# Lecture 2
# TENSOR PRODUCTS AND PARTIAL TRACES

Stéphane ATTAL

**Abstract** This lecture concerns special aspects of Operator Theory which are of much use in Quantum Mechanics, in particular in the theory of Quantum Open Systems. These are the concepts of trace-class operators, tensor products of Hilbert spaces and operators, and above all of partial traces. Note that a general treatment of partial traces in the infinite dimensional case, as we do in this lecture, is not at all common in the literature.



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Stéphane ATTAL
Institut Camille Jordan, Université Lyon 1, France
e-mail: attal@math.univ-lyon1.fr

Most of this lecture deals with bounded operators, hence the full theory of unbounded operators, Spectral Theorem, etc is not necessary here. We assume the reader is familiar with the usual theory of bounded operators and in particular with compact operators and their fundamental representation, with the notion of positive (bounded) operators, with continuous functional calculus for bounded operators, and with the polar decomposition. For all these notions, read the corresponding sections in Lecture 1 if necessary.

## 2.1 Trace-Class Operators

We first start with the notions of trace, trace-class operators and their properties. We shall explore the nice duality properties which are attached to these spaces. We end up with a closely related family of operators: the Hilbert-Schmidt operators.

### *2.1.1 Basic Definitions*

**Definition 2.1.** Let $\mathsf{T}$ be a bounded positive operator on a Hilbert space $\mathcal{H}$. For a fixed orthonormal basis $(e_n)$ of $\mathcal{H}$ we define the quantity

$$\mathrm{Tr}\,(\mathsf{T}) = \sum_{n\in\mathbb{N}} \langle e_n\,,\,\mathsf{T}e_n\rangle\ ,$$

which is positive (eventually infinite). It is called the *trace* of $\mathsf{T}$. We shall sometimes simply denote it by $\mathrm{Tr}\ \mathsf{T}$.

**Proposition 2.2.** *The quantity* $\mathrm{Tr}\ \mathsf{T}$ *is independent of the choice of the orthonormal basis* $(e_n)$*.*

*Proof.* Let $(f_n)$ be another orthonormal basis of $\mathcal{H}$. As $\mathsf{T}$ is a positive operator, it admits a square root $\sqrt{\mathsf{T}}$. We then have (the series below being all made of positive terms, interchanging the sums is allowed)

$$\begin{aligned}
\sum_n \langle f_n\,,\,\mathsf{T}\,f_n\rangle &= \sum_n \left\|\sqrt{\mathsf{T}}\,f_n\right\|^2 \\
&= \sum_n\sum_m \left|\left\langle \sqrt{\mathsf{T}}\,f_n\,,\,e_m\right\rangle\right|^2 \\
&= \sum_m\sum_n \left|\left\langle f_n\,,\,\sqrt{\mathsf{T}}\,e_m\right\rangle\right|^2 \\
&= \sum_m \left\|\sqrt{\mathsf{T}}\,e_m\right\|^2 \\
&= \sum_m \langle e_m\,,\,\mathsf{T}\,e_m\rangle\ .
\end{aligned}$$

This proves the independence property. □

We now define the trace-class operators for general bounded operators.

**Definition 2.3.** A bounded operator $\mathsf{T}$ on $\mathcal{H}$ is *trace-class* if

$$\mathrm{Tr}\ |\mathsf{T}| < \infty\,.$$

**Theorem 2.4.**

*1) Every trace-class operator is compact.*

*2) A compact operator $\mathsf{T}$, with singular values $(\lambda_n)_n$, is trace-class if and only if*

$$\sum_n \lambda_n < \infty\,. \tag{2.1}$$

*In that case we have*

$$\mathrm{Tr}\ |\mathsf{T}| = \sum_n \lambda_n\,. \tag{2.2}$$

*Proof.* Assume first that $\mathsf{T}$ is positive and trace-class. Let $(e_n)$ be an orthonormal family in $\mathcal{H}$, then

$$\sum_n \left\|\sqrt{\mathsf{T}}\,e_n\right\|^2 = \sum_n \langle e_n\,,\,\mathsf{T}e_n\rangle \leq \mathrm{Tr}\ \mathsf{T} < \infty\,.$$

Hence $\left\|\sqrt{\mathsf{T}}\,e_n\right\|$ converges to 0 and $\sqrt{\mathsf{T}}$ is compact. In particular the operator $\mathsf{T} = \sqrt{\mathsf{T}}\,\sqrt{\mathsf{T}}$ is also compact.

Now if $\mathsf{T}$ is a bounded operator which is trace-class, then $|\mathsf{T}|$ is trace-class by definition. Hence $|\mathsf{T}|$ is compact. By the Polar Decomposition the operator $\mathsf{T} = \mathsf{U}\,|\mathsf{T}|$ is also compact. We have proved 1).

If $\mathsf{T}$ is compact then $|\mathsf{T}| = \mathsf{U}^*\mathsf{T}$ is also compact. The operator $|\mathsf{T}|$ can thus be decomposed as

$$|\mathsf{T}| = \sum_n \lambda_n\,|v_n\rangle\langle v_n|\,,$$

for some positive $\lambda_n$'s and some orthonormal basis $(v_n)$.

Taking the trace of $|\mathsf{T}|$ with respect to the orthonormal basis $(v_n)$ gives

$$\mathrm{Tr}\ |\mathsf{T}| = \sum_n \lambda_n\,.$$

It is easy to conclude now. □

**Theorem 2.5.** *If* $\mathsf{T}$ *is a trace-class operator on* $\mathcal{H}$ *then for every orthonormal basis* $(e_n)$ *of* $\mathcal{H}$ *the series*

$$\mathrm{Tr}\ \mathsf{T} = \sum_n \langle e_n\,,\, \mathsf{T}e_n\rangle$$

*is absolutely convergent. Its sum is independent of the choice of the orthonormal basis* $(e_n)$*.*

*We always have*

$$|\mathrm{Tr}\ \mathsf{T}| \leq \mathrm{Tr}\ |\mathsf{T}|\,. \tag{2.3}$$

*Proof.* By Theorem 2.4 the operator $\mathsf{T}$ is compact, hence it admits a decomposition as

$$\mathsf{T} = \sum_{n=1}^\infty \lambda_n\,|u_n\rangle\langle v_n|\,,$$

for some positive $\lambda_n$'s and some orthonormal bases $(u_n)$, $(v_n)$ of $\mathcal{H}$. We then have

$$\begin{aligned}
\sum_k |\langle e_k\,,\, \mathsf{T}e_k\rangle| &\leq \sum_k\sum_n \lambda_n\ |\langle e_k\,,\, u_n\rangle|\ |\langle v_n\,,\, e_k\rangle| \\
&\leq \sum_n \lambda_n\ \left(\sum_k |\langle e_k\,,\, u_n\rangle|^2\right)^{1/2}\ \left(\sum_k |\langle v_n\,,\, e_k\rangle|^2\right)^{1/2} \\
&= \sum_n \lambda_n\ \|v_n\|\ \|u_n\| \\
&= \sum_n \lambda_n
\end{aligned}$$

which is finite by Theorem 2.4. We have proved the absolute convergence and the inequality (2.3).

Let us prove the independence with respect to the choice of the orthonormal basis. Let $(f_n)$ be any orthonormal basis of $\mathcal{H}$. We then have (using the absolute convergence above and Fubini's Theorem)

$$\sum_k \langle f_k \, , \, \mathsf{T} f_k \rangle = \sum_k \sum_n \lambda_n \, \langle f_k \, , \, u_n \rangle \, \langle v_n \, , \, f_k \rangle$$
$$= \sum_n \lambda_n \, \sum_k \langle v_n \, , \, f_k \rangle \, \langle f_k \, , \, u_n \rangle$$
$$= \sum_n \lambda_n \, \langle u_n \, , \, v_n \rangle \; .$$

This quantity does not depend on the choice of $(f_n)$. □

By the way, we have proved the following characterization.

**Corollary 2.6.** *A bounded operator* $\mathsf{T}$ *on* $\mathcal{H}$ *is trace-class if and only if it can be decomposed as*

$$\mathsf{T} = \sum_n \lambda_n \, |u_n\rangle\langle v_n|$$

*where* $(u_n)$ *and* $(v_n)$ *are orthonormal families in* $\mathcal{H}$ *and* $(\lambda_n)$ *is a sequence of positive numbers such that*

$$\sum_n \lambda_n < \infty \, .$$

**Definition 2.7.** The set of trace-class operators on $\mathcal{H}$ is denoted by $\mathcal{L}_1(\mathcal{H})$. We also put

$$\|\mathsf{T}\|_1 = \mathrm{Tr}\, |\mathsf{T}| \; .$$

### *2.1.2 Properties*

Let us start with the very fundamental properties of trace-class operators.

**Theorem 2.8.**

*1) The set* $\mathcal{L}_1(\mathcal{H})$ *is a vector space, the mapping* $\|\cdot\|_1$ *is a norm on* $\mathcal{L}_1(\mathcal{H})$*. When equipped with the norm* $\|\cdot\|_1$*, the space* $\mathcal{L}_1(\mathcal{H})$ *is a Banach space. This norm always satisfies*

$$\|\mathsf{T}\| \leq \|\mathsf{T}\|_1 \; .$$

*2) The space* $\mathcal{L}_1(\mathcal{H})$ *is two-sided ideal of* $\mathcal{B}(\mathcal{H})$*. For every* $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$ *and* $\mathsf{X} \in \mathcal{B}(\mathcal{H})$ *we have*

$$\|\mathsf{X}\,\mathsf{T}\|_1 \leq \|\mathsf{X}\| \, \|\mathsf{T}\|_1 \, . \tag{2.4}$$

*3) The space* $\mathcal{L}_0(\mathcal{H})$ *is* $\|\cdot\|_1$*-dense in* $\mathcal{L}_1(\mathcal{H})$*.*

*4) If* $\mathsf{T}$ *belongs to* $\mathcal{L}_1(\mathcal{H})$ *then so does* $\mathsf{T}^*$ *and we have*

$$\mathrm{Tr}\,(\mathsf{T}^*) = \overline{\mathrm{Tr}\,(\mathsf{T})} \, .$$

*5) For any two bounded operators* $\mathsf{S}$ *and* $\mathsf{T}$ *such that one of them is trace-class, we have*

$$\mathrm{Tr}\,(\mathsf{S}\,\mathsf{T}) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{S})\,.$$

*Proof.*

1) In order to prove that $\mathcal{L}_1(\mathcal{H})$ is a vector space and that $\|\cdot\|_1$ is a norm, the only non-trivial point to be proved is the following: for every positive bounded operators $\mathsf{S}$ and $\mathsf{T}$ on $\mathcal{H}$ we have

$$\mathrm{Tr}\,|\mathsf{S}+\mathsf{T}| \leq \mathrm{Tr}\,|\mathsf{S}| + \mathrm{Tr}\,|\mathsf{T}|\,.$$

Let us prove this fact. Let

$$\mathsf{S} = \mathsf{U}\,|\mathsf{S}|\,, \qquad \mathsf{T} = \mathsf{V}\,|\mathsf{T}|\,, \qquad \mathsf{S}+\mathsf{T} = \mathsf{W}\,|\mathsf{S}+\mathsf{T}|$$

be the polar decompositions of $\mathsf{S}, \mathsf{T}$ and $\mathsf{S}+\mathsf{T}$. Let $(e_n)$ be an orthonormal basis of $\mathcal{H}$. We have

$$\begin{aligned}
\sum_n \langle e_n\,,\,|\mathsf{S}+\mathsf{T}|\;e_n\rangle &= \sum_n \langle e_n\,,\,\mathsf{W}^*\,(\mathsf{S}+\mathsf{T})\,e_n\rangle \\
&= \sum_n \langle e_n\,,\,\mathsf{W}^*\,\mathsf{U}\;|\mathsf{S}|\;e_n\rangle + \sum_n \langle e_n\,,\,\mathsf{W}^*\,\mathsf{V}\;|\mathsf{T}|\;e_n\rangle \\
&\leq \sum_n \left\|\sqrt{|\mathsf{S}|}\,\mathsf{U}^*\,\mathsf{W}\,e_n\right\|\,\left\|\sqrt{|\mathsf{S}|}\,e_n\right\| + \\
&\qquad + \sum_n \left\|\sqrt{|\mathsf{T}|}\,\mathsf{V}^*\,\mathsf{W}\,e_n\right\|\,\left\|\sqrt{|\mathsf{T}|}\,e_n\right\| \\
&\leq \left(\sum_n \left\|\sqrt{|\mathsf{S}|}\,\mathsf{U}^*\,\mathsf{W}\,e_n\right\|^2\right)^{1/2} \left(\sum_n \left\|\sqrt{|\mathsf{S}|}\,e_n\right\|^2\right)^{1/2} + \\
&\quad + \left(\sum_n \left\|\sqrt{|\mathsf{T}|}\,\mathsf{V}^*\,\mathsf{W}\,e_n\right\|^2\right)^{1/2} \left(\sum_n \left\|\sqrt{|\mathsf{T}|}\,e_n\right\|^2\right)^{1/2}.
\end{aligned}$$

But we have

$$\begin{aligned}
\sum_n \left\|\sqrt{|\mathsf{S}|}\,\mathsf{U}^*\,\mathsf{W}\,e_n\right\|^2 &= \sum_n \langle e_n\,,\,\mathsf{W}^*\,\mathsf{U}\;|\mathsf{S}|\;\mathsf{U}^*\,\mathsf{W}\,e_n\rangle \\
&= \mathrm{Tr}\,(\mathsf{W}^*\,\mathsf{U}\;|\mathsf{S}|\;\mathsf{U}^*\,\mathsf{W})\,.
\end{aligned}$$

Choosing an orthonormal basis $(f_n)$ which is adapted to the decomposition $\mathcal{H} = \mathrm{Ker}\,\mathsf{W} \oplus (\mathrm{Ker}\,\mathsf{W})^\perp$, we see that

$$\mathrm{Tr}\,(\mathsf{W}^*\,\mathsf{U}\;|\mathsf{S}|\;\mathsf{U}^*\,\mathsf{W}) \leq \mathrm{Tr}\,(\mathsf{U}\;|\mathsf{S}|\;\mathsf{U}^*)\,.$$

Applying the same trick for $\mathsf{U}^*$ shows that

$$\mathrm{Tr}\,(\mathsf{U}\ |\mathsf{S}|\ \mathsf{U}^*) \leq \mathrm{Tr}\,(|\mathsf{S}|)\,.$$

Altogether, we have proved that

$$\sum_n \langle e_n\,,\ |\mathsf{S}+\mathsf{T}|\ e_n\rangle \leq \mathrm{Tr}\,(|\mathsf{S}|) + \mathrm{Tr}\,(|\mathsf{T}|)\,.$$

This proves the vector space structure of $\mathcal{L}_1(\mathcal{H})$ and also the norm property of $\|\cdot\|_1$.

By the usual properties of the operator norm we have

$$\|\,|\mathsf{T}|\,\|^2 = \left\|\,|\mathsf{T}|^*\ |\mathsf{T}|\,\right\| = \left\|\,|\mathsf{T}|^2\right\| = \|\mathsf{T}^*\,\mathsf{T}\| = \|\mathsf{T}\|^2\ .$$

As $|\mathsf{T}|$ is self-adjoint its norm is given by

$$\sup_{\|x\|=1}\ \langle x\,,\ |\mathsf{T}|\,x\rangle\ .$$

The above quantity is clearly smaller than Tr $|\mathsf{T}|$. We have proved that $\|\mathsf{T}\| \leq \|\mathsf{T}\|_1$.

As a consequence, if $(\mathsf{T}_n)$ is Cauchy sequence in $\mathcal{B}(\mathcal{H})$ for the norm $\|\cdot\|_1$ then it is a Cauchy sequence for the usual operator norm. Hence it converges in operator norm to a bounded operator $\mathsf{T}$. To each operator $\mathsf{T}_n$ is associated a sequence $\lambda^{(n)}_\cdot$ of singular values. By hypothesis this family of sequences is $\ell^1$-convergent. We have proved that it is also $\ell^\infty$-convergent and that it converges to the sequence $\lambda^{(\infty)}_\cdot$ of singular values of $\mathsf{T}$. Hence the $\ell^1$-limit of $\lambda^{(n)}_\cdot$ has to be $\lambda^{(\infty)}_\cdot$. This means that the sequence $(T_n)$ converges to $\mathsf{T}$ in $\|\cdot\|_1$. We have proved 1).

2) We shall need the following useful lemma.

**Lemma 2.9.** *Every bounded operator is a linear combination of four unitary operators*

*Proof.* Let $\mathsf{B} \in \mathcal{B}(\mathcal{H})$, then we can write $\mathsf{B}$ as a linear combination of two self-adjoint operators:

$$\mathsf{B} = \frac{1}{2}(\mathsf{B}+\mathsf{B}^*) - \frac{i}{2}(\mathsf{B}-\mathsf{B}^*)\,.$$

Now, if $\mathsf{A}$ is a bounded self-adjoint operator, with $\|\mathsf{A}\| \leq 1$ (which we can assume without loss of generality), the operators $\mathsf{A} \pm i\sqrt{\mathsf{I}-\mathsf{A}^2}$ are unitary and they sum up to $2\,\mathsf{A}$. □

We are now back to the proof of Property 2). By Lemma 2.9, it is sufficient to prove that for all $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$ and all unitary operator $\mathsf{U}$ we have that $\mathsf{U}\,\mathsf{T}$ and $\mathsf{T}\,\mathsf{U}$ belong to $\mathcal{L}_1(\mathcal{H})$. But we have $|\mathsf{U}\,\mathsf{T}| = |\mathsf{T}|$ and hence $\mathsf{U}\,\mathsf{T}$ belong to $\mathcal{L}_1(\mathcal{H})$. We also have $|\mathsf{T}\,\mathsf{U}| = \mathsf{U}^*\ |\mathsf{T}|\ \mathsf{U}$ and the same conclusion. We have proved the two-sided ideal property.

Let us prove the norm inequality now. When $\mathsf{U}$ is unitary, we see from the considerations above that $\|\mathsf{U}\,\mathsf{T}\|_1 \leq \|\mathsf{T}\|_1$. On the other hand, every bounded $\mathsf{B}$ operator is a linear combination of four unitary operators *with coefficients in the linear combination being smaller than* $\|\mathsf{B}\|$. It is easy to conclude.

The property 3) comes immediately from the canonical form of trace-class operators (Corollary 2.6).

4) Consider the polar decompositions $\mathsf{T} = \mathsf{U}\,|\mathsf{T}|$ and $\mathsf{T}^* = \mathsf{V}\,|\mathsf{T}^*|$. We then have $|\mathsf{T}^*| = \mathsf{V}^*\,|\mathsf{T}|\,\mathsf{U}^*$. By Property 2, the operator $|\mathsf{T}^*|$ is trace-class, and so is $\mathsf{T}^*$. Finally

$$\mathrm{Tr}\,(\mathsf{T}^*) = \sum_n \langle e_n\,,\,\mathsf{T}^* e_n\rangle = \sum_n \langle \mathsf{T} e_n\,,\,e_n\rangle = \overline{\mathrm{Tr}\,\mathsf{T}}\,.$$

5) Let $\mathsf{S}$ and $\mathsf{T}$ be such that $\mathsf{S}$ is bounded and $\mathsf{T}$ is trace-class, for example. By Lemma 2.9 we can reduce the problem to the case where $\mathsf{S}$ is unitary. Let $(e_n)$ be an orthonormal basis and $f_n = \mathsf{S}^* e_n$ for all $n$. We have

$$\begin{aligned}\mathrm{Tr}\,(\mathsf{S}\,\mathsf{T}) &= \sum_n \langle e_n\,,\,\mathsf{S}\,\mathsf{T}\,e_n\rangle\\ &= \sum_n \langle \mathsf{S}^*\,e_n\,,\,\mathsf{T}\,e_n\rangle\\ &= \sum_n \langle f_n\,,\,\mathsf{T}\,\mathsf{S}\,f_n\rangle\\ &= \mathrm{Tr}\,(\mathsf{T}\,\mathsf{S})\,. \qquad \square\end{aligned}$$

We end up this subsection with a useful characterization.

**Theorem 2.10.** *A bounded operator* $\mathsf{T}$ *on* $\mathcal{H}$ *is trace-class if and only if*

$$\sum_n |\langle g_n\,,\,\mathsf{T}\,h_n\rangle| < \infty$$

*for all orthonormal families* $(g_n)$ *and* $(h_n)$ *in* $\mathcal{H}$.

*Moreover there exists orthonormal families* $(g_n)$ *and* $(h_n)$ *in* $\mathcal{H}$ *such that*

$$\sum_n |\langle g_n\,,\,\mathsf{T}\,h_n\rangle| = \|\mathsf{T}\|_1\,.$$

*Proof.* Assume first that $\mathsf{T}$ is trace-class. Then $\mathsf{T}$ can be decomposed as (Corollary 2.6)

$$\mathsf{T} = \sum_n \lambda_n\,|u_n\rangle\langle v_n|\,,$$

for some orthonormal families $(u_n)$ and $(v_n)$ and a summable sequence $(\lambda_n)$ of positive scalars. Hence we have

$$\begin{aligned}\sum_n |\langle g_n\,,\, \mathsf{T}\, h_n\rangle| &= \sum_n \sum_k \lambda_k\ |\langle g_n\,,\, u_k\rangle|\ |\langle v_k\,,\, h_n\rangle| \\ &\leq \sum_k \lambda_k \left(\sum_n |\langle g_n\,,\, u_k\rangle|^2\right)^{1/2} \left(\sum_n |\langle v_k\,,\, h_n\rangle|^2\right)^{1/2} \\ &\leq \sum_k \lambda_k\ \|u_k\|\ \|v_k\| \\ &= \sum_k \lambda_k < \infty\,.\end{aligned}$$

We have proved the first part of theorem in one direction. Note that, in the above computation, if one had chosen $g_n = u_n$ and $h_n = v_n$ for all $n$ we would have

$$\sum_n |\langle g_n\,,\, \mathsf{T}\, h_n\rangle| = \sum_n \sum_k \lambda_k\ |\langle g_n\,,\, u_k\rangle|\ |\langle v_k\,,\, h_n\rangle| = \sum_k \lambda_k = \|\mathsf{T}\|_1\ .$$

This proves the second part of the theorem in that case.

Conversely, let $\mathsf{T}$ be a bounded operator on $\mathcal{H}$ satisfying

$$\sum_n |\langle g_n\,,\, \mathsf{T}\, h_n\rangle| < \infty$$

for all orthonormal families $(g_n)$ and $(h_n)$ in $\mathcal{H}$. Let $\mathsf{T} = \mathsf{U}\,|\mathsf{T}|$ be its polar decomposition. Choose an orthonormal sequence $(h_n)$ in $\overline{Ran\,|\mathsf{T}|}$ and put $g_n = \mathsf{U}\, h_n$ for all $n$. Since $\mathsf{U}$ is isometric on $\overline{Ran\,|\mathsf{T}|}$ we have $\mathsf{U}^* g_n = h_n$ for all $n$. Hence we have

$$\sum_n |\langle g_n\,,\, \mathsf{T}\, h_n\rangle| = \sum_n |\langle h_n\,,\, |\mathsf{T}|\ h_n\rangle| = \sum_n \langle h_n\,,\, |\mathsf{T}|\ h_n\rangle\,.$$

By our hypothesis on $\mathsf{T}$ this proves that $\sum_n \langle h_n\,,\, |\mathsf{T}|\ h_n\rangle$ is finite. Complete the family $(h_n)$ into an orthonormal basis $\left(\widetilde{h}_n\right)$ of $\mathcal{H}$, by completing with orthonormal vectors in $\overline{Ran\,|\mathsf{T}|}^{\perp} = \mathrm{Ker}\,|\mathsf{T}|$. Then

$$\sum_n \left\langle \widetilde{h}_n\,,\, |\mathsf{T}|\ \widetilde{h}_n\right\rangle = \sum_n \langle h_n\,,\, |\mathsf{T}|\ h_n\rangle$$

and hence is finite. This proves that $\mathrm{Tr}\ |\mathsf{T}| < \infty$ and that $\mathsf{T}$ is trace-class. □

### 2.1.3 Dualities

**Definition 2.11.** Let $\mathcal{H}$ be any separable complex Hilbert space. For any trace-class operator $\mathsf{T}$ and bounded operator $\mathsf{X}$ on $\mathcal{H}$ we write

$$\langle \mathsf{T}\,,\,\mathsf{X}\rangle = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})\,,$$

which is well-defined by Theorem 2.8.

**Theorem 2.12.**
*1) The Banach space* $(\mathcal{L}_1(\mathcal{H}), \|\cdot\|_1)$ *is isometrically isomorphic to the dual of* $(\mathcal{L}_\infty(\mathcal{H}), \|\cdot\|)$ *under the correspondence* $\mathsf{T} \mapsto \langle \mathsf{T}\,,\,\cdot\rangle|_{\mathcal{L}_\infty(\mathcal{H})}$ *.*
*2) The Banach space* $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ *is isometrically isomorphic to the dual of* $(\mathcal{L}_1(\mathcal{H}), \|\cdot\|_1)$ *under the correspondence* $\mathsf{X} \mapsto \langle\,\cdot\,,\mathsf{X}\rangle|_{\mathcal{L}_1(\mathcal{H})}$ *.*

*Proof.*
1) For all $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$ the mapping $\mathsf{X} \mapsto \langle \mathsf{T}\,,\,\mathsf{X}\rangle = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})$ define a linear form $\lambda_\mathsf{T}$ on $\mathcal{L}_\infty(\mathcal{H})$. By (2.3) and (2.4) we have

$$|\mathrm{Tr}\,(\mathsf{X}\,\mathsf{T})| \leq \mathrm{Tr}\;|\mathsf{X}\,\mathsf{T}| \leq \|\mathsf{X}\|\;\|\mathsf{T}\|_1\;.$$

Hence $\lambda_\mathsf{T}$ is a continuous linear form, with norm $\|\lambda_\mathsf{T}\| \leq \|\mathsf{T}\|_1$. Taking $\mathsf{X} = \mathsf{U}^*$ where $\mathsf{U}$ is the partial isometry in the polar decomposition of $\mathsf{T}$, gives the equality of norms: $\|\lambda_\mathsf{T}\| = \|\mathsf{T}\|_1$.

Conversely, let $\lambda$ be a continuous linear form on $\mathcal{L}_\infty(\mathcal{H})$. The mapping

$$B\;:\;(u,v) \longmapsto \lambda(|u\rangle\langle v|)$$

is a sesquilinear form on $\mathcal{H}$ and it satisfies $\|B(u,v)\| \leq \|\lambda\|\;\|u\|\;\|v\|$. Hence there exists a bounded operator $\mathsf{T}$ on $\mathcal{H}$ such that

$$\lambda(|u\rangle\langle v|) = \langle v\,,\,\mathsf{T}\,u\rangle = \mathrm{Tr}\,(\mathsf{T}\,|u\rangle\langle v|).$$

Consider the polar decomposition $\mathsf{T} = \mathsf{U}\;|\mathsf{T}|$ of $\mathsf{T}$ and choose an orthonormal basis $(u_n)$ in $\mathrm{Ran}\,|\mathsf{T}|$. As a consequence the family $\{v_n = \mathsf{U}\,u_n\,;\;n \in \mathbb{N}\}$ forms an orthonormal basis of $\mathrm{Ran}\,\mathsf{T}$. The operator

$$\mathsf{X}_n = \sum_{i=1}^n |u_i\rangle\langle v_i|$$

is finite rank and norm 1. We have

$$\lambda(\mathsf{X}_n) = \sum_{i=1}^n \langle v_i\,,\,\mathsf{T}\,u_i\rangle = \sum_{i=1}^n \langle u_i\,,\,|\mathsf{T}|\;u_i\rangle\;.$$

This shows that

$$\sum_{j=1}^{\infty} \langle u_i \,,\, |\mathsf{T}|\; u_i\rangle \leq \|\lambda\| \;.$$

Completing the orthonormal basis $(u_n)$ of $\mathrm{Ran}\,|\mathsf{T}|$ into an orthonormal basis of $\mathcal{H}$ (by choosing any orthonormal basis of $(\mathrm{Ran}\,|\mathsf{T}|)^\perp = \mathrm{Ker}\,|\mathsf{T}|$), we have found an orthonormal basis $(f_n)$ of $\mathcal{H}$ such that

$$\sum_{n\in\mathbb{N}} \langle f_n \,,\, |\mathsf{T}|\; f_n\rangle \leq \|\lambda\| < \infty\,.$$

This shows that $\mathsf{T}$ is trace-class and that $\|\mathsf{T}\|_1 \leq \|\lambda\|$.

As the finite rank operators are dense in the compact operators for the operator norm, passing to the limit on the identity

$$\lambda(\mathsf{X}_n) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X}_n)$$

shows that

$$\lambda(\mathsf{X}) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})$$

for all $\mathsf{X} \in \mathcal{L}_\infty(\mathcal{H})$. Finally, we have

$$|\lambda(\mathsf{X})| = |\mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})| \leq \|\mathsf{T}\|_1\; \|\mathsf{X}\|\,.$$

Hence $\|\lambda\| \leq \|\mathsf{T}\|_1$. This proves the equality of norms and the part 1) is completely proved.

2) The proof is very similar to the one of 1) above. Let $\mathsf{B} \in \mathcal{B}(\mathcal{H})$ and define the linear form $\lambda_\mathsf{B}$ on $\mathcal{L}_1(\mathcal{H})$ by $\lambda_\mathsf{B}(\mathsf{T}) = \mathrm{Tr}\,(\mathsf{B}\,\mathsf{T})$. Again, the inequality $|\lambda_\mathsf{B}(\mathsf{T})| \leq \|\mathsf{B}\|\;\|\mathsf{T}\|_1$ proves that $\lambda_\mathsf{B}$ is continuous with $\|\lambda_\mathsf{B}\| \leq \|\mathsf{B}\|$.

Take $\mathsf{T} = |x\rangle\langle y|$ for some $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$, this gives $\lambda_\mathsf{B}(\mathsf{T}) = \langle y \,,\, \mathsf{B}\,x\rangle$. But we have

$$\begin{aligned}\|\mathsf{B}\| &= \sup\{|\langle y \,,\, \mathsf{B}\,x\rangle| \;;\; x, y \in \mathcal{H}\,,\; \|x\| = \|y\| = 1\}\\ &= \sup\{|\lambda_\mathsf{B}(\mathsf{T})| \;;\; \mathsf{T} = |x\rangle\langle y|,\, \|x\| = \|y\| = 1\}\\ &\leq \sup\{|\lambda_\mathsf{B}(\mathsf{T})| \;;\; \mathsf{T} \in \mathcal{L}_1(\mathcal{H})\,,\; \|\mathsf{T}\|_1 = 1\}\\ &= \|\lambda_\mathsf{B}\|\;.\end{aligned}$$

Hence we have proved the equality $\|\lambda_\mathsf{B}\| = \|\mathsf{B}\|$.

Conversely, if $\lambda$ is a continuous linear form on $\mathcal{L}_1(\mathcal{H})$ then $(u, v) \mapsto \lambda(|u\rangle\langle v|)$ defines a bounded sesquilinear form on $\mathcal{H}$. Hence there exists a bounded operator $\mathsf{X}$ such that

$$\lambda(|u\rangle\langle v|) = \mathrm{Tr}\,(\mathsf{X}\,|u\rangle\langle v|).$$

If $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$ is of the form

$$\mathsf{T} = \sum_n \lambda_n \, |u_n\rangle\langle v_n|$$

then it is easy to check that

$$\sum_{n \leq N} \lambda_n \, |u_n\rangle\langle v_n|$$

converges to $\mathsf{T}$ in $\mathcal{L}_1(\mathcal{H})$ when $N$ tends to $+\infty$. As $\lambda$ is continuous we get

$$\lambda(\mathsf{T}) = \mathrm{Tr}\,(\mathsf{X}\,\mathsf{T})$$

for every $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$. It is easy to conclude now. □

**Definition 2.13.** As $\mathcal{B}(\mathcal{H})$ is the dual of $\mathcal{L}_1(\mathcal{H})$ it inherits a $*$-*weak topology*, that is, the topology which makes the mappings $\lambda_{\mathsf{T}}(\mathsf{X}) = \mathrm{Tr}\,(\mathsf{X}\,\mathsf{T})$ being continuous on $\mathcal{B}(\mathcal{H})$ for all $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$. It is easy to see that it coincides with the topology induced by the seminorms

$$n_{e,f}(\mathsf{X}) = \sum_n |\langle e_n \, , \, \mathsf{X}\, f_n\rangle|$$

where $(e_n)$ and $(f_n)$ run over all sequences in $\mathcal{H}$ such that

$$\sum_n \|e_n\|^2 < \infty \qquad \text{and} \qquad \sum_n \|f_n\|^2 < \infty \,.$$

This fact can be easily deduced from the canonical form of trace-class operators (Corollary 2.6). The topology above is often called the $\sigma$-*weak topology* on $\mathcal{B}(\mathcal{H})$.

The following result is a very classical consequence of the theory of Banach space duals (see [RS80], Theorem IV.20).

**Corollary 2.14.** *Every $\sigma$-weakly continuous linear form $\lambda$ on $\mathcal{B}(\mathcal{H})$ is of the form*

$$\lambda(\mathsf{X}) = \mathrm{Tr}\,(\mathsf{T}\,\mathsf{X})$$

*for some $\mathsf{T} \in \mathcal{L}_1(\mathcal{H})$. This trace-class operator $\mathsf{T}$ associated to $\lambda$ is unique.*

### *2.1.4 Hilbert-Schmidt Operators*

We end up this section with the another special family of operators which are closely connected to the trace-class operators: the *Hilbert-Schmidt operators*.

**Definition 2.15.** An operator $\mathsf{T} \in \mathcal{B}(\mathcal{H})$ is *Hilbert-Schmidt* if

$$\mathrm{Tr}\,(\mathsf{T}^*\,\mathsf{T}) < \infty \,.$$

The set of Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathcal{L}_2(\mathcal{H})$.

The following properties are obtained in a similar way as those of $\mathcal{L}_1(\mathcal{H})$. We leave the proofs to the reader.

**Theorem 2.16.**

*1) The space $\mathcal{L}_2(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$.*

*2) An operator $\mathsf{T} \in \mathcal{B}(\mathcal{H})$ is Hilbert-Schmidt if and only if there exists an orthonormal basis $(e_n)$ of $\mathcal{H}$ such that*

$$\sum_n \|\mathsf{T}\, e_n\|^2 < \infty\,.$$

*In that case the series converges to the same sum for all orthonormal basis of $\mathcal{H}$.*

*3) The mapping*

$$\langle \mathsf{S}\,,\, \mathsf{T}\rangle = \mathrm{Tr}\,(\mathsf{S}^*\, \mathsf{T})$$

*defines a scalar product on $\mathcal{L}_2(\mathcal{H})$ which gives it a Hilbert space structure.*

*The norm $\|\cdot\|_2$ associated to this scalar product satisfies*

$$\|\mathsf{A}\| \leq \|\mathsf{A}\|_2 \leq \|\mathsf{A}\|_1\ .$$

*4) Every Hilbert-Schmidt operator is a compact operator. Conversely, a compact operator is Hilbert-Schmidt if and only if the sequence of its singular values $(\lambda_n)$ satisfies*

$$\sum_n \lambda_n^2 < \infty\,.$$

*Every trace-class operator is Hilbert-Schmidt.*

## 2.2 Tensor Products

As already explained in the introduction of this chapter, we leave the trace-class operators for a while and start with a completely different topic: the tensor products of Hilbert spaces and of operators. The connections between the two notions appear in next section.

The notion of tensor product of Hilbert spaces and of tensor product of operators are key concepts in Quantum Mechanics, where they appear each time that several quantum systems are involved and interact with each other.

### *2.2.1 Tensor Products of Hilbert Spaces*

**Definition 2.17.** Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. Let $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$. We define $\varphi \otimes \psi$ to be the bi-antilinear form on $\mathcal{H}_1 \times \mathcal{H}_2$ given by

$$\varphi \otimes \psi\,(\varphi', \psi') = \langle \varphi'\,,\,\varphi\rangle\langle \psi'\,,\,\psi\rangle\,.$$

Let $E$ be the set of all finite linear combinations of such forms, acting in the obvious way on $\mathcal{H}_1 \otimes \mathcal{H}_2$. We equip $E$ with an internal product defined by

$$\begin{aligned}\langle \varphi \otimes \psi\,,\,\varphi' \otimes \psi'\rangle &= \langle \varphi\,,\,\varphi'\rangle\langle \psi\,,\,\psi'\rangle\\ &= \varphi' \otimes \psi'(\varphi, \psi)\,,\end{aligned}$$

and its natural extension to linear combinations.

**Proposition 2.18.** *The product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $E$ is well-defined and positive definite.*

*Proof.* To show that $\langle\,\cdot\,,\,\cdot\,\rangle$ is well-defined, we need to show that $\langle \lambda\,,\lambda'\,\rangle$ does not depend on the choice of the linear combinations representing $\lambda$ and $\lambda'$. It is thus sufficient to show that if $\mu$ is the null linear combination then $\langle \lambda\,,\mu\,\rangle = 0$ for all $\lambda \in E$. Assume $\lambda$ is of the form $\sum_{i=1}^n \alpha_i\,\varphi_i \otimes \psi_i$, we have

$$\begin{aligned}\langle \lambda\,,\,\mu\rangle = \Big\langle \sum_{i=1}^n \alpha_i\,\varphi_i \otimes \psi_i\,,\,\mu\Big\rangle &= \sum_{i=1}^n \overline{\alpha_i}\,\langle \varphi_i \otimes \psi_i\,,\,\mu\rangle\\ &= \sum_{i=1}^n \overline{\alpha_i}\,\mu\,(\varphi_i, \psi_i)\\ &= 0\,.\end{aligned}$$

This proves that the product is well-defined. Let us prove it is positive definite. Let $\lambda = \sum_{i=1}^n \alpha_i\,\varphi_i \otimes \psi_i$ again, and let $(e_i)_{i=1,\ldots,k}$ and $(f_i)_{i=1,\ldots,\ell}$ be orthonormal bases of the spaces generated by $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\psi_1, \ldots, \psi_n\}$ respectively. Then

$$\lambda = \sum_{i=1}^k \sum_{j=1}^\ell \alpha_{ij}\,e_i \otimes f_j\,,$$

for some coefficients $\alpha_{ij}$, and thus

$$\langle \lambda\,,\,\lambda\rangle = \sum_{i=1}^k \sum_{j=1}^\ell |\alpha_{ij}|^2\,.$$

Thus $\langle \lambda\,,\,\lambda\rangle$ is positive and $\langle \lambda\,,\,\lambda\rangle = 0$ if and only if $\lambda = 0$. $\square$

**Definition 2.19.** As a consequence $(E, \langle\, \cdot\,,\,\cdot\,\rangle)$ is a pre-Hilbert space. We denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$ its completion. It is called the *tensor product* of $\mathcal{H}_1$ by $\mathcal{H}_2$.

**Proposition 2.20.** *Let $\mathcal{N}$ and $\mathcal{M}$ be sets of indices in $\mathbb{N}$ such that $(e_i)_{i\in\mathcal{N}}$ and $(f_j)_{j\in\mathcal{M}}$ are any orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $(e_i \otimes f_j)_{(i,j)\in\mathcal{N}\times\mathcal{M}}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$ .*

*Proof.* The fact that the set $\{e_i \otimes f_j\,;\ (i,j) \in \mathcal{N} \times \mathcal{M}\}$ is an orthonormal family is clear for the definition of the scalar product on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let us check it forms a basis. Let $S$ be the vector space generated by $\{e_i \otimes f_j\,;\ (i,j) \in \mathcal{N} \times \mathcal{M}\}$. Let $\varphi \otimes \psi$ be an element of $E$, then we have decompositions of the form

$$\varphi = \sum_{i\in\mathcal{N}} c_i e_i \qquad \text{with} \qquad \sum_{i\in\mathcal{N}} |c_i|^2 < \infty$$

and

$$\psi = \sum_{j\in\mathcal{M}} d_j f_j \qquad \text{with} \qquad \sum_{j\in\mathcal{M}} |d_j|^2 < \infty\,.$$

As a consequence we have

$$\sum_{(i,j)\in\mathcal{N}\times\mathcal{M}} |c_i d_j|^2 < \infty$$

and the vector $\sum_{i,j} c_i d_j\, e_i \otimes f_j$ belongs to $\overline{S}$, the closure of $S$. But by direct computation one sees that, for all $N, M \in \mathbb{N}$

$$\Bigg\|\varphi\otimes\psi - \sum_{\substack{i\le N\\ i\in\mathcal{N}}} \sum_{\substack{j\le M\\ j\in\mathcal{M}}} c_i d_j\, e_i\otimes f_j\Bigg\|^2 = \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{M}} |c_i|^2\, |d_j|^2 - \sum_{\substack{i\le N\\ i\in\mathcal{N}}} \sum_{\substack{j\le M\\ j\in\mathcal{M}}} |c_i|^2\, |d_j|^2$$

and hence converges to 0 when $N$ and $M$ tend to $+\infty$. Thus $E$ is included in $\overline{S}$. This give the density of $S$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$. □

### *2.2.2 Tensor Products of Operators*

**Definition 2.21.** Let $\mathsf{A}$ be an operator on $\mathcal{H}_1$, with a dense domain $\mathrm{Dom}\,\mathsf{A}$, and $\mathsf{B}$ be an operator on $\mathcal{H}_2$, with dense domain $\mathrm{Dom}\,\mathsf{B}$. Let $D$ denote the space $\mathrm{Dom}\,\mathsf{A}\,\overline{\otimes}\,\mathrm{Dom}\,\mathsf{B}$, that is, the space of finite linear combinations of $\varphi\otimes\psi$ with $\varphi \in \mathrm{Dom}\,\mathsf{A}$ and $\psi \in \mathrm{Dom}\,\mathsf{B}$. Clearly $D$ is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. One defines the operator $\mathsf{A} \otimes \mathsf{B}$ on $D$ by

$$(\mathsf{A}\otimes\mathsf{B})(\varphi\otimes\psi) = \mathsf{A}\varphi\otimes\mathsf{B}\psi\,,$$

and its obvious linear extension to all $D$.

**Proposition 2.22.** *The operator* $\mathsf{A}\otimes\mathsf{B}$ *is well-defined. If* $\mathsf{A}$ *and* $\mathsf{B}$ *are closable operators then so is* $\mathsf{A}\otimes\mathsf{B}$.

*Proof.* Being well-defined in this case means that if

$$\sum_{i=1}^{n} c_i\,\varphi_i\otimes\psi_i=\sum_{j=1}^{m} d_j\,\varphi'_j\otimes\psi'_j \tag{2.5}$$

then

$$\sum_{i=1}^{n} c_i\,\mathsf{A}\varphi_i\otimes\mathsf{B}\psi_i=\sum_{j=1}^{m} d_j\,\mathsf{A}\varphi'_j\otimes\mathsf{B}\psi'_j\,. \tag{2.6}$$

But, passing through orthornormal bases $(e_i)_{i\in\mathcal{N}}$ and $(f_j)_{j\in\mathcal{M}}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, the identity (2.5) is equivalent to

$$\sum_{i=1}^{n} c_i\,\langle\varphi_i\,,\,e_j\rangle\,\langle\psi_i\,,\,f_k\rangle=\sum_{i=1}^{k} d_i\,\langle\varphi'_i\,,\,e_j\rangle\,\langle\psi'_i\,,\,f_k\rangle\,,$$

for all $j,k$. Applying $\mathsf{A}\otimes\mathsf{B}$ to $\sum_{i=1}^n c_i\,\varphi_i\otimes\psi_i$ but in the basis $\{e_j\otimes f_k\,;\ j\in\mathcal{N},k\in\mathcal{M}\}$ gives the identity (2.6) immediately.

Now, if $g\in\mathrm{Dom}\,\mathsf{A}^*\overline{\otimes}\,\mathrm{Dom}\,\mathsf{B}^*$ and $f\in\mathrm{Dom}\,\mathsf{A}\,\overline{\otimes}\,\mathrm{Dom}\,\mathsf{B}$ we have obviously

$$\langle(\mathsf{A}\otimes\mathsf{B})\,f,g\rangle=\langle f,(\mathsf{A}^*\otimes\mathsf{B}^*)g\rangle\,.$$

Thus $\mathrm{Dom}(\mathsf{A}\otimes\mathsf{B})^*$ contains $\mathrm{Dom}\,\mathsf{A}^*\otimes\mathrm{Dom}\,\mathsf{B}^*$ which is dense. This means that the operator $\mathsf{A}\otimes\mathsf{B}$ is closable. □

In particular we have also proved that

$$(\mathsf{A}\otimes\mathsf{B})^*\subset\mathsf{A}^*\otimes\mathsf{B}^*\,.$$

**Definition 2.23.** If $\mathsf{A}$ and $\mathsf{B}$ are two closable operators we call *tensor product* of $\mathsf{A}$ by $\mathsf{B}$ the closure of $\mathsf{A}\otimes\mathsf{B}$. It is still denoted by $\mathsf{A}\otimes\mathsf{B}$.

**Proposition 2.24.** *If* $\mathsf{A}$ *and* $\mathsf{B}$ *are bounded operators on* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *respectively then* $\mathsf{A}\otimes\mathsf{B}$ *is a bounded operator on* $\mathcal{H}_1\otimes\mathcal{H}_2$ *and we have*

$$\|\mathsf{A}\otimes\mathsf{B}\|=\|\mathsf{A}\|\,\|\mathsf{B}\|\,.$$

*Proof.* Assume that $f\in\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$ can be decomposed as $f=\sum_{i,j=1}^n c_{ij}\,e_i\otimes f_j$ in some orthonormal basis of $\mathcal{H}_1\otimes\mathcal{H}_2$. Then we have

$$\begin{aligned}
\|(\mathsf{A}\otimes \mathrm{I})f\|^2 &= \left\|(\mathsf{A}\otimes \mathrm{I})\sum_{i,j=1}^{n} c_{ij}\, e_i\otimes f_j\right\|^2\\
&= \left\|\sum_{i,j=1}^{n} c_{ij}\, \mathsf{A}e_i\otimes f_j\right\|^2\\
&= \sum_{j=1}^{n}\left\|\sum_{i=1}^{n} c_{ij}\, \mathsf{A}e_i\right\|^2\\
&\leq \|\mathsf{A}\|^2\sum_{j=1}^{n}\sum_{i=1}^{n}|c_{ij}|^2\\
&= \|\mathsf{A}\|^2\|f\|^2\,.
\end{aligned}$$

We have proved that $\mathsf{A}\otimes \mathrm{i}$ is bounded on $\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$, with $\|\mathsf{A}\otimes \mathrm{I}\|\leq\|\mathsf{A}\|$. Hence its closure is a bounded operator on $\mathcal{H}_1\otimes\mathcal{H}_2$, with the same norm estimate. In the same way, we obtain $\mathrm{I}\otimes\mathsf{B}$ is bounded, with $\|\mathrm{I}\otimes\mathsf{B}\|\leq\|\mathsf{B}\|$. As $\mathsf{A}\otimes\mathsf{B}=(\mathsf{A}\otimes\mathrm{I})(\mathrm{I}\otimes\mathsf{B})$ (at least obviously on $\mathcal{H}_1\overline{\otimes}\mathcal{H}_2$) we get that $\mathsf{A}\otimes\mathsf{B}$ is bounded and

$$\|\mathsf{A}\otimes\mathsf{B}\|\leq\|\mathsf{A}\|\,\|\mathsf{B}\|\,.$$

Let $\varepsilon>0$ be fixed, choose $\varphi$ and $\psi$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, such that

$$\begin{cases}\|\mathsf{A}\varphi\|\geq\|\mathsf{A}\|-\varepsilon\,,\\ \|\mathsf{B}\psi\|\geq\|B\|-\varepsilon\,,\\ \|\varphi\|=\|\psi\|=1\,.\end{cases}$$

Then

$$\|(\mathsf{A}\otimes\mathsf{B})(\varphi\otimes\psi)\|=\|\mathsf{A}\varphi\|\;\|\mathsf{B}\psi\|\geq\|\mathsf{A}\|\;\|\mathsf{B}\|-\varepsilon\|\mathsf{A}\|-\varepsilon\|\mathsf{B}\|+\varepsilon^2\,.$$

Making $\varepsilon$ go to 0, this proves that $\|(\mathsf{A}\otimes\mathsf{B})\|\geq\|\mathsf{A}\|\,\|\mathsf{B}\|$. The equality of norms is proved. □

Tensor products of operators preserve most of the classes of bounded operators.

**Theorem 2.25.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (separable) Hilbert spaces. Let $\mathsf{A}$ and $\mathsf{B}$ be bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*

*1) If $\mathsf{A}$ and $\mathsf{B}$ are compact operators then $\mathsf{A}\otimes\mathsf{B}$ is a compact operator on $\mathcal{H}_1\otimes\mathcal{H}_2$.*

*2) If $\mathsf{A}$ and $\mathsf{B}$ are trace-class operators then $\mathsf{A}\otimes\mathsf{B}$ is trace-class too. In particular we have*

$$\|\mathsf{A}\otimes\mathsf{B}\|_1=\|\mathsf{A}\|_1\;\|\mathsf{B}\|_1 \tag{2.7}$$

*and*

$$\mathrm{Tr}\,(\mathsf{A}\otimes\mathsf{B}) = \mathrm{Tr}\,(\mathsf{A})\ \mathrm{Tr}\,(\mathsf{B})\,. \tag{2.8}$$

*3) If* $\mathsf{A}$ *and* $\mathsf{B}$ *are Hilbert-Schmidt operators then* $\mathsf{A}\otimes\mathsf{B}$ *is Hilbert-Schmidt too. In particular we have*

$$\|\mathsf{A}\otimes\mathsf{B}\|_2 = \|\mathsf{A}\|_2\ \|\mathsf{B}\|_2\ . \tag{2.9}$$

*Proof.* We start by noticing that

$$|u_1\rangle\langle v_1|\otimes|u_2\rangle\langle v_2| = |u_1\otimes u_2\rangle\langle v_1\otimes v_2|$$

as can be easily seen by applying these operators to elements of the form $f\otimes g$ in $\mathcal{H}_1\otimes\mathcal{H}_2$, then passing to linear combinations and finally passing to the limit to general elements of $\mathcal{H}_1\otimes\mathcal{H}_2$.

Now the main characterizations of compact (*resp.* trace-class, *resp.* Hilbert-Schmidt) operators is that they are represented as

$$\sum_n \lambda_n|u_n\rangle\langle v_n|$$

for some orthonormal families $(u_n)$ and $(v_n)$ and a sequence of positive scalars $(\lambda_n)$ which converges to 0 (*resp.* is summable, *resp.* is square summable). It is now very easy to conclude in all the three cases. □

### *2.2.3 Countable Tensor Products*

All along this book we shall also make heavy use of the notion of countable tensor products of Hilbert spaces. Let us detail here how they are constructed.

Let $(\mathcal{H}_n)$ be a sequence of Hilbert spaces. All the finite tensor products

$$\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_n$$

are well-defined by the above constructions. But it is not clear how one can rigorously define the countable tensor product

$$\bigotimes_{n\in\mathbb{N}}\mathcal{H}_n\,.$$

The idea is the following: one defines $\otimes_{n\in\mathbb{N}}\mathcal{H}_n$ as the inductive limit of the spaces $\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_n$, when $n$ tends to $+\infty$. This can be achieved only if one finds a way to see the space $\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_n$ as a subspace of $\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_{n+1}$ for all $n$. The way this is obtained is by choosing a unit vector $u_n\in\mathcal{H}_n$ for each $n$, which will constitute a reference vector of $\mathcal{H}_n$. Then elements $f_0\otimes\ldots\otimes f_n$ of $\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_n$ are identified to $f_0\otimes\ldots\otimes f_n\otimes u_{n+1}$ as an element of $\mathcal{H}_0\otimes\ldots\otimes\mathcal{H}_{n+1}$. This embedding is isometric, it preserves the scalar product, etc ... One can easily go the limit $n \to +\infty$ in this construction. This gives rise to the following definition.

Let $(\mathcal{H}_n)$ be a sequence of Hilbert spaces. Choose a sequence $(u_n)$ such that $u_n \in \mathcal{H}_n$ and $\|u_n\| = 1$, for all $n$. This sequence is called a *stabilizing sequence* for $\otimes_{n\in\mathbb{N}} \mathcal{H}_n$. The space $\otimes_{n\in\mathbb{N}} \mathcal{H}_n$ is defined as the closure of the pre-Hilbert space of vectors of the form

$$\bigotimes_{n\in\mathbb{N}} f_n$$

such that $f_n \in \mathcal{H}_n$ for all $n$ and $f_n = u_n$ for all but a finite number of $n$. The scalar product on that space being obviously defined by

$$\left\langle \bigotimes_{n\in\mathbb{N}} f_n \,,\, \bigotimes_{n\in\mathbb{N}} g_n \right\rangle = \prod_{n\in\mathbb{N}} \langle f_n \,,\, g_n \rangle \ .$$

The above infinite product is finite for all but a finite number of its terms are equal to $\langle u_n \,,\, u_n \rangle = 1$.

This is all for the definition of a countable tensor product of Hilbert space. Note that the construction depends on the choice of the stabilizing sequence $(u_n)$ which has been chosen initially. Hence, when dealing with such tensor products, one should be clear about the choice of the stabilizing sequence.

## 2.3 Partial Traces

The notion of partial trace appears naturally with the tensor products of Hilbert spaces and of operators. This notion is essential in the study of open quantum systems.

### *2.3.1 Partial Trace with Respect to a Space*

**Definition 2.26.** Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For any $g \in \mathcal{K}$ consider the operator

$$\begin{aligned} |g\rangle_{\mathcal{K}} : \mathcal{H} &\longrightarrow \mathcal{H} \otimes \mathcal{K} \\ f &\longmapsto f \otimes g \,. \end{aligned}$$

It is clearly a bounded operator from $\mathcal{H}$ to $\mathcal{H}\otimes\mathcal{K}$, with norm $\|g\|$. Its adjoint is the operator defined by

$$\begin{aligned} {}_{\mathcal{K}}\langle g| : \mathcal{H} \otimes \mathcal{K} &\longrightarrow \mathcal{H} \\ u \otimes v &\longmapsto \langle g \,,\, v \rangle \ u \,, \end{aligned}$$

and its natural extension by linearity and continuity. Note that its operator norm is also equal to $\|g\|$.

In particular, if $\mathsf{T}$ is a bounded operator on $\mathcal{H}\otimes\mathcal{K}$ then the operator

$$ {}_{\mathcal{K}}\langle g|\,\mathsf{T}\,|g\rangle_{\mathcal{K}} $$

is a bounded operator on $\mathcal{H}$.

**Lemma 2.27.** *If $\mathsf{T}$ is a trace-class operator on $\mathcal{H}\otimes\mathsf{K}$ then for all $f\in\mathcal{K}$ the operator*

$$ {}_{\mathcal{K}}\langle f|\,\mathsf{T}\,|f\rangle_{\mathcal{K}} $$

*is a trace-class operator on $\mathcal{H}$.*

*Proof.* Let $f\in\mathcal{K}$ and assume it is norm 1, without loss of generality. Let $(g_n)$ and $(h_n)$ be any orthonormal family in $\mathcal{H}$. We have

$$ \sum_n \left|\left\langle g_n\,,\ {}_{\mathcal{K}}\langle f|\,\mathsf{T}\,|f\rangle_{\mathcal{K}}\,h_n\right\rangle\right| = \sum_n \left|\langle g_n\otimes f\,,\,\mathsf{T}\,h_n\otimes f\rangle\right|\,. $$

We now use the characterization of Theorem 2.10 in both directions. As $\mathsf{T}$ is trace-class on $\mathcal{H}\otimes\mathcal{K}$ the right hand side is finite for $(g_n\otimes f)$ and $(h_n\otimes f)$ are particular orthonormal sequences in $\mathcal{H}\otimes\mathcal{K}$. This means that the left hand side above is finite for all orthonormal sequences $(g_n)$ and $(h_n)$, which ensures that ${}_{\mathcal{K}}\langle f|\,\mathsf{T}\,|f\rangle_{\mathcal{K}}$ is trace-class. □

We can now prove the main theorem which defines and characterizes the *partial traces.*

**Theorem 2.28.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. Let $\mathsf{T}$ be a trace-class operator on $\mathcal{H}\otimes\mathcal{K}$. Then, for any orthonormal basis $(g_n)$ of $\mathcal{K}$, the series*

$$ \mathrm{Tr}_{\mathcal{K}}(\mathsf{T}) = \sum_n {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}} \tag{2.10} $$

*is $\|\cdot\|_1$-convergent. The operator $\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})$ defined this way does not depend on the choice of the orthonormal basis $(g_n)$.*

*The operator $\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})$ is the unique trace-class operator on $\mathcal{H}$ such that*

$$ \mathrm{Tr}\,\left(\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})\,\mathsf{B}\right) = \mathrm{Tr}\,(\mathsf{T}\,(\mathsf{B}\otimes \mathrm{I})) \tag{2.11} $$

*for all $\mathsf{B}\in\mathcal{B}(\mathcal{H})$.*

*Proof.* Let $(g_n)$ be an orthonormal basis of $\mathcal{K}$. As each ${}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}$ is trace-class on $\mathcal{H}$ (Lemma 2.27) then, by Theorem 2.10, there exist for all $n\in\mathbb{N}$ orthonormal families $(e^n_m)_{m\in\mathbb{N}}$ and $(f^n_m)_{m\in\mathbb{N}}$ in $\mathcal{H}$ such that

$$ \left\|{}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}\right\|_1 = \sum_m \left|\left\langle e^n_m\,,\ {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}\,f^n_m\right\rangle\right|\,. $$

Hence we have

$$\sum_n \left\| {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}} \right\|_1 = \sum_n \sum_m \left|\left\langle e_m^n\,,\ {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}\, f_m^n\right\rangle\right| = \sum_n \sum_m \left|\langle e_m^n \otimes g_n\,,\ \mathsf{T}\, f_m^n \otimes g_n\rangle\right| .$$

Note that the families $(e_m^n \otimes g_n)_{n,m\in\mathbb{N}}$ and $(f_m^n \otimes g_n)_{n,m\in\mathbb{N}}$ are orthonormal in $\mathcal{H}\otimes\mathcal{K}$, hence, by Theorem 2.10 again, the above quantity is finite, for $\mathsf{T}$ is trace-class on $\mathcal{H}\otimes\mathcal{K}$. We have proved that the series $\sum_n {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}$ is $\|\cdot\|_1$-convergent. The operator

$$\mathrm{Tr}_{\mathcal{K}}(\mathsf{T}) = \sum_n {}_{\mathcal{K}}\langle g_n|\,\mathsf{T}\,|g_n\rangle_{\mathcal{K}}$$

is a well-defined, trace-class operator on $\mathcal{H}$.

Let us check that this operator is independent of the choice of the basis $(g_n)$. Let $(h_n)$ be another orthonormal basis of $\mathcal{K}$, we have

$$\begin{aligned}
\langle x\,,\ \mathrm{Tr}_{\mathcal{K}}(\mathsf{T})\, y\rangle &= \sum_n \langle x\otimes g_n\,,\ \mathsf{T}\,(y\otimes g_n)\rangle \\
&= \sum_n \sum_k \sum_l \langle g_n\,,\ h_k\rangle\ \langle h_l\,,\ g_n\rangle\ \langle x\otimes h_k\,,\ \mathsf{T}\,(y\otimes h_l)\rangle \\
&= \sum_k \sum_l \langle h_l\,,\ h_k\rangle\ \langle x\otimes h_k\,,\ \mathsf{T}\,(y\otimes h_l)\rangle \\
&= \sum_k \langle x\otimes h_k\,,\ \mathsf{T}\,(y\otimes h_k)\rangle\ .
\end{aligned}$$

This proves that $\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})$ is also equal to $\sum_k {}_{\mathcal{K}}\langle h_k|\,\mathsf{T}\,|h_k\rangle_{\mathcal{K}}$ . We have proved the independence property.

We prove now the characterization (2.11). Let $\mathsf{B}$ be any bounded operator on $\mathcal{H}$, we have

$$\begin{aligned}
\mathrm{Tr}\,\left(\mathsf{T}\,(\mathsf{B}\otimes \mathrm{I})\right) &= \sum_n \sum_m \langle e_n\otimes g_m\,,\ \mathsf{T}\,(\mathsf{B}\otimes \mathrm{I})\, e_n\otimes g_m\rangle \\
&= \sum_n \sum_m \langle e_n\otimes g_m\,,\ \mathsf{T}\,(\mathsf{B}\, e_n\otimes g_m)\rangle \\
&= \sum_n \sum_m \left\langle e_n\,,\ {}_{\mathcal{K}}\langle g_m|\,\mathsf{T}\,|g_m\rangle_{\mathcal{K}}\ \mathsf{B}\, e_n\right\rangle \\
&= \mathrm{Tr}\,(\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})\mathsf{B})\,.
\end{aligned}$$

We have proved that $\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})$ satisfies the relation (2.10). We have to prove uniqueness now, and the theorem will be completely proved. If $\mathsf{S}$ and $\mathsf{S}'$ are two trace-class operators on $\mathcal{H}$ satisfying (2.10), then in particular $\mathrm{Tr}\,((\mathsf{S}-$

$\mathsf{S}')\,\mathsf{B}) = 0$ for all $\mathsf{B} \in \mathcal{B}(\mathcal{H})$. This is in particular true for all $\mathsf{B} \in \mathcal{L}_\infty(\mathcal{H})$ hence, by Theorem 2.12, this implies that $\mathsf{S} - \mathsf{S}' = 0$. □

We now give a list of the most useful properties of the partial trace. They are all straightforward applications of the definition and theorem above, we leave the proofs to the reader.

**Theorem 2.29.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces, let $\mathsf{T}$ be a trace-class operator on $\mathcal{H} \otimes \mathcal{K}$.*
*1) If $\mathsf{T}$ is of the form $\mathsf{A} \otimes \mathsf{B}$, with $\mathsf{A}$ and $\mathsf{B}$ being trace-class, then*

$$\mathrm{Tr}_{\mathcal{K}}(\mathsf{T}) = \mathrm{Tr}\,(\mathsf{B})\,\mathsf{A}\,.$$

*2) We always have*

$$\mathrm{Tr}\,\left(\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})\right) = \mathrm{Tr}\,(\mathsf{T})\,.$$

*3) If $\mathsf{A}$ and $\mathsf{B}$ are bounded operators on $\mathcal{H}$ then*

$$\mathrm{Tr}_{\mathcal{K}}\left((\mathsf{A} \otimes \mathrm{I})\,\mathsf{T}\,(\mathsf{B} \otimes \mathrm{I})\right) = \mathsf{A}\ \mathrm{Tr}_{\mathcal{K}}(\mathsf{T})\,\mathsf{B}\,.$$

### *2.3.2 Partial Trace with Respect to a State*

In applications to Quantum Mechanics one sometimes also needs the notion of partial trace with respect to a given trace-class operator, or more precisely with respect to a state. This partial trace is different, but related to the previous one.

Once again this partial trace is better defined through a theorem which characterizes it.

**Theorem 2.30.** *Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. Let $\mathsf{T}$ be a fixed trace-class operator on $\mathcal{K}$, with canonical form*

$$\mathsf{T} = \sum_n \lambda_n\,|u_n\rangle\langle v_n|\,.$$

*Then, for any bounded operator $\mathsf{X}$ on $\mathcal{H} \otimes \mathcal{K}$, the series*

$$\sum_n \lambda_n\,{}_{\mathcal{K}}\langle v_n|\,\mathsf{X}\,|u_n\rangle_{\mathcal{K}} \tag{2.12}$$

*is operator-norm convergent on $\mathcal{H}$. Its limit, denoted by $\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})$, is a bounded operator on $\mathcal{H}$.*

*The operator $\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})$ is the only bounded operator on $\mathcal{H}$ satisfying*

$$\mathrm{Tr}\,\left(\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})\,\mathsf{S}\right) = \mathrm{Tr}\,(\mathsf{X}\,(\mathsf{S} \otimes \mathsf{T})) \tag{2.13}$$

*for all* $\mathsf{S} \in \mathcal{L}_1(\mathcal{H})$.

*Proof.* The series $\sum_n \lambda_n \,{}_{\mathcal{K}}\langle v_n | \,\mathsf{X}\, | u_n\rangle_{\mathcal{K}}$ is operator-norm convergent for

$$\begin{aligned}\sum_n \left\| \lambda_n \,{}_{\mathcal{K}}\langle v_n | \,\mathsf{X}\, | u_n\rangle_{\mathcal{K}} \right\| &\leq \sum_n |\lambda_n| \, \left\| {}_{\mathcal{K}}\langle v_n | \,\mathsf{X}\, | u_n\rangle_{\mathcal{K}} \right\| \\ &\leq \sum_n |\lambda_n| \, \left\| {}_{\mathcal{K}}\langle v_n | \right\| \, \|\mathsf{X}\| \, \left\| | u_n\rangle_{\mathcal{K}} \right\| \\ &\leq \sum_n |\lambda_n| \, \|\mathsf{X}\| < \infty \,.\end{aligned}$$

Hence it defines a bounded operator, which we denote by $\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})$.

We shall now check that it satisfies the relation (2.13). Recal that the family $(v_n)$ appearing in the canonical form of $\mathsf{T}$ is an orthonormal family on $\mathcal{K}$. Hence we can extend the family $(v_n)$ into an orthonormal basis $(\widetilde{v}_n)$ of $\mathcal{K}$. Recall that $\mathsf{T}$ vanishes on the orthogonal complement of the family $(v_n)$. Finally, recall that

$$\mathsf{T} v_m = \lambda_m u_m$$

for all $m$. Let $\mathsf{S}$ be a trace-class operator on $\mathcal{H}$ and let $(e_n)$ be an orthonormal basis of $\mathcal{H}$. We have

$$\begin{aligned}\mathrm{Tr}\,(\mathsf{X}\,(\mathsf{S} \otimes \mathsf{T})) &= \sum_{n,m} \langle e_n \otimes \widetilde{v}_m \,,\, \mathsf{X}\,(\mathsf{S} \otimes \mathsf{T})\, e_n \otimes \widetilde{v}_m \rangle \\ &= \sum_{n,m} \langle e_n \otimes \widetilde{v}_m \,,\, \mathsf{X}\,(\mathsf{S}\, e_n \otimes \mathsf{T}\, \widetilde{v}_m) \rangle \\ &= \sum_{n,m} \langle e_n \otimes v_m \,,\, \mathsf{X}\,(\mathsf{S}\, e_n \otimes \mathsf{T}\, v_m) \rangle \\ &= \sum_{n,m} \lambda_m \, \langle e_n \otimes v_m \,,\, \mathsf{X}\,(\mathsf{S}\, e_n \otimes u_m) \rangle \\ &= \sum_{n,m} \lambda_m \, \left\langle e_n \,,\, {}_{\mathcal{K}}\langle v_m | \,\mathsf{X}\, | u_m\rangle_{\mathcal{K}} \, \mathsf{S}\, e_n \right\rangle \\ &= \sum_n \langle e_n \,,\, \mathrm{Tr}_{\mathsf{T}}(\mathsf{X})\, \mathsf{S}\, e_n \rangle \\ &= \mathrm{Tr}\,(\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})\, \mathsf{S}) \,.\end{aligned}$$

The required relation is proved.

We just have to prove uniqueness now. If $\mathsf{X}_1$ and $\mathsf{X}_2$ are two bounded operators on $\mathcal{H}$ such that

$$\mathrm{Tr}\, (\mathsf{X}_i\, \mathsf{S}) = \mathrm{Tr}\,(\mathsf{X}(\mathsf{S} \otimes \mathsf{T}))$$

for all $\mathsf{S} \in \mathcal{L}_1(\mathcal{H})$, then we must have

$$\mathrm{Tr}\,((\mathsf{X}_1 - \mathsf{X}_2)\mathsf{S}) = 0$$

for all $\mathsf{S} \in \mathcal{L}_1(\mathcal{H})$. By Theorem 2.12, this implies $\mathsf{X}_1 - \mathsf{X}_2 = 0$. □

### *2.3.3 Comments*

The second family of partial traces $\mathrm{Tr}_{\mathsf{T}}(\mathsf{X})$ are called "*with respect to a state*" for, in most of the quantum mechanical situations where they appear, the operator $\mathsf{T}$ is a quantum state, a *density matrix* (see Lecture 4).

The partial traces with respect to a space, $\mathrm{Tr}_{\mathcal{K}}(\mathsf{T})$, are in general simply called *partial traces* (if everyone is clear about the space which is concerned!). For those with respect to a state $\mathsf{T}$ one generally makes it more precise: "*partial trace with respect to* $\mathsf{T}$".

Note that the two partial traces could be formally related by the following formulas:

$$\mathrm{Tr}_{\mathsf{T}}(\mathsf{X}) = \mathrm{Tr}_{\mathcal{K}}(\mathsf{X}\,(\mathrm{I} \otimes \mathsf{T}))\,, \tag{2.14}$$

and

$$\mathrm{Tr}_{\mathcal{K}}(\mathsf{T}) = \mathrm{Tr}_{\mathrm{I}}(\mathsf{T})\,, \tag{2.15}$$

which can be obtained easily from (2.11) and (2.12), but which are not quite correct in general! Indeed, in full generality the operator $\mathsf{X}\,(\mathrm{I}\otimes\mathsf{T})$ is not trace-class, so its partial trace $\mathrm{Tr}_{\mathcal{K}}$ is not defined; the operator I is not trace-class in general neither, so the trace $\mathrm{Tr}_{\mathrm{I}}$ is not defined.

Actually these formula are correct once $\mathsf{X}\,(\mathrm{I} \otimes \mathsf{T})$ is trace-class, reap. I is trace-class. This is to say, they are meaningful and true in finite dimension typically.

Once can weaken the definitions so that the identities (2.14) and (2.15) are always true, by asking weaker convergences in the series that define these partial traces. But it is not worth developing this here.

## Notes

There is no true references which are specially dedicated to trace-class operators or to tensor products. The references we have given in Lecture 1 are still valid here.

Only the discussion on partial traces is not easy to find in the literature. It is most often not treated, or treated only in the finite-dimensional case. As a consequence the treatment we give here on partial trace is original, above all by the approach we take with partial "bras" and "kets", which is very close to the way the physicists usually understand partial traces.

## References

[RS80] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis.