii) for all n, there exists $\psi_n \in \mathcal{H}_n$ which is cyclic for T restricted to $\mathcal{H}_n$.

Proof. This just an application of Zorn lemma, let us detail it. Consider the set X of sub-Hilbert spaces $\mathcal{K}$ of $\mathcal{H}$ that can be decomposed as above. We give X a partial order structure by saying that $\mathcal{K}_1 \subset \mathcal{K}_2$ if they are included as subspaces of $\mathcal{H}$ and if their decompositions as direct sums are compatible. Consider any chain in X, that is, a totally ordered subset of X. As $\mathcal{H}$ is separable, the chain is at most countable, let us denote it by $(\mathcal{K}_n)$. This chain admits an upper bound by taking $\mathcal{K} = \vee_n \mathcal{K}_n$, the sub-Hilbert space generated by the $\mathcal{K}_n$'s. Hence by Zorn Lemma, there exists a maximal element in X. If that maximal element $\mathcal{A}$ were not the whole of $\mathcal{H}$, then take a $\psi \in \mathcal{A}^\perp$ and consider the Hilbert space $\mathcal{A}'$ generated by the $\mathsf{T}^n\psi$, $n \in \mathbb{N}$. This Hilbert space is orthogonal to $\mathcal{A}$, by the stability property i). Hence the space $\mathcal{A} \oplus \mathcal{A}'$ would be an element of X and would be larger than $\mathcal{A}$, which contradicts the maximality property. This means that $\mathcal{A}$ has to be the whole of $\mathcal{H}$, hence the announced result. □

Theorem 1.66 (Multiplication Operator Representation of Bounded Self-Adjoint Operators). *Let T be a bounded self-adjoint operator on $\mathcal{H}$. Then there exists a measured space $(E, \mathcal{E}, \mu)$ with $\mu(E) < \infty$, a unitary operator $\mathsf{U} : \mathcal{H} \to L^2(E, \mathcal{E}, \mu)$ and a real-valued bounded function f on E such that*

$$\boxed{\mathsf{U}\mathsf{T}\mathsf{U}^* = \mathsf{M}_f\,.}$$

Proof. Consider a decomposition of $\mathcal{H}$ as in Proposition 1.65. Let T_n denote the restriction of T to $\mathcal{H}_n$. Let E_n denote the spectrum of T_n. Define E to be the formal disjoint union of the sets E_n. Normalize each ψ_n such that $\|\psi_n\| = 2^{-n}$. Define the measure $\mu = \oplus_n \mu_{\psi_n}$ on E, it is a finite measure by construction. Let U_n be the associated unitary operators from $\mathcal{H}_n$ to $L^2(E_n, \mu_n)$, as given by Proposition 1.64. Define the unitary operator

$$\begin{aligned} \mathsf{U} : \mathcal{H} = \textstyle\bigoplus_n \mathcal{H}_n &\longrightarrow L^2(E, \mu) = \textstyle\bigoplus_n L^2(E_n, \mu_n) \\ y = \textstyle\bigoplus_n y_n &\longmapsto \qquad \mathsf{U}\, y = \textstyle\bigoplus_n \mathsf{U}_n\, y_n\,. \end{aligned}$$

It is now easy to conclude. □

1.6.5 Commuting Families of Bounded self-Adjoint Operators

An important extension of the spectral theorems we have established in Subsects. 1.6.1 – 1.6.4, is the case of a family of commuting bounded self-adjoint operators. In particular it will allow an extension of the previous theorems to bounded normal operators.

In this subsection we shall again be rather elliptic with the proofs, for they are once again repetitions of the same types of arguments as in the previous subsections.

The easiest theorem to extend to the commuting family case is the bounded Borel functional calculus.

Theorem 1.67 (Multivariable Bounded Borel Functional Calculus). *Let $\{\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n\}$ be a family of two-by-two commuting bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. Let σ_i be the spectrum of A_i for all $i \in \{1, \ldots, n\}$ and $\sigma = \prod_{i=1}^n \sigma_i$. Then there exists a unique map*

$$\begin{aligned} \Phi : \mathcal{B}_b(\sigma) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ f \quad &\longmapsto \Phi(f) = f(\mathsf{A}_1, \ldots, \mathsf{A}_n) \end{aligned}$$

such that

i) Φ is a unital $*$-algebra homomorphism, *ii)* Φ *is norm-continuous, more precisely* $\|\Phi(f)\| \leq \|f\|_{\infty,\sigma}$ *,*

iii) if $f : (x_1, \ldots, x_n) \mapsto x_i$ *then* $\Phi(f) = \mathsf{A}_i$*, for all* $i \in \{1, \ldots, n\}$*,*

iv) if (f_n) *is a bounded sequence in* $\mathcal{B}_b(\sigma)$ *converging pointwise to* f*, then* $\Phi(f_n)$ *converges strongly to* $\Phi(f)$*.*

We furthermore have the following properties, for all $f \in \mathcal{B}_b(\sigma)$ *.*

1) The operator $f(\mathsf{A}_1, \ldots, \mathsf{A}_n)$ *is normal; if* f *is real then* $f(\mathsf{A}_1, \ldots, \mathsf{A}_n)$ *is self-adjoint.*

2) If S *is a bounded operator commuting with all the operators* $\mathsf{A}_1, \ldots, \mathsf{A}_n$*, then* S *commutes with the operator* $\Phi(f)$*.*

Proof. For any Borel subsets $E_1, \ldots, E_n$ of $\sigma_1, \ldots, \sigma_n$ respectively, we consider the following function on σ

$$h(x_1, \ldots, x_n) = \mathbb{1}_{E_1}(x_1) \ldots \mathbb{1}_{E_n}(x_n)$$

and we put

$$h(\mathsf{A}_1, \ldots, \mathsf{A}_n) = \mathbb{1}_{E_1}(\mathsf{A}_1) \ldots \mathbb{1}_{E_n}(\mathsf{A}_n)$$

in the sense of the bounded functional calculus. Note that, by Property 3) of Theorem 1.61, the operators $\mathbb{1}_{E_i}(\mathsf{A}_i)$ two-by-two commute.

Now, if f is any function on σ which is a linear combination of indicator functions h_k as above, with the functions h_k having zero products, we define $f(\mathsf{A}_1, \ldots, \mathsf{A}_n)$ by linearity. It is then easy to check (with this zero product condition) that

$$\|f(\mathsf{A}_1, \ldots, \mathsf{A}_n)\| \leq \|f\|_{\infty,\sigma} .$$

Once we have this norm-continuity relation it is again standard approximation arguments to extend these definitions to a multi-variable bounded functional calculus for the family $\{\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n\}$. □

We then have the extension of the multiplication operator form.

Theorem 1.68 (Multivariable Multiplication Operator Representation). *Let* $\mathsf{A}_1, \ldots, \mathsf{A}_n$ *be two-by-two commuting bounded self-adjoint operators on* $\mathcal{H}$*. Then there exists a measured space* $(E, \mathcal{E}, \mu)$ *with* $\mu(E) < \infty$*, a unitary operator* $\mathsf{U} : \mathcal{H} \to L^2(E, \mathcal{E}, \mu)$ *and real-valued bounded functions* $f_1, \ldots, f_n$ *on* E *such that*

$$\boxed{\mathsf{U}\mathsf{A}_i\mathsf{U}^* = \mathsf{M}_{f_i} \,,}$$

for all $i = 1, \ldots, n$*.*

Proof. With the bounded Borel functional calculus on $\mathsf{A}_1, \ldots, \mathsf{A}_n$ we have the continuous functional calculus. We can then define, as in Subsect. 1.6.3, for every $\psi \in \mathcal{H}$, the measure μ_ψ on $\sigma = \prod_{i=1}^n \sigma(\mathsf{A}_i)$ by

$$\langle \psi \,,\, f(\mathsf{A}_1, \ldots, \mathsf{A}_n)\, \psi \rangle = \int_\sigma f(\lambda_1, \ldots, \lambda_n)\, \mathrm{d}\mu_\psi(\lambda_1, \ldots, \lambda_n)$$

for all continuous function f on σ.

We then have the same construction of the multiplication operators as in Subsect. 1.6.4: starting with the case of cyclic vectors and extending it to the general case by Zorn's Lemma. □

It is now easy to extend the Spectral Theorem to bounded normal operators. It is actually the multiplication operator form which is the most useful.

Theorem 1.69 (Multiplication Operator Representation of Bounded Normal Operators). *Let* T *be a bounded normal operator on* $\mathcal{H}$. *Then there exists a measured space* $(E, \mathcal{E}, \mu)$ *with* $\mu(E) < \infty$, *a unitary operator* $\mathsf{U} : \mathcal{H} \to L^2(E, \mathcal{E}, \mu)$ *and a complex-valued bounded function* g *on* E *such that*

$$\boxed{\mathsf{U}\mathsf{T}\mathsf{U}^* = \mathsf{M}_g\,.}$$

Proof. If T is a normal operator on $\mathcal{H}$, we put

$$\mathsf{A}_1 = \frac{\mathsf{T} + \mathsf{T}^*}{2} \qquad \text{and} \qquad \mathsf{A}_2 = -\mathrm{i}\,\frac{\mathsf{T} - \mathsf{T}^*}{2}\,.$$

We then check easily that the operators $\mathsf{A}_1, \mathsf{A}_2$ are bounded self-adjoint operators, they commute with each other and we have

$$\mathsf{T} = \mathsf{A}_1 + \mathrm{i}\mathsf{A}_2\,.$$

By Theorem 1.68 there exists a finitely measured space $(E, \mathcal{E}, \mu)$, a unitary operator U from $\mathcal{H}$ to $L^2(E, \mathcal{E}, \mu)$ and bounded real functions f_1, f_2 on E such that

$$\mathsf{U}\mathsf{A}_i\mathsf{U}^* = \mathsf{M}_{f_i}$$

for each i. Putting $g = f_1 + \mathrm{i} f_2$ we get $\mathsf{U}\mathsf{T}\mathsf{U}^* = \mathsf{M}_g$ and the theorem is proved. □

1.7 Positivity and Polar Decomposition

1.7.1 Positive Operators

Positive operators are a fundamental class of operators, they are used every now and then in Operator Theory, in Quantum Mechanics, etc.

Definition 1.70. A bounded operator T on $\mathcal{H}$ is *positive* if it is self-adjoint and $\sigma(\mathsf{T}) \subset \mathbb{R}^+$.

There are many equivalent characterization of the positivity for a bounded operator. They are summarized in the following very important theorem.

Theorem 1.71. *Let* T *be a bounded operator on* $\mathcal{H}$*. The following assertions are equivalent.*

i) The operator T *is positive.*

ii) $\langle x\,,\,\mathsf{T}x\rangle$ *is positive for all* $x\in\mathcal{H}$*.*

iii) There exists a bounded operator B *from* $\mathcal{H}$ *to some Hilbert space* $\mathcal{K}$ *such that* $\mathsf{T}=\mathsf{B}^*\mathsf{B}$*.*

iv) There exists a positive bounded operator C *on* $\mathcal{H}$ *such that* $\mathsf{T}=\mathsf{C}^2$*.*

v) The operator T *is self-adjoint and* $\|t\,\mathrm{I}-\mathsf{T}\|\leq t$ *for some* $t\geq\|\mathsf{T}\|$*.*

vi) The operator T *is self-adjoint and* $\|t\,\mathrm{I}-\mathsf{T}\|\leq t$ *for all* $t\geq\|\mathsf{T}\|$*.*

When these properties are realized, the positive operator C *satisfying iv) is unique.*

Proof. We first prove that ii) implies i). As $\langle x\,,\,\mathsf{T}x\rangle$ is real for all $x\in\mathcal{H}$ we have

$$\langle \mathsf{T}x\,,\,x\rangle=\overline{\langle x\,,\,\mathsf{T}x\rangle}=\langle x\,,\,\mathsf{T}x\rangle\ ,$$

for all $x\in\mathcal{H}$. But on the other hand we have the polarization identity

$$\begin{aligned}\langle y\,,\,\mathsf{T}x\rangle&=\frac{1}{4}\big[\,\langle x+y\,,\,\mathsf{T}(x+y)\rangle-\langle x-y\,,\,\mathsf{T}(x-y)\rangle+\mathrm{i}\,\langle x+\mathrm{i}y\,,\,\mathsf{T}(x+\mathrm{i}y)\rangle\\&\qquad-\mathrm{i}\,\langle x-\mathrm{i}y\,,\,\mathsf{T}(x-\mathrm{i}y)\rangle\,\big]\\&=\frac{1}{4}\big[\,\langle \mathsf{T}(x+y)\,,\,x+y\rangle-\langle \mathsf{T}(x-y)\,,\,x-y\rangle+\mathrm{i}\,\langle \mathsf{T}(x+\mathrm{i}y)\,,\,x+\mathrm{i}y\rangle\\&\qquad-\mathrm{i}\,\langle \mathsf{T}(x-\mathrm{i}y)\,,\,x-\mathrm{i}y\rangle\,\big]\\&=\langle \mathsf{T}y\,,\,x\rangle\ .\end{aligned}$$

This shows that T is self-adjoint. Hence, by Proposition 1.50, the spectrum $\sigma(\mathsf{T})$ of T is real. We have now to prove that $\sigma(\mathsf{T})$ is actually included in $\mathbb{R}^+$. If λ belongs to $\mathbb{R}$ we have

$$\lambda\,\|x\|^2=\langle\lambda x\,,\,x\rangle\leq\langle(\mathsf{T}+\lambda\mathrm{I})x\,,\,x\rangle\leq\|(\mathsf{T}+\lambda\mathrm{I})x\|\ \|x\|\ .$$

Hence, for $x\neq 0$ we have

$$\|(\mathsf{T}+\lambda\mathrm{I})x\|\geq\lambda\,\|x\|\ . \tag{1.13}$$

Now, if $\lambda>0$ the above implies obviously that $\mathsf{T}+\lambda\mathrm{I}$ is injective. As

$$(\mathrm{Ran}(\mathsf{T}+\lambda\mathrm{I}))^\perp=\mathrm{Ker}(\mathsf{T}+\lambda\mathrm{I})^*=\mathrm{Ker}(\mathsf{T}+\lambda\mathrm{I})=\{0\}\,,$$

then $\mathrm{Ran}(\mathsf{T}+\lambda\mathrm{I})$ is dense. Furthermore, the inverse of $\mathsf{T}+\lambda\mathrm{I}$ is bounded for

$$\left\|(\mathsf{T}+\lambda\mathrm{I})^{-1}y\right\|\leq\frac{1}{\lambda}\left\|(\mathsf{T}+\lambda\mathrm{I})(\mathsf{T}+\lambda\mathrm{I})^{-1}\,y\right\|=\frac{1}{\lambda}\,\|y\|\ .$$

We have proved that $-\lambda$ belongs to $\rho(\mathsf{T})$, that is, $\sigma(\mathsf{T})$ is included in $\mathbb{R}^+$.

We now prove that i) implies iv). If T is self-adjoint and $\sigma(\mathsf{T})$ is included in $\mathbb{R}^+$ then the continuous functional calculus (Theorem 1.57) applies to T. The function $x \mapsto \sqrt{x}$ is continuous on $\sigma(\mathsf{T})$, hence define

$$\mathsf{C} = \sqrt{\mathsf{T}}\,.$$

This operator is positive for it is self-adjoint and its spectrum is included in $\mathbb{R}^+$ (Theorem 1.57). We have $\mathsf{C}^2 = \mathsf{T}$ by the morphism property of functional calculus. This gives iv).

Obviously iv) implies iii) by taking $\mathcal{K} = \mathcal{H}$ and $\mathsf{B} = \mathsf{C}$.

It is clear that iii) implies ii) for $\mathsf{B}^*\mathsf{B}$ is a bounded operator on $\mathcal{H}$ and

$$\langle x\,,\,\mathsf{B}^*\mathsf{B}x\rangle = \langle \mathsf{B}x\,,\,\mathsf{B}x\rangle = \|\mathsf{B}x\|^2 \geq 0$$

for all $x \in \mathcal{H}$.

We have proved the equivalence between i), ii), iii) and iv).

Let us prove that i) implies vi). If i) is satisfied then $t\,\mathrm{I} - \mathsf{T}$ is a self-adjoint bounded operator and, by Theorem 1.53, we have

$$\begin{aligned}\|t\,\mathrm{I} - \mathsf{T}\| &= r(t\,\mathrm{I} - \mathsf{T}) = \sup\{|\lambda|\ ;\ \lambda \in \sigma(t\,\mathrm{I} - \mathsf{T})\}\\ &= \sup\{|\lambda - t|\ ;\ \lambda \in \sigma(\mathsf{T})\}\,.\end{aligned}$$

But as $\sigma(\mathsf{T})$ is included in $\mathbb{R}^+$, by hypothesis, we have $|\lambda - t| \leq t$ for all $t \geq \|\mathsf{T}\| \geq \lambda$. This gives vi).

Obviously vi) implies v).

Let us prove that v) implies i). If v) is satisfied and if $\lambda \in \sigma(\mathsf{T})$ then $t - \lambda \in \sigma(t\,\mathrm{I} - \mathsf{T})$ and with the same computation as above $|t - \lambda| \leq \|t\,\mathrm{I} - \mathsf{T}\| \leq t$. But as $\lambda \leq t$ we must have $\lambda \geq 0$. This proves i).

We have proved that the assertions are all equivalent.

We need to prove the uniqueness property now. If C' is another bounded positive operator such that $\mathsf{C}'^2 = \mathsf{T}$, then

$$\mathsf{C}'\mathsf{T} = \mathsf{C}'^3 = \mathsf{T}\mathsf{C}'\,.$$

This is to say that C' commutes with T. It is easy to deduce that C' commutes with any polynomial P applied to T. Passing to the limit to any continuous function h on $\sigma(\mathsf{T})$, we get $\mathsf{C}'h(\mathsf{T}) = h(\mathsf{T})\mathsf{C}'$. In particular C' commutes with the square root C constructed above.

As a consequence we have

$$(\mathsf{C} - \mathsf{C}')\mathsf{C}(\mathsf{C} - \mathsf{C}') + (\mathsf{C} - \mathsf{C}')\mathsf{C}'(\mathsf{C} - \mathsf{C}') = (\mathsf{C}^2 - \mathsf{C}'^2)(\mathsf{C} - \mathsf{C}') = 0\,.$$

Both the operators $(\mathsf{C} - \mathsf{C}')\mathsf{C}(\mathsf{C} - \mathsf{C}')$ and $(\mathsf{C} - \mathsf{C}')\mathsf{C}'(\mathsf{C} - \mathsf{C}')$ are positive operators, as can be easily checked. Hence their sum can be 0 if and only if they are both 0 (exercise). As a consequence their difference, which is the operator $(\mathsf{C}-\mathsf{C}')^3$, is the null operator. But $\mathsf{C}-\mathsf{C}'$ is a self-adjoint operator, hence

$$\left\|(\mathsf{C}-\mathsf{C}')^4\right\| = \|\mathsf{C}-\mathsf{C}'\|^4$$

by (1.2) and this proves that $\|\mathsf{C}-\mathsf{C}'\| = 0$. Hence $\mathsf{C} = \mathsf{C}'$. □

Definition 1.72. This unique operator C given by iv) above is called the *square root* of T and is denoted by $\sqrt{\mathsf{T}}$.

Definition 1.73. If T is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, then the operator $\mathsf{T}^*\mathsf{T}$ is a positive operator on $\mathcal{H}_1$, by iii) above. Hence, by iv) above, we may consider the positive operator $|\mathsf{T}|$ on $\mathcal{H}_1$ given by

$$|\mathsf{T}| = \sqrt{\mathsf{T}^*\mathsf{T}}\,.$$

It is called the *absolute value* of T.

Note that if T is self-adjoint then this operator $|\mathsf{T}|$ coincides with the definition of $|\mathsf{T}|$ obtained by applying the continuous function $x \mapsto |x|$ to the operator T, in the sense of the functional calculus above (exercise).

Note that $\mathsf{T}\mathsf{T}^*$ is also positive and hence admits a square root too, but it has no reason to be the same operator as $|\mathsf{T}|$ (even if $\mathcal{H}_1 = \mathcal{H}_2$). In the definition of $|\mathsf{T}|$ a choice has been made. It is this choice which is usually understood in the mathematical community when defining the operator $|\mathsf{T}|$.

Be careful with this notion of absolute value for an operator, most of the usual properties that one knows for the modulus of complex numbers do not apply for the absolute value of operators. For example it is in general *not true* that $|\mathsf{A}^*| = |\mathsf{A}|$, nor that $|\mathsf{AB}| = |\mathsf{A}|\,|\mathsf{B}|$, nor that $|\mathsf{A}+\mathsf{B}| \leq |\mathsf{A}| + |\mathsf{B}|$ (in the sense that the difference is a positive operator), etc.

We end this section with an interesting decomposition for self-adjoint bounded operators.

Proposition 1.74. *Every bounded self-adjoint operator* T *on a Hilbert space* $\mathcal{H}$ *can be decomposed in a unique way into*

$$\mathsf{T} = \mathsf{T}^+ - \mathsf{T}^-$$

where T^+ *and* T^- *are bounded positive operators satisfying* $T\omega^+\mathsf{T}^- = 0$*.*

In that case we also have

$$|\mathsf{T}| = \mathsf{T}^+ + \mathsf{T}^-$$

and

$$\left\|\mathsf{T}^+\right\| \leq \|\mathsf{T}\|\ , \qquad \left\|\mathsf{T}^-\right\| \leq \|\mathsf{T}\|\ .$$

Proof. Consider the functions $f^+(x) = x \vee 0$ and $f^-(x) = -(x \wedge 0)$, they are both continuous positive functions, they satisfy

$$\begin{aligned} f^+(x) - f^-(x) &= x\,, \\ f^+(x) + f^-(x) &= |x|\,, \\ f^+(x) f^-(x) &= 0 \end{aligned}$$

for all x. By the Bounded Spectral Theorem 1.57, applying the functions f^+ and f^- to T we get two positive operators T^+ and T^- satisfying

$$\begin{aligned} \mathsf{T}^+ - \mathsf{T}^- &= \mathsf{T}\,, \\ \mathsf{T}^+ + \mathsf{T}^- &= |\mathsf{T}|\,, \\ \mathsf{T}^+\mathsf{T}^- &= 0\,. \end{aligned}$$

By the same Bounded Spectral Theorem, we also have

$$\left\|\mathsf{T}^+\right\| = \sup\{\left|f^+(x)\right|\,;\ x \in \sigma(\mathsf{T})\} \leq \sup\{|x|\,;\ x \in \sigma(\mathsf{T})\} = \|\mathsf{T}\|\,.$$

The same holds in the same way for T^-.

There only remains to prove the uniqueness of such a decomposition. If $\mathsf{T} = \mathsf{A} - \mathsf{B}$ where A and B are positive and $\mathsf{AB} = 0$, then we get $\mathsf{T}^2 = \mathsf{A}^2 + \mathsf{B}^2$ by immediate computation. Note that $\mathsf{A}+\mathsf{B}$ is positive and $(\mathsf{A}+\mathsf{B})^2 = \mathsf{A}^2+\mathsf{B}^2$ too. This says that $\mathsf{A}+\mathsf{B}$ is the square root of T^2 (uniqueness of the square root, Theorem 1.71), that is, $\mathsf{A}+\mathsf{B} = |\mathsf{T}|$. As a consequence we have $\mathsf{A} = (\mathsf{T} + |\mathsf{T}|)/2 = \mathsf{T}^+$ and thus $\mathsf{B} = \mathsf{T}^-$. This proves uniqueness. □

Definition 1.75. These operators T^+ and T^- are called the *positive part* and the *negative part* of T.

1.7.2 Polar Decomposition

This notion of absolute value of an operator leads to a very important decomposition theorem for bounded operator. This decomposition has to be thought of as similar to the decomposition $z = |z|\, e^{i\theta}$ for complex numbers, with the limitations explained above on the notion of absolute value.

Theorem 1.76 (Polar Decomposition). *For every bounded operator* T *from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *there exists a partial isometry* U *with initial space* $\overline{\mathrm{Ran}\,|\mathsf{T}|} \subset \mathcal{H}_1$ *and final space* $\overline{\mathrm{Ran}\,\mathsf{T}} \subset \mathcal{H}_2$*, such that*

$$\mathsf{T} = \mathsf{U}\,|\mathsf{T}|\,.$$

The partial isometry U *is uniquely determined by the condition* $\mathrm{Ker}\,\mathsf{U} = \mathrm{Ker}\,\mathsf{T}$*.*

Proof. For every $f, g \in \mathcal{H}_1$ we have

$$\langle |\mathsf{T}|\, f\,,\, |\mathsf{T}|\, g\rangle = \left\langle f\,,\, |\mathsf{T}|^2\, g\right\rangle = \langle f\,,\, \mathsf{T}^*\mathsf{T}g\rangle = \langle \mathsf{T}f\,,\, \mathsf{T}g\rangle\ .$$

This shows that the mapping

$$\begin{array}{rcl} \mathsf{U} : \mathrm{Ran}\,|\mathsf{T}| \subset \mathcal{H}_1 & \longrightarrow & \mathrm{Ran}\,\mathsf{T} \subset \mathcal{H}_2 \\ |\mathsf{T}|\, g & \longmapsto & \mathsf{T}g \end{array}$$

is isometric. It extends to a unitary operator from $\overline{\mathrm{Ran}\,|\mathsf{T}|}$ to $\overline{\mathrm{Ran}\,\mathsf{T}}$ and to a partial isometry from $\mathcal{H}_1$ to $\mathcal{H}_2$ by assigning the value 0 to $\overline{\mathrm{Ran}\,|\mathsf{T}|}^{\perp}$. Clearly, by definition, we have

$$\mathsf{T} = \mathsf{U}\,|\mathsf{T}|\,.$$

The kernel of U is by definition

$$\overline{\mathrm{Ran}\,|\mathsf{T}|}^{\perp} = \mathrm{Ran}\,|\mathsf{T}|^{\perp} = \mathrm{Ker}\,|\mathsf{T}| = \mathrm{Ker}\,\mathsf{T}\,.$$

Uniqueness is easy and left to the reader. □

1.8 Compact Operators

Compact operators are a special class of bounded operators which is of much importance in applications. This is due in particular to their very simple spectral decomposition. A special case of compact operators, the trace-class operators (that we explore in next Lecture) are central objects in Quantum Mechanics of Open Systems.

1.8.1 Finite Rank and Compact Operators

Definition 1.77. A bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is *finite rank* if its range is finite dimensional. The space of finite rank operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{L}_0(\mathcal{H}_1, \mathcal{H}_2)$. The space of finite rank operators on $\mathcal{H}$ is denoted by $\mathcal{L}_0(\mathcal{H})$.

Definition 1.78. A bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is *compact* if the image by T of the unit ball of $\mathcal{H}_1$ has a compact closure in $\mathcal{H}_2$. Equivalently, T is compact if and only if, for every bounded sequence (x_n) in $\mathcal{H}_1$, the sequence $(\mathsf{T}x_n)$ admits a convergent subsequence in $\mathcal{H}_2$.

The space of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{L}_\infty(\mathcal{H}_1, \mathcal{H}_2)$. The space of compact operators on $\mathcal{H}$ is denoted by $\mathcal{L}_\infty(\mathcal{H})$.

Theorem 1.79.

1) The space $\mathcal{L}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is a closed two-sided ideal of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

2) The space $\mathcal{L}_0(\mathcal{H}_1, \mathcal{H}_2)$ is dense in $\mathcal{L}_\infty(\mathcal{H}_1, \mathcal{H}_2)$.

Proof.

1) Let (A_n) be a sequence of compact operators in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which converges in operator norm to $\mathsf{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Let B_1 denote the unit ball of $\mathcal{H}_1$. The set $\mathsf{A}_n(B_1)$ is precompact and hence can be covered by a finite union of balls:

$$\mathsf{A}_n(B_1) \subset \bigcup_{x \in S} B(\mathsf{A}_n x, \varepsilon)\,,$$

where $S \subset B_1$ is finite. For every $y \in B_1$ there exists a $x \in S$ such that

$$\|\mathsf{A}y - \mathsf{A}x\| \leq \|\mathsf{A}y - \mathsf{A}_n y\| + \|\mathsf{A}_n y - \mathsf{A}_n x\| + \|\mathsf{A}_n x - \mathsf{A}x\| \leq 3\varepsilon$$

by choosing n large enough. This says that

$$\mathsf{A}(B_1) \subset \bigcup_{x \in S} B(\mathsf{A}x, 3\varepsilon).$$

Hence $\mathsf{A}(B_1)$ is precompact. This proves that $\mathcal{L}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is closed. The two-sided ideal property is easy to check and is left to the reader.

2) Let $\mathsf{A} \in \mathcal{L}_\infty(\mathcal{H}_1, \mathcal{H}_2)$. As above, for every $\varepsilon > 0$, there exists a finite set $S \subset B_1$ such that

$$\mathsf{A}(B_1) \subset \bigcup_{x \in S} B(\mathsf{A}x, \varepsilon).$$

Let E be the finite dimensional space generated by the $\mathsf{A}x$, $x \in S$ and let P be the orthogonal projection onto E. The operator PA is then finite rank and we claim that $\|\mathsf{A} - \mathsf{PA}\| \leq \varepsilon$. Indeed, for all $y \in \mathcal{H}_1$ with $\|y\| = 1$ and all $x \in S$ we have

$$\begin{aligned}
\|\mathsf{A}y - \mathsf{A}x\|^2 &= \|\mathsf{A}y - \mathsf{PA}x\|^2 \\
&= \|\mathsf{A}y - \mathsf{PA}y + \mathsf{PA}y - \mathsf{PA}x\|^2 \\
&= \|\mathsf{A}y - \mathsf{PA}y\|^2 + \|\mathsf{PA}x - \mathsf{PA}y\|^2\,,
\end{aligned}$$

hence $\|\mathsf{A}y - \mathsf{PA}y\| \leq \|\mathsf{A}y - \mathsf{A}x\| \leq \varepsilon$. This proves our claim and ends the proof. □

Proposition 1.80. *A bounded operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is compact if and only if for every orthonormal family (e_n) in $\mathcal{H}_1$ we have*

$$\lim_{n \to +\infty} \|\mathsf{T}e_n\| = 0\,.$$

Proof. For the “only if” part, assume that T is compact. If (e_n) is an orthonormal family in $\mathcal{H}_1$ then the sequence (e_n) converges weakly to 0, for

$$\sum_n |\langle y\,,\, e_n\rangle|^2 = \|y\|^2 < \infty$$

for all $y \in \mathcal{H}_1$. For all $z \in \mathcal{H}$ we have

$$\langle z\,,\, \mathsf{T}e_n\rangle = \langle \mathsf{T}^* z\,,\, e_n\rangle \longrightarrow 0\,.$$

Hence the sequence $(\mathsf{T}e_n)_n$ converges weakly to 0. If $(\mathsf{T}e_n)_n$ were not converging strongly to 0, there would exist a $\varepsilon > 0$ and a subsequence $(e_{n_k})_k$ such that $\|\mathsf{T}e_{n_k}\| > \varepsilon$ for all k. As $(e_{n_k})_k$ is obviously bounded and T is compact, the sequence $(\mathsf{T}e_{n_k})_k$ admits a convergent subsequence in $\mathcal{H}_2$, with limit y satisfying $\|y\| \geq \varepsilon$. This contradict the fact that $(\mathsf{T}e_n)_n$ converges weakly to 0. As a consequence $(\mathsf{T}e_n)$ converges strongly to 0.

Conversely, let us prove the “if” part. Assume that for every orthonormal family (e_n) in $\mathcal{H}_1$ we have $\lim \|\mathsf{T}e_n\| = 0$. Let us prove that T belongs to the closure of $\mathcal{L}_0(\mathcal{H}_1, \mathcal{H}_2)$ (this would give the conclusion, by the previous theorem). If it were not the case, there would exist a $\varepsilon > 0$ such that $\|\mathsf{T} - \mathsf{R}\| > \varepsilon$ for every $\mathsf{R} \in \mathcal{L}_0(\mathcal{H}_1, \mathcal{H}_2)$. In particular $\|\mathsf{T}\| > \varepsilon$ and there exists a $e_0 \in \mathcal{H}_1$ such that $\|e_0\| = 1$ and $\|\mathsf{T}e_0\| > \varepsilon$.

Assume that there exists an orthonormal family $e_0, \ldots, e_n$ in $\mathcal{H}_1$ such that $\|\mathsf{T}e_k\| > \varepsilon$ for all $0 \leq k \leq n$. Let P be the orthogonal projector on the space E generated by $e_0, \ldots, e_n$. The operator TP is finite rank, hence $\|\mathsf{T} - \mathsf{TP}\| > \varepsilon$. As a consequence there exists a $y \in \mathcal{H}_1$ with $\|y\| = 1$ and such that $\|(\mathsf{T} - \mathsf{TP})y\| > \varepsilon$. Obviously y does not belong to E for otherwise $(\mathsf{T} - \mathsf{TP})y = \mathsf{T}(\mathrm{I} - \mathsf{P})y = 0$. Define $z = (\mathrm{I} - \mathsf{P})y$ and $e_{n+1} = \|z\|^{-1}\, z$. This element of $\mathcal{H}_1$ is orthogonal to E and satisfies

$$\|\mathsf{T}e_{n+1}\| > \|z\|^{-1}\,\varepsilon \geq \varepsilon\,.$$

By induction we have constructed an orthonormal family (e_n) such that $\|\mathsf{T}e_n\| > \varepsilon$ for all n. This contradicts the initial hypothesis. □

1.8.2 Spectrum of Compact Operators

The structure of the spectrum of compact self-adjoint operators is very simple, as we shall prove in this section. But before hand we need a useful lemma, which shows that the spectral radius is always attained for compact self-adjoint operators.

Lemma 1.81. *If T is a compact self-adjoint operator then $\|\mathsf{T}\|$ or $-\|\mathsf{T}\|$ is an eigenvalue of T.*

Proof. We can assume that $\|\mathsf{T}\| = 1$ without loss of generality. We claim that $\|\mathsf{T}^3\| = 1$ also, for, using (1.2), we have

$$\|\mathsf{T}\| \, \|\mathsf{T}^3\| \geq \|\mathsf{T}^4\| = \|\mathsf{T}\|^4 \;,$$

which shows that $\|\mathsf{T}^3\| \geq 1$, while the converse inequality is obvious. Hence, by Proposition 1.37, there exists a sequence (x_n) of norm 1 vectors in $\mathcal{H}$ such that

$$\lim_{n\to+\infty} \left|\langle x_n \,,\, \mathsf{T}^3 x_n\rangle\right| = \lim_{n\to+\infty} \left|\langle \mathsf{T}x_n \,,\, \mathsf{T}(\mathsf{T}x_n)\rangle\right| = 1.$$

By the compacity of T, up to taking a subsequence of $(\mathsf{T}x_n)$, we can assume that $\mathsf{T}x_n$ converges to a $y \in \mathcal{H}$. Hence we have proved that $|\langle y \,,\, \mathsf{T}y\rangle| = 1$. It is easy to see that this implies $\mathsf{T}y = \pm y$ (exercise). □

We can now come to the main result on the structure of the spectrum of compact self-adjoint operators.

Theorem 1.82. *Let* T *be a compact self-adjoint operator on* $\mathcal{H}$*. Then the spectrum of* T *is at most countable. It is made of a sequence converging to 0, if* $\mathcal{H}$ *is infinite dimensional. All the elements of the spectrum, except maybe 0, are eigenvalues for* T*; they are all of finite multiplicity. There exists an orthonormal basis of* $\mathcal{H}$ *made of eigenvectors for* T*.*

Proof. Let $\mathsf{T}_0 = \mathsf{T}$ and let y_0 be an eigenvector of T for the eigenvalue $\lambda_0 \in \{\|\mathsf{T}\|\,, -\|\mathsf{T}\|\}$ (by the lemma above). Then T is the orthogonal sum of the operator $\lambda_0 \mathrm{I}$ on $\mathbb{C}y_0$ and a compact operator T_1 on $(\mathbb{C}y_0)^\perp$ (which is stable for T is self-adjoint). Furthermore, the operator T_1 clearly satisfies $\|\mathsf{T}_1\| \leq \|\mathsf{T}\|$.

By induction, we construct an orthonormal sequence (y_n) of eigenvectors of T for some eigenvalues (λ_n). Let $\varepsilon > 0$ be fixed, we denote by E_ε the vector space generated by the vectors y_n, $n \in \mathbb{N}$ such that $|\lambda_n| \geq \varepsilon$. We claim that $\overline{E_\varepsilon}$ is a closed subspace of $\operatorname{Ran} \mathsf{T}$. Indeed, let $g_n = \mathsf{T}f_n$, $n \in \mathbb{N}$, be a sequence in E_ε converging to $g \in \overline{E_\varepsilon}$. Clearly, all element $h \in E_\varepsilon$ satisfies $\|\mathsf{T}h\| \geq \varepsilon\,\|h\|$. Hence, the fact that $(\mathsf{T}f_n)$ is a Cauchy sequence implies that (f_n) is also a Cauchy sequence; it converges to a $f \in \mathcal{H}$. By boundedness of T we get $g = \mathsf{T}f$. This proves our claim.

Now, let P be the orthogonal projector onto $\overline{E_\varepsilon}$, then PT is a compact operator (Theorem 1.79) which is onto from $\mathcal{H}$ to $\overline{E_\varepsilon}$. By the Open Mapping Theorem (Theorem 1.44), there exists a non empty ball in $\overline{E_\varepsilon}$ which is included in $\mathsf{PT}(B_1)$. As the only Hilbert spaces admitting precompact balls are the finite dimensional ones, we get that $\overline{E_\varepsilon}$ is finite dimensional.

This proves that the eigenspaces of T associated to non-null eigenvalues are all finite dimensional.

We have also proved that, for every $\varepsilon > 0$, there is only a finite number of eigenvalues λ_n such that $|\lambda_n| \geq \varepsilon$. Hence, the sequence (λ_n) converges to 0.

Let E be the closed subspace spanned by the y_n's. It is easy to see that E and $E^\perp$ are invariant subspaces of T. On E the operator T is diagonal. On

$E^\perp$ the operator T has to be the null operator, for otherwise it would admit a non-null eigenvalue (Lemma 1.81) and an associated eigenvector. This would contradict the fact that E is generated by all the eigenvectors with non-null eigenvalues.

This proves that T admits an orthonormal basis of eigenvectors (by extending the y_n's with an orthonormal basis of $E^\perp$).

The only point which remains to be proved now is the fact that the spectrum of T is reduced to the set of eigenvalues of T. Let (e_n) be an orthonormal basis of $\mathcal{H}$ made of eigenvectors for T. Let $\lambda \in \mathbb{C} \setminus \{0\}$ be different of all the eigenvalues (λ_n) of T. Then the sequence $((\lambda_n - \lambda)^{-1})_n$ is well-defined and bounded. The operator

$$\mathsf{S}f = \sum_n (\lambda_n - \lambda)^{-1} \, \langle e_n \,,\, f \rangle \; e_n$$

is bounded and is clearly the inverse of $\mathsf{T} - \lambda \mathrm{I}$. Hence λ does not belong to $\sigma(\mathsf{T})$. All the claims of the theorem are now proved. □

1.8.3 Fundamental Decomposition

The previous theorem on the spectrum of compact self-adjoint operators is now the key for a very important theorem giving the canonical form of any compact operator. We first start with a small technicality.

Lemma 1.83. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. If (u_n) and (v_n) are orthonormal families on $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively and if (λ_n) is a sequence in $\mathbb{C}$ converging to 0, then the series*

$$\sum_{n \in \mathbb{N}} \lambda_n \, |u_n\rangle\langle v_n|$$

is norm-convergent in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

Proof. For $\varepsilon > 0$ being fixed there exists a $N \in \mathbb{N}$ such that $|\lambda_n| < \varepsilon$ for all $n \geq N$. Then, for all $\psi \in \mathcal{H}_1$ we have

$$\begin{aligned} \left\| \sum_{n \geq N} \lambda_n \, \langle v_n \,,\, \psi \rangle \; u_n \right\|^2 &= \sum_{n \geq N} |\lambda_n|^2 \; |\langle v_n \,,\, \psi \rangle|^2 \\ &\leq \varepsilon^2 \sum_{n \geq N} |\langle v_n \,,\, \psi \rangle|^2 \\ &\leq \varepsilon^2 \|\psi\|^2 . \end{aligned}$$

This proves the norm-convergence. □

We can now state the fundamental representation for compact operators.

Theorem 1.84. *If* T *is a compact operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *then*

$$\mathsf{T} = \sum_{n=0}^{\infty} \lambda_n \, |u_n\rangle\langle v_n| \tag{1.14}$$

where
i) the sequence $(\lambda_n) \subset \mathbb{R}^+ \setminus \{0\}$ *is either finite or converges to 0,*
ii) the sequences (u_n) *and* (v_n) *are orthonormal families in* $\mathcal{H}_2$ *and* $\mathcal{H}_1$ *respectively.*
In that case, the polar decomposition of T *is* $\mathsf{U}\,|\mathsf{T}|$ *with*

$$|\mathsf{T}| = \sum_{n=0}^{\infty} \lambda_n \, |v_n\rangle\langle v_n| \tag{1.15}$$

and

$$u_n = \mathsf{U} v_n \tag{1.16}$$

for all n.

Conversely, if T *is a bounded operator from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *of the form*

$$\mathsf{T} = \sum_{n=0}^{\infty} \lambda_n \, |u_n\rangle\langle v_n| \tag{1.17}$$

where
i) the sequence $(\lambda_n) \subset \mathbb{C} \setminus \{0\}$ *is either finite or converges to 0,*
ii) the sequences (u_n) *and* (v_n) *are orthonormal families in* $\mathcal{H}_2$ *and* $\mathcal{H}_1$ *respectively,*
then T *is a compact operator.*

In that case, putting $\lambda_n = |\lambda_n| \, \mathrm{e}^{\mathrm{i}\theta_n}$ *for all* n*, the polar decomposition of* T *is* $\mathsf{U}\,|\mathsf{T}|$ *with*

$$|\mathsf{T}| = \sum_{n=0}^{\infty} |\lambda_n| \; |v_n\rangle\langle v_n| \,, \tag{1.18}$$

with

$$\mathsf{U} v_n = \mathrm{e}^{\mathrm{i}\theta_n} u_n \tag{1.19}$$

for all n *and* U *vanishes on the orthogonal complement of the* v_n*'s.*

Proof. If T is compact then so is $\mathsf{T}^*\mathsf{T}$ (Theorem 1.79). By Theorem 1.82 the positive compact operator $\mathsf{T}^*\mathsf{T}$ admits an orthonormal basis of eigenvectors (v_n) associated to positive (or null) eigenvalues (μ_n), which is either a finite set or which converges to 0 (Theorem 1.82). The operator $|\mathsf{T}| = \sqrt{\mathsf{T}^*\mathsf{T}}$ also admits the same eigenvector basis with eigenvalues $\lambda_n = \sqrt{\mu_n}$, $n \in \mathbb{N}$. For all n such that $\lambda_n > 0$, put $u_n = \mathsf{U}v_n$, where U is the partial isometry of the polar decomposition of T (Theorem 1.76).

It is easy to check that (u_n) is an orthonormal family. Furthermore, for all $\psi \in \mathcal{H}$ we have

$$\mathsf{T}\psi = \sum_n \langle v_n\,,\,\psi\rangle\ \mathsf{T}v_n = \sum_n \langle v_n\,,\,\psi\rangle\ \mathsf{U}\,|\mathsf{T}|\,v_n = \sum_n \langle v_n\,,\,\psi\rangle\ \lambda_n\,u_n\,.$$

This proves the decomposition (1.17) and the properties i) and ii). We have proved the theorem in one direction.

Conversely if T is an operator of the form (1.17), then

$$\mathsf{T}^* = \sum_{n=1}^{\infty} \overline{\lambda_n}\,|v_n\rangle\langle u_n|$$

where this series is also norm-convergent. We then easily see that

$$\mathsf{T}^*\mathsf{T} = \sum_{n=1}^{\infty} |\lambda_n|^2\ |v_n\rangle\langle v_n|$$

where the series is again norm-convergent. This is to say that

$$|\mathsf{T}| = \sum_{n=1}^{\infty} |\lambda_n|\ |v_n\rangle\langle v_n|$$

as can be checked easily. But this operator is the operator-norm limit of the finite rank operators

$$\mathsf{S}_N = \sum_{n\leq N} \lambda_n\,|v_n\rangle\langle v_n|\,.$$

Hence it is a compact operator by Theorem 1.79.

The partial isometry U appearing in the polar decomposition of T is now easy to compute (recall Theorem 1.76). On the space generated by the v_n's we have $\mathsf{U}\,|\mathsf{T}|\,v_n = \mathsf{T}v_n$ by definition, hence $\mathsf{U}v_n = \mathrm{e}^{\mathrm{i}\theta_n}u_n$. The operator U vanishes on Ker T, that is, on the orthogonal complement of the space generated by the v_n's. The whole theorem is proved now. □

Definition 1.85. The positive scalars (λ_n) which constitute the eigenvalues of $|T|$ are called the *singular values* of the compact operator T.

The theorem we have established above gets into a nice form when dealing only with compact self-adjoint operators. This theorem is so useful that we have isolated it below.

Corollary 1.86. *A bounded self-adjoint operator* T *on* $\mathcal{H}$ *is compact if and only if there exists an orthonormal basis* (v_n) *in* $\mathcal{H}$ *and a sequence of real numbers* (λ_n) *converging to 0, whenever the sequence is infinite, such that*

$$\mathsf{T} = \sum_{n\in\mathbb{N}} \lambda_n \, |v_n\rangle\langle v_n| \, .$$

In that case the above series converges in operator norm.

1.9 The Spectral Theorem

From now on we are back to the general theory of operators, that is, we are not anymore considering only bounded operators. In this section, our goal is to present the main constructions and the main results concerning the spectral representation of unbounded self-adjoint operators: the spectral measures, the spectral integrals and the Spectral Theorem.

1.9.1 Multiplication Operator Form

In the case of bounded operators, recall that we have established two forms of the spectral theorem: a functional calculus form and a multiplication operator form. The multiplication operator form is the easiest one to extend to unbounded operators.

Theorem 1.87 (Spectral Theorem, Multiplication Operator Form). *Let* T *be a self-adjoint operator on* $\mathcal{H}$*. Then there exist a measured space* $(E, \mathcal{E}, \mu)$ *with* $\mu(E) < \infty$ *, a unitary operator* $\mathsf{U} : \mathcal{H} \to L^2(E, \mathcal{E}, \mu)$ *and a function* $g : E \to \mathbb{R}$ *such that:*

i) $\psi \in \operatorname{Dom} \mathsf{T}$ *if and only if* $\mathsf{U}\psi \in \operatorname{Dom} \mathsf{M}_g$ *,*

ii) $\mathsf{U}\mathsf{T}\mathsf{U}^* = \mathsf{M}_g$ *on* $\operatorname{Dom} \mathsf{M}_g$ *.*

Proof. As T is self-adjoint its spectrum is real (Proposition 1.50), hence the operators $(\mathsf{T}+\mathrm{i}\,\mathrm{I})^{-1}$ and $(\mathsf{T}-\mathrm{i}\,\mathrm{I})^{-1}$ are bounded, they commute and they are adjoint of each other (Proposition 1.22). That is, the operator $(\mathsf{T}+\mathrm{i}\,\mathrm{I})^{-1}$ is a bounded normal operator (Proposition 1.43). We apply the Spectral Theorem for bounded normal operators (Theorem 1.69): there exists a measured space $(E, \mathcal{E}, \mu)$, a unitary operator $\mathsf{U} : \mathcal{H} \to L^2(E, \mathcal{E}, \mu)$ and a bounded function $f : M \to \mathbb{C}$ such that

$$\mathsf{U}(\mathsf{T} + \mathrm{i}\,\mathrm{I})^{-1}\mathsf{U}^* = \mathsf{M}_f \, .$$

As $\operatorname{Ker} (\mathsf{T} + \mathrm{i}\,\mathrm{I})^{-1} = \{0\}$ (Theorem 1.40), we have that the function f never vanishes μ-a.s. (exercise). We put $g = (1/f) - i$.

If ψ belongs to $\operatorname{Dom} \mathsf{T}$ then there exists $\varphi \in \mathcal{H}$ such that $\psi = (\mathsf{T} + \mathrm{iI})^{-1}\varphi$. This gives $\mathsf{U}\psi = \mathsf{U}(\mathsf{T} + \mathrm{iI})^{-1}\varphi = f\,\mathsf{U}\varphi$ and $g\,\mathsf{U}\psi = fg\,\mathsf{U}\varphi$. The function fg being bounded then $fg\,U\phi$ belongs to $L^2(E, \mathcal{E}, \mu)$ and thus $g\,\mathsf{U}\psi$ also belongs to $L^2(M, \mathcal{M}, \mu)$. This exactly means that $\mathsf{U}\psi$ belongs to $\operatorname{Dom} \mathsf{M}_g$.

Conversely, if $\mathsf{U}\psi$ belongs to $\mathrm{Dom}\, M_g$ then $g\,\mathsf{U}\psi$ belongs to $L^2(E,\mathcal{E},\mu)$. As a consequence, there exists $\varphi \in \mathcal{H}$ such that $\mathsf{U}\varphi = (g+i)\,\mathsf{U}\psi$, that is, $f\,\mathsf{U}\varphi = \mathsf{U}\psi$, or else $\mathsf{U}(\mathsf{T}+\mathrm{i}\,\mathrm{I})^{-1}\varphi = \mathsf{U}\psi$. This proves that $\psi = (\mathsf{T}+\mathrm{i}\,\mathrm{I})^{-1}\varphi$ and ψ belongs to $\mathrm{Dom}\,\mathsf{T}$. We have proved *i)*.

If ψ belongs to $\mathrm{Dom}\,\mathsf{T}$ then put $\varphi = (\mathsf{T}+\mathrm{i}\,\mathrm{I})\psi$, that is, $\mathsf{T}\psi = \varphi - \mathrm{i}\psi$. We then get

$$\begin{aligned}\mathsf{UT}\psi &= \mathsf{U}\varphi - \mathrm{i}\,\mathsf{U}\psi\\ &= \left(\frac{1}{f} - \mathrm{i}\right)\mathsf{U}\psi\\ &= g\,\mathsf{U}\psi\,.\end{aligned}$$

This proves ii). □

1.9.2 Bounded Borel Functional Calculus

The second form of the Spectral Theorem is the functional calculus one.

Theorem 1.88 (Spectral Theorem, Bounded Functional Calculus Version). *Let T be a self-adjoint operator on $\mathcal{H}$. Then there exists a unique mapping*

$$\begin{aligned}\Phi : \mathcal{B}_b(\mathbb{R}) &\longrightarrow \mathcal{B}(\mathcal{H})\\ f\quad &\longmapsto \Phi(f) = f(\mathsf{T})\end{aligned}$$

such that:

i) Φ is a unital $*$-algebra morphism,

ii) Φ is norm-continuous,

iii) if $h_n(x)$ tends to x and $|h_n(x)| \leq |x|$ for all $x \in \mathbb{R}$, all $n \in \mathbb{N}$, then $h_n(\mathsf{T})\psi$ tends to $\mathsf{T}\psi$, for all $\psi \in \mathrm{Dom}\,\mathsf{T}$,

iv) if $h_n(x)$ tends to $h(x)$ for all $x \in \mathbb{R}$ and if the sequence $(\|h_n\|_\infty)$ is bounded, then $h_n(\mathsf{T})\psi$ tends to $h(\mathsf{T})\psi$ for all $\psi \in \mathrm{Dom}\,\mathsf{T}$.

Proof. Existence is very easy with Theorem 1.87. Indeed, if h belongs to $\mathcal{B}_b(\mathbb{R})$ we put

$$h(\mathsf{T}) = \mathsf{U}^{-1}\mathsf{M}_{h\circ f}\mathsf{U}\,.$$

Properties i) and ii) are then obvious from the definition of $h(\mathsf{T})$. Properties iii) and iv) are easy consequences of Lebesgue's Theorem.

We shall now prove uniqueness of the functional calculus under the conditions i) to iv). Let Φ and Ψ be two such functional calculus.

Let z_0 be a complex number such that $\mathrm{Im} z_0 \neq 0$. The function

$$f_{z_0}(x) = \frac{1}{x - z_0}\,,$$

from $\mathbb{R}$ to $\mathbb{C}$ is bounded. Let $h_n(x) = x\,\mathbb{1}_{[-n,n]}(x)$. For all $\psi \in \operatorname{Dom} \mathsf{T}$ we have

$$\Phi(f_{z_0})\Phi(h_n - z_0)\psi = \Phi(f_{z_0}(h_n - z_0))\psi\,.$$

By hypothesis iii) we have that $\Phi(h_n)\psi$ tends to $\mathsf{T}\psi$; hence $\Phi(h_n - z_0)\psi$ tends to $(\mathsf{T} - z_0\,\mathrm{I})\psi$. As $\Phi(f_{z_0})$ is a bounded operator, we have

$$\lim_{n\to+\infty} \Phi(f_{z_0})\Phi(h_n - z_0)\psi = \Phi(f_{z_0})(\mathsf{T} - z_0\,\mathrm{I})\psi\,.$$

On the other hand, the function $f_{z_0}(x)(h_n(x) - z_0)$ is equal to

$$\begin{cases} 1 & \text{if } x \in [-n, n]\,, \\ \frac{-z_0}{x-z_0} & \text{if } x \notin [-n, n]\,. \end{cases}$$

It is a sequence of bounded functions, converging to the constant function $\mathbb{1}$ and admitting a bound independent of n. Hence by hypothesis iv) we have

$$\lim_{n\to+\infty} \Phi(f_{z_0}(h_n - z_0))\psi = \Phi(\mathbb{1})\psi = \psi\,.$$

We have proved that

$$\Phi(f_{z_0})(\mathsf{T} - z_0\,\mathrm{I})\psi = \psi$$

for all $\psi \in \operatorname{Dom} \mathsf{T}$. As T is self-adjoint then z_0 is in the resolvent set $\rho(\mathsf{T})$ (Proposition 1.50) and $(\mathsf{T} - z_0\,\mathrm{I})^{-1}$ is a bounded operator. For any $\varphi \in \mathcal{H}$ the element $(\mathsf{T} - z_0\,\mathrm{I})^{-1}\varphi$ belongs to $\operatorname{Dom} \mathsf{T}$ and hence we have

$$\Phi(f_{z_0})(\mathsf{T} - z_0\,\mathrm{I})(\mathsf{T} - z_0\,\mathrm{I})^{-1}\varphi = (\mathsf{T} - z_0\,\mathrm{I})^{-1}\varphi\,,$$

that is,

$$\Phi(f_{z_0}) = (\mathsf{T} - z_0\,\mathrm{I})^{-1}\,.$$

We have proved that the two functional calculus Φ and Ψ coincide on the functions f_{z_0}. By the morphism property, for every polynomial function P on $\mathbb{C}$ we have

$$\Phi(P \circ f_{z_0}) = \Psi(P \circ f_{z_0})\,.$$

If h is any continuous function on $\mathbb{C}$ then $h \circ f_{z_0}$ is a bounded function. One can approximate h uniformly on $B(0, \|f_{z_0}\|_\infty)$ by polynomial functions. Hence, by the approximation property iv) we have

$$\Phi(h \circ f_{z_0}) = \Psi(h \circ f_{z_0})\,.$$

Again, by approximation, the same equality holds for any bounded Borel function h on $\mathbb{C}$, approximating h by a sequence (h_n) of continuous functions satisfying $|h_n| \le |h|$.

Now, let g be any Borel bounded function on $\mathbb{R}$. Define h on $\mathbb{C}$ by

$$h(z) = g\left(\mathrm{Re}\left(\frac{1}{z} + z_0\right)\right) ,$$

when $z \neq 0$, and $h(0)$ taking any fixed value in $\mathbb{C}$. Then

$$h(f_{z_0}(x)) = g(x) ,$$

for all $x \in \mathbb{R}$. We have proved that $\Phi(g) = \Psi(g)$ for all bounded Borel function g on $\mathbb{R}$. We have proved the uniqueness property. □

1.9.3 Spectral Measures and Spectral Integration

The most interesting form of the Spectral Theorem is the one which makes use of spectral measures and spectral integrals.

Definition 1.89. Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{P}(\mathcal{H})$ the set of orthogonal projectors on $\mathcal{H}$.

Definition 1.90. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{H}$ a Hilbert space. A $\mathcal{H}$-valued *spectral measure* on $(\Omega, \mathcal{F})$ is a mapping

$$\begin{aligned} \xi : \mathcal{F} &\longrightarrow \mathcal{P}(\mathcal{H}) \\ E &\longmapsto \xi(E) \end{aligned}$$

which satisfies:

i) $\xi(\Omega) = \mathrm{I}$

ii) $\xi\big(\bigcup_n E_n\big) = \sum_n \xi(E_n)$,

for every sequence of *disjoint* sets (E_n) and where the convergence of the series is understood for the strong topology.

Lemma 1.91. *If ξ is a spectral measure on $(\Omega, \mathcal{F})$, then for all $E, F \subset \Omega$ we have*

$$\xi(E)\,\xi(F) = \xi(E \cap F) \tag{1.20}$$

and

$$\xi(\emptyset) = 0\,. \tag{1.21}$$

Proof. If E and F are disjoint subsets then $\xi(E) + \xi(F) = \xi(E \cup F)$ and in particular $\xi(E) + \xi(F)$ is an orthogonal projector. By Proposition 1.29, this means that $\xi(E)\,\xi(F) = 0$.

Now, for general $E, F \subset \Omega$ we have

$$\xi(E)\,\xi(F) = \left(\xi(E \setminus F) + \xi(E \cap F)\right)\left(\xi(F \setminus E) + \xi(E \cap F)\right) .$$

Developing and using the property established above in the case of disjoint sets, this quantity is clearly equal to $\xi(E \cap F)^2 = \xi(E \cap F)$. This proves (1.20). The second property is obvious. □

Let us see now how such spectral measures are able to integrate Borel functions. Fix $\psi \in \mathcal{H}$ and consider the mapping

$$\begin{aligned} \mathcal{F} &\longrightarrow \qquad \mathbb{R}^+ \\ E &\longmapsto \langle \psi\,,\, \xi(E)\,\psi\rangle = \|\xi(E)\,\psi\|^2\,. \end{aligned}$$

By the properties i) and ii) above, this mapping clearly defines a measure μ_ψ on $\mathcal{F}$ such that $\mu_\psi(\Omega) = \|\psi\|^2$.

Definition 1.92. Assume first that f is a simple function from Ω to $\mathbb{C}$, that is $f = \sum_{i=1}^n \alpha_i\, \mathbb{1}_{E_i}$ where the Borel sets E_i are 2 by 2 disjoint. Define the *spectral integral*

$$\int_\Omega f(\lambda)\,\mathrm{d}\xi(\lambda) = \sum_{i=1}^n \alpha_i\,\xi(E_i)\,,$$

which we shall also simply denote by

$$\int_\Omega f\,\mathrm{d}\xi\,.$$

Definition 1.93. In the following we denote by $\mathrm{essup}_\xi\, f$ the quantity

$$\inf\{\sup_{x\in E} f(x)\,;\ E \in \mathcal{F}\,,\ \xi(E) = \mathrm{I}\}\,.$$

It is easy to check that $f \mapsto \mathrm{essup}_\xi\, |f|$ defines a norm on the ξ-almost sure equivalence classes of bounded functions f on Ω (that is, those classes of functions which coincide on sets $E \in \mathcal{F}$ such that $\xi(E) = 0$). It is also easy to check that

$$\mathrm{essup}_\xi\, |fg| \le \mathrm{essup}_\xi\, |f|\ \mathrm{essup}_\xi\, |g|\ .$$

All these properties are left to the reader.

We then have the following properties for this integral, at that stage.

Proposition 1.94. *The integral defined above satisfies the following properties.*

i) $$\left\|\int_\Omega f\,\mathrm{d}\xi\right\| = \mathit{essup}_\xi\, |f|\ ,$$

ii) $$\left(\int_\Omega f\,\mathrm{d}\xi\right)\left(\int_\Omega g\,\mathrm{d}\xi\right) = \int_\Omega fg\,\mathrm{d}\xi\,,$$

iii) $$\left(\int_\Omega f\,\mathrm{d}\xi\right)^\star = \int \overline{f}\,\mathrm{d}\xi\,,$$

iv) $$\langle\psi, \int f\,\mathrm{d}\xi\,\psi\rangle = \int f(\lambda)\,\mathrm{d}\mu_\psi(\lambda)\,.$$

Proof. The properties ii), iii) and iv) come very easily from the definitions and Lemma 1.91. Let us detail the proof of i). Let $E \in \mathcal{F}$ be such that $\xi(E) = I$. Then we have

$$\int_\Omega f \,\mathrm{d}\xi = \xi(E) \int_\Omega f \,\mathrm{d}\xi = \sum_{i=1}^n \alpha_i \,\xi(E \cap E_i)\,.$$

In particular, we have the estimate

$$\left\| \int_\Omega f \,\mathrm{d}\xi \right\| \leq \sum_{i=1}^n |\alpha_i| \; \|\xi(E \cap E_i)\| = \sum_{i\,;\; E_i \cap E \neq \emptyset} |\alpha_i| = \sup_{x \in E} |f(x)| \;.$$

This proves that

$$\left\| \int_\Omega f \,\mathrm{d}\xi \right\| \leq \mathrm{essup}_\xi \; |f| \;.$$

Now, taking $\Psi \in \mathcal{H}$, with $\xi(E_i)\,\Psi = \Psi$ and $\|\Psi\| = 1$, we have

$$\left(\int_\Omega f \,\mathrm{d}\xi \right) \Psi = \alpha_i \,\Psi$$

so that

$$\left\| \left(\int_\Omega f \,\mathrm{d}\xi \right) \Psi \right\| = |\alpha_i| \;.$$

This proves that

$$\left\| \int_\Omega f \,\mathrm{d}\xi \right\| \geq \max_i |\alpha_i| \geq \mathrm{essup}_\xi \; |f| \;.$$

The proposition is proved. □

Now we pass to the second step of the construction of the spectral integral.

Proposition 1.95. *Assume that f is a bounded Borel function on Ω. Then there exists a sequence (f_n) of simple functions on Ω such that*

$$\lim \mathrm{essup}_\xi \,|f_n - f| = 0\,.$$

Furthermore the sequence $(\int f_n \,d\xi)$ converges in operator norm to a limit, denoted by

$$\int_\Omega f \,\mathrm{d}\xi\,.$$

This limit does not depend on the choice of $(f_n)_{n \in \mathbb{N}}$.

The spectral integral $\int_\Omega f \,\mathrm{d}\xi$ *defined this way satisfies the same properties* i), ii), iii) *and* iv) *as in Proposition 1.94.*

Proof. The existence of a sequence (f_n) satisfying $\lim \mathrm{essup}_\xi \,|f_n - f| = 0$ is obvious for the essup_ξ norm is clearly dominated by the usual sup norm. It is then an usual approximation argument for bounded functions by means of simple functions.

As we have

$$\| \int f_n \, d\xi - \int f_m \, d\xi \| = \text{essup}_\xi \, |f_n - f_m| \ ,$$

which tends to 0 as n and m tend to $+\infty$, we have the norm convergence of $\int_\Omega f \, \mathrm{d}\xi$.

Let us check that the limit does not depend on the choice of the approximating sequence. If (g_n) is another sequence of simple functions, with $\lim \text{essup}_\xi |g_n - f| = 0$, then denote by A the bounded operator, limit of $\int_\Omega g_n \, \mathrm{d}\xi$. We have

$$\begin{aligned}\left\| \mathsf{A} - \int_\Omega f \, \mathrm{d}\xi \right\| &\leq \left\| \mathsf{A} - \int_\Omega g_n \, \mathrm{d}\xi \right\| + \left\| \int_\Omega (g_n - f_n) \, \mathrm{d}\xi \right\| + \left\| \int_\Omega f_n \, \mathrm{d}\xi - \int_\Omega f \, \mathrm{d}\xi \right\| \\ &= \left\| \mathsf{A} - \int_\Omega g_n \, \mathrm{d}\xi \right\| + \text{essup}_\xi \, |g_n - f_n| + \left\| \int_\Omega f_n \, \mathrm{d}\xi - \int_\Omega f \, \mathrm{d}\xi \right\| .\end{aligned}$$

All the terms of the right hand side converge to 0, we conclude easily.

The spectral integral $\int_\Omega f \, \mathrm{d}\xi$ is now well-defined, we have to check that it satisfies the properties i)-iv). The property i) is obvious from the construction of the integral itself. Properties ii) and iii) are rather easy to obtain by usual approximation arguments and using the properties of the essup_ξ-norm. For proving Property iv) one needs to notice the following: if E is such that $\xi(E) = 0$ then $\mu_\psi(E) = 0$. This implies that

$$\text{essup}_\xi \ |f| \geq \text{essup}_{\mu_\psi} \ |f|$$

for all f. With this remark it is easy to obtain Property iv) by an approximation argument again (left to the reader). □

Finally, we go to the last step of the construction of the spectral integrals. Let f be any Borel function. Define

$$D_f = \left\{ \psi \in \mathcal{H} \, ; \ \int_\Omega |f(\lambda)|^2 \, \mathrm{d}\mu_\psi(\lambda) < \infty \right\} .$$

Lemma 1.96. *The set D_f is a dense subspace of $\mathcal{H}$.*

Proof. The set D_f is a space for $\mu_{\varphi+\psi} \leq 2(\mu_\varphi + \mu_\psi)$.

Let $E_n = \{x \, ; \ |f(x)| \leq n\}$ and let $\psi \in \mathcal{H}$. The sequence $(\xi(E_n)\psi)$ converges to ψ and we have

$$\int_\Omega |f(\lambda)|^2 \, \mathrm{d}\mu_{\xi(E_n)\psi}(\lambda) = \int_\Omega |f(\lambda)|^2 \mathbb{1}_{E_n}(\lambda) \, \mathrm{d}\mu_\psi(\lambda) < \infty \, .$$

This shows that $\xi(E_n)\psi$ belongs to D_f and proves the density property. □

Let $\psi \in D_f$ and let (f_n) be a sequence of bounded functions such that

$$\int_\Omega |f_n(\lambda) - f(\lambda)|^2 \,\mathrm{d}\mu_\psi(\lambda)$$

tends to 0 when n tends to $+\infty$.

Let $v_n = \int_\Omega f_n(\lambda)\,\mathrm{d}\mu_\psi(\lambda)$. We have

$$\begin{aligned}
\|v_n - v_m\|^2 &= \left\langle \left(\int_\Omega f_n - f_m \,\mathrm{d}\xi\right) \psi, \left(\int f_n - f_m \,\mathrm{d}\xi\right) \psi \right\rangle \\
&= \left\langle \psi, \left(\int_\Omega \overline{f_n} - \overline{f_m} \,\mathrm{d}\xi\right) \left(\int_\Omega f_n - f_m \;\mathrm{d}\xi\right) \psi \right\rangle \\
&= \left\langle \psi, \left(\int_\Omega |f_n - f_m|^2 \,\mathrm{d}\xi\right) \psi \right\rangle \\
&= \int_\Omega |f_n - f_m|^2 \,\mathrm{d}\mu_\psi
\end{aligned}$$

which tends to 0 as n and m tend to $+\infty$.

Hence, the sequence $(v_n)_{n\in\mathbb{N}}$ converges to a limit denoted by

$$\int_\Omega f \,\mathrm{d}\xi\, \psi \,.$$

It is immediate that

$$\| \int_\Omega f \,\mathrm{d}\xi\, \psi\|^2 = \int_\Omega |f|^2 \,\mathrm{d}\mu_\psi \,.$$

Lemma 1.97. *If g is a bounded Borel function and f is a Borel function, then for every $\psi \in D_f$ we have $\psi \in D_{fg}$. We also have $\int_\Omega g \,\mathrm{d}\xi\, \psi \in D_f$ and*

$$\int_\Omega f \,\mathrm{d}\xi \int_\Omega g \,\mathrm{d}\xi\, \psi = \int_\Omega fg \,\mathrm{d}\xi\, \psi \,. \tag{1.22}$$

Proof. The fact that $\psi \in D_{fg}$ is obvious since g is bounded. The fact that $\int_\Omega g \,\mathrm{d}\xi\, \psi \in D_f$ comes from the identity

$$\int_\Omega |f(\lambda)|^2 \;\mathrm{d}\mu_{\int_\Omega g\,\mathrm{d}\xi\,\psi} = \int_\Omega |f(\lambda)|^2 \; |g(\lambda)|^2 \;\mathrm{d}\mu_\psi \,,$$

which can be checked easily, starting with f being a simple function, then passing to the limit to bounded functions and finally passing to the limit to a general f.

Finally the identity (1.22) is also easily obtained by passing to the limit on the identity

$$\int_\Omega f_n \,\mathrm{d}\xi \int_\Omega g \,\mathrm{d}\xi\, \psi = \int_\Omega f_n\, g \,\mathrm{d}\xi\, \psi \,.$$

□

Proposition 1.98. *The operator* $\int_\Omega f\,\mathrm{d}\xi$ *defined on* D_f *satisfies the following properties.*

1) The operator $\int_\Omega f\,\mathrm{d}\xi$ *is closed.*

2) The adjoint of $\int_\Omega f\,\mathrm{d}\xi$ *is* $\int_\Omega \overline{f}\,\mathrm{d}\xi$ *.*

3) If f *is real then* $\int_\Omega f\,\mathrm{d}\xi$ *is self-adjoint.*

Proof. Let us first prove 2). Let $E_n = \{x\,;\ |f(x)| \leq n\}$ again. Let u and v belong to D_f, we have

$$\left\langle u\,,\,\int_\Omega f \mathbb{1}_{E_n}\,\mathrm{d}\xi\, v\right\rangle = \left\langle \int_\Omega \overline{f}\mathbb{1}_{E_n}\,\mathrm{d}\xi\, u\,,\, v\right\rangle$$

thus, passing to the limit,

$$\left\langle u\,,\,\int_\Omega f\,\mathrm{d}\xi\, v\right\rangle = \left\langle \int_\Omega \overline{f}\,\mathrm{d}\xi\, u\,,\, v\right\rangle$$

which means

$$\int_\Omega \overline{f}\,\mathrm{d}\xi \subset \left(\int_\Omega f\,\mathrm{d}\xi\right)^* .$$

Now, let $u \in \mathrm{Dom}(\int_\Omega f\,\mathrm{d}\xi)^*$, we have

$$\begin{aligned}
\int_\Omega |f|^2\, \mathbb{1}_{E_n}\,\mathrm{d}\mu_u &= \left\|\int_\Omega \overline{f}\,\mathbb{1}_{E_n}\,\mathrm{d}\xi\, u\right\|^2\\
&= \left\|\left(\int_\Omega f\,\mathrm{d}\xi\ \xi(E_n)\right)^* u\right\|^2\\
&= \left\|\xi(E_n)\left(\int_\Omega f\,\mathrm{d}\xi\right)^* u\right\|^2\\
&\leq \left\|\left(\int_\Omega f\,\mathrm{d}\xi\right)^* u\right\|^2 .
\end{aligned}$$

Thus $\int_\Omega |f|^2\,\mathrm{d}\mu_u$ is finite and $u \in D_{\overline{f}}$. We have proved 2).

Property 1) is now immediate for $\int_\Omega f\,\mathrm{d}\xi$ is the adjoint of $\int_\Omega \overline{f}\,\mathrm{d}\xi$. Hence, by Theorem 1.21, it is a closed operator.

Property 3) is also immediate from 2). □

Here we are! The *spectral integral* has been constructed with its largest possible domain.

1.9.4 von Neumann's Spectral Theorem

We are now able to prove what is certainly the most important theorem of Operator Theory.

Theorem 1.99 (Von Neumann's Spectral Theorem). *The formula*

$$\mathsf{T} = \int_{\mathbb{R}} \lambda \, \mathrm{d}\xi(\lambda)$$

establishes a bijection between the self-adjoint operators T *on* $\mathcal{H}$ *and the spectral measures* $\xi : \mathrm{Bor}(\mathbb{R}) \to \mathcal{P}(\mathcal{H})$*. We have furthermore*

$$f(\mathsf{T}) = \int_{\mathbb{R}} f(\lambda) \, \mathrm{d}\xi(\lambda)$$

for all bounded Borel function f *on* $\mathcal{B}$*. In particular*

$$\xi(E) = \mathbb{1}_E(\mathsf{T}) \, .$$

Proof. If ξ is a spectral measure on $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$, we know by Proposition 1.98 that

$$\mathsf{T} = \int_{\mathbb{R}} \lambda \, \mathrm{d}\xi(\lambda)$$

is a self-adjoint operator. The mapping $f \mapsto \int_\Omega f \, \mathrm{d}\xi$, for f bounded, satisfies all the hypothesis of the bounded functional calculus version of the Spectral Theorem (Theorem 1.88), as can be checked easily from the properties we have established on the spectral integrals and by several applications of Lebesgue's Theorem (left to the reader). By uniqueness of this functional calculus we have

$$\int_{\mathbb{R}} f \, \mathrm{d}\xi = f(\mathsf{T}) \, .$$

Conversely, let T be a self-adjoint operator. Put $\xi(E) = \mathbb{1}_E(\mathsf{T})$, in the sense of the bounded functional calculus. Then ξ is a spectral measure and

$$\mathbb{1}_E(\mathsf{T}) = \xi(E) = \int_{\mathbb{R}} \mathbb{1}_E(\lambda) \, \mathrm{d}\xi(\lambda) \, .$$

Thus, by the functional calculus

$$f(\mathsf{T}) = \int_{\mathbb{R}} f(\lambda) \, \mathrm{d}\xi(\lambda) \, .$$ □

Definition 1.100. The spectral measure associated to a self-adjoint operator T on $\mathcal{H}$ is denoted by ξ_{T}.

1.9.5 Unitary Conjugation

A situation which appears very often is the following. We are given a self-adjoint operator T on a Hilbert space $\mathcal{H}$, with its functional calculus and its spectral measure ξ_{T}. We are also given a unitary operator U from $\mathcal{H}$ to $\mathcal{K}$ and we consider the operator

$$\mathsf{S} = \mathsf{U}\,\mathsf{T}\,\mathsf{U}^*$$

on $\mathcal{K}$. The following result relates the properties of self-adjointness, the functional calculus and the spectral messures of T and S.

Theorem 1.101. *Under the conditions and notations above, the operator* S *is self-adjoint on* $\mathcal{K}$*. Its bounded functional calculus is given by*

$$\boxed{f(\mathsf{S}) = \mathsf{U}\, f(\mathsf{T})\, \mathsf{U}^*\,,} \tag{1.23}$$

for all $f \in \mathcal{B}_b(\mathbb{R})$*. Its spectral measure is given by*

$$\boxed{\xi_{\mathsf{S}}(A) = \mathsf{U}\, \xi_{\mathsf{T}}(A)\, \mathsf{U}^*\,,} \tag{1.24}$$

for all $A \in \mathrm{Bor}(\mathbb{R})$ *.*

Proof. By Proposition 1.20 we have $\mathsf{S}^* = \mathsf{U}\,\mathsf{T}^*\,\mathsf{U}^* = \mathsf{U}\,\mathsf{T}\,\mathsf{U}^* = \mathsf{S}$, with equality of domains. Hence S is self-adjoint.

Consider the bounded functional calculus Φ_{S} associated to S by Theorem 1.88. Define $\Psi(f) = \mathsf{U}\, f(\mathsf{T})\, \mathsf{U}^*$, for all $f \in \mathcal{B}_b(\mathbb{R})$. It is easy to check (and left to the reader) that Ψ satisfies the four conditions i)-iv) of Theorem 1.88 associated to the operator $\mathsf{U}\,\mathsf{T}\,\mathsf{U}^* = \mathsf{S}$. Hence, by uniqueness of the Bounded Function Calculus we have $f(\mathsf{S}) = \mathsf{U}\, f(\mathsf{T})\, \mathsf{U}^*$ for all $f \in \mathcal{B}_b(\mathbb{R})$.

By Theorem 1.99 the spectral measure ξ_{S} is given by $\xi_{\mathsf{S}}(A) = \mathbb{1}_A(\mathsf{S})$ in the sense of the Bounded Functional Calculus. This gives the relation (1.24). □

Notes

This lecture has been elaborated with the help of many references. The main one, which we highly recommend, is the first volume [RS80] of the series of four books by Reed and Simon ([RS80], [RS75], [RS79], [RS78]). They constitute a true reference in Operator and Spectral Theory.

We also appreciated and used the book by J. Weidmann [Wei80] which is very pedagogical, except maybe for the construction of the spectral measure, which we took from K.R. Parthasarathy's book [Par92].

A very interesting reference, for it is very concise, is the course by De la Harpe and Jones [dlHJ95] from which we took most of Section 1.8. But this reference does not cover all the topics of this chapter.

There are plenty other well-known references on Operator Theory, which we did not use here but which are considered as references by many mathematicians. Among them we think of Dunford and Schwartz' famous two volumes ([DS88a], [DS88b]), Yoshida's book ([Yos80]) or Kato's book ([Kat76]).

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