Flows, coalescence and noise

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Summary. We are interested in stationary "fluid" random evolutions with independent increments. Under some mild assumptions, we show they are solutions of a stochastic differential equation (SDE). There are situations where these evolutions are not described by flows of diffeomorphisms, but by coalescing flows or by flows of probability kernels.

In an intermediate phase, for which there exists a coalescing flow and a flow of kernels solution of the SDE, a classification is given : All solutions of the SDE can be obtained by filtering a coalescing motion with respect to a sub-noise containing the Gaussian part of its noise. Thus, the coalescing motion cannot be described by a white noise.

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Contents

Introduction.

A stationary motion on the real line with independent increments is described by a Levy process, or equivalently by a convolution semigroup of probability measures. This naturally extends to "rigid" motions represented by Levy processes on Lie groups. If one assumes the continuity of the paths, a convolution semigroup on a Lie group G is determined by an element of the Lie algebra $\mathfrak{g}$ (the drift) and a scalar product on $\mathfrak{g}$ (the diffusion matrix) (see for example [21]). We call them the local characteristics of the convolution semigroup.

We will be interested in stationary "fluid" random evolutions which have independent increments. Strong solutions of Lipschitz stochastic differential equations (SDEs) define such evolutions. Those are of a regular type, namely

(a) The probability that two points thrown in the fluid at the same time and at distance ε separate at distance one in one unit of time tends to 0 as ε tends to 0.

(b) Such points will never hit each other.

Their laws can be viewed as convolution semigroups on the group of diffeomorphisms. The local characteristics are given by a drift vector field and a covariance function, which determine the SDE. But it can be seen that covariance functions which are not smooth on the diagonal (e.g. covariance associated with Sobolev norms of order between $d/2$ and $(d+2)/2$, d being the dimension of the space) can produce strong solutions, which define random evolutions of different type :

- diffusive (or turbulent) evolutions where **(a)** is not satisfied, which means that two points thrown initially at the same place separate,

- coalescing evolutions where **(b)** does not hold.

Among diffusive evolutions, we can distinguish the intermediate ones where two points thrown in the fluid at the same place separate but can meet after, i.e. where **(a)** and **(b)** are both not satisfied. Regular or coalescing evolutions are represented by flows of maps. Turbulent evolutions by flows of probability kernels obtained by dividing infinitely the initial point.

In the intermediate phase, we will see that the evolution can be modified in order to get a coalescing motion, which solves the SDE on an extended probability space. The associated noise, in Tsirelson sense (see [30]), is not linearizable, i.e. cannot be generated by a white noise.

A complete classification of the solutions can be given : They are obtained by filtering a coalescing motion defined on an extended probability space with respect to a sub-noise containing the Gaussian part of its noise.

The original purpose of this work was actually to get a better understanding of coalescing solutions of SDEs. In a previous work [18] we have shown that, given a Brownian motion W on vector fields, a strong solution $(S_{s,t}(W),\ s \leq t)$ of the SDE driven by W can be defined as a stochastic flow of kernels under very general circumstances. All the possible behaviours were shown to hold in examples of special interest, namely isotropic stationary Brownian vector fields associated with Sobolev norms.

The intermediate phase, where the diagonal can be hit and left by the two-point process, also occurs. It has been shown in [8] (for gradient fields) and (at a physical level) in [9, 10, 12] that in such cases, a coalescing solution of the SDE can be defined in law, i.e. in the sense of the martingale problems for the n-point motion. We present a construction of a coalescing flow in the intermediate phase. This flow obviously differs from $(S_{s,t},\ s \leq t)$ and corresponds to an absorbing boundary condition on the diagonal for the two-point motion.

This flow generates a vector field valued white noise W and we can identify $(S_{s,t},\ s \leq t)$ as the coalescing flow $(\varphi_{s,t},\ s \leq t)$ filtered by $\sigma(W)$.

Let us explain in more details the contents of the paper. We give in section 1 a construction result of a stochastic flow of kernels $(K_{s,t},\ s \leq t)$ associated with a general compatible family $(\mathsf{P}_t^{(n)},\ n \geq 1)$ of Feller semigroups, which represents the motion of n points thrown in the fluid. The two notions are shown to be equivalent. This is related to a recent result of Ma and Xiang [19] where an associated measure valued process was constructed in a special case (the flow can actually be viewed as giving the genealogy of this process, i.e. as its "historical process") and to a result of Darling [8]. Note however that Darling did not get flows of measurable maps except in very special cases.

In section 2, coalescing flows are constructed and briefly studied. They can be obtained from any flow whose two-point motion hits the diagonal. Then the original flow is shown to be recovered by filtering.

In section 3 we restrict our attention to diffusions generators. We define the vector field valued white noise W associated with the stochastic flow of kernels $(K_{s,t},\ s \leq t)$ and prove that the flow solves the SDE driven by the white noise W.

In section 4, under some off diagonal uniqueness assumption for the law of the n-point motion, we show there is only one strong solution. In the intermediate phase described above, the classification of other solutions by filtering of the coalescing solution is established. Then we identify the linear part of the noise generated by these solutions to the noise generated by W.

The examples related to our previous work (see [18]) are presented in section 5, with an emphasis on the verification of the Feller property for the semigroups $\mathsf{P}_t^{(n)}$, the classification of the solutions and the appearance of predictable noises which cannot be generated by white noises.

1 Stochastic flow of kernels, Feller convolution semigroup and compatible family of Feller semigroups.

1.1 Presentation of the results.

Let M be a separable compact metric space and d a distance on M. We denote by $\mathcal{P}(M)$ the space of probability measures on M, equipped with the weak convergence topology. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions dense in $\{f \in C(M),\ \|f\|_\infty \le 1\}$. We will equip $\mathcal{P}(M)$ with the distance $\rho(\mu,\nu) = (\sum_n 2^{-n}(\int f_n\, d\mu - \int f_n\, d\nu)^2)^{1/2}$ for all μ and ν in $\mathcal{P}(M)$. Thus $\mathcal{P}(M)$ is a separable compact metric space.

Definition 1.1.1 *Let $(\mathsf{P}_t^{(n)},\ n \ge 1)$ be a family of Feller semigroups* [1]*, respectively defined on M^n and acting on $C(M^n)$. We say that this family is compatible as soon as for all $k \le n$,*

$$\mathsf{P}_t^{(k)} f(x_1,\ldots,x_k) = \mathsf{P}_t^{(n)} g(y_1,\ldots,y_n) \tag{1.1}$$

where f and g are any continuous functions such that

$$g(y_1,\ldots,y_n) = f(y_{i_1},\ldots,y_{i_k}) \tag{1.2}$$

with $\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$ and $(x_1,\ldots,x_k) = (y_{i_1},\ldots,y_{i_k})$.

We will denote by $\mathsf{P}^{(n)}_{(x_1,\ldots,x_n)}$ the law of the Markov process associated with $\mathsf{P}_t^{(n)}$ starting from $(x_1,\ldots,x_n)$. This Markov process will be called the n-point motion of this family of semigroups. It is defined on the set of càdlàg paths on M^n which is a Polish space (see [20]).

Let us recall that a kernel on M is a measurable mapping from M into $\mathcal{P}(M)$, M and $\mathcal{P}(M)$ being equipped with their Borel σ-fields. For all kernel K, $f \in C(M)$ and $x \in M$, $Kf(x)$ denotes $\int f(y)\, K(x,dy)$. We denote by E the space of all kernels on M and we equip E with the σ-field generated by the mappings $K \mapsto Kf(x)$, for all $f \in C(M)$ and $x \in M$. We denote this σ-field by $\mathcal{E}$.

[1] $\mathsf{P}_t^{(n)}$ is a Feller semigroup on M^n if and only if $\mathsf{P}_t^{(n)}$ is positive (i.e. $\mathsf{P}_t^{(n)} f \ge 0$ for all $f \ge 0$), $\mathsf{P}_t^{(n)} 1 = 1$ and for all continuous function f, $\lim_{t\to 0} \mathsf{P}_t^{(n)} f(x) = f(x)$ which implies the uniform convergence of $\mathsf{P}_t^{(n)} f$ towards f (see theorem 9.4 in chapter I of [6]).

Let E^m denote the space of measurable mappings on M. Note that the inclusion map of E^m in E is measurable with respect to $\mathcal{E}^m = \{A \cap E^m,\ A \in \mathcal{E}\}$, though E^m is not in $\mathcal{E}$.

Definition 1.1.2 *A family $(\nu_t)_{t\geq 0}$ of probability measures on $(E, \mathcal{E})$ is a convolution semigroup if for all nonnegative s and t, on the probability space $(E^2, \mathcal{E}^{\otimes 2}, \nu_s \otimes \nu_t)$, there exists a $(E, \mathcal{E})$-valued random variable K of law ν_{s+t} such that for all $x \in M$,*

$$K(x) = K_1 K_2(x) \qquad \nu_s \otimes \nu_t(dK_1, dK_2) - a.s., \tag{1.3}$$

where $K_1K_2(x) = \int K_2(y) K_1(x, dy)$. (Note that in general, $(K_1, K_2) \mapsto K_1K_2$ is not measurable.)

We say that a convolution semigroup $(\nu_t)_{t\geq 0}$ is a Feller convolution semigroup as soon as

(i) $\forall f \in C(M)$, $\lim_{t\to 0} \sup_{x\in M} \int (Kf(x) - f(x))^2 \nu_t(dK) = 0.$

(ii) $\forall f \in C(M)$, $\forall t \geq 0$, $\lim_{d(x,y)\to 0} \int (Kf(x) - Kf(y))^2 \nu_t(dK) = 0.$

Definition 1.1.3 *Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space. Then a family of $(E, \mathcal{E})$-valued random variables $(K_{s,t},\ s \leq t)$ is a stochastic flow of kernels if and only if*

(a) *For all $s < u < t$, for all $x \in M$, P-almost surely, for all continuous function f, $K_{s,t}f(x) = K_{s,u}(K_{u,t}f)(x)$. (cocycle property).*

(b) *For all $s \leq t$, the law of $K_{s,t}$ only depends of $t - s$. (Stationarity)*

(c) *The flow has independent increments, i.e. for all $t_1 < t_2 < \cdots < t_n$, the family $\{K_{t_i,t_{i+1}},\ 1 \leq i \leq n-1\}$ is independent.*

(d) *For all continuous function f,*

$$\lim_{(u,v)\to(s,t)} \sup_{x\in M} \mathsf{E}[(K_{s,t}f(x) - K_{u,v}f(x))^2] = 0. \tag{1.4}$$

(e) *For all $s < t$, for all continuous function f,*

$$\lim_{d(x,y)\to 0} \mathsf{E}[(K_{s,t}f(x) - K_{s,t}f(y))^2] = 0. \tag{1.5}$$

Let $(\Omega^0, \mathcal{A}^0)$ (respectively $(\Omega^{0,m}, \mathcal{A}^{0,m})$) denote the measurable space $(\prod_{s\leq t} E, \otimes_{s\leq t}\mathcal{E})$ (respectively $(\prod_{s\leq t} E^m, \otimes_{s\leq t}\mathcal{E}^m)$). For $s \leq t$, let $K_{s,t}$ denote the random variable $\omega \mapsto \omega(s,t)$. Let also K (respectively φ) be the random variable $(K_{s,t},\ s \leq t)$ (respectively $(\varphi_{s,t},\ s \leq t)$). Then $K(\omega) = \omega$ (respectively $\varphi(\omega) = \omega$).

We say that a probability measure on $(E, \mathcal{E})$ (or on $(\Omega^0, \mathcal{A}^0)$) is carried by E^m (or by $\Omega^{0,m}$) if and only if it is the image of a probability measure on E^m (or on $\Omega^{0,m}$) by the inclusion map. We will use the same notation.

Let $(T_h)_{h\in\mathbb{R}}$ be the one-parametric group of transformations of Ω^0 (and of $\Omega^{0,m}$) defined by $T_h(\omega)(s,t) = \omega(s+h, t+h)$, for all $s \leq t$, $h \in \mathbb{R}$ and ω.

Theorem 1.1.4 **1-** *For all compatible family* $(\mathsf{P}_t^{(n)},\ n \geq 1)$ *of Feller semigroups on* M*, there exists a unique Feller convolution semigroup* $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$ *such that for all* $n \geq 1$*,* $t \geq 0$*,* $f \in C(M^n)$ *and* $x \in M^n$*,*

$$\mathsf{P}_t^{(n)} f(x) = \int K^{\otimes n} f(x)\ \nu_t(dK). \tag{1.6}$$

Conversely, for all Feller convolution semigroup $(\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$*, equation (1.6) defines a compatible family of Feller semigroups on* M*.*

2- *For all Feller convolution semigroup* $\nu = (\nu_t)_{t\geq 0}$ *on* $(E, \mathcal{E})$*, there exists a unique* $(T_h)_{h\in\mathbb{R}}$*-invariant probability measure* P_ν *on* $(\Omega^0, \mathcal{A}^0)$ *such that the family of random variables* $(K_{s,t},\ s \leq t)$ *is a stochastic flow of kernels and for all* $s \leq t$*, the law of* $K_{s,t}$ *is* ν_{t-s}*.*

This flow of kernels is called the canonical stochastic flow of kernels associated with ν *(or equivalently with* $(\mathsf{P}_t^{(n)},\ n \geq 1)$*).*

Conversely every stochastic flow of kernels (defined on some probability space $(\Omega, \mathcal{A}, \mathsf{P})$*) defines a canonical stochastic flow of kernels associated with a (unique) compatible family of Feller semigroups on* M *and a Feller convolution semigroup on* $(E, \mathcal{E})$*.*

Remark 1.1.5 *We will say that a stochastic flow of kernels* $K = (K_{s,t},\ s \leq t)$ *(defined on some probability space* $(\Omega, \mathcal{A}, \mathsf{P})$*) is a stochastic flow of mappings if and only if the law of* K *is carried by* $\Omega^{0,m}$*, i.e. if there exists a family of* $(E^m, \mathcal{E}^m)$*-valued random variables* $(\varphi_{s,t},\ s \leq t)$ *such that for all* $x \in M$*,* P*-almost surely,* $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$*. Then this family of random mappings satisfies*

(a) *For all* $s < u < t$, *for all* $x \in M$, P*-almost surely,* $\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x)$. *(cocycle property).*

(b) *For all* $s \le t$, *the law of* $\varphi_{s,t}$ *only depends of* $t - s$. *(Stationarity)*

(c) *The flow has independent increments, i.e. for all* $t_1 < t_2 < \cdots < t_n$, *the family* $\{\varphi_{t_i,t_{i+1}},\ 1 \le i \le n-1\}$ *is independent.*

(d) $\lim_{(u,v)\to(s,t)} \sup_{x\in M} \mathsf{E}[(d(\varphi_{s,t}(x), \varphi_{u,v}(x)))^2] = 0$.

(e) *For all* $s \le t$, $\lim_{d(x,y)\to 0} \mathsf{E}[(d(\varphi_{s,t}(x), \varphi_{s,t}(y)))^2] = 0$.

Note that every stochastic flow of measurable mappings defines a compatible family of Feller semigroups and a Feller convolution semigroup.

If $(\mathsf{P}_t^{(n)},\ n \ge 1)$ *denotes the associated compatible family of Feller semigroups, then for all* $x_1, \ldots, x_n$ *in* M, *the law of* $X_t^{(n)} = (\varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_n))$ *is* $\mathsf{P}_t^{(n)}(x_1, \ldots, x_n)$. *It follows that the law of* $(X_t^{(n)},\ t \ge 0)$ *is* $\mathsf{P}^{(n)}_{(x_1,\ldots,x_n)}$.

Theorem 1.1.6 *Let* $(\mathsf{P}_t^{(n)},\ n \ge 1)$ *be a compatible family of Feller semigroups on* M, *then the associated canonical stochastic flow of kernels is a stochastic flow of mappings if and only if for all* $f \in C(M)$, $x \in M$ *and* $t \ge 0$,

$$\mathsf{P}_t^{(2)} f^{\otimes 2}(x,x) = \mathsf{P}_t f^2(x). \tag{1.7}$$

Remark 1.1.7 *The theorems of this section are also satisfied when* M *is a locally compact separable metric space. In this case,* $(\mathsf{P}_t^{(n)},\ n \ge 1)$ *is a compatible family of Markovian semigroups acting continuously on* $C_0(M^n)$, *the set of continuous functions on* M^n *converging towards 0 at* ∞ *(we call them Feller semigroups). In the previous definitions (1.1.2 and 1.1.3) and in the statement of the theorems the function* f *has to be taken in* $C_0(M)$ *or in* $C_0(M^n)$. *Moreover* **(ii)** *of definition 1.1.2 must be modified by: for all* $x \in M$, $f \in C_0(M)$ *and* $t \ge 0$,

$$\begin{cases} \lim_{y\to x} \int (Kf(y) - Kf(x))^2\ \nu_t(dK) &= 0 \\ \textit{and} \qquad \lim_{y\to\infty} \int (Kf(y))^2\ \nu_t(dK) &= 0. \end{cases} \tag{1.8}$$

In definition 1.1.3, **(e)** *must be modified by: for all* $x \in M$, $f \in C_0(M)$ *and* $s \le t$,

$$\begin{cases} \lim_{y\to x} \mathsf{E}[(K_{s,t}f(x) - K_{s,t}f(y))^2] &= 0 \\ \textit{and} \qquad \lim_{y\to\infty} \mathsf{E}[(K_{s,t}f(y))^2] &= 0. \end{cases} \tag{1.9}$$

In remark 1.1.5, **(e)** *must be modified by: for all $x \in M$ and $s \leq t$,*

$$\begin{cases} \lim_{y\to x} \mathsf{E}[(d(\varphi_{s,t}(x),\varphi_{s,t}(y)))^2] & = & 0 \\ \textit{and} \qquad \lim_{y\to\infty} \mathsf{E}[(d(\varphi_{s,t}(y),K^c))^2] & = & 0 \end{cases} \tag{1.10}$$

for all compact K.

Proof. In order to prove this remark, note that the one-point compactification of M, $\hat{M} = M \cup \{\infty\}$, is a separable compact metric space. On $\hat{M}$, we define the compatible family of Feller semigroups, $(\hat{\mathsf{P}}_t^{(n)},\ n \geq 1)$, by the following relations,

for all $n \geq 2$ and all family of continuous functions on $\hat{M}$, $\{f_i,\ i \geq 1\}$,

$$\begin{aligned} \hat{\mathsf{P}}_t^{(n)} f_1 \otimes \cdots \otimes f_n &= \mathsf{P}_t^{(n)} g_1 \otimes \cdots \otimes g_n \\ &+ \sum_{i=1}^n f_i(\infty) \hat{\mathsf{P}}_t^{(n-1)} f_1 \otimes \cdots \otimes f_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n \end{aligned} \tag{1.11}$$

and

$$\hat{\mathsf{P}}_t^{(1)} f_1 = f_1(\infty) + \mathsf{P}_t^{(1)} g_1, \tag{1.12}$$

where $g_i = f_i - f_i(\infty) \in C_0(M)$ and with the convention $\mathsf{P}_t^{(n)} g_1 \otimes \cdots \otimes g_n(x_1,\ldots,x_n) = 0$ if there exists i such that $x_i = \infty$. We apply theorem 1.1.4 to $\hat{M}$ and to the family $(\hat{\mathsf{P}}_t^{(n)},\ n \geq 1)$ to construct Feller convolution semigroup $\hat{\nu}$ and a stochastic flow of kernels $(\hat{K}_{s,t},\ s \leq t)$ on $\hat{M}$. This stochastic flow of kernels satisfies

(i) $\hat{K}_{s,t}(\infty) = \delta_\infty$ for all $s \leq t$ and

(ii) $\hat{K}_{s,t}(x)(\infty) = 0$ for all $x \in M$ and $s \leq t$.

Proof of (i). For all $f \in C(\hat{M})$,

$$\begin{aligned} \mathsf{E}[(\hat{K}_{s,t}f(\infty) - f(\infty))^2] &= \hat{\mathsf{P}}_{t-s}^{(2)} f^{\otimes 2}(\infty,\infty) - 2f(\infty)\hat{\mathsf{P}}_{t-s}^{(1)} f(\infty) + f(\infty)^2 \\ &= 0 \end{aligned}$$

since $\hat{\mathsf{P}}_{t-s}^{(2)} f^{\otimes 2}(\infty,\infty) = f(\infty)^2$ and $\hat{\mathsf{P}}_{t-s}^{(1)} f(\infty) = f(\infty)$. This implies **(i)**. □

Proof of (ii). Let g_n be a sequence in $C_0(M)$ such that $g_n \in [0,1]$ and simply converging towards 1. Then $f_n = 1 - g_n \in C(\hat{M})$ is such that $f_n(\infty) = 0$ and

$$\mathsf{E}[(\hat{K}_{s,t}f_n(x))^2] = \hat{\mathsf{P}}_{t-s}^{(2)} g_n^{\otimes 2}(x,x) + 1 - 2\hat{\mathsf{P}}_{t-s}^{(1)} g_n(x).$$

This implies that

$$\lim_{n\to\infty} \mathsf{E}[(\hat{K}_{s,t}f_n(x))^2] = 0.$$

Assertion **(ii)** follows since $\hat{K}_{s,t}(x)(\infty) = \lim_{n\to\infty} \hat{K}_{s,t}f_n(x)$. □

For all $x \in M$, let us denote $\hat{K}_{s,t}(x)$ by $K_{s,t}(x)$. Assertions **(i)** and **(ii)** implies that $K_{s,t}$ is a kernel on M and that $(K_{s,t},\ s \le t)$ is a stochastic flow of kernels on M. In a similar way, one can show that $\hat{\nu}$ induces a Feller convolution semigroup on $(E, \mathcal{E})$. Note that the converse statements are easy to prove. It is also easy to see that theorem 1.1.6 holds. □

On a first reading, we advise to skip the long proof of theorem 1.1.4 and go directly to section 1.5.

1.2 Proof of the first part of theorem 1.1.4.

Let us explain briefly the method we employ to prove theorem 1.1.4. We first suppose we are given a compatible family of Feller semigroups. We begin by defining a convolution semigroup $(\mathsf{Q}_t,\ t \ge 0)$ on measurable mappings on $\mathcal{P}(M)$. For all t, to define Q_t, we first define the law of $(\hat{K}(\mu_i),\ i \in \mathbb{N})$, where the law of $\hat{K}$ is Q_t, for some dense family $(\mu_i,\ i \in \mathbb{N})$ in $\mathcal{P}(M)$ and get Q_t by an approximation. This convolution semigroup induces the Feller convolution semigroup $(\nu_t)_{t\ge 0}$ on $(E, \mathcal{E})$.

The approximation used to construct this convolution semigroup allows us to define a stochastic flow of mappings on $\mathcal{P}(M)$ in such a way that these mappings are measurable, defining it first on the dyadic numbers. Note that a difficulty to get this measurability comes from the fact that the composition of mappings from $\mathcal{P}(M)$ onto $\mathcal{P}(M)$ is not measurable with respect to the natural σ-field. This stochastic flow of measurable mappings on $\mathcal{P}(M)$ induces a stochastic flow of kernels on M.

In the following we assume we are given a compatible family of Feller semigroups, $(\mathsf{P}_t^{(n)},\ n \ge 1)$. And we intend to construct a Feller convolution semigroup $(\nu_t)_{t\ge 0}$ on $(E, \mathcal{E})$ satisfying (1.6). Note that the uniqueness of such a convolution semigroup is immediate since (1.6) characterizes ν_t.

1.2.1 A measurable choice of limit points in $\mathcal{P}(M)$.

It is known that, as a separable compact metric space, $\mathcal{P}(M)$ is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}}$ (see corollaire 1 §6.1 of chapter 9 in [7]).

A probability y can be represented by a sequence $(y^n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$. Let $y = (y_i)_{i\in\mathbb{N}}$ be a sequence of elements of $\mathcal{P}(M)$.

Let $y^1 = \limsup_{i\to\infty} y_i^1$. Let $i_k^1 = \inf\{i,\ |y^1 - y_i^1| < 1/k\}$. By induction, for all integer j, we construct y^j and $\{i_k^j,\ k\in\mathbb{N}\}$ by the relations

$$y^j = \limsup_{k\to\infty} y^j_{i_k^{j-1}} \quad \text{and} \quad i_k^j = \inf\{i \in \{i_k^{j-1},\ k\in\mathbb{N}\},\ |y^j - y_i^j| < 1/k\}.$$

We denote $(y^n)_{n\in\mathbb{N}}$ by $l(y)$. Note that $l(y)^j = \lim_{n\to\infty} y^j_{i_n^n}$. Hence $l(y)$ belongs to $\mathcal{P}(M)$. It is easy to see that l satisfies the following lemma.

Lemma 1.2.1 *$l : \mathcal{P}(M)^{\mathbb{N}} \to \mathcal{P}(M)$ is a measurable mapping, $\mathcal{P}(M)$ being equipped with the Borel σ-field $\mathcal{B}(\mathcal{P}(M))$ and $\mathcal{P}(M)^{\mathbb{N}}$ with the product σ-field $\mathcal{B}(\mathcal{P}(M))^{\otimes\mathbb{N}}$. Moreover $l((y_i)_{i\in\mathbb{N}}) = y_\infty$ when y_i converges towards y_∞.*

1.2.2 Notations and definitions.

Let Γ denote the space of measurable functions $\hat{K} : \mathcal{P}(M) \to \mathcal{P}(M)$ and $\mathcal{G}$ the σ-field generated by the sets of the form $\{\hat{K} \in \Gamma,\ \hat{K}(\mu) \in A\}$ where $\mu \in \mathcal{P}(M)$ and $A \in \mathcal{B}(\mathcal{P}(M))$.

Let $\{\mu_l,\ l\in\mathbb{N}\}$ be a dense family in $\mathcal{P}(M)$, which will be fixed in the following. We wish to define a measurable mapping $i\ :\ \mathcal{P}(M)^{\mathbb{N}} \to \Gamma$ such that $i((y_j)_{j\in\mathbb{N}})(\mu_l) = y_l$ for all integer l.

Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a positive sequence decreasing towards 0 (this sequence will be fixed later). Let $i\ :\ \mathcal{P}(M)^{\mathbb{N}} \to \Gamma$ be the injective mapping defined by

$$i(y)(\mu) = l((y_{n_k^\mu})_{k\in\mathbb{N}}) \tag{1.13}$$

where

$$n_k^\mu = \inf\{n,\ \rho(\mu_n, \mu) \le \varepsilon_k\}, \tag{1.14}$$

for $(y,\mu) \in \mathcal{P}(M)^{\mathbb{N}} \times \mathcal{P}(M)$. Note that $i(y)$ defined this way is a measurable mapping since l is measurable and $\mu \mapsto (y_{n_k^\mu})_{k\in\mathbb{N}}$ is measurable. Note also that the relation $i(y)(\mu_l) = y_l$ is satisfied for all integer l.

Lemma 1.2.2 *For $n \ge 1$, the mapping $\Psi_n : \mathcal{P}(M) \times (\mathcal{P}(M)^{\mathbb{N}})^n \to \mathcal{P}(M)$, defined by*

$$\Psi_n(\mu, y^1, \ldots, y^n) = i(y^n) \circ i(y^{n-1}) \circ \cdots \circ i(y^1)(\mu), \tag{1.15}$$

is measurable. ($\mathcal{P}(M) \times (\mathcal{P}(M)^{\mathbb{N}})^n$ is equipped with the product σ-field.)

Proof. Note that Ψ_1 is the composition of the mappings l and $(\mu, y) \mapsto (y_{n_k^\mu})_{k\in\mathbb{N}}$. Since these mappings are measurable, Ψ_1 is measurable.

By induction, we prove that Ψ_n is measurable since, for $n \geq 2$,

$$\Psi_n(\mu, y^1, \dots, y^n) = \Psi_1(\Psi_{n-1}(\mu, y^1, \dots, y^{n-1}), y^n). \qquad \square$$

Lemma 1.2.3 *For $n \geq 1$, the mapping $\Phi_n : (\mathcal{P}(M)^{\mathbb{N}})^n \to \Gamma$, defined by*

$$\Phi_n(y^1, \dots, y^n) = i(y^n) \circ i(y^{n-1}) \circ \dots \circ i(y^1), \tag{1.16}$$

is measurable. In particular, i is measurable.

Proof. Note that for all $A \in \mathcal{P}(M)$,

$$\Phi_n^{-1}(\{\hat{K} \in \Gamma,\ \hat{K}(\mu) \in A\}) = \{y \in (\mathcal{P}(M)^{\mathbb{N}})^n,\ (\mu, y) \in \Psi_n^{-1}(A)\}.$$

This event belongs to $(\mathcal{B}(\mathcal{P}(\mathcal{M}))^{\otimes\mathbb{N}})^{\otimes n}$ since Ψ_n is measurable. This shows the measurability of Φ_n. $\square$

We need to introduce the functions Φ_n and Ψ_n because the composition application $\Gamma^n \to \Gamma$, $(\hat{K}_1, \dots, \hat{K}_n) \mapsto \hat{K}_n \circ \dots \circ \hat{K}_1$ is not $\mathcal{G}^{\otimes n}$-measurable in general.

Let $j : \Gamma \to \mathcal{P}(M)^{\mathbb{N}}$ be the mapping defined by

$$j(\hat{K}) = (\hat{K}(\mu_l))_{l\in\mathbb{N}}. \tag{1.17}$$

Lemma 1.2.4 *The mapping j is measurable and satisfies $j \circ i(y) = y$ for all $y \in \mathcal{P}(M)^{\mathbb{N}}$.*

Proof. We have for all $A \in \mathcal{P}(M)^{\otimes n}$,

$$j^{-1}(\{y \in \mathcal{P}(M)^{\mathbb{N}},\ (y_1, \dots, y_n) \in A\}) = \{\hat{K} \in \Gamma,\ (\hat{K}(\mu_1), \dots, \hat{K}(\mu_n)) \in A\}.$$

This set belongs to $\mathcal{G}$. $\square$

Note that for all $l \in \mathbb{N}$ and $\hat{K} \in \Gamma$, $i \circ j(\hat{K})(\mu_l) = \hat{K}(\mu_l)$.

1.2.3 Constructions of probabilities on $\mathcal{P}(M)^{\mathbb{N}}$ and on Γ.

The method we employ to construct the probability Q_t on $(\Gamma, \mathcal{G})$ consists in constructing a probability $\Pi_t^{(\infty)}$ on $(\mathcal{P}(M)^{\mathbb{N}}, \mathcal{B}(\mathcal{P}(M))^{\otimes\mathbb{N}})$ such that $\Pi_t^{(\infty)}$ is the law of $(\hat{K}(\mu_l),\ l \in \mathbb{N})$, when the law of $\hat{K}$ is Q_t, and then in defining Q_t using the mapping i.

In order to construct $\Pi_t^{(\infty)}$, we first construct for all integer k a Feller semigroup $\Pi_t^{(k)}$ acting on the continuous functions on $\mathcal{P}(M)^k$ (see Ma-Xiang [19] for a similar construction when $k = 1$).

Let $\mathcal{A}_k$ denote the algebra of functions $g : \mathcal{P}(M)^k \to \mathbb{R}$ such that [2]

$$g(\mu_1, \ldots, \mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle \tag{1.18}$$

for $f \in C(M^n)$ and $n_1, \ldots, n_k$ integers such that $n = n_1 + \cdots + n_k$ ($\mathcal{A}_k$ is the union of an increasing family of algebras $\mathcal{A}_{n_1,\ldots,n_k}$). For all $g \in \mathcal{A}_k$, given by equation (1.18), let

$$\Pi_t^{(k)} g(\mu) = \langle \mathsf{P}_t^{(n)} f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{\otimes n_k} \rangle. \tag{1.19}$$

with $\mu = (\mu_1, \ldots, \mu_k) \in \mathcal{P}(M)^k$. Note that since the family of semigroups $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is compatible, (1.19) is independent of the expression of g in (1.18).

Let us notice that, by the theorem of Stone-Weierstrass, the algebra $\mathcal{A}_k$ is dense in $C(\mathcal{P}(M)^k)$ and that $\Pi_t^{(k)}$ acts on $\mathcal{A}_k$.

Lemma 1.2.5 $\Pi_t^{(k)}$ *is a Markovian operator acting on* $\mathcal{A}_k$.

Proof. The only thing to be proved is the positivity property (it is obvious that $\Pi_t^{(k)} 1 = 1$).

For all integer N, let $((X^{1,i}, \ldots, X^{N,i}),\ 1 \leq i \leq k)$ be a family of independent Markov processes associated with the Markovian semigroup $\mathsf{P}_t^{(N)}$ such that the law of $(X_0^{1,i}, \ldots, X_0^{N,i})$ is $\mu_i^{\otimes N}$. Let us introduce the following Markov process on $\mathcal{P}(M)^k$, $\mu_t^N = (\mu_t^{N,1}, \ldots, \mu_t^{N,k})$ where

$$\mu_t^{N,i} = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^{j,i}}, \quad \text{for } 1 \leq i \leq k. \tag{1.20}$$

[2]Here and in the following, for all measure μ and $f \in L^1(\mu)$, we denote $\int f\, d\mu$ by $\langle f, \mu \rangle$, $\langle \mu, f \rangle$ or μf.

For $g(\mu_1,\ldots,\mu_k) = \langle f, \mu_1^{\otimes n_1} \otimes \cdots \otimes \mu_k^{n_k}\rangle$, a simple computation (when $k=1$, see (3.2) in [19]) gives

$$\mathsf{E}[g(\mu_t^N)] = \mathsf{E}[\langle \mathsf{P}_t^{(n)} f, (\mu_0^{N,1})^{\otimes n_1} \otimes \cdots \otimes (\mu_0^{N,k})^{n_k}\rangle]. \tag{1.21}$$

The law of large numbers implies that $\mu_0^{N,i}$ converges weakly towards μ_i. Therefore we get

$$\Pi_t^{(k)} g(\mu_1,\ldots,\mu_k) = \lim_{N\to\infty} E[g(\mu_t^{N,k})]. \tag{1.22}$$

This shows that $\Pi_t^{(k)}$ is positive. □

Using this lemma, it is easy to define $\Pi_t^{(k)} g$ for all continuous function g and to show that $\Pi_t^{(k)}$ is a Markovian semigroup acting on $C(M^n)$.

Let us remark that since the semigroups $\mathsf{P}_t^{(n)}$ are Feller, the semigroups $\Pi_t^{(k)}$ are also Feller : for all g in $\mathcal{A}_k$, then $\Pi_t^{(k)} g$ is continuous and $\lim_{t\to 0} \Pi_t^{(k)} g = g$ and these properties extend to every continuous functions.

The family of semigroups $(\Pi_t^{(k)},\ k \geq 1)$ is compatible (in the sense given in section 1.1). Thus $(\Pi_t^{(k)},\ k \geq 1)$ is a compatible family of Feller semigroups on $\mathcal{P}(M)$. We will denote $\Pi_{(\mu,\nu)}^{(2)}$ the law of the Markov process associated with $\Pi_t^{(2)}$ starting from (μ,ν) and we will denote this process by (μ_t,ν_t).

By Kolmogorov's theorem, we construct on $\mathcal{P}(M)^{\mathbb{N}}$ a probability measure $\Pi_t^{(\infty)}$ such that $\Pi_t^{(\infty)}(A \times \mathcal{P}(M)^{\mathbb{N}}) = \Pi_t^{(n)} 1_A(\mu_1,\ldots,\mu_n)$, for any $A \in \mathcal{B}(\mathcal{P}(M))^{\otimes n}$. We now prove useful lemmas satisfied by $\Pi_t^{(\infty)}$:

Lemma 1.2.6 *For all positive T, there exists a positive function $\varepsilon_T(r)$ converging towards 0 as r goes to 0 such that*

$$\sup_{t\in[0,T]} \mathsf{E}_{(\mu,\nu)}^{(2)}[(\rho(\mu_t,\nu_t))^2] \leq \varepsilon_T(\rho(\mu,\nu)). \tag{1.23}$$

Proof. Note that for all continuous function f,

$$\mathsf{E}_{(\mu,\nu)}^{(2)}[(\mu_t f - \nu_t f)^2] = \Pi_t^{(2)} g^{\otimes 2}(\mu,\mu) + \Pi_t^{(2)} g^{\otimes 2}(\nu,\nu) - 2\Pi_t^{(2)} g^{\otimes 2}(\mu,\nu),$$

where $g(\mu) = \mu f = \int f\ d\mu$. We conclude the lemma after remarking that this function is uniformly continuous in (t,μ,ν) on $[0,T]\times\mathcal{P}(M)^2$. □

From now on we fix T and define the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ (which defines the sequence $(n_k^\mu)_{k\in\mathbb{N}}$ for all $\mu \in \mathcal{P}(M)$ by equation (1.14)) such that $0 \leq r \leq 2\varepsilon_k$ implies $\varepsilon_T(r) \leq 2^{-3k}$. The sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ is well defined since $\lim_{r\to 0} \varepsilon_T(r) = 0$.

Lemma 1.2.7 *For all $t \in [0,T]$ and for any independent random variables X and Y respectively in $\mathcal{P}(M)$ and $\mathcal{P}(M)^{\mathbb{N}}$, such that the law of Y is $\Pi_t^{(\infty)}$, then $Y_{n_k^X}$ converges almost surely towards $l((Y_{n_k^X})_{k\in\mathbb{N}}) = i(Y)(X)$ as k goes to ∞.*

Proof. Note that $(Y_{n_k^X})_{k\in\mathbb{N}}$ is a random variable (the mapping $(\mu, y) \mapsto (y_{n_k^\mu})_{k\in\mathbb{N}}$ is measurable). For all integer k,

$$\mathsf{P}[\rho(Y_{n_k^X}, Y_{n_{k+1}^X}) > 2^{-k}] \leq 2^{2k}\mathsf{E}[\varepsilon_T(\rho(\mu_{n_k^X}, \mu_{n_{k+1}^X}))] \leq 2^{-k}. \tag{1.24}$$

Using Borel-Cantelli's lemma, we prove that almost surely, $(Y_{n_k^X})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore converges. Its limit can only be $l((Y_{n_k^X})_{k\in\mathbb{N}})$. □

Lemma 1.2.8 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathcal{P}(M)$ converging in probability towards a random variable X and Y a random variable in $\mathcal{P}(M)^{\mathbb{N}}$ of law $\Pi_t^{(\infty)}$ independent of $(X_n)_{n\in\mathbb{N}}$, then $i(Y)(X_n) = l((Y_{n_k^{X_n}})_{k\in\mathbb{N}})$ converges in probability towards $i(Y)(X) = l((Y_{n_k^X})_{k\in\mathbb{N}})$ as n tends to ∞.*

Proof. Let $Z_n = l((Y_{n_k^{X_n}})_{k\in\mathbb{N}})$ and $Z = l((Y_{n_k^X})_{k\in\mathbb{N}})$. For all integer k, we have

$$\begin{aligned} \mathsf{P}[\rho(Z_n, Z) > \varepsilon] &\leq \mathsf{P}[\rho(Z_n, Y_{n_k^{X_n}}) > \varepsilon/3] + \mathsf{P}[\rho(Y_{n_k^{X_n}}, Y_{n_k^X}) > \varepsilon/3] \\ &+ \mathsf{P}[\rho(Y_{n_k^X}, Z) > \varepsilon/3]. \end{aligned}$$

Lemma 1.2.7 implies that the first and last terms of the right hand side of the preceding equation converge towards 0 as k goes to ∞. The second term is lower than $\frac{9}{\varepsilon^2}\mathsf{E}[\varepsilon_T(\rho(\mu_{n_k^{X_n}}, \mu_{n_k^X})) \wedge 8]$. Since for all positive α, there exists a positive η such that $|r| < \eta$ implies $\frac{9}{\varepsilon^2}|\varepsilon_T(r)| < \alpha$, we get

$$\begin{aligned} \mathsf{P}[\rho(Y_{n_k^{X_n}}, Y_{n_k^X}) > \varepsilon/3] &\leq \alpha + C\mathsf{P}[\rho(\mu_{n_k^{X_n}}, \mu_{n_k^X}) > \eta] \\ &\leq \alpha + C\mathsf{P}[\rho(X_n, X) > \eta - 2^{-(k-1)}], \end{aligned}$$

where $C = 72/\varepsilon^2$. Therefore, we get $\mathsf{P}[\rho(Z_n, Z) > \varepsilon] \leq \alpha + C\mathsf{P}[\rho(X_n, X) \geq \eta]$ and for all positive α, $\limsup_{n\to\infty} \mathsf{P}[\rho(Z_n, Z) > \varepsilon] \leq \alpha$. We therefore prove that Z_n converges in probability towards Z. □

For all $t \in [0,T]$, let $\mathsf{Q}_t = i^*(\Pi_t^{(\infty)})$. It is a probability measure on $(\Gamma, \mathcal{G})$ and it satisfies the following proposition.

Proposition 1.2.9 Q_t *is the unique probability measure on* $(\Gamma, \mathcal{G})$ *such that for any continuous function* f *on* $\mathcal{P}(M)^n$ *and any* $(\nu_1, \ldots, \nu_n) \in \mathcal{P}(M)^n$,

$$\int_\Gamma f(\hat{K}(\nu_1), \ldots, \hat{K}(\nu_n)) \mathsf{Q}_t(d\hat{K}) = \Pi_t^{(n)} f(\nu_1, \ldots, \nu_n). \tag{1.25}$$

Proof. The unicity is obvious since (1.25) characterizes Q_t. Let us check that $\mathsf{Q}_t = i^*(\Pi_t^{(\infty)})$ satisfies (1.25). Let Y be a random variable of law $\Pi_t^{(\infty)}$ then for all $f \in C(\mathcal{P}(M)^n)$ and all $(\nu_1, \ldots, \nu_n) \in \mathcal{P}(M)^n$,

$$\begin{aligned}
\int_\Gamma f(\hat{K}(\nu_1), \ldots, \hat{K}(\nu_n)) \mathsf{Q}_t(d\hat{K}) &= \mathsf{E}[f(i(Y)(\nu_1), \ldots, i(Y)(\nu_n))] \\
&= \lim_{k\to\infty} \mathsf{E}[f(Y_{n_k^{\nu_1}}, \ldots, Y_{n_k^{\nu_n}})] \\
&= \lim_{k\to\infty} \Pi_t^{(n)} f(\mu_{n_k^{\nu_1}}, \ldots, \mu_{n_k^{\nu_n}}) \\
&= \Pi_t^{(n)} f(\nu_1, \ldots, \nu_n),
\end{aligned}$$

using first dominated convergence theorem and lemma 1.2.7, then the definition of $\Pi_t^{(\infty)}$ and the fact that $\Pi_t^{(n)}$ is Feller. □

Remark 1.2.10 **(i)** *Then it is obvious that* $j^*(\mathsf{Q}_t) = \Pi_t^{(\infty)}$ *and that* $(i \circ j)^*(\mathsf{Q}_t) = i^*(\Pi_t^{(\infty)}) = \mathsf{Q}_t$.

(ii) *Note that, since* T *can be taken arbitrarily large, we can define* Q_t *for all positive* t *and the definition of* Q_t *is independent of the chosen* T, *since* Q_t *satisfies proposition 1.2.9.*

1.2.4 A convolution semigroup on $(\Gamma, \mathcal{G})$.

The following lemma will be very useful in the following.

Lemma 1.2.11 *For* $t \in [0, T]$, *let* $\hat{K}$ *and* X *be independent random variables respectively* Γ*-valued and* $\mathcal{P}(M)$*-valued such that the law of* $\hat{K}$ *is* Q_t, *then*

$$i \circ j(\hat{K})(X) = \hat{K}(X) \quad \textit{almost surely.} \tag{1.26}$$

Proof. Note that, since $\mathsf{Q}_t = i^*(\Pi_t^{(\infty)})$, if Y is a random variable of law $\Pi_t^{(\infty)}$,

$$\begin{aligned}
\mathsf{P}[\rho(\hat{K}(\mu_{n_k^X}), \hat{K}(X)) > 2^{-k}] &= \mathsf{P}[\rho(Y_{n_k^X}, i(Y)(X)) \geq 2^{-k}] \\
&= \lim_{l\to\infty} \mathsf{P}[\rho(Y_{n_k^X}, Y_{n_l^X}) \geq 2^{-k}] \leq 2^{-k}
\end{aligned}$$

(see equation (1.24)). Using Borel-Cantelli's lemma, we prove $\hat{K}(\mu_{n_k^X})$ converges almost surely towards $\hat{K}(X)$. This proves the lemma. □

Lemma 1.2.12 *For all $t_1,\dots,t_n$ in $[0,T]$,*

$$\Phi_n^*(\Pi_{t_1}^{(\infty)}\otimes\cdots\otimes\ \Pi_{t_n}^{(\infty)})=\mathsf{Q}_{t_1+\cdots+t_n}. \tag{1.27}$$

Proof. Let us prove that $\Phi_n^*(\Pi_{t_1}^{(\infty)}\otimes\cdots\otimes\Pi_{t_n}^{(\infty)})$ satisfies (1.25) for all $f\in C(\mathcal{P}(M)^k)$, all $\nu\in\mathcal{P}(M)^k$ and $t=t_1+\cdots+t_n$. To simplify we prove this for $k=1$. Let $f\in C(\mathcal{P}(M))$ and $\nu\in\mathcal{P}(M)$, then applying Fubini's theorem,

$$\begin{aligned}
&\int_\Gamma f(\hat{K}(\nu))\ \Phi_n^*(\Pi_{t_1}^{(\infty)}\otimes\cdots\otimes\ \Pi_{t_n}^{(\infty)})(d\hat{K})\\
&=\ \int f(i(y^n)\circ i(y^{n-1})\circ\cdots\circ i(y^1)(\nu))\ \Pi_{t_1}^{(\infty)}(dy^1)\otimes\cdots\otimes\Pi_{t_n}^{(\infty)}(dy^n)\\
&=\ \int \Pi_{t_n}^{(1)}f(i(y^{n-1})\circ\cdots\circ i(y^1)(\nu))\ \Pi_{t_1}^{(\infty)}(dy^1)\otimes\cdots\otimes\Pi_{t_{n-1}}^{(\infty)}(dy^{n-1})\\
&=\ \cdots\ =\ \Pi_{t_1+\cdots+t_n}^{(1)}f(\nu).
\end{aligned}$$

The proof is similar for $f\in C(\mathcal{P}(M))^k$ and $\nu\in\mathcal{P}(M)^k$. We conclude using proposition 1.2.9. □

Let $c_\Gamma=\Phi_2\circ j^{\otimes 2}$. Then c_Γ is a measurable mapping from $\Gamma\times\Gamma$ into Γ and for all s and t in $[0,T]$ (the definition of c_Γ depends on T), $c_\Gamma^*(\mathsf{Q}_s\otimes\mathsf{Q}_t)=\mathsf{Q}_{s+t}$.

Proposition 1.2.13 $(\mathsf{Q}_t)_{t\geq 0}$ *is a convolution semigroup on $(\Gamma,\mathcal{G})$, i.e. for all nonnegative s and t, on the probability space $(\Gamma^2,\mathcal{G}^{\otimes 2},\mathsf{Q}_s\otimes\mathsf{Q}_t)$, there exists a $(\Gamma,\mathcal{G})$-valued random variable $\hat{K}$ of law Q_{s+t} such that for all $\mu\in\mathcal{P}(M)$,*

$$\hat{K}(\mu)=\hat{K}_2\circ\hat{K}_1(\mu)\quad \mathsf{Q}_s\otimes\mathsf{Q}_t(d\hat{K}_1,d\hat{K}_2)-a.s. \tag{1.28}$$

(Note that in general, $(\hat{K}_1,\hat{K}_2)\mapsto\hat{K}_2\circ\hat{K}_1$ is not measurable.)

Proof. We fix $T=s\vee t$ to define c_Γ. We set $\hat{K}=c_\Gamma(\hat{K}_1,\hat{K}_2)$, then $\hat{K}$ is a $(\Gamma,\mathcal{G})$-valued random variable $\hat{K}$ of law Q_{s+t} and for all $\mu\in\mathcal{P}(M)$, $\mathsf{Q}_s\otimes\mathsf{Q}_t(d\hat{K}_1,d\hat{K}_2)$-almost surely,

$$c_\Gamma(\hat{K}_1,\hat{K}_2)(\mu)=(i\circ j)(\hat{K}_2)\circ(i\circ j)(\hat{K}_1)(\mu)=\hat{K}_2\circ\hat{K}_1(\mu) \tag{1.29}$$

from lemma 1.2.11. □

1.2.5 A convolution semigroup on $(E, \mathcal{E})$.

Let $\delta : (\Gamma, \mathcal{G}) \to (E, \mathcal{E})$ be the measurable mapping defined by $\delta(\hat{K})(x) = \hat{K}(\delta_x)$ (δ is measurable since the restriction to Dirac measures of a measurable function on $\mathcal{P}(M)$ induces a measurable function on M). For all $t \geq 0$, we set $\nu_t = \delta^*(\mathsf{Q}_t)$.

Lemma 1.2.14 *For all $t \geq 0$ and $\mu \in \mathcal{P}(M)$,*

$$\hat{K}(\mu) = \int \delta(\hat{K})(x)\ \mu(dx) \qquad \mathsf{Q}_t(d\hat{K}) - a.s. \tag{1.30}$$

Proof. Take $f \in C(M)$ and $\mu \in \mathcal{P}(M)$, then

$$\begin{aligned}\int \langle \hat{K}(\mu) - K\mu, f\rangle^2\ \mathsf{Q}_t(d\hat{K}) \ &= \ \Pi_t^{(2)} g^{\otimes 2}(\mu,\mu) - 2\int \Pi_t^{(2)} g^{\otimes 2}(\mu, \delta_x)\ \mu(dx) \\ &+ \ \int \Pi_t^{(2)} g^{\otimes 2}(\delta_x, \delta_y)\ \mu(dx)\mu(dy),\end{aligned}$$

with $g(\mu) = \int f\ d\mu$, $K = \delta(\hat{K})$ and $K\mu = \int K(x)\ \mu(dx)$. Since, for all μ and ν in $\mathcal{P}(M)$,

$$\Pi_t^{(2)} g^{\otimes 2}(\mu,\nu) = \int \mathsf{P}_t^{(2)} f^{\otimes 2}(x,y)\ \mu(dx)\nu(dy),$$

we get $\mathsf{E}[\langle \hat{K}_t(\mu) - \int K_t(x)\ \mu(dx), f\rangle^2] = 0$. This proves the lemma. □

Then one can easily check that the family $(\nu_t)_{t\geq 0}$ satisfies (1.6).

Lemma 1.2.15 *Let $(y_n)_{n\in\mathbb{N}} \in \mathcal{P}(M)^{\mathbb{N}}$ converging towards y in $\mathcal{P}(M)$. Then*

$$\int \rho(\hat{K}(y_n), \hat{K}(y))^2\ \mathsf{Q}_t(d\hat{K}) = \int \rho\left(\int K\ dy_n, \int K\ dy\right)^2\ \nu_t(dK) \tag{1.31}$$

converges towards 0 as n tends to ∞ and there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that as k tends to ∞,

$$\begin{cases} \hat{K}(y_{n_k}) \to \hat{K}(y) & \mathsf{Q}_t(d\hat{K}) - a.s. \\ \int K\ dy_{n_k} \to \int K dy & \nu_t(dK) - a.s. \end{cases} \tag{1.32}$$

Proof. The equality (1.31) is a consequence of lemma 1.2.14 and of the fact that $\nu_t = \delta^*(Q_t)$. Now, take $f \in C(M)$,

$$\begin{aligned}\int \langle \hat{K}(y_n) - \hat{K}(y), f\rangle^2 \ \mathsf{Q}_t(d\hat{K}) &= \Pi_t^{(2)} g^{\otimes 2}(y_n, y_n) + \Pi_t^{(2)} g^{\otimes 2}(y, y) \\ &\quad -2\Pi_t^{(2)} g^{\otimes 2}(y_n, y) \\ &= \langle \mathsf{P}_t^{(2)} f^{\otimes 2}, (y_n - y)^{\otimes 2}\rangle\end{aligned}$$

where $g(y) = \int f \ dy$. We conclude since $\mathsf{P}_t^{(2)}$ is a Feller semigroup. The last statement is just a classical application of Borel Cantelli's lemma. □

Let $U \ : \ (E, \mathcal{E}) \to (\Gamma, \mathcal{G})$ be the measurable mapping defined by

$$U(K) = i\left((\int K \ d\mu_l)_{l\in\mathbb{N}} \right). \tag{1.33}$$

Then for all $t \in [0, T]$, $U^*(\nu_t) = \mathsf{Q}_t$ (the definition of U depends on T).

Lemma 1.2.16 *For all $t \in [0, T]$ and $y \in \mathcal{P}(M)$,*

$$\int K \ dy = U(K)(y) \qquad \nu_t(dK) - a.s. \tag{1.34}$$

Proof. From the definition of U, we have for all integer l,

$$\int K \ d\mu_l = U(K)(\mu_l) \qquad \nu_t(dK) - \text{a.s.} \tag{1.35}$$

(recall that $(\mu_l)_{l\in\mathbb{N}}$ is the dense sequence in $\mathcal{P}(M)$ we used to define i).

For all $y \in \mathcal{P}(M)$, using lemma 1.2.15, there exists a sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ converging towards y such that

$$\begin{cases} \int K \ d\mu_{n_k} \to \int K dy & \nu_t(dK) - \text{a.s.} \\ U(K)(\mu_{n_k}) \to U(K)(y) & \nu_t(dK) - \text{a.s.} \end{cases} \tag{1.36}$$

Thus the lemma is proved. □

Proposition 1.2.17 *$(\nu_t)_{t\geq 0}$ is a Feller convolution semigroup on $(E, \mathcal{E})$.*

Proof. Take s and t in $[0,\infty[$ and $T = s \vee t$ to define i. Let $c_E = \delta \circ c_\Gamma \circ U^{\otimes 2}$, then $c_E : E \times E \to E$ is measurable and $c_E^*(\nu_s \otimes \nu_t) = \nu_{s+t}$. Note that for all $x \in M$,

$$c_E(K_1, K_2)(x) = \int K_2(y)\ K_1(x, dy) \qquad \nu_s \otimes \nu_t(dK_1, dK_2) - \text{a.s.} \tag{1.37}$$

since lemma 1.2.16 implies that $\nu_s \otimes \nu_t(dK_1, dK_2)$-almost surely

$$\begin{aligned} c_E(K_1, K_2)(x) = c_\Gamma(U(K_1), U(K_2))(\delta_x) &= U(K_2) \circ U(K_1)(\delta_x) \\ &= \int K_2(y)\ K_1(x, dy). \end{aligned}$$

This proves that $(\nu_t)_{t\geq 0}$ is a convolution semigroup. Let us now prove it is a Feller convolution semigroup. Properties **(i)** and **(ii)** in definition 1.1.2 are satisfied since for all $f \in C(M)$ and $(x, y) \in M^2$,

$$\begin{aligned} \int (Kf(x) - f(x))^2\ \nu_t(dK) &= \mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) - 2f(x)\mathsf{P}_t^{(1)} f(x) + f^2(x), \\ \int (Kf(x) - Kf(y))^2\ \nu_t(dK) &= \mathsf{P}_t^{(2)} f^{\otimes 2}(x, x) + \mathsf{P}_t^{(2)} f^{\otimes 2}(y, y) \\ &\quad -2\mathsf{P}_t^{(2)} f^{\otimes 2}(x, y) \end{aligned}$$

and we use the Feller property of the semigroups $\mathsf{P}_t^{(n)}$ for $n \in \{1, 2\}$. □

Thus we have proved the direct statement of the first part of theorem 1.1.4. The converse statement will be proved at the end of the next section.

1.3 Proof of the second part of theorem 1.1.4.

1.3.1 Construction of a probability space.

For all $n \in \mathbb{N}$, let $D_n = \{j2^{-n},\ j \in \mathbb{Z}\}$ and $D = \cup_{n\in\mathbb{N}} D_n$ the set of the dyadic numbers. To define the functions i and Φ_2, we let $T = 1$.

For all integer $n \geq 1$, let $(S_n, \mathcal{S}_n, \mathsf{P}_n)$ denote the probability space $(\mathcal{P}(M)^{\mathbb{N}}, \mathcal{B}(\mathcal{P}(M))^{\otimes\mathbb{N}}, \Pi_{2^{-n}}^{(\infty)})^{\otimes\mathbb{Z}}$. Let $\pi_{n-1,n} : S_n \to S_{n-1}, \quad \omega^n \mapsto \omega^{n-1}$, where

$$\omega^{n-1}_{\frac{i}{2^{n-1}}} = j \circ \Phi_2(\omega^n_{\frac{2i-1}{2^n}}, \omega^n_{\frac{2i}{2^n}}) = j(i(\omega^n_{\frac{2i}{2^n}}) \circ i(\omega^n_{\frac{2i-1}{2^n}})). \tag{1.38}$$

From lemma 1.2.12, $\pi^*_{n-1,n}(\mathsf{P}_n) = \mathsf{P}_{n-1}$.

Let $\Omega = \{(\omega^n)_{n\in\mathbb{N}} \in \prod S_n,\ \pi_{n-1,n}(\omega^n) = \omega^{n-1}\}$ and $\mathcal{A}$ be the σ-field on Ω generated by the mappings $\pi_n : \Omega \to S_n$, with $\pi_n((\omega^k)_{k\in\mathbb{N}}) = \omega^n$. Let P be the unique probability on $(\Omega, \mathcal{A})$ such that $\pi_n^*(\mathsf{P}) = \mathsf{P}_n$ (see theorem 3.2 in [24]).

For all dyadic numbers $s < t$, let $\mathcal{F}_{s,t}$ be the σ-field generated by the mappings $(\omega^k)_{k\in\mathbb{N}} \mapsto \omega_u^n$ for all $n \in \mathbb{N}$ and $u \in D_n \cap [s,t[$.

1.3.2 A stochastic flow of mappings on $\mathcal{P}(M)$.

Definition 1.3.1 *On $(\Omega, \mathcal{A}, \mathsf{P})$, we define the following random variables*

1. *For all $s < t \in D_n$, let $\hat{K}^n_{s,t}((\omega^k)_{k\in\mathbb{N}}) = \Phi_{(t-s)2^n}(\omega_s^n, \dots, \omega^n_{t-2^{-n}})$,*

2. *For all $s < t \in D$, let $\hat{K}_{s,t} = \hat{K}^n_{s,t}$ where $n = \inf\{k,\ (s,t) \in D_k^2\}$.*

Let us remark that for all $s < t \in D_n$, the law of $\hat{K}_{s,t}$ and of $\hat{K}^n_{s,t}$ is Q_{t-s} (this is a consequence of lemma 1.2.12). Note also that for all $s < u < t \in D_n$, we have $\hat{K}^n_{s,t} = \hat{K}^n_{u,t} \circ \hat{K}^n_{s,u}$.

Proposition 1.3.2 *For all $s < t \in D_n$ and all $\mathcal{P}(M)$-valued random variable X independent of $\mathcal{F}_{s,t}$,*

$$\hat{K}^n_{s,t}(X) = \hat{K}_{s,t}(X) \qquad \mathsf{P}\text{-almost surely.}$$

Proof. It is enough to prove that for all $s < t \in D_n$, $\hat{K}^n_{s,t}(X) = \hat{K}^{n+1}_{s,t}(X)$ almost surely. This holds since

$$\begin{aligned} \hat{K}^n_{s,t}(X) &= i(\omega^n_{t-2^{-n}}) \circ \cdots \circ i(\omega^n_s)(X) \\ &= (i \circ j)(\hat{K}^{n+1}_{t-2^{-n},t}) \circ \cdots \circ (i \circ j)(\hat{K}^{n+1}_{s,s+2^{-n}})(X). \end{aligned}$$

Using lemma 1.2.11 and the independence of the family of random variables $\{\omega_u^{n+1},\ u \in D_{n+1}\}$, we prove that the last term is almost surely equal to $\hat{K}^{n+1}_{t-2^{-n},t} \circ \cdots \circ \hat{K}^{n+1}_{s,s+2^{-n}}(X) = \hat{K}^{n+1}_{s,t}(X)$. □

Remark 1.3.3 *The preceding proposition implies that for all $s < u < t \in D$ and all $\mathcal{P}(M)$-valued random variable X independent of $\mathcal{F}_{s,t}$,*

$$\hat{K}_{s,t}(X) = \hat{K}_{u,t} \circ \hat{K}_{s,u}(X) \qquad \text{almost surely.} \tag{1.39}$$

We now intend to define by approximation for all $s < t$ in $\mathbb{R}$ a $(\Gamma, \mathcal{G})$-valued random variable $\hat{K}_{s,t}$ of law Q_{t-s}. In order to do this, we prove the following lemma.

Lemma 1.3.4 *For all continuous function f on $\mathcal{P}(M)^2$, the mapping*

$$(s, t, u, v, \mu, \nu) \mapsto \mathsf{E}[f(\hat{K}_{s,t}(\mu), \hat{K}_{u,v}(\nu))] \tag{1.40}$$

is continuous on $\{(s,t) \in D^2,\ s \leq t\}^2 \times \mathcal{P}(M)^2$. (And therefore uniformly continuous on every compact.)

Proof. For all $s \leq u \leq t \leq v$ in D, using the cocycle property, we have

$$\begin{aligned}\mathsf{E}[f(\hat{K}_{s,t}(\mu), \hat{K}_{u,v}(\nu))] &= \mathsf{E}[f(\hat{K}_{u,t} \circ \hat{K}_{s,u}(\mu), \hat{K}_{t,v} \circ \hat{K}_{u,t}(\nu))] \\ &= (\Pi^{(1)}_{u-s} \otimes I)\Pi^{(2)}_{t-u}(I \otimes \Pi^{(1)}_{v-t}) f(\mu, \nu).\end{aligned}$$

For all $s \leq u \leq v \leq t$ in D, using the cocycle property, we have

$$\begin{aligned}\mathsf{E}[f(\hat{K}_{s,t}(\mu), \hat{K}_{u,v}(\nu))] &= \mathsf{E}[f(\hat{K}_{v,t} \circ \hat{K}_{u,v} \circ \hat{K}_{s,u}(\mu), \hat{K}_{u,v}(\nu))] \\ &= (\Pi^{(1)}_{u-s} \otimes I)\Pi^{(2)}_{v-u}(\Pi^{(1)}_{t-v} \otimes I) f(\mu, \nu).\end{aligned}$$

For all $s \leq t \leq u \leq v$ in D,

$$\mathsf{E}[f(\hat{K}_{s,t}(\mu), \hat{K}_{u,v}(\nu))] = (\Pi^{(1)}_{t-s} \otimes \Pi^{(1)}_{v-u}) f(\mu, \nu).$$

All these functions are continuous. This implies the lemma. □

For all real t and all integer n, let $t_n = \sup\{u \in D_n,\ u \leq t\}$. For all $s < t \in \mathbb{R}$, we define the increasing sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$. Using lemma 1.3.4 for $f(\mu, \nu) = \rho(\mu, \nu)$ and the Markov inequality, we have

$$\lim_{n\to\infty} \sup_{k>n} \sup_{\mu\in\mathcal{P}(M)} \mathsf{P}[\rho(\hat{K}_{s_n,t_n}(\mu), \hat{K}_{s_k,t_k}(\mu)) > \varepsilon] = 0 \tag{1.41}$$

for all positive ε. Let $(n_k)_{k\in\mathbb{N}}$ be the increasing sequence depending only on s and t defined by induction by the relations :

$$\begin{aligned} n_0 &= 0, \\ n_{k+1} &= \inf\{n > n_k,\ \sup_{k>n} \sup_{\mu\in\mathcal{P}(M)} \mathsf{P}[\rho(\hat{K}_{s_n,t_n}(\mu), \hat{K}_{s_k,t_k}(\mu)) > 2^{-k}] < 2^{-k}\}.\end{aligned}$$

Using Borel-Cantelli's lemma, we prove that for all $\mu \in \mathcal{P}(M)$, as k tends to ∞, $\hat{K}_{s_{n_k},t_{n_k}}(\mu)$ converges almost surely towards a limit we denote by $\hat{K}_{s,t}(\mu)$. Then $\hat{K}_{s,t}$ is a $(\Gamma, \mathcal{G})$-valued random variable.

Lemma 1.3.5 *For all positive ε, for all $s \le t$,*

$$\lim_{n\to\infty} \sup_{\mu\in\mathcal{P}(M)} \mathsf{P}[\rho(\hat{K}_{s_n,t_n}(\mu), \hat{K}_{s,t}(\mu)) > \varepsilon] = 0. \tag{1.42}$$

Proof. This follows from the definition of $\hat{K}_{s,t}$ and equation (1.41). □

Proposition 1.3.6 *For all $s < t \in \mathbb{R}$, the law of $\hat{K}_{s,t}$ is Q_{t-s}.*

Proof. For all positive integer k, all continuous function f on $\mathcal{P}(M)^k$ and all $(\nu_1, \dots, \nu_k)$ in $\mathcal{P}(M)^k$, lemma 1.3.5 and dominated convergence theorem implies that

$$\begin{aligned}
\mathsf{E}[f(\hat{K}_{s,t}(\nu_1), \dots, \hat{K}_{s,t}(\nu_k))] &= \lim_{n\to\infty} \mathsf{E}[f(\hat{K}_{s_n,t_n}(\nu_1), \dots, \hat{K}_{s_n,t_n}(\nu_k))] \\
&= \lim_{n\to\infty} \Pi^{(k)}_{t_n - s_n} f(\nu_1, \dots, \nu_k) \\
&= \Pi^{(k)}_{t-s} f(\nu_1, \dots, \nu_k)
\end{aligned}$$

since $\Pi^{(k)}_t$ is Feller. □

Let us now prove the cocycle property.

Proposition 1.3.7 *For all $s < u < t$ and all $\mu \in \mathcal{P}(M)$, P-almost surely,*

$$\hat{K}_{s,t}(\mu) = \hat{K}_{u,t} \circ \hat{K}_{s,u}(\mu). \tag{1.43}$$

Proof. Almost surely, we have $\hat{K}_{s_n,t_n}(\mu) = \hat{K}_{u_n,t_n} \circ \hat{K}_{s_n,u_n}(\mu)$ since $s_n < u_n < t_n$ belongs to D. On one hand, $\hat{K}_{s_n,t_n}(\mu)$ converges in probability towards $\hat{K}_{s,t}(\mu)$. On the other hand,

$$\begin{aligned}
&\mathsf{P}[\rho(\hat{K}_{u_n,t_n} \circ \hat{K}_{s_n,u_n}(\mu), \hat{K}_{u,t} \circ \hat{K}_{s,u}(\mu)) > \varepsilon] \\
&\qquad \le \mathsf{P}[\rho(\hat{K}_{u_n,t_n} \circ \hat{K}_{s_n,u_n}(\mu), \hat{K}_{u,t} \circ \hat{K}_{s_n,u_n}(\mu)) > \varepsilon/2] \\
&\qquad + \mathsf{P}[\rho(\hat{K}_{u,t} \circ \hat{K}_{s_n,u_n}(\mu), \hat{K}_{u,t} \circ \hat{K}_{s,u}(\mu)) > \varepsilon/2].
\end{aligned}$$

Lemma 1.3.5 shows that the first term converges towards 0 and lemma 1.2.8 shows that the second term converges towards 0 (with $X_n = \hat{K}_{s_n,u_n}(\mu)$, $X = \hat{K}_{s,u}(\mu)$, $Y = j(\hat{K}_{u,t})$ and we use the fact that $i \circ j(\hat{K}_{u,t})(\mu) = \hat{K}_{u,t}(\mu)$ almost surely for every $\mu \in \mathcal{P}(M)$). □

Thus we have constructed a stochastic flow of measurable mappings on $\mathcal{P}(M)$ associated with the compatible family of Feller semigroups $(\Pi^{(k)}_t,\ k \ge 1)$.

1.3.3 Construction of a stochastic flow of kernels on M.

For all $s \leq t$, let $K_{s,t}$ be defined by $\delta(\hat{K}_{s,t})$. Note that when $s = t$, $K_{s,t}(x) = \delta_x$.

Proposition 1.3.8 *$(K_{s,t},\ s \leq t)$ is a cocycle* P*-almost surely, i.e. for $s < u < t$ and $x \in M$,* P*-almost surely, for all continuous function f,*

$$K_{s,t}f(x) = K_{s,u}(K_{u,t}f)(x). \tag{1.44}$$

Proof. This is a consequence of propositions 1.3.7 and lemma 1.2.14. □

Note that the family $(K_{s,t},\ s \leq t)$ is a stochastic flow of kernels associated with the convolution semigroup $\nu = (\nu_t)_{t\in\mathbb{R}}$. Indeed, **(d)** and **(e)** in definition 1.1.3 are satisfied since the function (for $f(\mu,\nu) = (\rho(\mu,\nu))^2$) defined in lemma 1.3.4 is also continuous on $\{(s,t) \in \mathbb{R}^2,\ s \leq t\}^2 \times \mathcal{P}(M)^2$.

Let K be the $(\Omega^0, \mathcal{A}^0)$-valued random variable defined by $K = (K_{s,t},\ s \leq t)$. Let $\mathsf{P}_\nu = K^*(\mathsf{P})$ be the law of K. Then by a monotone class argument we show that $T_h^*(\mathsf{P}_\nu) = \mathsf{P}_\nu$ for all $h \in \mathbb{R}$. Thus, we have constructed the canonical stochastic flow of kernels on M associated with the Feller convolution semigroup ν.

Proof of the reciprocal. Let $(K_{s,t},\ s \leq t)$ be a stochastic flow of kernels. For all positive integer n, let us define the operator $\mathsf{P}_t^{(n)}$ acting on $C(M)$ such that for all $f \in C(M^n)$ and $x \in M$,

$$\mathsf{P}_t^{(n)} f(x) = \mathsf{E}[(K_{0,t})^{\otimes n} f(x)]. \tag{1.45}$$

It is clear that $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is a compatible family of Markovian semigroups. It remains to prove the Feller property.

For all $h \in C(M^n)$ in the form $f_1 \otimes \cdots \otimes f_n$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have for M large enough

$$|\mathsf{P}_t^{(n)} h(y) - \mathsf{P}_t^{(n)} h(x)| \leq M \sum_{k=1}^{n} \mathsf{E}[(K_{0,t}f_k(y_k) - K_{0,t}f_k(x_k))^2]^{\frac{1}{2}} \tag{1.46}$$

which converges towards 0 as $d(x,y)$ goes to 0 since **(e)** in definition 1.1.3 is satisfied. We also have

$$|\mathsf{P}_t^{(n)} h(x) - h(x)| \leq M \sum_{k=1}^{n} \mathsf{E}[(K_{0,t}f_k(x_k) - f_k(x_k))^2]^{\frac{1}{2}} \tag{1.47}$$

which converges towards 0 as t goes to 0 since **(d)** in definition 1.1.3 is satisfied. These properties extend to all function h in $C(M^n)$ by an approximation argument. This proves the Feller property of the Markovian semigroups $\mathsf{P}_t^{(n)}$. This ends the proof of the reciprocal since the stochastic flow of kernels $(K_{s,t},\ s \le t)$ is obviously associated with the family $(\mathsf{P}_t^{(n)},\ n \ge 1)$.

In a similar way, we prove that starting with a Feller convolution semigroup on $(E, \mathcal{E})$, the family of semigroups defined by equation (1.6) is compatible family of Feller semigroups on M. □

1.4 Proof of theorem 1.1.6.

Let us remark that a probability measure μ in $\mathcal{P}(M)$ is a Dirac measure if and only if for all $f \in C(M)$, $(\int f\ d\mu)^2 = \int f^2\ d\mu$.

The fact that $\mathsf{P}_t^{(2)} f^{\otimes 2}(x,x) = \mathsf{P}_t^{(1)} f^2(x)$ holds for all positive t, $x \in M$ and $f \in C(M)$, implies that for all $s \le t$, $x \in M$ and $f \in C(M)$, $\mathsf{E}[(K_{s,t}f)^2(x)] = \mathsf{E}[K_{s,t}f^2(x)]$ and

$$(K_{s,t}f)^2(x) = K_{s,t}f^2(x) \qquad \mathsf{P}_\nu - \text{a.s.}, \tag{1.48}$$

where $(K_{s,t},\ s \le t)$ is the canonical flow associated with $(\mathsf{P}_t^{(n)},\ n \ge 1)$, of law P_ν.

Let $A_{s,t} = \{(x,\omega) \in M \times \Omega,\ \forall f \in C(M),\ (K_{s,t}f)^2(x,\omega) = K_{s,t}f^2(x,\omega)\}$. The set $A_{s,t}$ belongs to $\mathcal{B}(M) \otimes \mathcal{A}$ and for all (x,ω) in $A_{s,t}$, $K_{s,t}(x,\omega)$ is a Dirac measure we denote $\delta_{\varphi_{s,t}(x,\omega)}$. For (x,ω) not in $A_{s,t}$, let $\varphi_{s,t}(x,\omega) = x$. Thus we have defined a mapping $\varphi_{s,t}\ :\ M \times \Omega \to M$. This mapping is measurable since for all $A \in \mathcal{B}(M)$,

$$\varphi_{s,t}^{-1}(A) = (K_{s,t}^{-1}(\{\delta_y,\ y \in A\}) \cap A_{s,t}) \cup (A \times \Omega \cap A_{s,t}^c) \tag{1.49}$$

is a measurable set (note that $\{\delta_y,\ y \in A\} \in \mathcal{B}(\mathcal{P}(M))$ holds since $C(M)$ is separable and when A is open, $\{\delta_y,\ y \in A\} = \{\mu \in \mathcal{P}(M),\ \mu(A^c) = 0 \text{ and } \mu f^2 = (\mu f)^2, f \in C(M)\}$ and use monotone class theorem). Then it is easy to check that the family $(\varphi_{s,t},\ s \le t)$ is a stochastic flow of measurable mappings with $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$ P_ν-a.s. for all $s \le t$ and $x \in M$. This proves that P_ν, the law of K, is carried by $\Omega^{0,m}$. □

1.5 The noise generated by a stochastic flow of kernels.

We recall the definition of the noise given by Tsirelson in [30].

Definition 1.5.1 *A noise consists of a probability space* $(\Omega, \mathcal{A}, \mathsf{P})$*, a one-parametric group* $(T_h)_{h\in\mathbb{R}}$ *of* P*-preserving transformations of* Ω *and a family* $\{\mathcal{F}_{s,t},\ -\infty \le s \le t \le \infty\}$ *of sub-*σ*-fields of* $\mathcal{A}$ *such that*

(a) T_h *sends* $\mathcal{F}_{s,t}$ *onto* $\mathcal{F}_{s+h,t+h}$ *for all* $h \in \mathbb{R}$ *and all* $s \le t$*,*

(b) $\mathcal{F}_{s,t}$ *and* $\mathcal{F}_{t,u}$ *are independent for all* $s \le t \le u$*,*

(c) $\mathcal{F}_{s,t} \vee \mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ *for all* $s \le t \le u$*.*

Moreover, we will assume that, for all $s \le t$*,* $\mathcal{F}_{s,t}$ *contains all* P*-negligible sets of* $\mathcal{F}_{-\infty,\infty}$*, denoted* $\mathcal{F}$*.*

Let $K = (K_{s,t},\ s \le t)$, defined on the probability space $(\Omega^0, \mathcal{A}^0, \mathsf{P}_\nu)$, be the canonical flow associated with a Feller convolution semigroup ν.

For all $-\infty \le s \le t \le \infty$, let $\mathcal{F}^\nu_{s,t}$ be the sub-σ-field of $\mathcal{A}^0$ generated by the random variables $K_{u,v}$ for all $s \le u \le v \le t$ completed by all P_ν-negligible sets of $\mathcal{A}^0$. Then the cocycle property of K implies that $N_\nu := (\Omega^0, \mathcal{A}^0, (\mathcal{F}^\nu_{s,t})_{s\le t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ is a noise. We call it the noise generated by the canonical flow $(K_{s,t},\ s \le t)$.

Definition 1.5.2 *Given a Feller convolution semigroup* ν*, let* N *be a noise* $(\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s\le t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ *and a stochastic flow of kernels* K *of law* P_ν *defined on* $(\Omega, \mathcal{A}, \mathsf{P})$ *such that for all* $s < t$*,* $K_{s,t}$ *is* $\mathcal{F}_{s,t}$*-measurable and for all* $h \in \mathbb{R}$*,*

$$K_{s+h,t+h} = K_{s,t} \circ T_h, \qquad a.s. \tag{1.50}$$

We will call (N, K) *an extension of the noise* N_ν*.*

Let (N_1, K_1) and (N_2, K_2) be two extensions of the noise N_ν. Let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and P be a probability measure on Ω defined by

$$\mathsf{E}[Z] = \int \mathsf{E}_1[Z_1 | K_1 = K] \mathsf{E}_2[Z_2 | K_2 = K]\ \mathsf{P}_\nu(dK), \tag{1.51}$$

for any bounded random variable $Z(\omega_1, \omega_2) = Z_1(\omega_1) Z_2(\omega_2)$. Let $(T_h)_{h\in\mathbb{R}}$ be the one-parametric group of P-preserving transformations of Ω defined by $T_h(\omega_1, \omega_2) = (T^1_h(\omega_1), T^2_h(\omega_2))$. For all $s < t$, let $\mathcal{F}_{s,t} = \mathcal{F}^1_{s,t} \otimes \mathcal{F}^2_{s,t}$. Then

$N := (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s\leq t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a noise. And if K denotes the random variable $K(\omega_1,\omega_2) = K_1(\omega_1)(= K_2(\omega_2)$ P-a.s.), then (N, K) is an extension of N_ν. We will call (N, K) the product of the extensions (N_1, K_1) and (N_2, K_2). Note that N_1 and N_2 are isomorphic to sub-noises of N.

1.6 Filtering by a sub-noise.

Let $\bar{N}$ be a sub-noise of an extension (N, K) of N_ν, i.e. $\bar{N}$ is a noise $(\Omega, \mathcal{A}, (\bar{\mathcal{F}}_{s,t})_{s\leq t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}}))$ such that $\bar{\mathcal{F}}_{s,t} \subset \mathcal{F}_{s,t}$ for all $s \leq t$.

Remark 1.6.1 *A sub-noise is characterized by $\bar{\mathcal{F}}_{-\infty,\infty}$, denoted $\bar{\mathcal{F}}$. It has to be stable under T_h, to contain all P-negligible sets of $\mathcal{F}$, and be such that $\bar{\mathcal{F}} = (\bar{\mathcal{F}} \cap \mathcal{F}_{-\infty,0}) \vee (\bar{\mathcal{F}} \cap \mathcal{F}_{0,\infty})$.*

For all $n \geq 1$, let $\bar{\mathsf{P}}_t^{(n)}$ be the operator acting on $C(M^n)$ defined by

$$\bar{\mathsf{P}}_t^{(n)}(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = \mathsf{E}[\prod_{i=1}^n \mathsf{E}[K_{0,t}f_i(x_i)|\bar{\mathcal{F}}_{0,t}]], \tag{1.52}$$

for all $x_1, \ldots, x_n$ in M and all $f_1, \ldots, f_n$ in $C(M)$.

Lemma 1.6.2 *The family $(\bar{\mathsf{P}}_t^{(n)},\ n \geq 1)$ is a compatible family of Feller semigroups.*

Proof. Note that the semigroup property of $\bar{\mathsf{P}}_t^{(n)}$ follows directly from the independence of the increments of the flow. The Markovian property and in particular the positivity property holds since for all $h \in C(M^n)$,

$$\bar{\mathsf{P}}_t^{(n)}h(x_1, \ldots, x_n) = \mathsf{E}[\langle h, \otimes_{i=1}^n \mathsf{E}[K_{0,t}(x_i)|\bar{\mathcal{F}}_{0,t}]\rangle]. \tag{1.53}$$

From this, it is clear that $(\bar{\mathsf{P}}_t^{(n)},\ n \geq 1)$ is a compatible family of Markovian semigroups respectively acting on $C(M^n)$.

It remains to prove the Feller property. Note that for all continuous functions $f_1, \ldots, f_n$, $h = f_1 \otimes \cdots \otimes f_n$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in M^n, for M large enough,

$$\begin{aligned} |\bar{\mathsf{P}}_t^{(n)}h(x) - \bar{\mathsf{P}}_t^{(n)}h(y)\| &\leq M\sum_{i=1}^n \mathsf{E}[(\mathsf{E}[K_{0,t}f_i(x_i) - K_{0,t}f_i(y_i)|\bar{\mathcal{F}}_{0,t}])^2]^{\frac{1}{2}} \\ &\leq M\sum_{i=1}^n \mathsf{E}[(K_{0,t}f_i(x_i) - K_{0,t}f_i(y_i))^2]^{\frac{1}{2}} \end{aligned} \tag{1.54}$$

which converges towards 0 as y tends to x since **(e)** in definition 1.1.3 is satisfied.

We also have, for all $h = f_1 \otimes \cdots \otimes f_n$ and $x = (x_1, \ldots, x_n)$ in M^n, for M large enough,

$$\begin{aligned} |\bar{\mathsf{P}}_t^{(n)} h(x) - h(x)| &\leq M \sum_{i=1}^n \mathsf{E}[(\mathsf{E}[(K_{0,t} f_i(x_i) - f_i(x_i) | \bar{\mathcal{F}}_{0,t}])^2]^{\frac{1}{2}} \\ &\leq M \sum_{i=1}^n \mathsf{E}[(K_{0,t} f_i(x_i) - f_i(x_i))^2]^{\frac{1}{2}} \end{aligned} \tag{1.55}$$

which converges towards 0 as t tends to 0 since **(d)** in definition 1.1.3 is satisfied. Hence, for all function $h \in C(M^n)$ such that h is a linear combinaison of functions of the type $f_1 \otimes \cdots \otimes f_n$, we have $\bar{\mathsf{P}}_t^{(n)} h$ is continuous and $\lim_{t \to 0} \bar{\mathsf{P}}_t^{(n)} h(x) = h(x)$ for all $x \in M^n$. By an approximation argument, we conclude. □

Let us denote by $\bar{\nu} = (\bar{\nu}_t)_{t \geq 0}$ the Feller convolution semigroup on $(E, \mathcal{E})$ associated with $(\bar{\mathsf{P}}_t^{(n)},\ n \geq 1)$. Note that the one point motion of ν and of $\bar{\nu}$ is the same, i.e. $\bar{\mathsf{P}}_t^{(1)} = \mathsf{P}_t^{(1)}$.

Lemma 1.6.3 **1-** *Let K be a $(E, \mathcal{E})$-valued random variable of law ν defined on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Assume that ν satisfies*

$$\lim_{d(x,y) \to 0} \int \rho(K(x), K(y))^2 \ \nu(dK) = 0. \tag{1.56}$$

Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{A}$. Then there exists a $(E, \mathcal{E})$-valued random variable $K^{\mathcal{G}}$ which is $\mathcal{G}$-measurable and such that

$$K^{\mathcal{G}} f(x) = \mathsf{E}[K f(x) | \mathcal{G}] \qquad \mathsf{P} - a.s. \tag{1.57}$$

for all $f \in C(M)$ and $x \in M$. Thus $K^{\mathcal{G}} = \mathsf{E}[K | \mathcal{G}]$. Note that $K^{\mathcal{G}} = K^{\tilde{\mathcal{G}}}$ where $\tilde{\mathcal{G}} = \sigma(K^{\mathcal{G}})$.

2- *Let (N, K) be an extension of N_ν and $\bar{N}$ be a sub-noise of N. Then there exists $\bar{K} = (\bar{K}_{s,t},\ s \leq t)$ a stochastic flow of kernels of law $\mathsf{P}_{\bar{\nu}}$ such that $(\bar{N}, \bar{K})$ is an extension of $N_{\bar{\nu}}$ and*

$$\bar{K}_{s,t} f(x) = \mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}] = \mathsf{E}[K_{s,t} f(x) | \bar{\mathcal{F}}] \qquad \mathsf{P} - a.s. \tag{1.58}$$

for all $s \leq t$, $x \in M$ and $f \in C(M)$. We say $\bar{K}$ is obtained by filtering K with respect to $\bar{N}$.

Proof. **1-** Let $(x_i)_{i\in\mathbb{N}}$ be a dense sequence in M. Equation (1.56) implies the existence of a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that if $d(x,y)\le\varepsilon_k$ then

$$\mathsf{E}[\rho(K(x),K(y))^2]\le 2^{-3k}. \tag{1.59}$$

For all $x\in M$, let $(n_k^x)_{k\in\mathbb{N}}$ be defined by $n_k^x=\inf\{n\in\mathbb{N},\ d(x_n,x)\le\varepsilon_k\}$. Then $\lim_{k\to\infty}x_{n_k^x}=x$ and Borel Cantelli's lemma shows that

$$\lim_{k\to\infty}K(x_{n_k^x})=K(x)\qquad \mathsf{P}-\text{a.s.} \tag{1.60}$$

(since $\nu[\rho(K(x_{n_k^x}),K(x))\ge 2^{-k}]\le 2^{-k}$). Then by dominated convergence, $\lim_{k\to\infty}\mathsf{E}[K(x_{n_k^x})|\mathcal{G}]=\mathsf{E}[K(x)|\mathcal{G}]$, P-a.s. Let us chose an everywhere defined $\mathcal{G}$-measurable version of $\mathsf{E}[K(x_i)|\mathcal{G}]$ for all $i\in\mathbb{N}$.

Let $K^{\mathcal{G}}$ be defined by $K^{\mathcal{G}}(x)=l\left(\left(\mathsf{E}[K(x_{n_k^x})|\mathcal{G}]\right)_{k\in\mathbb{N}}\right)$. Then $K^{\mathcal{G}}$ is a $(E,\mathcal{E})$-valued $\mathcal{G}$-measurable random variable and for all $x\in M$,

$$K^{\mathcal{G}}(x)=\lim_{k\to\infty}\mathsf{E}[K(x_{n_k^x})|\mathcal{G}]=\mathsf{E}[K(x)|\mathcal{G}]\quad \mathsf{P}-\text{a.s.} \tag{1.61}$$

2- Since for all $t\ge 0$, ν_t satisfies (1.56), **1-** shows that for all $s\le t$ there exists $\bar{K}_{s,t}$ a $(E,\mathcal{E})$-valued $\bar{\mathcal{F}}_{s,t}$-measurable random variable such that

$$\bar{K}_{s,t}f(x)=\mathsf{E}[K_{s,t}f(x)|\bar{\mathcal{F}}_{s,t}]\qquad \mathsf{P}-\text{a.s.} \tag{1.62}$$

for all $s\le t$, $x\in M$ and $f\in C(M)$.

It is easy to see that $\bar{K}=(\bar{K}_{s,t},\ s\le t)$ is a stochastic flow of kernels of law $\mathsf{P}_{\bar{\nu}}$ and that $(\bar{N},\bar{K})$ is an extension of $N_{\bar{\nu}}$. Let us just show the cocycle property. For all $s\le u\le t$, $x\in M$ and $f\in C(M)$, P_ν-a.s.,

$$\begin{aligned}\mathsf{E}[K_{s,t}f(x)|\bar{\mathcal{F}}_{s,t}] &= \mathsf{E}[K_{s,u}K_{u,t}f(x)|\bar{\mathcal{F}}_{s,t}]\\ &= \mathsf{E}[\mathsf{E}[K_{s,u}K_{u,t}f(x)|\bar{\mathcal{F}}_{s,u}\vee\bar{\mathcal{F}}_{u,t}]|\bar{\mathcal{F}}_{s,t}]\\ &= \bar{K}_{s,u}\bar{K}_{u,t}f(x).\end{aligned}$$

Thus the lemma is proved. □

Definition 1.6.4 *Given two Feller convolution semigroups on $(E,\mathcal{E})$, ν^1 and ν^2, we say that ν^1 dominates (respectively weakly dominates) ν^2, denoted $\nu^1\succeq\nu^2$ (respectively $\nu^1\succeq^w\nu^2$), if and only if there exists a sub-noise of N_{ν^1} (respectively of an extension (N^1,K^1) of N_{ν^1}) such that P_{ν^2} is the law of the flow obtained by filtering the canonical flow of law P_{ν^1} (respectively by filtering K^1) with respect to this sub-noise.*

Notice that lemma 1.6.3 implies that ν weakly dominates $\bar{\nu}$ and that ν dominates $\bar{\nu}$ if $\bar{N}$ is a sub-noise of N_ν. Note that the domination relation is in fact an extension of the notion of barycenters.

Lemma 1.6.5 *Let ν and $\bar{\nu}$ be two Feller convolution semigroups such that ν dominates $\bar{\nu}$. Let (N, K) be an extension of N_ν. Let $\tilde{N}_\nu$ be the sub-noise (isomorphic to N_ν) of N generated by K. Then there exists a sub-noise $\bar{N}$ of $\tilde{N}_\nu$ such that $\mathsf{P}_{\bar{\nu}}$ is the law of the flow obtained by filtering K with respect $\bar{N}$.*

Proof. Let $N_\nu := (\Omega^0, \mathcal{A}^0, (\mathcal{F}^\nu_{s,t})_{s\le t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ be the noise generated by the canonical flow associated with ν. Notice that $\nu \succeq \bar{\nu}$ means the existence of $\bar{N}^0$ a sub-noise of N_ν such that $\mathsf{P}_{\bar{\nu}}$ is the law of $\bar{K}^0$, the flow obtained by filtering the canonical flow of law P_ν with respect to $\bar{N}^0$.

Note that $K : (\Omega, \mathcal{A}) \to (\Omega^0, \mathcal{A}^0)$ is measurable. Let $\bar{\mathcal{F}}$ be the completion of $K^{-1}(\bar{\mathcal{F}}^0)$ by all P-negligible sets of $\mathcal{A}$ and, for all $s \le t$, set $\bar{\mathcal{F}}_{s,t} = \bar{\mathcal{F}} \cap \mathcal{F}_{s,t}$. Then $\bar{N} = (\Omega, \mathcal{A}, (\bar{\mathcal{F}}_{s,t})_{s\le t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a sub-noise of N. Lemma 1.6.3 allows us to define $\bar{K}$ the flow obtained by filtering K with respect to $\bar{N}$. One can check that $\bar{K} = \bar{K}^0(K)$. This implies that the law of $\bar{K}$ is $\mathsf{P}_{\bar{\nu}}$. Thus the proposition is proved. □

Proposition 1.6.6 *The domination relation and the weak domination relation are partial orders on the class of Feller convolution semigroups.*

Proof. 1) The transitivity of the domination relation follows from lemma 1.6.5 by the chain rule for conditional expectations.

Let us observe that if $\nu^1 \preceq \nu^2$ and $\nu^2 \preceq \nu^1$ then $\nu^1 = \nu^2$. Indeed, if $\nu^1 \succeq \nu^2$, Jensen's inequality shows that for all $f_1, \dots, f_n$ in $C(M)$, $x_1, \dots, x_n$ in M and $t \ge 0$,

$$\mathsf{E}_{\nu^1}[\exp(\sum_{i=1}^n K_{0,t} f_i(x_i))] \ge \mathsf{E}_{\nu^2}[\exp(\sum_{i=1}^n K_{0,t} f_i(x_i))]. \tag{1.63}$$

Therefore, if moreover $\nu^1 \preceq \nu^2$, the preceding inequality becomes an equality. This proves $\nu^1 = \nu^2$.

2) For the weak domination relation, the proof is similar. We prove the transitivity using the product of extensions. Indeed, if $\nu \preceq^w \bar{\nu}$, given any extension (N^1, K^1) of N_ν, there exist a larger extension (N, K) and a subnoise $\bar{N}$ of N such that $\bar{K}$ has law $\mathsf{P}_{\bar{\nu}}$: let $\bar{N}^2$ be a sub-noise of an extension (N^2, K^2) of N_ν such that $\bar{K}^2$ has law $\mathsf{P}_{\bar{\nu}}$, then (N, K) is taken as the product of the extensions (N^1, K^1) and (N^2, K^2), and $\bar{N}$ is induced by $\bar{N}^1$. □

1.7 Sampling the flow.

Let $(K_{s,t},\ s \le t)$ be a stochastic flow of kernels defined on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$. In this section, we construct on an extension of $(\Omega, \mathcal{A}, \mathsf{P})$ a random path X_t starting at x such that for all positive t,

$$K_{0,t}f(x) = \mathsf{E}[f(X_t)|\mathcal{A}]. \tag{1.64}$$

For $x \in M$ and $\omega \in \Omega$, by Kolmogorov's theorem, we define on $M^{\mathbb{R}^+}$, a probability $\mathsf{P}^0_{x,\omega}$ such that

$$\mathsf{E}^0_{x,\omega}[\prod_{i=1}^n f_i(X^0_{t_i})] = K_{0,t_1}(f_1(K_{t_1,t_2}f_2(\cdots(f_{n-1}K_{t_{n-1},t_n}f_n))))(x), \tag{1.65}$$

for all $f_1, \ldots, f_n$ in $C(M)$, $0 < t_1 < t_2 < \cdots < t_n$ and where $\mathsf{E}^0_{x,\omega}$ denotes the expectation with respect to $\mathsf{P}^0_{x,\omega}$.

With P and $\mathsf{P}^0_{x,\omega}$, we construct a probability $\mathsf{P}^0_x(d\omega, d\omega') = \mathsf{P}(d\omega) \otimes \mathsf{P}^0_{x,\omega}(d\omega')$ on $\Omega \times M^{\mathbb{R}^+}$. Then, on the probability space $(\Omega \times M^{\mathbb{R}^+}, \mathcal{A} \otimes \mathcal{B}(M)^{\otimes \mathbb{R}^+}, \mathsf{P}^0_x)$, the random process $(X^0_t,\ t \ge 0)$, defined by $X^0_t(\omega, \omega') = \omega'(t)$, is a Markov process starting at x with semigroup $\mathsf{P}^{(1)}_t$ since

$$\mathsf{E}^0_x[\prod_{i=1}^n f_i(X^0_{t_i})] = \mathsf{P}^{(1)}_{t_1}(f_1(\mathsf{P}^{(1)}_{t_2-t_1}f_2(\cdots(f_{n-1}\mathsf{P}^{(1)}_{t_n-t_{n-1}}f_n))))(x), \tag{1.66}$$

for all $f_1, \ldots, f_n$ in $C(M)$, $0 < t_1 < t_2 < \cdots < t_n$ and where E^0_x denotes the expectation with respect to P^0_x.

Therefore, there is a càdlàg (or continuous when $\mathsf{P}^{(1)}_t$ is the semigroup of a continuous Markov process) modification $X = (X_t,\ t \ge 0)$ of $(X^0_t,\ t \ge 0)$. Let now $\mathsf{P}_{x,\omega}$ be the law of X knowing $\mathcal{A}$. It is a law on $D(\mathbb{R}^+, M)$, the space of càdlàg functions (or $C(\mathbb{R}^+, M)$ when $\mathsf{P}^{(1)}_t$ is the semigroup of a continuous Markov process). Equipped with the Skorohod topology (see [20] or [4]), $D(\mathbb{R}^+, M)$ becomes a Polish space (respectively $C(\mathbb{R}^+, M)$ is equipped with the topology of uniform convergence).

On the probability space $(\Omega \times D(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(D(\mathbb{R}^+, M)), \mathsf{P}_x)$ (respectively on $(\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), \mathsf{P}_x)$), where $\mathsf{P}_x(d\omega, d\omega') = \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega')$, let X be the random process $X(\omega, \omega') = \omega'$. Then X is a càdlàg (respectively continuous) process and

$$\begin{aligned} \mathsf{E}_x[\prod_{i=1}^n f_i(X_{t_i})|\mathcal{A}] &= \mathsf{E}_{x,\omega}[\prod_{i=1}^n f_i(X_{t_i})] \\ &= K_{0,t_1}(f_1(K_{t_1,t_2} f_2(\cdots(f_{n-1} K_{t_{n-1},t_n} f_n))))(x), \end{aligned} \tag{1.67}$$

where E_x denotes the expectation with respect to P_x.

Let $(K'_{s,t},\ s \leq t)$ be the stochastic flow of kernels defined on $(\Omega, \mathcal{A}, \mathsf{P})$ by

$$K'_{s,t} f(x, \omega) = K'_{0,t-s} f(x, T_s \omega) \tag{1.68}$$

where

$$K'_{0,t} f(x) = \mathsf{E}_x[f(X_t)|\mathcal{A}] = \int f(X_t(\omega, \omega')) \mathsf{P}_{x,\omega}(d\omega') \tag{1.69}$$

for $f \in C(M)$, $x \in M$. Then $(K'_{s,t},\ s \leq t)$ is a càdlàg in t (respectively continuous in t) modification of $(K_{s,t},\ s \leq t)$.

1.7.1 Continuous martingales.

Let $\mathcal{F}$ be the filtration $(\mathcal{F}_{0,t})_{t \geq 0}$ (recall that $\mathcal{F}_{s,t} = \sigma(K_{u,v},\ s \leq u \leq v \leq t)$). Let $\mathcal{M}(\mathcal{F})$ be the space of locally square integrable $\mathcal{F}$-martingales.

Proposition 1.7.1 *Suppose that $\mathsf{P}_t^{(1)}$ is the semigroup of a Markov process with continuous paths, then $\mathcal{M}(\mathcal{F})$ is constituted of continuous martingales.*

Proof. Let $M \in \mathcal{M}(\mathcal{F})$ be a martingale in the form $\mathsf{E}[F|\mathcal{F}_{0,t}]$ where $F = \prod_{i=1}^n K_{s_i,t_i} f_i(x_i)$, with $f_1, \ldots, f_n$ in $C(M)$, $x_1, \ldots, x_n$ in M and $0 \leq s_i < t_i$ (we take here the continuous modification in t of the stochastic flow of kernels). Since martingales of this form are dense in $\mathcal{M}(\mathcal{F})$, it is enough to prove the continuity of these martingales.

For all t, let $\tilde{K}_t$ be the kernel defined on $\mathbb{R}^+ \times M$ by

$$\tilde{K}_t(s, x) = \begin{cases} \delta_{s-t} \otimes \delta_x & \text{for } \ s \geq t, \\ \delta_0 \otimes K_{s,t}(x) & \text{for } \ s \leq t. \end{cases} \tag{1.70}$$

Then we can rewrite F in the form $\prod_{i=1}^n \tilde{K}_{t_i} \tilde{f}_i(s_i, x_i)$, where $\tilde{f}_i(s, x) = f_i(x)$.

Note that $(\tilde{K}_{t_i}(s_i,x_i),\ 1 \leq i \leq n)$ is a Markov process on $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{P}(M))^n$. This Markov process is continuous and Feller (the Feller property follows from the Feller property of the semigroups $(\Pi_t^{(k)},\ k \geq 1)$). It is well known that the martingales relative to the filtration denoted here $(\mathcal{F}_t^{\{s_i,x_i\}_{1\leq i\leq n}},\ t \geq 0)$ generated by such a process are continuous (see [28] tome II).

This proves that $\mathsf{E}[F|\mathcal{F}_t^{\{s_i,x_i\}_{1\leq i\leq n}}]$ is a continuous martingale. We conclude after remarking that $M_t = \mathsf{E}[F|\mathcal{F}_t^{\{s_i,x_i\}_{1\leq i\leq n}}]$, which holds since the σ-field $\mathcal{F}_t^{\{s_i,x_i\}_{1\leq i\leq n}}$ is a sub-σ-field of $\mathcal{F}_t$ and M_t is $\mathcal{F}_t^{\{s_i,x_i\}_{1\leq i\leq n}}$-measurable. □

1.8 The example of Lipschitz SDEs.

We first show a sufficient condition for a compatible family of Markovian kernels semigroups to be constituted of Feller semigroups.

Lemma 1.8.1 *A compatible family* $(\mathsf{P}_t^{(n)},\ n \geq 1)$ *of Markovian kernels semigroups is constituted of Feller semigroups when the following condition is satisfied*

(F) *For all* $f \in C(M)$ *and* $x \in M$, $\lim_{t\to 0} \mathsf{P}_t^{(1)} f(x) = f(x)$ *and for all* $x \in M$, $\varepsilon > 0$ *and* $t > 0$, $\lim_{y\to x} \mathsf{P}_t^{(2)} f_\varepsilon(x,y) = 0$, *where* $f_\varepsilon(x,y) = 1_{d(x,y)>\varepsilon}$.

Proof. Let $h \in C(M^n)$ be in the form $f_1 \otimes \cdots \otimes f_n$ and $x = (x_1,\ldots,x_n)$ in M^n. We have for M large enough

$$|\mathsf{P}_t^{(n)} h(x) - h(x)| \leq M \sum_{k=1}^{n} (\mathsf{P}_t^{(1)} f_k^2 + f_k^2 - 2 f_k \mathsf{P}_t^{(1)} f_k)^{\frac{1}{2}}(x_k) \tag{1.71}$$

which converges towards 0 as t goes to 0 since for all $f \in C(M)$ and all $x \in M$, $\lim_{t\to 0} \mathsf{P}_t^{(1)} f(x) = f(x)$. We also have for $y = (y_1,\ldots,y_n)$ in M^n,

$$|\mathsf{P}_t^{(n)} h(y) - \mathsf{P}_t^{(n)} h(x)| \leq M \sum_{k=1}^{n} \mathsf{P}_t^{(2)}(|1 \otimes f_k - f_k \otimes 1|)(y_k,x_k) \tag{1.72}$$

which converges towards 0 as y tends to x since for all $f \in C(M)$ and $x \in M$, $\lim_{y\to x} \mathsf{P}_t^{(2)}(|1 \otimes f - f \otimes 1|)(y,x) = 0$. Indeed, $\forall \alpha > 0$, $\exists \varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $|f(y) - f(x)| < \alpha$. This implies

$$\mathsf{P}_t^{(2)}(|1 \otimes f - f \otimes 1|)(y,x) \leq \alpha + 2\|f\|_\infty \mathsf{P}_t^{(2)} f_\varepsilon(x,y). \tag{1.73}$$

This implies $\limsup_{y\to x} \mathsf{P}_t^{(2)}(|1\otimes f - f\otimes 1|)(y,x) \le \alpha$ for all $\alpha > 0$. □

Remark 1.8.2 • *The previous result extends to the locally compact case (using the fact that $C_0(M)$ is constituted of uniformly continuous functions).*

• *When* **(F)** *is satisfied, for all positive t, $f \in C_0(M)$ and $x \in M$, $\mathsf{P}_t^{(2)} f^{\otimes 2}(x,x) = \mathsf{P}_t^{(1)} f^2(x)$. Therefore a stochastic flow of kernels associated with this family of semigroups is a stochastic flow of mappings. This implies that* **(F)** *is not a necessary condition.*

Let $V, V_1, \ldots, V_k$ be bounded Lipschitz vector fields on a smooth locally compact manifold M. Let $W^1, \ldots, W^k$ be k independent real white noises [3]. We consider the SDE on M

$$dX_t = \sum_{i=1}^{k} V_i(X_t) \circ dW_t^i + V(X_t), \qquad t \in \mathbb{R}. \tag{1.74}$$

From the usual theory of strong solutions of SDEs (see for example [15]), it is possible to construct a stochastic flow of diffeomorphisms $(\varphi_{s,t},\ s \le t)$ such that for all $x \in M$, $\varphi_{s,t}(x)$ is a strong solution of the SDE (1.74) with $\varphi_{s,s}(x) = x$.

Using this stochastic flow, it is possible to construct a compatible family of Markovian semigroups $(\mathsf{P}_t^{(n)},\ n \ge 1)$ with

$$\mathsf{P}_t^{(n)} h(x_1, \ldots, x_n) = \mathsf{E}[h(\varphi_{0,t}(x_1), \ldots, \varphi_{0,t}(x_n))] \tag{1.75}$$

for $h \in C(M^n)$ and $x_1, \ldots, x_n$ in M. Using lemma 1.8.1, it is easy to check that these semigroups are Feller (these properties were previously observed by P. Baxendale in [2]).

It can also easily be shown that the canonical stochastic flow of kernels associated with this family of semigroups is a stochastic flow of measurable (actually, here, continuous) mappings equal in law to $(\varphi_{s,t},\ s \le t)$. Moreover, we will show that the noise (in Tsirelson's sense) generated by this stochastic flow is the noise generated by the family of white noises $W^1, \ldots, W^k$ (see section 4).

[3] By a real white noise, we mean a two-parametric family $(W_{s,t},\ s \le t)$ of centered Gaussian variables respectively with variance $t-s$ such that for all $((s_i,t_i),\ 1 \le i \le n)$ with $s_i \le t_i \le s_{i+1}$, the random variables $(W_{s_i,t_i},\ 1 \le i \le n)$ are independent and for all $s \le t \le u$, $W_{s,u} = W_{s,t} + W_{u,t}$.

2 Stochastic coalescing flows.

In this section we study stochastic coalescing flows, we denote by $(\varphi_{s,t},\ s \leq t)$. In section 2.2, it is shown that for $s < t$, $\varphi^*_{s,t}(\lambda)$ is atomic (where λ denotes any positive Radon measure on M). We study this point measure valued process which provides a description of the coalescing flow.

In section 2.3, starting from a compatible family of Feller semigroups, under the hypothesis that starting close to the diagonal the two-point motion hits the diagonal with a probability close to 1, we construct another compatible family of Feller semigroups to which is associated with a stochastic coalescing flow. We then show that the stochastic flow of kernels associated with the first family of semigroups can be defined by filtering the stochastic coalescing flow with respect to a sub-noise of an extension of its canonical noise.

Finally, we give three examples. The first one, due to Arratia [1], describes the flow of independent Brownian motions sticking together when they meet. The second one is due to Propp and Wilson [25] in the context of perfect simulation of the invariant distribution of a finite state irreducible Markov chain, their stochastic flows being indexed by the integers. The third one is the construction of a stochastic coalescing flow solution of Tanaka's SDE

$$dX_t = \operatorname{sgn}(X_t)dW_t, \tag{2.1}$$

where W is a real white noise. This coalescing flow was first constructed by Watanabe in [33]. In [18], a stochastic flow of kernels solution of this SDE was constructed as the only strong solution of this SDE.

2.1 Definition.

Let M be a locally compact separable metric space.

Definition 2.1.1 *A stochastic flow of mappings on M, $(\varphi_{s,t},\ s \leq t)$, is a stochastic coalescing flow if and only if for all $(x,y) \in M^2$, $T_{x,y} = \inf\{t \geq 0,\ \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite with a positive probability and for all $t \geq T_{x,y}$, $\varphi_{0,t}(x) = \varphi_{0,t}(y)$. In other words, every pair of points stick together after a finite time with a positive probability.*

Let $(\mathsf{P}^{(n)}_t,\ n \geq 1)$ be a compatible family of Feller semigroups. We denote by $\mathsf{P}^{(2)}_{(x,y)}$ the law of the Markov process associated with $\mathsf{P}^{(2)}_t$ starting from (x,y) and we denote this process (X_t, Y_t) or $X^{(2)}_t$.

Remark 2.1.2 *A compatible family* $(\mathsf{P}^{(n)}_t,\ n \geq 1)$ *of Feller semigroups defines a stochastic coalescing flow if and only if for all* $(x,y) \in M^2$*,* $T_{x,y} = \inf\{t \geq 0,\ X_t = Y_t\}$ *satisfies* $\mathsf{P}^{(2)}_{(x,y)}[T_{x,y} < \infty] > 0$ *and for all* $t \geq T_{x,y}$*,* $X_t = Y_t$ $\mathsf{P}^{(2)}_{(x,y)}$*-almost surely.*

2.2 A point measures valued process associated with a stochastic coalescing flow.

In this subsection, we suppose we are given a compatible family of Feller semigroups $(\mathsf{P}^{(n)}_t,\ n \geq 1)$ such that

$$\begin{cases} \forall x \in M,\ \forall t > 0, & \lim_{y\to x} \mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t] & = & 0, \\ \forall (x,y) \in M^2, & \limsup_{t\to\infty} \mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t] & < & 1. \end{cases} \tag{2.2}$$

Remark 2.2.1 *The assumption (2.2) is verified in all the examples of coalescing flows we will study except for the example presented in section 2.4.3, where* $\mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t]$ *does not converge towards 0 as* y *tends to* x *when* $x \neq 0$*.*

Proposition 2.2.2 *A compatible family* $(\mathsf{P}^{(n)}_t,\ n \geq 1)$ *of Feller semigroups satisfying (2.2) defines a stochastic coalescing flow.*

Proof. Note that for all $(x,y) \in M^2$, $\mathsf{P}^{(2)}_{(x,y)}[T_{x,y} = \infty] \leq \mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t]$ for all positive t. Hence $\mathsf{P}^{(2)}_{(x,y)}[T_{x,y} = \infty] \leq \limsup_{t\to\infty} \mathsf{P}^{(2)}_{(x,y)}[X_t \neq Y_t] < 1$. We conclude using the strong Markov property and the fact that $\mathsf{P}^{(2)}_{(x,x)}[X_t \neq Y_t] = 0$. □

For all $s < t \in \mathbb{R}$, let $\mu_{s,t} = \varphi^*_{s,t}(\lambda)$, where λ is any positive Radon measure on M.

Proposition 2.2.3 **(a)** *For all* $s < t \in \mathbb{R}$*, almost surely,* $\mu_{s,t}$ *is atomic.*

(b) *For all* $s < u < t \in \mathbb{R}$*, almost surely,* $\mu_{s,t}$ *is absolutely continuous with respect to* $\mu_{u,t}$*.*

Proof. Fix $s < t \in \mathbb{R}$. For all positive ε and all $x \in M$, let $m^x_\varepsilon = \int_{B(x,\varepsilon)} 1_{\varphi_{s,t}(x)=\varphi_{s,t}(y)}\ \lambda(dy)$ (m^x_ε is well defined since $(x,\omega) \mapsto \varphi_{s,t}(x,\omega)$ is measurable). For all $\alpha \in]0,1[$ and $x \in M$, let

$$A^{\alpha,x}_n = \{m^x_{\varepsilon^x_n} < (1-\alpha)\lambda(B(x,\varepsilon^x_n))\}, \tag{2.3}$$

where ε_n^x is a positive sequence such that $d(x,y) \leq \varepsilon_n^x$ implies

$$\mathsf{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}] \leq 2^{-n}.$$

Lemma 2.2.4 *For all positive α, $x \in M$ and $n \in \mathbb{N}$, $\mathsf{P}(A_n^{\alpha,x}) \leq \frac{1}{\alpha 2^n}$.*

Proof. For all integer n, we have

$$\begin{aligned}
\mathsf{E}[m^x_{\varepsilon_n^x}] &= \int_{B(x,\varepsilon_n^x)} \mathsf{P}^{(2)}_{(x,y)}[X_{t-s} = Y_{t-s}] \; \lambda(dy) \\
&\geq \int_{B(x,\varepsilon_n^x)} (1 - \mathsf{P}^{(2)}_{(x,y)}[X_{t-s} \neq Y_{t-s}]) \; \lambda(dy) \\
&\geq (1 - 2^{-n})\lambda(B(x,\varepsilon_n^x)).
\end{aligned}$$

And we conclude since

$$\mathsf{E}[m^x_{\varepsilon_n^x}] \leq \mathsf{P}(A_n^{\alpha,x})(1-\alpha)\lambda(B(x,\varepsilon_n^x)) + (1 - \mathsf{P}(A_n^{\alpha,x}))\lambda(B(x,\varepsilon_n^x))$$

(we use the fact that $m^x_{\varepsilon_n^x} \leq \lambda(B(x,\varepsilon_n^x))$). □

Lemma 2.2.5 *For all $x \in M$, almost surely, $m^x_{\varepsilon_n^x} \sim \lambda(B(x,\varepsilon_n^x))$ as $n \to \infty$.*

Proof. Using Borel-Cantelli's lemma, for all $\alpha \in]0,1[$

$$1 - \alpha \leq \liminf_{n\to\infty} \frac{m^x_{\varepsilon_n^x}}{\lambda(B(x,\varepsilon_n^x))} \leq \limsup_{n\to\infty} \frac{m^x_{\varepsilon_n^x}}{\lambda(B(x,\varepsilon_n^x))} \leq 1$$

almost surely. This implies $\lim_{n\to\infty} \frac{m^x_{\varepsilon_n^x}}{\lambda(B(x,\varepsilon_n^x))} = 1$ a.s. □

Since for all $(x,\omega) \in M \times \Omega$,

$$\begin{aligned}
\mu_{s,t}(\{\varphi_{s,t}(x)\}) &= \lambda(\{y, \; \varphi_{s,t}(y) = \varphi_{s,t}(x)\}) \\
&\geq \lambda(\{y \in B(x,\varepsilon_n), \; \varphi_{s,t}(y) = \varphi_{s,t}(x)\}),
\end{aligned}$$

lemma 2.2.5 implies that for all $x \in M$,

$$\mu_{s,t}(\{\varphi_{s,t}(x)\}) > 0 \tag{2.4}$$

almost surely. Since $(x,\omega) \mapsto \mu_{s,t}(\{\varphi_{s,t}(x)\})$ is measurable,

$$\lambda(dx) \otimes \mathsf{P}(d\omega)\text{-a.e.}, \qquad \mu_{s,t}(\{\varphi_{s,t}(x)\}) > 0. \tag{2.5}$$

This equation implies (since $\mu_{s,t} = \varphi^*_{s,t}(\lambda)$)

$$\mu_{s,t}(dy)\text{-a.e.}, \qquad \mu_{s,t}(\{y\}) > 0 \tag{2.6}$$

almost surely. This last equation is one characterization of the atomic nature of $\mu_{s,t}$ and **(a)** is proved.

To prove **(b)**, note first that for any Dirac measure δ_x, almost surely, $\varphi^*_{u,t}(\delta_x)$ is absolutely continuous with respect to $\varphi^*_{u,t}(\lambda) = \mu_{u,t}$ since (2.4) holds. Note also that $\lambda(dx) \otimes \mathsf{P}(d\omega)$-a.e, $\varphi_{s,t}(x) = \varphi_{u,t} \circ \varphi_{s,u}(x)$. This implies

$$\mu_{s,t} = \varphi^*_{u,t}(\mu_{s,u}) \qquad \text{a.s.} \tag{2.7}$$

Since $\mu_{s,u}$ is atomic and independent of $\varphi_{s,t}$, it follows that $\mu_{s,t}$ is absolutely continuous with respect to $\mu_{u,t}$. This proves **(b)**. □

Remark 2.2.6 • $(\mu_{s,t},\ s \le t)$ is Markovian in t.

• Since $\mu_{s,t}$ is atomic for $t > s$, there exists a point process $\xi_{s,t} = \{\xi^i_{s,t}\}$ and weights $\{\alpha^i_{s,t}\} \in \mathbb{R}^{\mathbb{N}}$ such that $\mu_{s,t} = \sum_i \alpha^i_{s,t}\delta_{\xi^i_{s,t}}$. The point process $\xi_{s,t}$ and the marked point process $(\xi_{s,t}, \alpha_{s,t})$ are Markovian in t since for all $s < u < t$, $\xi_{s,t} = \varphi_{u,t}(\xi_{s,u})$ and $\alpha^i_{s,t} = \sum_{\{j,\ \xi^i_{s,t} = \varphi_{u,t}(\xi^j_{s,u})\}} \alpha^j_{s,u}$.

• Let $A^i_{s,t} = \varphi^{-1}_{s,t}(\xi^i_{s,t})$ and $\Pi_{s,t}$ be the collection of the sets $A^i_{s,t}$. Note that $\cup_i A^i_{s,t} = M$ λ-a.e, the union being disjoint. Note also that being given $\xi_{s,t}$ and $\Pi_{s,t}$ determines $\varphi_{s,t}$ λ-a.e. Note finally that $\Pi_{s,t}$ is Markovian in s when s decreases, since for all $s < u < t$, $\Pi_{s,t} = \{\varphi^{-1}_{s,u}(A^i_{u,t})\}$. This Markov process has also a coalescence property : one can have for $i \neq j$, $\varphi^{-1}_{s,u}(A^i_{u,t}) = \varphi^{-1}_{s,u}(A^j_{u,t})$. When s decreases, the partition $\Pi_{s,t}$ becomes coarser.

2.3 Construction of a family of coalescent semigroups.

Let $(\mathsf{P}^{(n)}_t,\ n \ge 1)$ be a compatible family of Feller semigroups on a separable locally compact metric space M and $\nu = (\nu_t)_{t\in\mathbb{R}}$ the associated Feller convolution semigroup on $(E, \mathcal{E})$. Let $\Delta_n = \{x \in M^n,\ \exists i \neq j,\ x_i = x_j\}$ and $T_{\Delta_n} = \inf\{t \ge 0,\ X^{(n)}_t \in \Delta_n\}$, where $X^{(n)}_t$ denotes the n-point motion, i.e. the Markov process on M^n associated with the semigroup $\mathsf{P}^{(n)}_t$. We will denote Δ_2 by Δ.

Theorem 2.3.1 *There exists a unique compatible family,* $(\mathsf{P}^{(n),c}_t,\ n \ge 1)$ *of Markovian semigroups on* M *such that if* $X^{(n),c}$ *is the associated* n*-point motion and* $T^c_{\Delta_n} = \inf\{t \ge 0,\ X^{(n),c}_t \in \Delta_n\}$*, then*

- *$(X_t^{(n),c},\ t \le T_{\Delta_n}^c)$ is equal in law to $(X_t^{(n)},\ t \le T_{\Delta_n})$,*
- *for $t \ge T_{\Delta_n}^c$, $X_t^{(n),c} \in \Delta_n$.*

Moreover, this family is constituted of Feller semigroups if condition **(C)** *below is satisfied,*

(C) *For all $t > 0$, $\varepsilon > 0$ and $x \in M$,*

$$\lim_{y \to x} \mathsf{P}^{(2)}_{(x,y)}[\{T_\Delta > t\} \cap \{d(X_t, Y_t) > \varepsilon\}] = 0$$

where $(X_t, Y_t) = X_t^{(2)}$. And for all x and y in M, $\mathsf{P}^{(2)}_{(x,y)}[T_\Delta < \infty] > 0$.

In this case, the associated canonical stochastic flow of kernels is a coalescing flow of maps (we will call it the canonical coalescing flow).

Proof. For $1 \le k \le n$, denote

$$\partial_k M^n = \{x \in M^n, \quad \exists i_1 < i_2 < \cdots < i_{n-k+1}, \quad x_{i_1} = \cdots = x_{i_{n-k+1}}\}. \quad (2.8)$$

Then $\partial_1 M^n \simeq M$, $\partial_n M^n = M^n$ and $\partial_k M^n$ is the union of C_n^{k-1} copies E_α of M^k indexed by $c(n,k)$, the set of subsets of $\{1, \dots, n\}$ with $(n-k+1)$ elements.

For $\alpha \in c(n,k)$, denote j_α the isometry between E_α and M^k. We denote also $\partial E_\alpha = j_\alpha^{-1}(\partial_{k-1} M^k)$. We remark that for $\alpha \in c(n,k)$, $\partial E_\alpha = E_\alpha \cap (\partial_{k-1} M^n)$ and $\cup_{\alpha \in c(n,k)} (E_\alpha \backslash \partial E_\alpha) = \partial_k M^n \backslash \partial_{k-1} M^n$.

By induction on k, we define a Markov process $\hat{X}^{(n,k)}$ on $\partial_k M^n$. For $k=1$ and $\alpha \in c(n,1)$, $\hat{X}^{(n,1)} = j_\alpha^{-1}(X^{(1)})$. We obtain $\hat{X}^{(n,k)}$ concatenating the process (when the starting point is in E_α) $j_\alpha^{-1}(X^{(k)})$ stopped at the entrance time in $\partial_{k-1} M^n$ (or equivalently in ∂E_α) with the process $\hat{X}^{(n,k-1)}$ starting from the corresponding point.

For all integer n, let $\mathsf{P}_t^{(n),c}$ be the Markovian semigroup associated with the Markov process $\hat{X}^{(n,n)}$. In the following we will denote this Markov process by $X^{(n),c}$. From the above construction, it is clear that the family $(\mathsf{P}_t^{(n),c},\ n \ge 1)$ of Markovian semigroups is compatible.

It remains to prove that when **(C)** is satisfied, this family of Markovian semigroups is constituted of Feller semigroups. This holds since **(C)** implies **(F)** in lemma 1.8.1 : for all positive ε, $\mathsf{P}^{(2),c}_{(x,y)}[d(X_t, Y_t) > \varepsilon] \le \mathsf{P}^{(2)}_{(x,y)}[\{T_\Delta >$

$t\} \cap \{d(X_t, Y_t) > \varepsilon\}]$ which converges towards 0 as $y \to x$. Note that when **(C)** holds, then condition (2.2) is satisfied so that the canonical flow is a coalescing flow. □

We now suppose that $(\mathsf{P}_t^{(n),c},\ n \geq 1)$ is constituted of Feller semigroups (which is true when **(C)** holds). We denote by ν^c the associated Feller convolution semigroup.

Theorem 2.3.2 *The convolution semigroup ν^c weakly dominates ν.*

Proof. The idea of the proof is to define a coupling between the flows of kernels K and K^c respectively defined by ν and ν^c.

In a way similar to the construction of the Markov process $X^{(n),c}$ in the proof of theorem 2.3.1, for all integer $n \geq 1$, we construct a Markov process $\hat{X}^{(n)}$ on $(M \times M)^n$ identified with $M^n \times M^n$ such that:

- $(\hat{X}_1^{(n)}, \ldots, \hat{X}_n^{(n)})$ is the n-point motion of ν^c,
- $(\hat{X}_{n+1}^{(n)}, \ldots, \hat{X}_{2n}^{(n)})$ is the n-point motion of ν,
- between the coalescing times, $\hat{X}^{(n)}$ is described by the $(k+n)$-point motion of ν (when $(\hat{X}_1^{(n)}, \ldots, \hat{X}_n^{(n)})$ belongs to $\partial_k M^n$).

Let $\hat{\mathsf{P}}_t^{(n)}$ be the Markovian semigroup associated with $\hat{X}^{(n)}$. One easily checks that this semigroup is Feller using the fact that $\mathsf{P}_t^{(n)}$ and $\mathsf{P}_t^{(n),c}$ are Feller. Then $(\hat{\mathsf{P}}_t^{(n)},\ n \geq 1)$ is a compatible family of Feller semigroups, associated with a Feller convolution semigroup $\hat{\nu}$.

Let $\hat{K}$ be the canonical stochastic flow associated with this family of semigroups. Straightforward computations show that for all $s < t$, $(f, g) \in C(M)^2$ and $(x, y) \in M^2$,

$$\begin{aligned}
\mathsf{E}[(\hat{K}_{s,t}(f \otimes g)(x,y))^2] &= \mathsf{P}_{t-s}^{(3)} f^2 \otimes g \otimes g(x,y,y), \\
\mathsf{E}[(\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))^2(x,y)] &= \mathsf{P}_{t-s}^{(3)} f^2 \otimes g \otimes g(x,y,y), \\
\mathsf{E}[(\hat{K}_{s,t}(f \otimes g)\hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))(x,y)] &= \mathsf{P}_{t-s}^{(3)} f^2 \otimes g \otimes g(x,y,y).
\end{aligned}$$

This implies that

$$\mathsf{E}[(\hat{K}_{s,t}(f \otimes g) - \hat{K}_{s,t}(f \otimes 1)\hat{K}_{s,t}(1 \otimes g))^2(x,y)] = 0. \tag{2.9}$$

Thus we have $\hat{K}_{s,t}(x,y) = K^c_{s,t}(x) \otimes K_{s,t}(y)$ and it is easy to check that the laws of K^c and of K are respectively P_{ν^c} and P_ν. Thus $(N_{\hat{\nu}}, K^c)$ is an extension of N_{ν^c}. Let $\tilde{N}_\nu$ be the sub-noise of $N_{\hat{\nu}}$ generated by K.

Let us notice now that for all $g, f_1, \dots, f_n$ in $C_0(M)$, all $y, x_1, \dots, x_n$ in M and all $s < t$, we have (setting $y_i = x_{n+1} = y$ and for $i \leq n$, $h_i = f_i \otimes 1$ and $h_{n+1} = 1 \otimes g$)

$$\begin{aligned} \mathsf{E}[K^c_{s,t}g(y)\prod_{i=1}^n K_{s,t}f_i(x_i)] &= \mathsf{E}[\prod_{i=1}^{n+1} \hat{K}_{s,t}h_i(x_i, y_i)] \\ &= \mathsf{P}^{(n+1)}_{t-s} f_1 \otimes \cdots \otimes f_n \otimes g(x_1, \cdots, x_n, y). \end{aligned}$$

More generally one can prove in a similar way that for all $g, f_1, \dots, f_n$ in $C_0(M)$, all $y, x_1, \dots, x_n$ in M, all $s < t$ and all $(s_i, t_i)_{1 \leq i \leq n}$ with $s_i \leq t_i$ that

$$\mathsf{E}[K^c_{s,t}g(y)\prod_{i=1}^n K_{s_i,t_i}f_i(x_i)] = \mathsf{E}[K_{s,t}g(y)\prod_{i=1}^n K_{s_i,t_i}f_i(x_i)]. \tag{2.10}$$

This implies that $K_{s,t}g(y) = \mathsf{E}[K^c_{s,t}(y)|\sigma(K)]$ and therefore that $\nu^c \succeq^w \nu$. □

2.4 Examples.

2.4.1 Arratia's coalescing flow of independent Brownian motions.

The first example of coalescing flows was given by Arratia [1]. On $\mathbb{R}$, let P_t be the semigroup of a Brownian motion. With this semigroup we define the compatible family $(\mathsf{P}_t^{\otimes n},\ n \geq 1)$ of Feller semigroups. Note that the n-point motion of this family of semigroups is given by n independent Brownian motions. Let us also remark that the canonical stochastic flow of kernels associated with this family of semigroups is not random and is given by $(\mathsf{P}_{t-s},\ s \leq t)$.

Let $(\mathsf{P}_t^{(n)},\ n \geq 1)$ be the compatible family of Markovian coalescent semigroups associated with $(\mathsf{P}_t^{\otimes n},\ n \geq 1)$ (see section 2.3). Note that the n-point motion of this family of semigroups is given by n independent Brownian motions who stick together when they meet.

Proposition 2.4.1 *The family $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is constituted of Feller semigroups and the associated canonical flow of kernels is a coalescing flow.*

Proof. It is obvious after remarking that two real independent Brownian motions meet each other almost surely (condition **(C)** is verified). □

2.4.2 Propp-Wilson algorithm.

Similarly to Arratia's coalescing flow, let P_t be the semigroup of an irreducible aperiodic Markov process on a finite set M, with invariant probability measure m. Let $(\mathsf{P}_t^{(n)},\ n \geq 1)$ be the compatible family of Markovian coalescent semigroups associated with $(\mathsf{P}_t^{\otimes n},\ n \geq 1)$.

Proposition 2.4.2 *The canonical flow of kernels associated with* $(\mathsf{P}_t^{(n)},\ n \geq 1)$ *is a coalescing flow.*

Proof. It is obvious since the two-point motion defined by $\mathsf{P}_t^{\otimes 2}$ hits the diagonal almost surely. □

Let $(\varphi_{s,t},\ s \leq t)$ denote this coalescing flow. Then almost surely, for all x, y in M, $\tau_{x,y} = \inf\{t > 0,\ \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is almost surely finite. Therefore, after a finite time $\mathrm{Card}\{\varphi_{0,t}(x),\ x \in M\} = 1$.

In Propp-Wilson [25], an algorithm to exactly simulate a random variable distributed according to the invariant probability measure of a Markov chain with finite state space is given. The method consists in constructing a stochastic coalescing flow. We explain this in our context.

Let $\tau = \inf\{t > 0,\ \varphi_{-t,0}(x) = \varphi_{-t,0}(y) \text{ for all } (x,y) \in M^2\}$.

Proposition 2.4.3 *τ is almost surely finite and the law of X_τ, the random variable $\varphi_{-\tau,0}(x)$ (independent of $x \in M$), is m.*

Proof. Let us remark that for $t > \tau$ and all $x \in M$, the cocycle property implies that $\varphi_{-t,0}(x) = X_\tau$.

Since for all positive t,

$$\begin{aligned} \mathsf{P}[\tau \geq t] &= \mathsf{P}[\exists x, y,\ \varphi_{-t,0}(x) \neq \varphi_{-t,0}(y)] \\ &\leq \textstyle\sum_{(x,y)\in M^2} \mathsf{P}[\tau_{x,y} \geq t] \end{aligned} \tag{2.11}$$

which converges towards 0 as t goes to ∞, we have $\tau < \infty$ a.s.

For all function f on M and all $x \in M$, $\lim_{t\to\infty} \mathsf{P}_t f(x) = \sum_{y\in M} f(y)m(y)$ and

$$\mathsf{P}_t f(x) = \mathsf{E}[f(\varphi_{-t,0}(x))] = \mathsf{E}[f(\varphi_{-t,0}(x))1_{t\leq\tau}] + \mathsf{E}[f(X_\tau)1_{\tau<t}]. \tag{2.12}$$

Since τ is almost surely finite, as t goes to ∞, the first term of the right hand side of the preceding equation converges towards 0 and the second term converges towards $\mathsf{E}[f(X_\tau)]$. Therefore we prove that $\mathsf{E}[f(X_\tau)] = \sum_{y\in M} f(y)m(y)$. □

2.4.3 Tanaka's SDE.

In [18], starting from a real Brownian motion B, we constructed the stochastic flow of kernels $(S_t,\ t \geq 0)$, strong solution of the SDE

$$dX_t = \text{sgn}(X_t)dB_t, \quad t \geq 0. \tag{2.13}$$

For f continuous,

$$S_t f(x) = f(R_t^x)1_{t<T_x} + \frac{1}{2}(f(R_t^x) + f(-R_t^x))1_{t \geq T_x}, \tag{2.14}$$

where R_t^x is the Brownian motion $x + B_t$ reflected at 0 and T_x the first time it hits 0. For all continuous functions $f_1, \dots, f_n$, let

$$\mathsf{P}_t^{(n)}(f_1 \otimes \cdots \otimes f_n)(x_1, \dots, x_n) = \mathsf{E}[\prod_{i=1}^n S_t f_i(x_i)]. \tag{2.15}$$

Then it is easy to see that $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is a compatible family of Feller semigroups. Let $(\mathsf{P}_t^{(n),c},\ n \geq 1)$ be the family of semigroups constructed by theorem 2.3.1.

Let us describe the n-point motion associated with $(\mathsf{P}_t^{(n),c},\ n \geq 1)$. Let $(X_t,\ t \geq 0)$ be a Brownian motion starting at 0. Let $B_t = \int_0^t \text{sgn}(X_s)\ dX_s$, $(B_t,\ t \geq 0)$ is also a Brownian motion starting at 0. For all $x \in \mathbb{R}$, let $\tau_x = \inf\{t \geq 0,\ |x| + B_t = 0\}$. Note that $X_{\tau_x} = 0$. For all $x \in \mathbb{R}$, let

$$X_t^x = \begin{cases} x + \text{sgn}(x)B_t & \text{if} \quad t < \tau_x, \\ X_t & \text{if} \quad t \geq \tau_x. \end{cases} \tag{2.16}$$

Note that for all $x \in \mathbb{R}$, we have $B_t = \int_0^t \text{sgn}(X_t^x)\ dX_t^x$ and X^x is a solution of the SDE

$$dX_t^x = \text{sgn}(X_t^x)\ dB_t, \quad t \geq 0, \quad X_0^x = x. \tag{2.17}$$

Then for all $x_1, \dots, x_n$ in M, $((X_t^{x_1}, \dots, X_t^{x_n}),\ t \geq 0)$ is the n-point motion of the family of semigroups $(\mathsf{P}_t^{(n),c},\ n \geq 1)$.

Proposition 2.4.4 *The family* $(\mathsf{P}_t^{(n),c},\ n \geq 1)$ *is constituted of Feller semigroups and the associated canonical flow of kernels is a coalescing flow.*

Proof. It is easy to see that $(\mathsf{P}_t^{(n),c},\ n \geq 1)$ is constituted of Feller semigroups since for all t, $x \mapsto X_t^x$ is continuous (it implies that **(F)** in lemma 1.8.1 is satisfied). This also implies that the associated stochastic flow of kernels is a flow of maps. And it is a coalescing flow since almost surely, every pair of point meets after a finite time. Note that condition **(C)** is verified. □

3 Stochastic flows of kernels and SDE's.

3.1 Hypotheses.

In this section, M is a smooth locally compact manifold and we suppose we are given $(\mathsf{P}_t^{(n)},\ n \geq 1)$, a compatible family of Feller semigroups, or equivalently a Feller convolution semigroup $\nu = (\nu_t)_{t\geq 0}$ on $(E, \mathcal{E})$. For all positive integer n, we will denote by $X_t^{(n)}$ the Markov process associated with the semigroup $\mathsf{P}_t^{(n)}$ and we will call it the n-point motion. We assume that

(i) The space $C_K^2(M) \otimes C_K^2(M)$ [4] of functions of the form $f(x)g(y)$, with f, g in $C_K^2(M)$ and x, y in M, is included in the domain [5] $\mathcal{D}(A^{(2)})$ of the infinitesimal generator $A^{(2)}$ of $\mathsf{P}_t^{(2)}$.

(ii) The one-point motion $X_t^{(1)}$ has continuous paths.

In that case, we say that ν is a diffusion convolution semigroup on $(E, \mathcal{E})$ and that the $\mathsf{P}_t^{(n)}$ are diffusion semigroups.

Let us denote by A the infinitesimal generator of $\mathsf{P}_t^{(1)}$ in $C_K^2(M)$. Note that it follows easily from **(i)** and **(ii)** that for all $f \in C_K^2(M)$,

$$M_t^f = f(X_t^{(1)}) - f(X_0^{(1)}) - \int_0^t Af(X_s^{(1)})\ ds \tag{3.1}$$

is a martingale. Since f^2 also belongs to $C_K^2(M)$, using the martingale M^{f^2}, it is easy to see that

$$\langle M^f \rangle_t = \int_0^t \Gamma(f)(X_s^{(1)})\ ds \tag{3.2}$$

where

$$\Gamma(f) = Af^2 - 2fAf. \tag{3.3}$$

In the following $\Gamma(f, g)$ will denote $A(fg) - fAg - gAf$, for f and g in $C_K^2(M)$.

[4] $C_K(M)$ (respectively $C_K^2(M)$) denotes the set of continuous (respectively C^2) functions with compact support.

[5] f is in the domain of the infinitesimal generator A of a Feller semigroup P_t if and only if $\frac{\mathsf{P}_t f - f}{t}$ converges uniformly as t goes towards 0. Its limit is denoted Af.

Lemma 3.1.1 *On a smooth local chart on an open set $U \subset M$, there exist continuous functions on U, $a^{i,j}$ and b^i such that for all $f \in C_K^2(M)$,*

$$Af = \frac{1}{2} a^{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} + b^i \frac{\partial f}{\partial x^i}. \tag{3.4}$$

Proof. For all $x \in U$, let $\varphi^i(x) = x^i$ denote the coordinate functions of the local chart. We can extend φ^i into an element of $C_K^2(M)$. For $f \in C_K^2(M)$, using Itô's formula, for $t < T_U$, the exit time of U,

$$f(X_t^{(1)}) - f(X_0^{(1)}) - \int_0^t \left(\frac{1}{2} a^{i,j}(X_s^{(1)}) \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s^{(1)}) + b^i(X_s^{(1)}) \frac{\partial f}{\partial x^i}(X_s^{(1)}) \right) ds$$

is a martingale, where $b^i(x) = A\varphi^i(x)$ and $a^{i,j}(x) = \Gamma(\varphi^i, \varphi^j)(x)$. And we get (3.4) since for all $x \in U$, $Af(x) = \lim_{t \to 0} \frac{\mathsf{P}_t^{(1)} f(x) - f(x)}{t}$. □

Note that the two-point motion $X_t^{(2)}$ has also continuous trajectories and these results also apply to functions in $C_K^2(M) \otimes C_K^2(M)$. For all f, g in $C_K^2(M)$, let

$$C(f,g) = A^{(2)}(f \otimes g) - f \otimes Ag - Af \otimes g. \tag{3.5}$$

It is clear that on a local chart on $U \times V \subset M \times M$,

$$C(f,g)(x,y) = c^{i,j}(x,y) \frac{\partial f}{\partial x^i}(x) \frac{\partial g}{\partial y^j}(y), \tag{3.6}$$

where $c^{i,j} \in C(U \times V)$. Note that we can shortly write $A^{(2)} = A \otimes I + I \otimes A + C$. On a local chart on $U \times V$, for all $h \in C_K^2(M) \otimes C_K^2(M)$,

$$\begin{aligned} A^{(2)} h(x,y) &= \frac{1}{2} a^{i,j}(x) \frac{\partial^2}{\partial x^i \partial x^j} h(x,y) + b^i(x) \frac{\partial}{\partial x^i} h(x,y) \\ &+ \frac{1}{2} a^{i,j}(y) \frac{\partial^2}{\partial y^i \partial y^j} h(x,y) + b^i(y) \frac{\partial}{\partial y^i} h(x,y) \\ &+ C^{i,j}(x,y) \frac{\partial^2}{\partial x^i \partial y^j} h(x,y). \end{aligned} \tag{3.7}$$

We will call $\Gamma(f,g)(x) - C(f,g)(x,x) = \frac{1}{2} A^{(2)}(1 \otimes f - g \otimes 1)^2(x,x) - (1 \otimes f - g \otimes 1)(1 \otimes Af - Ag \otimes g)(x,x)$ the pure diffusion form. It can easily be seen that it vanishes if the associated canonical flow is a flow of maps. The converse is not true (see examples in section 5). Diffusive flows for which the pure diffusion form vanishes may be called turbulent.

Note that the two-point motion $X_t^{(2)} = (X_t, Y_t)$ solves the following martingale problem associated with $A^{(2)}$:

$$M_t^{f\otimes g} := f(X_t)g(Y_t) - f(X_0)g(Y_0) - \int_0^t A^{(2)}(f\otimes g)(X_s, Y_s)\, ds \tag{3.8}$$

is a martingale for all f and g in $C_K^2(M)$.

Note also that for all functions h_1 and h_2 in $C_K^2(M)\otimes C_K^2(M)$, the martingale bracket $\langle h_1(X^{(2)}), h_2(X^{(2)})\rangle_t$ is equal to

$$\int_0^t (A^{(2)}(h_1h_2) - h_1A^{(2)}h_2 - h_2A^{(2)}h_1)(X_s^{(2)})\, ds \tag{3.9}$$

and for all functions f and g in $C_K(M)$,

$$\langle f(X), g(Y)\rangle_t = \int_0^t C(f,g)(X_s, Y_s)\, ds. \tag{3.10}$$

Proposition 3.1.2 **(a)** *C is a covariance function on the space of vector fields* [6].

(b) *For all $f_1, \ldots, f_n$ in $C_K^2(M)$, then $g = f_1\otimes\cdots\otimes f_n \in \mathcal{D}(A^{(n)})$ and for $x = (x_1, \ldots, x_n) \in M^n$,*

$$A^{(n)}g(x) = \sum_i \prod_{j\neq i} f_j(x_i)Af_i(x_i) + \sum_{i<j} C(f_i, f_j)(x_i, x_j) \prod_{k\neq i,j} f_k(x_k). \tag{3.11}$$

Proof. Note that for all f and g in $C_K^2(M)$, $C(f,g)(x,y)$ is a function of $df(x)$ and of $dg(y)$ we denote $C(df(x), dg(y))$. Hence C is a symmetric map from T^*M^2 in $\mathbb{R}$ and its restriction to $T_x^*M\times T_y^*M$ is bilinear. To prove **(a)**, it remains to prove $\sum_{i,j} C(\xi_i, \xi_j) \geq 0$ for all $\xi_1, \ldots, \xi_n$ in T^*M^2. This holds since, for all $f_1, \ldots, f_n$ in $C^2(M)$ and all $x_1, \ldots, x_n$ in M,

$$\sum_{i,j} C(f_i, f_j)(x_i, x_j) = (A^{(n)}g^2 - 2gA^{(n)}g)(x_1, \ldots, x_n) \tag{3.12}$$

[6] C is a covariance function on the space of vector fields if and only if C is a symmetric map from T^*M^2 in $\mathbb{R}$ such that C restricted to $T_x^*M\times T_y^*M$ is bilinear and for any n-uples $(\xi_1, \ldots, \xi_n)$ of T^*M^2, $\sum_{1\leq i,j\leq n} C(\xi_i, \xi_j) \geq 0$, (see [18]). And for f and g in $C_K^1(M)$, we denote $C(df(x), dg(y))$ by $C(f,g)(x,y)$.

where $g(x_1,\dots,x_n)=\sum_{i=1}^n f_i(x_i)\in\mathcal{D}(A^{(n)})$. This expression is nonnegative since $A^{(n)}g^2-2gA^{(n)}g=\lim_{t\to 0}\frac{1}{t}(\mathsf{P}_t^{(n)}g^2-(\mathsf{P}_t^{(n)}g)^2+(\mathsf{P}_t^{(n)}g-g)^2)$.

The proof of **(b)** is an application of Itô's formula. □

Definition 3.1.3 *The restriction of A to $C_K^2(M)$ and the covariance function C are called the local characteristics of the family $(\mathsf{P}_t^{(n)},\ n\geq 1)$ or of the diffusion convolution semigroup.*

When there is no pure diffusion, to give the local characteristics (A,C) is equivalent to give a drift b and C (this corresponds to the usual definition of a local characteristics of a stochastic flow) since in this case $a^{i,j}(x)=c^{i,j}(x,x)$.

Remark 3.1.4 *Note that when $(\mathsf{P}_t^{(n)},\ n\geq 1)$ satisfies* **(C)**, **(i)** *and* **(ii)**, *then $(\mathsf{P}_t^{(n),c},\ n\geq 1)$ also satisfies* **(i)** *if and only if for all x in M and all f, g in $C_K^2(M)$, $C(f,g)(x,x)=\Gamma(f,g)(x)$ (this holds since we have $C(f,g)(x,x)-\Gamma(f,g)(x)=\lim_{t\to 0}\frac{1}{t}(\mathsf{P}_t^{(2),c}(f\otimes g)(x,x)-\mathsf{P}_t^{(1)}(fg)(x))$), i.e. when there is no pure diffusion. So that the results of this section also apply to $(\mathsf{P}_t^{(n),c},\ n\geq 1)$.*

Then in this case $(\mathsf{P}_t^{(n)},\ n\geq 1)$ and $(\mathsf{P}_t^{(n),c},\ n\geq 1)$ have the same local characteristics.

Let $(K_{s,t},\ s\leq t)$ be a stochastic flow of kernels associated with $(\mathsf{P}_t^{(n)},\ n\geq 1)$ defined on a probability space $(\Omega,\mathcal{A},\mathsf{P})$.

In this section, we intend to define on $(\Omega,\mathcal{A},\mathsf{P})$ a vector field valued white noise W of covariance C such that $(K_{s,t},\ s\leq t)$ solves a SDE driven by W. Let us recall that $(W_{s,t},\ s\leq t)$ is a vector field valued white noise of covariance C if and only if for all $s_i\leq t_i\leq s_{i+1}$, the random variables variables $(W_{s_i,t_i},\ 1\leq i\leq n)$ are independent, for all $s\leq t\leq u$, $W_{s,u}=W_{s,t}+W_{u,t}$ a.s. and for all $s\leq t$, $\{\langle W_{s,t},\xi\rangle,\ \xi\in T^*M\}$ [7] is a centered Gaussian process of covariance given by

$$\mathsf{E}[\langle W_{s,t},\xi\rangle\langle W_{s,t},\xi'\rangle]=(t-s)C(\xi,\xi'), \tag{3.13}$$

for ξ and ξ' in T^*M.

In section 4, under an additional assumption, we will prove that the linear (or Gaussian) part of the noise generated by $(K_{s,t},\ s\leq t)$ (in the case it is the canonical flow) is the noise generated by the vector field valued white noise W.

[7] when $\xi=(x,u)$, $\langle W_{s,t},\xi\rangle=\langle W_{s,t}(x),u\rangle$.

3.2 Construction of W.

For all $s \leq t$, all $f \in C_K^2(M)$ and all $x \in M$, let

$$M_{s,t}f(x) = K_{s,t}f(x) - f(x) - \int_s^t K_{s,u}(Af)(x)\, du. \tag{3.14}$$

Lemma 3.2.1 *For all $s \in \mathbb{R}$, $f \in C_K^2(M)$ and $x \in M$, $M_s^{f,x} = (M_{s,t}f(x),\ t \geq s)$ is a martingale with respect to the filtration $\mathcal{F}^s = (\mathcal{F}_{s,t},\ t \geq s)$ and*

$$\frac{d}{dt}\langle M_{s,\cdot}f(x), M_{s,\cdot}g(y)\rangle_t = K_{s,t}^{\otimes 2}C(f,g)(x,y), \tag{3.15}$$

for all f, g in $C_K^2(M)$ and all x, y in M.

Proof. Since $(K_{s,t},\ s \leq t)$ is a stochastic flow of kernels and that for all positive h and all f in $C_K^2(M)$,

$$M_{s,t+h}f(x) - M_{s,t}f(x) = K_{s,t}(M_{t,t+h}f)(x), \tag{3.16}$$

$M_s^{f,x}$ is a martingale. Note that equation (3.16) also implies that for all positive h, all f, g in $C_K^2(M)$ and all x, y in M,

$$\begin{aligned} &\mathsf{E}[(M_{s,t+h}f(x) - M_{s,t}f(x))(M_{s,t+h}g(y) - M_{s,t}g(y))|\mathcal{F}_{s,t}] \\ &\qquad = K_{s,t}^{\otimes 2}(\mathsf{E}[M_{t,t+h}f \otimes M_{t,t+h}g])(x,y). \end{aligned}$$

The stationarity implies that $\mathsf{E}[M_{t,t+h}f(x)M_{t,t+h}g(y)] = \mathsf{E}[M_{0,h}f(x)M_{0,h}g(y)]$. Computation using the fact that $\mathsf{P}_t^{(1)}f - f = \int_0^t \mathsf{P}_s^{(1)}Af\, ds$ and $\mathsf{P}_t^{(2)}(f \otimes g) - f \otimes g = \int_0^t \mathsf{P}_s^{(2)}A^{(2)}(f \otimes g)\, ds$ gives

$$\mathsf{E}[M_{0,h}f(x)M_{0,h}g(y)] = \int_0^h \mathsf{P}_s^{(2)}(C(f,g))(x,y)\, ds. \tag{3.17}$$

Since $\mathsf{P}_t^{(2)}$ is Feller and $C(f,g)$ is continuous with compact support,

$$\mathsf{E}[M_{0,h}f(x)M_{0,h}g(y)] = h\ C(f,g)(x,y) + o(h), \tag{3.18}$$

uniformly in $(x,y) \in M^2$.

Therefore $\mathsf{E}[(M_{s,t+h}f(x) - M_{s,t}f(x))(M_{s,t+h}g(y) - M_{s,t}g(y))|\mathcal{F}_{s,t}]$ is equivalent as h tends to 0 to $h\ K_{s,t}^{\otimes 2}C(f,g)(x,y)$. This proves the lemma. □

Remark 3.2.2 *Note that in the case of Arratia's coalescing flow* $(\varphi_{s,t},\ s \le t)$, $C = 0$ *but* $\frac{d}{dt}\langle M_{s,\cdot}f(x), M_{s,\cdot}g(y)\rangle_t = 1_{\{\varphi_{s,t}(x)=\varphi_{s,t}(y)\}}$. *In this case,* $C^2_K(M)\otimes C^2_K(M)$ *is not included in* $\mathcal{D}(A^{(2)})$. *This property also fails for the coalescing flow associated with Tanaka's SDE.*

For all $s < t$, $n \ge 1$ and $0 \le k \le 2^n - 1$, let $t^n_k = s + k2^{-n}(t-s)$. For all $f \in C^2_K(M)$ and all $x \in M$, let $W^n_{s,t} = \sum_{k=0}^{2^n-1} M_{t^n_k,t^n_{k+1}}$. Note that $(M_{t^n_k,t^n_{k+1}})_{0\le k\le 2^n-1}$ are independent equidistributed random variables taking their values in the space of vector fields.

3.2.1 Convergence in law.

Lemma 3.2.3 *For all* $s < t$ *and all* $((x_i, f_i),\ 1 \le i \le m) \in (M \times C^2_K(M))^m$, *then* $\sum_{i=1}^m W^n_{s,t} f_i(x_i)$ *converges in law towards* $\sum_{i=1}^m W_{s,t} f_i(x_i)$ *as* n *tends to* ∞, *where* W *is a vector field valued white noise of covariance* C.

Proof. Using lemma 3.2.1, we have for all f, g in $C^2_K(M)$ and all x, y in M,

$$\begin{aligned} \mathsf{E}[M_{t^n_k,t^n_{k+1}}f(x)M_{t^n_k,t^n_{k+1}}g(y)] &= \int_0^{2^{-n}(t-s)} \mathsf{P}^{(2)}_u C(f,g)(x,y)\,du \\ &= 2^{-n}(t-s)C(f,g)(x,y) + o(2^{-n}) \qquad (3.19) \end{aligned}$$

and this developement is uniform in x and y in M.

We will only prove the proposition when $m = 1$ (the proof being the same for $m > 1$). The proposition is just an application of the central limit theorem for arrays (see [5]), which one we can apply since (3.19) is satisfied provided the Lyapounov condition

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}[|M_{t^n_k,t^n_{k+1}}f(x)|^{2+\delta}] = 0 \qquad (3.20)$$

for some positive δ, is satisfied.

Using Burkholder-Davies-Gundy's inequality and lemma 3.2.1,

$$\begin{aligned} \mathsf{E}[|M_{t^n_k,t^n_{k+1}}f(x)|^{2+\delta}] &\le C\mathsf{E}\left[\left(\int_0^{2^{-n}(t-s)} K^{\otimes 2}_{0,u}(C(f,f))(x,x)\,du\right)^{\frac{2+\delta}{2}}\right] \\ &\le C2^{-\frac{(2+\delta)n}{2}}, \end{aligned}$$

where C is a constant (changing every lines) depending only on f, $(t-s)$ and δ. This implies

$$\sum_{k=0}^{2^n-1} \mathsf{E}[|M_{t_k^n,t_{k+1}^n} f(x)|^{2+\delta}] \le C 2^n 2^{-\frac{(2+\delta)n}{2}} \le C 2^{-\frac{n\delta}{2}}. \quad \square \tag{3.21}$$

Remark 3.2.4 *For Arratia's coalescing flow, one can show the convergence in law of* $(W_{s,t}^n(x_1), \ldots, W_{s,t}^n(x_k))$ *towards* $(B_{s,t}^1, \ldots, B_{s,t}^k)$*, where* $(B^1, \ldots, B^k)$ *is a* k*-dimensional white noise.*

3.2.2 Convergence in $L^2(\mathsf{P})$.

In the preceding section, we proved the convergence in law of W^n towards a vector field valued white noise of covariance C. In this section, we prove that this convergence holds in $L^2(\mathsf{P})$.

Lemma 3.2.5 *For all* $s < t$ *and all* $(x, f) \in M \times C_K^2(M)$*,* $W_{s,t}^n f(x)$ *converges in* $L^2(\mathsf{P})$*.*

Proof. For all $f \in C_K^2(M)$, all $x \in M$ and all $s < t$,

$$\begin{aligned} \mathsf{E}[(W_{s,t}^n f(x) - W_{s,t}^{n+k} f(x))^2] &= \mathsf{E}[(W_{s,t}^n f(x))^2] + \mathsf{E}[(W_{s,t}^{n+k} f(x))^2] \\ &- 2\mathsf{E}[W_{s,t}^n f(x) W_{s,t}^{n+k} f(x)] \end{aligned} \tag{3.22}$$

Elementary computations using equation (3.18) implies

$$\mathsf{E}[(W_{s,t}^n f(x))^2] = (t-s)\ C(f,f)(x,x) + o(1) \tag{3.23}$$

$$\mathsf{E}[(W_{s,t}^{n+k} f(x))^2] = (t-s)\ C(f,f)(x,x) + o(1) \tag{3.24}$$

as n goes to ∞ and this uniformly in $k \in \mathbb{N}$. Using the independence of the increments, the last term (3.22) rewrites

$$\mathsf{E}[W_{s,t}^n f(x) W_{s,t}^{n+k} f(x)] = \sum_{i=0}^{2^n-1} \sum_{j=i2^k}^{(i+1)2^k-1} \mathsf{E}[M_{t_i^n,t_{i+1}^n} f(x) M_{t_j^{n+k},t_{j+1}^{n+k}} f(x)]. \tag{3.25}$$

Note that for $s \le u \le v \le t$, using first the martingale property, then equation (3.18) and the uniform continuity of $C(f,f)$, we have

$$\begin{aligned} \mathsf{E}[M_{s,t} f(x) M_{u,v} f(x)] &= \mathsf{E}[M_{s,v} f(x) M_{u,v} f(x)] \\ &= \mathsf{E}[(K_{s,u} \otimes I)(M_{u,v} f \otimes M_{u,v} f)(x,x)] \\ &= \mathsf{E}[(K_{s,u} \otimes I)(\mathsf{E}[M_{u,v} f \otimes M_{u,v} f])(x,x)] \\ &= (v-u)\ C(f,f)(x,x) + o(v-u), \end{aligned} \tag{3.26}$$

uniformly in $x \in M$. This implies

$$\mathsf{E}[W^n_{s,t}f(x)W^{n+k}_{s,t}f(x)] = (t-s)\ C(f,f)(x,x) + o(1) \tag{3.27}$$

as n tends to ∞ and uniformly in $k \in \mathbb{N}$. We therefore have

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \mathsf{E}[(W^n_{s,t}f(x) - W^{n+k}_{s,t}f(x))^2] = 0, \tag{3.28}$$

i.e. $(W^n_{s,t}f(x),\ n \in \mathbb{N})$ is a Cauchy sequence in $L^2(\mathsf{P})$. This proves the lemma.
□

Remark 3.2.6 *For Arratia's coalescing flow, this lemma is not satisfied since* $(W^n_{s,t}f(x),\ n \in \mathbb{N})$ *fails to be a Cauchy sequence in* $L^2(\mathsf{P})$.

Thus, for all $s < t$, we have defined the vector field valued random variable $W_{s,t}$ such that $W_{s,t}f(x)$ is the $L^2(\mathsf{P})$-limit of $W^n_{s,t}f(x)$ for all $x \in M$ and all $f \in C(M)$. Then, using lemma 3.2.3, it is easy to see that $W = (W_{s,t},\ s \le t)$ is a vector field valued white noise of covariance C.

3.3 The stochastic flow of kernels solves a SDE.

In [18], it is shown that a vector field valued white noise W of covariance C can be constructed with a sequence of independent real white noises $(W^\alpha)_\alpha$ by the formula $W = \sum_\alpha V_\alpha W^\alpha$, where $(V^\alpha)_\alpha$ is an orthonormal basis of H_C, the self-reproducing space associated with C.

For all predictable (with respect to the filtration $(\mathcal{F}_{-\infty,t},\ t \in \mathbb{R})$) process $(H_t)_{t\in\mathbb{R}}$ taking its values in the dual of H_C, we define the stochastic integral of H with respect to W by the formula

$$\int_s^t H_u(W(du)) = \sum_\alpha \int_s^t \langle H_u, V_\alpha\rangle\ W^\alpha(du), \tag{3.29}$$

for $s < t$. Note that the above definition is independent of the choice of the orthonormal basis $(V^\alpha)_\alpha$.

In particular this applies to $H_u(V) = K_{s,u}(Vf)(x)1_{s\le u<t}$ for $f \in C_K(M)$ and $x \in M$. Then the stochastic integral $\sum_\alpha \int_s^t K_{s,u}(V^\alpha f)W^\alpha(du)$ is denoted

$$\int_s^t K_{s,u}(Wf(du))(x). \tag{3.30}$$

Remark 3.3.1 Note that the stochastic integral (3.30) is equal to the limit in $L^2(\mathsf{P})$ of $\sum_{k=0}^{2^n-1} K_{s,t_k^n}(W_{t_k^n t_{k+1}^n} f)(x)$ as n tends to ∞, where $t_k^n = s+k2^{-n}(t-s)$. Indeed,

$$\mathsf{E}\left[\left(\int_s^t K_{s,u}(Wf(du))(x) - \sum_{k=0}^{2^n-1} K_{s,t_k^n}(W_{t_k^n t_{k+1}^n} f)(x)\right)^2\right] =$$

$$= \sum_{k=0}^{2^n-1} \int_{t_k^n}^{t_{k+1}^n} \mathsf{P}^{(2)}_{t_k^n - s}(I + \mathsf{P}^{(2)}_{u-t_k^n} - 2I \otimes \mathsf{P}^{(1)}_{u-t_k^n})C(f,f)(x,x)\, du$$

which tends to 0 as n tends to ∞.

Proposition 3.3.2 *W is the unique vector field valued white noise such that for all $s < t$, $x \in M$ and $f \in C^2_K(M)$, P-almost surely,*

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(Wf(du))(x) + \int_s^t K_{s,u}(Af)(x)\, du. \qquad (3.31)$$

Note that giving the local characteristics of the flow is equivalent to giving this SDE. This SDE will be called the (A,C)-SDE. By a solution of the (A,C)-SDE, we mean a stochastic flow of kernels K and a vector field valued white noise W of covariance C such that (3.31) holds. Under some additional assumptions, we will give in section 4 a representation of all solutions of the (A,C)-SDE.

Proof. For all $s < t$, from remark 3.3.1,

$$\int_s^t K_{s,u}(Wf(du))(x) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} K_{s,t_k^n}(W_{t_k^n,t_{k+1}^n} f)(x) \qquad (3.32)$$

in $L^2(\mathsf{P})$, where $t_k^n = s + k2^{-n}(t-s)$.

Note that for all integers i,l,k and n such that $l \geq n$ and $k2^{l-n} \leq i \leq (k+1)2^{l-n} - 1$, the development (3.26) implies

$$\mathsf{E}[M_{t_i^l,t_{i+1}^l} f(x) M_{t_k^n,t_{k+1}^n} f(x)] = 2^{-l}(t-s)C(f,f)(x,x) + o(2^{-l}), \qquad (3.33)$$

uniformly in $x \in M$. This implies that for $l \geq n$,

$$\mathsf{E}\left[\left(\sum_{i=k2^{l-n}}^{(k+1)2^{l-n}-1} M_{t_i^l,t_{i+1}^l} f(x) - M_{t_k^n,t_{k+1}^n} f(x)\right)^2\right] = o(2^{-n}), \qquad (3.34)$$

uniformly in $x \in M$. Taking the limit as l goes to ∞, we get

$$\mathsf{E}[(W_{t_k^n,t_{k+1}^n}f(x) - M_{t_k^n,t_{k+1}^n}f(x))^2] = o(2^{-n}), \tag{3.35}$$

uniformly in $x \in M$. We use this estimate to prove that

$$\int_s^t K_{s,u}(W(du)f)(x) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} K_{s,t_k^n}(M_{t_k^n,t_{k+1}^n}f)(x) \tag{3.36}$$

in $L^2(\mathsf{P})$. This holds since

$$\begin{aligned}
\mathsf{E}\left[\left(\sum_{k=0}^{2^n-1} K_{s,t_k^n}(W_{t_k^n,t_{k+1}^n}f) - \sum_{k=0}^{2^n-1} K_{s,t_k^n}(M_{t_k^n,t_{k+1}^n}f)\right)^2(x)\right] &= \\
&= \sum_{k=0}^{2^n-1} \mathsf{E}[(K_{s,t_k^n}(W_{t_k^n,t_{k+1}^n}f - M_{t_k^n,t_{k+1}^n}f))^2(x)] \\
&\leq \sum_{k=0}^{2^n-1} \mathsf{P}^{(1)}_{t_k^n-s}\left(\mathsf{E}[(W_{t_k^n,t_{k+1}^n}f - M_{t_k^n,t_{k+1}^n}f)^2]\right)(x) \\
&\leq 2^n\, o(2^{-n}) = o(1).
\end{aligned}$$

Note now that $\sum_{k=0}^{2^n-1} K_{s,t_k^n}(M_{t_k^n,t_{k+1}^n}f)(x) \quad =$

$$\begin{aligned}
&= \sum_{k=0}^{2^n-1} K_{s,t_k^n}\left(K_{t_k^n,t_{k+1}^n}f - f - \int_{t_k^n}^{t_{k+1}^n} K_{t_k^n,u}(Af)\, du\right)(x) \\
&= K_{s,t}f(x) - f(x) - \int_s^t K_{s,u}(Af)(x)\, du.
\end{aligned}$$

This proves that $(K_{s,t},\ s \leq t)$ solves the (A,C)-SDE driven by W. Finally, note that if $(K_{s,t},\ s \leq t)$ solves the (A,C)-SDE driven by a vector field valued white noise W' then we must have $W' = W$. □

Let $X = (X_t,\ t \geq 0)$ be the Markov process defined in section 1.7 on $(\Omega \times C(\mathbb{R}^+, M), \mathcal{A} \otimes \mathcal{B}(C(\mathbb{R}^+, M)), \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega'))$ by $X(\omega,\omega') = \omega'$.

Proposition 3.3.3 *Assume there is no pure diffusion (i.e. for all* $f \in C_K^2(M)$ *and all* $x \in M$, $\Gamma(f)(x) = C(f,f)(x,x)$*). Then, for all* $t \geq 0$, $x \in M$ *and* $f \in C_K^2(M)$, $\mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega')$*-almost surely,*

$$f(X_t) = f(x) + \int_0^t W(du)f(X_u) + \int_0^t Af(X_u)\ du, \tag{3.37}$$

i.e. X *is a weak solution of this SDE.*

Proof. Like in the proof of (3.36) in proposition 3.3.2, we show that

$$\int_0^t W(du)f(X_u) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} M_{t_k^n,t_{k+1}^n} f(X_{t_k^n}) \tag{3.38}$$

in $L^2(\mathsf{P}_x)$, with $\mathsf{P}_x = \mathsf{P}(d\omega) \otimes \mathsf{P}_{x,\omega}(d\omega'))$. Let $M_t^f = f(X_t) - f(x) - \int_0^t Af(X_u)\ du$, then $(M_t^f,\ t \geq 0)$ is a martingale relative to the filtration $(\mathcal{F}_t^X,\ t \geq 0)$ generated by the Markov process X. We now prove that $\mathsf{E}_x[(M_t^f - \int_0^t W(du)f(X_u))^2] = 0$, where E_x denotes the expectation with respect to P_x. It is easy to see that, since there is no pure diffusion,

$$\mathsf{E}_x[(M_t^f)^2] = \mathsf{E}_x[(\int_0^t W(du)f(X_u))^2] = \mathsf{E}_x[\int_0^t C(f,f)(X_u,X_u)\ du]. \tag{3.39}$$

Equation (3.38) and the martingale property of M_t^f implies that

$$\begin{aligned}\mathsf{E}_x[M_t^f \int_0^t W(du)f(X_u)] &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x[M_{t_{k+1}^n}^f \times M_{t_k^n,t_{k+1}^n} f(X_{t_k^n})]. \qquad (3.40)\\ &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x[(M_{t_{k+1}^n}^f - M_{t_k^n}^f) \times M_{t_k^n,t_{k+1}^n} f(X_{t_k^n})].\end{aligned}$$

Since for all $0 \leq s < t$, $\mathsf{E}_x[M_t^f - M_s^f | \mathcal{A} \vee \mathcal{F}_s^X] = M_{s,t}f(X_s)$, we get

$$\begin{aligned}\mathsf{E}_x[M_t^f \int_0^t W(du)f(X_u)] &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \mathsf{E}_x[(M_{t_k^n,t_{k+1}^n} f(X_{t_k^n}))^2]\\ &= \mathsf{E}_x[(\int_0^t W(du)f(X_u))^2]. \qquad (3.41)\end{aligned}$$

Therefore $\mathsf{E}_x[(M_t^f - \int_0^t W(du)f(X_u))^2] = 0$. □

3.4 Strong solutions.

We say that $K = (K_{s,t},\ s \leq t)$ is a strong solution of the (A,C)-SDE if and only if it solves the (A,C)-SDE and for every $s \leq t$, $K_{s,t}$ is $\mathcal{F}^W_{s,t}$-measurable, where $\mathcal{F}^W_{s,t}$ is the completion by all P-negligible sets of $\mathcal{A}$ of the σ-field $\sigma(W_{u,v},\ s \leq u \leq v \leq t)$. This obviously only depends of P_ν the law of K, i.e. of $(\mathsf{P}^{(n)}_t,\ n \geq 1)$ or of $\nu = (\nu_t)_{t\geq 0}$. We will say they define a strong solution of the (A,C)-SDE.

Let us now consider the canonical flow associated with ν, a diffusion convolution semigroup. Let $N^W_\nu := (\Omega^0, \mathcal{A}^0, (\mathcal{F}^W_{s,t})_{s\leq t}, \mathsf{P}_\nu, (T_h)_{h\in\mathbb{R}})$ be the noise generated by the vector field valued white noise W. Note that N^W_ν is a linear or Gaussian [8] sub-noise of N_ν, the noise generated by the canonical flow.

Let $(\bar{K}_{s,t},\ s \leq t)$ be the stochastic flow of kernels obtained by filtering the canonical flow with respect to the sub-noise N^W (see section 1.6). It is easy to see that $(\bar{K}_{s,t},\ s \leq t)$ also solves the (A,C)-SDE (see the proof of lemma 3.7 in [18]) and has the same local characteristics as the canonical flow. Since, for all $s \leq t$, $\bar{K}_{s,t}$ is $\mathcal{F}^W_{s,t}$-measurable, it is a strong solution of the (A,C)-SDE. Let ν^W denote the associated diffusion convolution semigroup.

For any $f \in C_0(M)$ and $x \in M$, $\bar{K}_{s,t}f(x)$ can be expanded into a sum of Wiener chaos elements, i.e. iterated Wiener integrals of the form $\sum_{\alpha_1,\dots,\alpha_n} \int C^{\alpha_1,\dots,\alpha_n}(s_1,\dots,s_n)\ dW^{\alpha_n}_{s_n}\cdots dW^{\alpha_1}_{s_1}$. Since W was constructed from the flow, it is clear that the functions $C^{\alpha_1,\dots,\alpha_n}$ are determined by the law of the flow (we will give, under some additional assumptions, an explicit form of them in the following section).

Definition 3.4.1 *We say that strong uniqueness holds for the (A,C)-SDE when there is only one diffusion convolution semigroup with local characteristics (A,C) defining a strong solution.*

3.5 The Krylov-Veretennikov expansion.

In addition to our previous assumptions, we suppose in this section that there exists a Radon measure m on M such that $\mathsf{P}^{(1)}_t$ is symmetric with respect

[8] The noise $(\mathcal{G}_{s,t})_{s\leq t}$ is Gaussian if and only if there exists a countable family of independent Brownian motions $\{W^\alpha_t\}$ such that, up to negligible sets, $\mathcal{G}_{s,t}$ is generated by the random variables $W^\alpha_v - W^\alpha_u$ for all $s \leq u \leq v \leq t$ and all α.

to m (this would also apply if $\mathsf{P}^{(1)}_t$ was associated with a non-symmetric Dirichlet form). Let $(K_{s,t},\ s\le t)$ be a strong solution of the (A,C)-SDE and W the associated white noise (of covariance C). In [18], starting from such a vector field valued white noise W, we defined $(S_{s,t},\ s\le t)$ a stochastic flow of Markovian operators (acting on $L^2(m)$).

Proposition 3.5.1 *For all $s\le t$, $m\otimes\mathsf{P}$-a.e., for all $f\in C_K(M)$,*

$$K_{s,t}f=S_{s,t}f. \tag{3.42}$$

Thus strong uniqueness holds for the (A,C)-SDE.

Proof. Since $(K_{s,t},\ s\le t)$ is a strong solution, applying theorem 3.2 in [18] where strong uniqueness [9] is proved, equation (3.42) is satisfied for all continuous function with compact support f, $m\otimes\mathsf{P}$-a.e. □

Following [18], by induction, we define for all $f\in C_K(M)$, $x\in M$ and $s\le t$ a family of random variables $(\tilde S^n_{s,t}f(x),\ n\ge 0)$ in $L^2(\mathsf{P})$ such that $\mathsf{E}[(\tilde S^n_{s,t}f(x))^2]\le \mathsf{P}^{(1)}_{t-s}f^2(x)$: we take $\tilde S^0_{s,t}f(x)=\mathsf{P}^{(1)}_{t-s}f(x)$ and

$$\tilde S^{n+1}_{s,t}f(x)=\mathsf{P}^{(1)}_{t-s}f(x)+\int_s^t \tilde S^n_{s,u}(W(\mathsf{P}^{(1)}_{t-u}f)(du))(x). \tag{3.43}$$

Indeed, if the inequality $\mathsf{E}[(\tilde S^n_{s,t}f(x))^2]\le \mathsf{P}^{(1)}_{t-s}f^2(x)$ is satisfied for all $f\in C_K(M)$, then (using $C(f,f)(x,x)\le\Gamma(f)(x)$)

$$\mathsf{E}[(\tilde S^{n+1}_{s,t}f(x))^2]\le(\mathsf{P}^{(1)}_{t-s}f)^2(x)+\int_s^t \mathsf{P}^{(1)}_{u-s}(\Gamma(\mathsf{P}^{(1)}_{t-u}f))(x)\,du=\mathsf{P}^{(1)}_{t-s}f^2(x).$$

Then, for all $f\in C_K(M)$ and $x\in M$, we let $J^0_{s,t}f(x)$ be $\mathsf{P}^{(1)}_{t-s}f(x)$ and for $n\ge 1$, $J^n_{s,t}f(x)$ be $\tilde S^n_{s,t}f(x)-\tilde S^{n-1}_{s,t}f(x)$. Then $J^n_{s,t}f(x)$ can be written by the multiple stochastic integral

$$J^n_{s,t}f(x)=\int_{s\le s_1\le\cdots\le s_n\le t}\mathsf{P}^{(1)}_{s_1-s}W(ds_1)\mathsf{P}^{(1)}_{s_2-s_1}\cdots\mathsf{P}^{(1)}_{s_n-s_{n-1}}W(ds_n)\mathsf{P}^{(1)}_{t-s_n}f(x). \tag{3.44}$$

Note that we have the following Wiener chaos decomposition

$$S_{s,t}f=\sum_{n\ge 0}J^n_{s,t}f \quad\text{in}\quad L^2(m\otimes\mathsf{P}). \tag{3.45}$$

[9] In [18], strong uniqueness means that $(S_{s,t},\ s\le t)$ is the only flow of Markovian operators acting on $L^2(m)$ being a strong solution of the (A,C)-SDE.

Proposition 3.5.2 *Assume that* $\mathsf{P}_t^{(1)}$ *is absolutely continuous with respect to* m. *Then, for all* $x \in M$ *and* $f \in C_K(M)$, *the Wiener chaos decomposition* $K_{s,t}f(x) = \sum_{n\geq 0} J_{s,t}^n f(x)$ *holds* P-*a.s.*

Proof. Note that for all $s < t$, $\mathsf{P}_\varepsilon^{(1)} S_{s+\varepsilon,t} f(x)$ is a martingale as ε decreases. Indeed, since $\mathsf{P}_\varepsilon^{(1)}$ is absolutely continuous with respect to m and the fact that (3.42) holds,

$$\mathsf{P}_\varepsilon^{(1)} S_{s+\varepsilon,t} f(x) = \mathsf{P}_\varepsilon^{(1)} K_{s+\varepsilon,t} f(x) = \mathsf{E}[K_{s,t}f(x)|\mathcal{F}_{s+\varepsilon,t}]. \tag{3.46}$$

This martingale converges and its limit is $K_{s,t}f(x)$. Let us denote by $\bar{J}_{s,t}^n f(x)$ the n-th Wiener chaos of $K_{s,t}f(x)$. Then $\bar{J}_{s,t}^n f = J_{s,t}^n f$ in $L^2(m \otimes \mathsf{P})$. Finally, since

$$J_{s,t}^n f(x) = \lim_{\varepsilon \to 0} \mathsf{P}_\varepsilon^{(1)} J_{s+\varepsilon,t}^n f(x), \tag{3.47}$$

we must have $\bar{J}_{s,t}^n f(x) = J_{s,t}^n f(x)$. This proves the proposition. □

This proposition improves the results in [18] in the sense that (3.45) holds only in $L^2(m \otimes \mathsf{P})$ and here we get a modification of $(S_{s,t},\ s \leq t)$ such that its Wiener chaos decomposition holds in $L^2(\mathsf{P})$ for all $x \in M$.

4 Noise and classification.

4.1 Assumptions.

In this section, we fix a pair of local characteristics (A, C). We suppose there exists at least one diffusion convolution semigroup $\nu = (\nu_t)_{t\in\mathbb{R}}$ with these local characteristics.

Let $\mathcal{M}(n, x)$ be the following martingale problem associated with $A^{(n)}$ and $x \in M^n$:

There exists a probability space on which is constructed a stochastic process $X^{(n)} = (X_t^{(n)},\ t \geq 0)$ such that

$$f(X_t^{(n)}) - f(x) - \int_0^t A^{(n)} f(X_s^{(n)})\, ds \tag{4.48}$$

is a martingale for all test function f in $C_K^2(M) \otimes \cdots \otimes C_K^2(M)$ and where $A^{(n)}$ is the operator acting on $C_K^2(M) \otimes \cdots \otimes C_K^2(M)$ defined by (3.11).

In addition we suppose that the local characteristics (A, C) verify the following assumption

(U) For all $n \geq 1$, the martingale problem $\mathcal{M}(n, x)$ has a unique solution in law in the set of trajectories stopped at Δ_n.

Remark 4.1.1 Condition **(U)** is satisfied when the coefficients of the local characteristics are C^2 outside of Δ_n (see theorem 12.12 and section V.19 in [28]) or when $A^{(n)}$ is elliptic outside of Δ_n (see section V.24 in [28]).

Our purpose is to classify Feller convolution semigroups associated with these local characteristics. We will treat two cases

(a) The non coalescing case where the solution of the martingale problem $\mathcal{M}(2, x)$ does not hit the diagonal when $x = (x_1, x_2)$ with $x_1 \neq x_2$.

(b) The coalescing case where assumption **(C)** holds for $X_t^{(2)} = (X_t, Y_t)$ the solution of $\mathcal{M}(2, x)$ and there is no pure diffusion (i.e. $\Gamma(f)(x) = C(f, f)(x, x)$ for all $f \in C_K^2(M)$ and $x \in M$).

When the local characteristics are non coalescing, assumption **(U)** implies that these local characacteristics are associated with a unique convolution semigroup and a unique canonical flow. From section 3.4 we know the latter has to be a strong solution of the SDE (otherwise uniqueness would be violated).

Note that the family of semigroups given in the example of Lipschitz SDE's (see section 1.8) satisfies these assumptions.

4.2 The coalescing case : classification.

We assume **(C)** and that there is no pure diffusion.

Following Harris [13], $\mathcal{M}(n, x)$ has a unique solution in the set of coalescing trajectories, i.e. $X^{(n)}(\omega) \in C^{(n)}$ where $C^{(n)}$ is the set of continuous functions $f : \mathbb{R}^+ \to M^n$ such that if $f_i(s) = f_j(s)$ for $1 \leq i, j \leq n$ and $s \geq 0$ then for all $t \geq s$, $f_i(t) = f_j(t)$ (In [13], this martingale problem is solved when $M = \mathbb{R}$, but the proof can obviously be adapted to our framework).

Hence all coalescing flows with these local characteristics have the same law P_{ν^c}. They induce the same family of semigroups $(\mathsf{P}_t^{(n),c},\ n \geq 1)$ and the same diffusion convolution semigroup ν^c.

Let N_{ν^c} be the noise generated by the canonical coalescing flow $(\varphi_{s,t},\ s \leq t)$ associated with the local characteristics (A, C).

Let W be the vector field valued white noise defined on $(\Omega^0, \mathcal{A}^0, \mathsf{P}_{\nu^c})$ in section 3 and $N^W_{\nu^c}$ the sub-noise of N_{ν^c} generated by W. Then $N^W_{\nu^c}$ is a Gaussian sub-noise of N and it is possible to represent it by a countable family of independent real white noises $\{W^\alpha\}$ and $W = \sum_\alpha V_\alpha W^\alpha$, where $\{V_\alpha\}$ is a countable family of vector fields on M.

We denote by $\nu^{c,W}$ the diffusion convolution semigroup associated with the flow obtained by filtering the canonical coalescing flow of law P_{ν^c} with respect to $N^W_{\nu^c}$.

The following theorem gives a representation of all flows with the same local characteristics. They lie "between" the strong solution and the coalescing solution of the SDE which are distinct when the coalescing solution is not a strong solution of the SDE.

Theorem 4.2.1 *Suppose we are given a set of local characteristics* (A, C) *verifying* **(U)** *and* **(C)** *associated with at least one diffusion convolution semigroup.*

(a) ν^c *is the unique diffusion convolution semigroup associated with* (A, C) *and defining a flow of maps (which is coalescing).*

(b) $\nu^{c,W}$ *is the unique diffusion convolution semigroup associated with* (A, C) *and defining a strong solution of the* (A, C)*-SDE.*

(c) *The diffusion convolution semigroups associated with* (A, C) *are all the Feller convolution semigroups weakly dominated by* ν^c *and dominating* $\nu^{c,W}$.

Proof. We have already proved **(a)**. Theorem 2.3.2 implies that every diffusion convolution semigroup $\bar{\nu}$ with local characteristics (A, C) is weakly dominated by ν^c so that a stochastic flow $\bar{K}$ of law $\mathsf{P}_{\bar{\nu}}$ can be obtained by filtering on an extension (N, φ) of N_{ν^c} the coalescing flow φ with respect to a sub-noise $\bar{N}$ of N.

Proposition 3.3.2 shows that $\bar{K}$ solves the (A, C)-SDE driven by $\bar{W}$ a vector field valued white noise of covariance C. Notice that $\bar{W}$ can be obtained by filtering W with respect to $\bar{N}$. Indeed section 3.2 shows that $\bar{W}^n_{s,t}$ (defined from $\bar{K}$) converges (in L^2) towards $\bar{W}_{s,t}$ and we have that for all $s \leq t$,

$f \in C^2_K(M)$ and $x \in M$, $\bar{W}^n_{s,t} f(x) = \mathsf{E}[W^n_{s,t} f(x)|\bar{\mathcal{F}}_{s,t}]$ a.s. and therefore that $\bar{W}_{s,t} f(x) = \mathsf{E}[W_{s,t} f(x)|\bar{\mathcal{F}}_{s,t}]$ a.s. Since $\bar{W}$ and W have the same law, we must have $W_{s,t} = \bar{W}_{s,t}$ a.s. This proves that $\bar{\nu}$ dominates $\nu^{c,W}$.

Let us now suppose that $\bar{K}$ is a strong solution of the (A, C)-SDE driven by $\bar{W}$ (i.e. that $\bar{\nu}$ defines a strong solution). Then, since $\bar{W} = W$, we must have $N^W_{\nu^c} = \bar{N}$ (since $\bar{K}_{s,t}$ is $\mathcal{F}^W_{s,t}$-measurable) and thus $\nu^{c,W} = \bar{\nu}$. This proves the strong uniqueness for the (A, C)-SDE.

Finally let $\bar{\nu}$ be a Feller convolution semigroup weakly dominated by ν^c and dominating $\nu^{c,W}$. The fact that $\bar{\nu} \preceq^w \nu^c$ implies that a stochastic flow $\bar{K}$ of law $\mathsf{P}_{\bar{\nu}}$ can be obtained by filtering on an extension (N, φ) of N_{ν^c} the coalescing flow φ with respect to a sub-noise $\bar{N}$ of N. Then section 3.2 shows that $\bar{W}^n_{s,t}$ (defined from $\bar{K}$) converges (in L^2) towards $\bar{W}_{s,t} = \mathsf{E}[W_{s,t}|\bar{\mathcal{F}}_{s,t}]$. Now, since $\bar{\nu} \succeq \nu^{c,W}$, there exists (see lemma 1.6.5) a sub-noise $\bar{\bar{N}}$ of $\bar{N}$ such that the flow obtained by filtering $\bar{K}$ or equivalently, the coalescing flow, with respect to $\bar{\bar{N}}$ has law $\mathsf{P}_{\nu^{c,W}}$. The associated white noise $\bar{\bar{W}}$ verifies for all $s \le t$, $x \in M$ and $f \in C^2_K(M)$

$$\bar{\bar{W}}_{s,t} f(x) = \mathsf{E}[\bar{W}_{s,t} f(x)|\bar{\bar{\mathcal{F}}}_{s,t}] = \mathsf{E}[W_{s,t} f(x)|\bar{\bar{\mathcal{F}}}_{s,t}]. \tag{4.49}$$

Since $\bar{\bar{W}}$ has covariance C, it has to coincide with W and $\bar{W} = W$.

Thus, $\bar{K}$ solves the (A, C)-SDE driven by W so that $\bar{\nu}$ is a diffusion convolution semigroup whose local characteristics are (A, C). □

4.3 The coalescing case : martingale representation.

On the probability space $(\Omega^0, \mathcal{A}^0, \mathsf{P}_{\nu^c})$, let $\mathcal{F}^{\nu^c}$ be the filtration $(\mathcal{F}^{\nu^c}_{0,t})_{t\ge 0}$ and $\mathcal{M}(\mathcal{F}^{\nu^c})$ be the space of locally square integrable $\mathcal{F}^{\nu^c}$-martingales.

Proposition 4.3.1 *All $\mathcal{F}^{\nu^c}$-martingale $M = (M_t)_{t\in\mathbb{R}^+}$ has the predictable representation property : There exist predictable processes $\Phi^\alpha = (\Phi^\alpha_s)_{s\ge 0}$ such that*

$$M_t = \sum_\alpha \int_0^t \Phi^\alpha_s \; W^\alpha(ds). \tag{4.50}$$

Proof. We follow an argument by Dellacherie (see Rogers-Williams (V-25) [28]). Suppose there exists $F \in L^2(\mathcal{F}^{\nu^c}_{0,\infty})$ orthogonal in $L^2(\mathcal{F}^{\nu^c}_{0,\infty})$ to all stochastic integrals of $(W^\alpha)_\alpha$ of the form (4.50), then $M_t = \mathsf{E}[F|\mathcal{F}^{\nu^c}_{0,t}]$ is orthogonal to W^α for all α, i.e. $\langle M, W^\alpha_{0,\cdot}\rangle_t = 0$.

Let $\tau = \inf\{t,\ |M_t| = 1/2\}$ and $\hat{\mathsf{P}}_{\nu^c} = (1 + M_\tau)\cdot \mathsf{P}_{\nu^c}$. Since M is a uniformly integrable martingale and τ a stopping time (with $1 + M_\tau \geq 1/2$), $\hat{\mathsf{P}}_{\nu^c}$ is a probability measure on $(\Omega, \mathcal{A})$. Since $\langle M, W^\alpha_{0,\cdot}\rangle_t = 0$, we get that under $\hat{\mathsf{P}}_{\nu^c}$, $(W^\alpha_{0,t})_\alpha$ is a family of independent Brownian motions.

We are now going to prove that since **(U)** is satisfied, we must have $\mathsf{P}_{\nu^c} = \hat{\mathsf{P}}_{\nu^c}$, which implies $M_t = 0$ and a contradiction.

Let $F = \prod_{i=1}^n f_i(\varphi_{0,t_i}(x_i))$, for $f_1, \dots, f_n$ in $C^2_K(M)$, $t_1, \dots, t_n$ in $\mathbb{R}^+$ and $x_1, \dots, x_n$ in M. We know that under P_{ν^c}, for all $1 \leq i \leq n$, $(\varphi_{0,t}(x_i),\ t \geq 0)$ is a solution of the SDE

$$dg_i(\varphi_{0,t}(x_i)) = \sum_\alpha V_\alpha g_i(\varphi_{0,t}(x_i)) W^\alpha(dt) + Af(\varphi_{0,t}(x_i))dt, \tag{4.51}$$

for all $g_1, \dots, g_n$ in $C^2_K(M)$. Note that under $\hat{\mathsf{P}}_{\nu^c}$, these SDEs are also satisfied. Since under $\hat{\mathsf{P}}_{\nu^c}$, $(W^\alpha)_\alpha$ is a family of independent Brownian motions, $((\varphi_{0,t}(x_i),\ t \geq 0),\ 1 \leq i \leq n)$ is a coalescing solution of the martingale problem associated with $A^{(n)}$ and **(U)** implies that the law of $((\varphi_{0,t}(x_i),\ t \geq 0),\ 1 \leq i \leq n)$ is the same under P_{ν^c} and under $\hat{\mathsf{P}}_{\nu^c}$. Therefore $\hat{\mathsf{E}}[F] = \mathsf{E}[F]$, where $\hat{\mathsf{E}}$ denotes the expectation with respect to $\hat{\mathsf{P}}_{\nu^c}$.

To conclude that $\hat{\mathsf{P}}_{\nu^c} = \mathsf{P}_{\nu^c}$, we need to prove $\hat{\mathsf{E}}[F] = \mathsf{E}[F]$ with $F = \prod_{i=1}^n f_i(\varphi_{s_i,t_i}(x_i))$ for all $f_1, \dots, f_n$ in $C^2_K(M)$, $0 \leq s_i < t_i$ in $\mathbb{R}^+$ and $x_1, \dots, x_n$ in M. This can be proved the same way but using the kernel $\tilde{K}_t$ introduced in section 1.7. In this case $\tilde{K}_t = \delta_{\tilde{\varphi}_t}$, where $\tilde{\varphi}_t : \mathbb{R}^+ \times M \to \mathbb{R}^+ \times M$ is measurable. Then $F = \prod_{i=1}^n \tilde{f}_i(\tilde{\varphi}_{t_i}(s_i, x_i))$ and $(\tilde{\varphi}_t(s_i, x_i),\ t \geq 0)$ is a solution of an SDE on $\mathbb{R}^+ \times M$. □

4.4 The coalescing case : the linear noise.

Note that for all diffusion convolution semigroup ν, N_ν is a predictable noise (see proposition 1.7), i.e. $\mathcal{M}(\mathcal{F}^\nu)$ is formed of continuous martingales (in particular, a Gaussian noise is predictable). Following Tsirelson [30], a linear representation of a predictable noise $N = (\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{s\leq t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a family of real random variables $(X_{s,t},\ s \leq t)$ such that

(a) $X_{s,t} \circ T_h = X_{s+h,t+h}$ for all $s \leq t$ and all $h \in \mathbb{R}$,

(b) $X_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for all $s \leq t$,

(c) $X_{r,s} + X_{s,t} = X_{r,t}$ a.s., for all $r \leq s \leq t$.

The space of linear representations is a vector space. Equipped with the norm $\|(X_{s,t})_{s\leq t}\| = (\mathsf{E}[|X_{0,1}|^2])^{\frac{1}{2}}$, it is a Hilbert space we denote by H_{lin}. Let H^0_{lin} be the orthogonal in H_{lin} of the one-dimensional vector space constituted of the representation $X_{s,t} = v(t-s)$ for $v \in \mathbb{R}$, then H^0_{lin} is constituted with the centered linear representations. Note that if $(X_{s,t})_{s\leq t} \in H^0_{\text{lin}}$ with $\|(X_{s,t})_{s\leq t}\| = 1$, then $(X_{0,t})_{t\geq 0}$ is a standard Brownian motion. The Hilbert space H^0_{lin} is a Gaussian system and every $(X_{s,t})_{s\leq t} \in H^0_{\text{lin}}$ is a real white noise.

Note that if X and Y are orthogonal linear representations then X and Y are independent.

For all $-\infty \leq s \leq t \leq \infty$, let $\mathcal{F}^{\text{lin}}_{s,t}$ be the σ-field generated by the random variables $X_{u,v}$ for all $X \in H^0_{\text{lin}}$ and $s \leq u \leq v \leq t$, and completed by all P-negligible sets of $\mathcal{F}_{-\infty,+\infty}$. Then $N_{\text{lin}} := (\Omega, \mathcal{A}, (\mathcal{F}^{\text{lin}}_{s,t})_{s\leq t}, \mathsf{P}, (T_h)_{h\in\mathbb{R}})$ is a noise. It is called the linearizable part of the noise N. The noise N_{lin} is a maximal Gaussian sub-noise of N, hence N is Gaussian if and only if $N_{\text{lin}} = N$. When N_{lin} is trivial (i.e. constituted of trivial σ-fields), one says that N is a black noise (when N is not trivial).

Theorem 4.4.1 $N^W_{\nu^c} = N^{\text{lin}}_{\nu^c}$.

Proof. Let H^W be the space of centered linear representations of the noise $N^W_{\nu^c}$. Then H^W is an Hilbert space (an orthonormal basis of H^W is given by $\{(W^\alpha_{s,t})_{s\leq t}\}$) and we have $H^W \subset H^0_{\text{lin}}$. This implies that $N^W_{\nu^c}$ is a Gaussian sub-noise of $N^{\text{lin}}_{\nu^c}$.

If $N^W_{\nu^c} \neq N^{\text{lin}}_{\nu^c}$ then there exists a linear representation $(X_{s,t})_{s\leq t} \neq 0 \in H^0_{\text{lin}}$ orthogonal to H^W and therefore independent of $\{W^\alpha\}$. Since $(X_{0,t})_{t\geq 0} \in \mathcal{M}(\mathcal{F})$, proposition 4.3.1 implies that the martingale bracket of $X_{0,t}$ equals 0. This is a contradiction. □

In section 5, we give an example of a stochastic coalescing flow whose noise is predictable but not Gaussian, i.e. an example of non-uniqueness of the diffusion convolution semigroup associated with a set of local characteristics.

Remark 4.4.2 In example 2.4.3, although **(i)** is not satisfied, it is still possible to construct a white noise W from the coalescing flow $(\varphi_{s,t},\ s \leq t)$. For all $s < t$, we set $W_{s,t} = \int_s^t \operatorname{sgn}(\varphi_{s,u}(0))\ d\varphi_{s,u}(0)$. Then we have

$W_{s,t} = \int_s^t \mathrm{sgn}(\varphi_{s,u}(x))\, d\varphi_{s,u}(x)$ for all $x \in \mathbb{R}$. Therefore one can check that $W = (W_{s,t},\ s \le t)$ is a real white noise.

The coalescing flow $(\varphi_{s,t},\ s \le t)$ solves the SDE

$$\varphi_{s,t}(x) = \int_s^t \mathrm{sgn}(\varphi_{s,u}(x))\, dW_u, \text{ for } s < t \text{ and } x \in \mathbb{R}. \tag{4.52}$$

The results of this subsection apply since proposition 4.3.1 is also satisfied if we only assume the uniqueness in law of the coalescing solutions [10] of the SDE satisfied by the n-point motion (i.e. the SDE (4.51)), which here is almost obvious. Therefore, the linear part of the noise generated by the coalescing flow is given by the noise generated by W. But since the strong solution of the SDE (4.52) is not a flow of mappings, the coalescing flow is not a strong solution. Therefore, we recover the result of Watanabe [33] that the noise of this stochastic coalescing flow is predictable but not Gaussian.

The strong solution given in section 2.4.3 can be recovered by filtering the coalescing solution with respect to the noise generated by W.

5 Isotropic Brownian flows.

In this section, we give examples of compatible families of Feller semigroups. They are constructed on M, an homogeneous space, with C an isotropic covariance function on the space of vector fields and the semigroup of a Brownian motion on M.

5.1 Isotropic covariance functions.

Let $M = G/K$ be an homogeneous space. Then a covariance function C is said isotropic if and only if

$$C(g\cdot\xi, g\cdot\xi') = C(\xi,\xi') \tag{5.1}$$

for all $g \in G$ and $(\xi,\xi') \in (T^*M)^2$ and where $g\cdot\xi = Tg(\xi)$ (or $g\cdot(x,u) = (gx, Tg_x u)$ for $(x,u) \in T^*M$).

Examples of isotropic covariances are given by Monin and Yaglom in [22] on $\mathbb{R}^d$ and by Raimond [26, 27] on the sphere and on the hyperbolic plane. In

[10] i.e. such that if $(X^1,\dots,X^n)$ solves the SDE then if for $i \ne j$ and $s \ge 0$, $X^i_s = X^j_s$ then $X^i_t = X^j_t$ for all $t \ge s$.

these examples, the group G of isometries on $\mathbb{R}^d$ (making $\mathbb{R}^d$ homogeneous) is generated by $O(d)$ and by the translations. For the sphere $\mathbb{S}^d$, this group is $O(d+1)$ and for the hyperbolic space, it is $SO(d,1)$.

5.2 A compatible family of Markovian semigroups.

Let C be an isotropic covariance on $\mathcal{X}(M)$, the space of vector fields on an homogeneous space $M = G/K$. To this isotropic covariance function is associated a Brownian vector field on M (i.e. a $\mathcal{X}(M)$-valued Brownian motion W such that $\mathsf{E}[\langle W_t, \xi\rangle\langle W_s, \xi'\rangle] = t \wedge s\ C(\xi, \xi')$). Let P be the associated Wiener measure, constructed on the canonical space $\Omega = \{\omega : \mathbb{R}^+ \to \mathcal{X}(M)\}$, equipped with the σ-field $\mathcal{A}$ generated by the coordinate functions.

We denote by W the random variable $W(\omega) = \omega$, then W is a Brownian vector field of covariance C. W is isotropic in the sense that for all $g \in G$, $(dg_x^{-1} W_t(gx),\ t \in \mathbb{R}^+,\ x \in M)$ is a Brownian vector field of covariance C.

Notice that the restriction of C to the diagonal defines a G-invariant metric on M, we assume to be non-degenerate. Let P_t be the associated heat semigroup, m the associated volume element and Δ the associated Laplacian.

Let $(S_t,\ t \geq 0)$ be the family of random operators defined in [18], associated with W and to the heat semigroup P_t. Following [18], we define the associated semigroups of the n-point motion, $\mathsf{P}_t^{(n)} = \mathsf{E}[S_t^{\otimes n}]$ (with $\mathsf{P}_t^{(1)} = \mathsf{P}_t$). Then, it is obvious that $(\mathsf{P}_t^{(n)},\ n \geq 1)$ is a compatible family of Markovian semigroups of operators acting on $L^2(m)$. We now prove that these semigroups are induced by Feller semigroups (the question was raised in [19]).

One can extend $(W_t)_{t\geq 0}$ into a vector field valued white noise $(W_{s,t},\ s \leq t)$ of covariance C such that $W_t = W_{0,t}$ for $t \geq 0$ and associate to it a stationary cocycle of random operators $(S_{s,t},\ s \leq t)$ such that $S_{0,t} = S_t$ for $t \geq 0$.

5.3 Verification of the Feller property.

For all $g \in G$, let $L_g : \Omega \to \Omega$ defined by $L_g\omega_t(\cdot) = Tg^{-1}(\omega_t(g\cdot))$, for all $t \in \mathbb{R}$ and $x \in M$. Then L_g is linear and for all g_1 and g_2 in G, $L_{g_1 g_2} = L_{g_1} L_{g_2}$ (i.e. $g \mapsto L_g$ is a representation of G). It is easy to check that for all $g \in G$, $(L_g)^*\mathsf{P} = \mathsf{P}$. Note that this last condition is also a characterization that C is isotropic.

For all $g \in G$, L_g induces a linear transformation on $L^2(\Omega, \mathcal{A}, \mathsf{P})$ we will also denote by L_g. Then for all $f \in L^2(\Omega, \mathcal{A}, \mathsf{P})$, we have $L_g f(\omega) = f(L_g\omega)$.

This transformation is unitary since

$$\|L_g f\|^2 = \int f^2(L_g\omega)\ \mathsf{P}(d\omega) = \int f^2(\omega)\ ((L_g)^*\mathsf{P})(d\omega) = \|f\|^2,$$

(where $\|\cdot\|$ denotes the $L^2(\mathsf{P})$-norm), i.e. L_g is an unitary representation of G with carrier space $L^2(\Omega, \mathcal{A}, \mathsf{P})$.

Proposition 5.3.1 *For all $v \in L^2(\Omega, \mathcal{A}, \mathsf{P})$, the mapping $g \mapsto L_g v$ is continuous.*

Proof. Note that, since L is a representation, it is enough to prove the continuity at e, the identity element in G.

Remark 5.3.2 Note that if $(v_n,\ n \in \mathbb{N})$ is a sequence in $L^2(\Omega, \mathcal{A}, \mathsf{P})$ converging towards $v \in L^2(\Omega, \mathcal{A}, \mathsf{P})$ as $n \to \infty$ such that $\lim_{g\to e} L_g v_n = v_n$ for all integer n, then $\lim_{g\to e} L_g v = v$. Indeed, since for all $g \in G$, L_g is unitary, $\|L_g v - v\| \leq 2\|v_n - v\| + \|L_g v_n - v_n\|$. Hence $\limsup_{g\to e} \|L_g v - v\| \leq 2\|v_n - v\|$ for all integer n and we conclude using the convergence of v_n towards v.

We first prove that $\lim_{g\to e} L_g v = v$ for every v of the form $\sum_i W_{t_i}(\xi_i)$ (with $W_t(x,u) = \langle W_t(x), u\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Riemannian metric) :

$$\|L_g(\sum_i W_{t_i}(\xi_i)) - \sum_i W_{t_i}(\xi_i)\|^2 = 2\sum_{i,j} t_i \wedge t_j (C(\xi_i, \xi_j) - C(g\cdot\xi_i, \xi_j))$$

which converges towards 0 as g tends to e.

Let H denote the closure (in $L^2(\Omega, \mathcal{A}, \mathsf{P})$) of the class of all v of the form $\sum_i W_{t_i}(\xi_i)$. Remark 5.3.2 implies that $\lim_{g\to e} L_g v = v$ holds for all $v \in H$.

It is well known that $L^2(\Omega, \mathcal{A}, \mathsf{P})$ is the orthogonal sum of the Wick powers H^n of H (See [29]), also called the n-th Wiener chaos (see [23]), H^0 is constituted by the constants. The space H^n is isometric to the symmetric tensor product spaces $H^{\otimes^s n}$. We now prove that $\lim_{g\to e} L_g v = v$ holds for all $v \in H^n$. For all $v = v_1 \otimes^s \cdots \otimes^s v_n \in H^n$ (or : $v_1 v_2 \cdots v_n$:), with $v_1, \ldots, v_n$ in H,

$$\begin{aligned}
&\|L_g v - v\| \\
&\quad\leq \sum_j \|L_g v_1 \otimes^s \cdots \otimes^s L_g v_{j-1} \otimes^s (L_g v_j - v_j) \otimes^s v_{j+1} \otimes^s \cdots \otimes^s v_n\| \\
&\quad\leq \sqrt{n!} \sum_j \|L_g v_j - v_j\| \times \prod_{i\neq j} \|v_i\|
\end{aligned}$$

which converges towards 0 as g tends to e. Since the class of linear combinaisons of elements of the form $v_1 \otimes^s \cdots \otimes^s v_n$ is dense in H^n, we have $\lim_{g\to e} L_g v = v$ for all v in H^n. And we conclude since L_g is linear and $L^2(\Omega, \mathcal{A}, \mathsf{P}) = \oplus_{n\geq 0} H^n$. □

For all $x \in M$, $s \leq t$ and $f \in C_0(M)$, let $K_{s,t}f(x) = \lim_{\varepsilon\to 0} \mathsf{P}_\varepsilon S_{s+\varepsilon,t} f(x) = \sum_{n\geq 0} J^n_{s,t} f(x)$, where $J^n_{s,t}f(x)$ is an element of H^n defined by equation (3.44) (the proof of proposition 3.5.2 applies). Then, $S_{s,t}f = K_{s,t}f$ in $L^2(m \otimes \mathsf{P})$ and $\mathsf{P}^{(n)}_t = \tilde{P}^{(n)}_t$, where $\tilde{P}^{(n)}_t$ denotes $\mathsf{E}[K^{\otimes n}_{s,t}]$.

Lemma 5.3.3 *The mapping $x \mapsto K_{s,t}f(x)$ is continuous for all Lipschitz function f and all $s \leq t$.*

Proof. Note that for all $g \in G$ and all $x \in M$,

$$L_g K_{s,t} f(x) = K_{s,t} f^{g^{-1}}(gx) \tag{5.2}$$

where $f^{g^{-1}}(x) = f(g^{-1}x)$. We then have

$$\begin{aligned} \|K_{s,t}f(gx) - K_{s,t}f(x)\| \;&\leq\; \|K_{s,t}f(gx) - K_{s,t}f^{g^{-1}}(gx)\| \\ &\quad + \|L_g K_{s,t}f(x) - K_{s,t}f(x)\|. \end{aligned} \tag{5.3}$$

Hence $\lim_{g\to e} K_{s,t}f(gx) = K_{s,t}f(x)$ since $\lim_{g\to e} L_g K_{s,t}f(x) = K_{s,t}f(x)$ and $\|K_{s,t}f(gx) - K_{s,t}f^{g^{-1}}(gx)\| \leq \|f - f^{g^{-1}}\|_\infty$ which converges towards 0 (since $|f(x) - f^{g^{-1}}(x)| \leq C d(x, g^{-1}x) = \varepsilon(g^{-1})$). This implies the proposition. □

Proposition 5.3.4 **(a)** *$(\tilde{\mathsf{P}}^{(n)}_t,\ n \geq 1)$ is a compatible family of Feller semigroups.*

(b) *The associated convolution semigroup $\nu^W = (\nu^W_t)_{t\geq 0}$ is a diffusion convolution semigroup with local characteristics $(\frac{1}{2}\Delta, C)$.*

Proof. For all bounded Lipschitz functions $f_1, \ldots, f_n$, lemma 5.3.3 implies that $(x_1, \ldots, x_n) \mapsto \tilde{\mathsf{P}}^{(n)}_t f_1 \otimes \cdots \otimes f_n(x_1, \ldots, x_n) = \mathsf{E}[\prod_{i=1}^n K_{s,t} f_i(x_i)]$ is continuous. This suffices to prove **(a)** (the proof that $\lim_{t\to 0} \mathsf{P}^{(n)}_t h(x) = h(x)$ for all $h \in C(M^n)$ is the same as in lemma 1.8.1).

To prove **(b)**, notice that Itô's formula for $(S_{s,t},\ s \leq t)$ (see theorem 3.2 in [18]) implies that for all $f \in C^2_K(M)$ and $s \leq t$,

$$K_{s,t}f(x) = f(x) + \int_s^t K_{s,u}(Wf(du))(x) + \frac{1}{2}\int_s^t K_{s,u}(\Delta f)(x)\, du, \tag{5.4}$$

i.e. $(K_{s,t},\ s \leq t)$ solves the $(\frac{1}{2}\Delta, C)$-SDE. Thus **(ii)** is satisfied. □

5.4 Classification.

Let ν^W be the diffusion convolution semigroup constructed above. It defines a strong solution of the $(\frac{1}{2}\Delta, C)$-SDE. Note that there is no pure diffusion.

Let $(d_t)_{t\geq 0}$ denote the distance process induced by the 2-point motion $X_t^{(2)} = (X_t, Y_t)$ (then $d_t = d(X_t, Y_t)$). The isotropy condition implies that it is a real diffusion. We denote in the following the law of this diffusion starting from $x \geq 0$ by P_x. Let $H_x = \inf\{t > 0,\ d_t = x\}$.

Proposition 5.4.1 (a) *ν^W defines a non-coalescing flow of maps (i.e. such that the 2-point motion starting outside of the diagonal never hits the diagonal) if and only if 0 is a natural boundary point, i.e. if*

$$\forall x > 0,\ \mathsf{P}_x[H_0 < \infty] = 0 \textit{ and } \mathsf{P}_0[H_x < \infty] = 0. \tag{5.5}$$

(b) *ν^W defines a coalescing flow of maps if and only if 0 is a closed exit boundary point, i.e. if*

$$\exists x > 0,\ \mathsf{P}_x[H_0 < \infty] > 0 \textit{ and } \forall x > 0,\ \mathsf{P}_0[H_x < \infty] = 0. \tag{5.6}$$

(c) *ν^W defines a turbulent flow* [11] *without hitting (i.e. such that the 2-point motion starting outside of the diagonal never hits the diagonal) if and only if 0 is an open entrance boundary point, i.e. if*

$$\forall x > 0,\ \mathsf{P}_x[H_0 < \infty] = 0 \textit{ and } \exists x > 0,\ \mathsf{P}_0[H_x < \infty] > 0. \tag{5.7}$$

(d) *ν^W defines a turbulent flow with hitting (i.e. such that the 2-point motion starting outside of the diagonal hits the diagonal with a positive probability) if and only if 0 is a reflecting regular boundary point, i.e. if*

$$\exists x > 0,\ \mathsf{P}_x[H_0 < \infty] > 0 \textit{ and } \exists x > 0,\ \mathsf{P}_0[H_x < \infty] > 0. \tag{5.8}$$

In all cases except **(d)**, *ν^W is the unique diffusion convolution semigroup with local characteristics $(\frac{1}{2}\Delta, C)$.*

In case **(d)**, *called the intermediate phase, $\nu^c \neq \nu^W$ and theorems 4.2.1 and 4.4.1 apply. Thus N_{ν^c} is a predictable non-Gaussian noise.*

---

[11] We recall that a turbulent flow was defined as a stochastic flow of kernels which is not a flow of maps and without pure diffusion.

Proof. The proof of **(a)**, **(b)**, **(c)** and **(d)** is straightforward. Notice that the local characteristics satisfy **(U)**. In all cases, ν^W defines a strong solution of the $(\frac{1}{2}\Delta, C)$-SDE. This with proposition 5.4.1 implies that in the coalescing case **(b)**, since $\nu^W = \nu^c = \nu^{c,W}$, ν^W is the unique diffusion convolution semigroup whose local characteristics are $(\frac{1}{2}\Delta, C)$.

In the non-coalescing case **(a)** and in the turbulent case without hitting **(c)**, the fact that ν is the unique diffusion convolution semigroup whose local characteristics are $(\frac{1}{2}\Delta, C)$ follows directly from **(U)**.

In the intermediate phase **(d)**, **(C)** holds so that we can conclude using theorems 4.2.1 and 4.4.1. □

Remark 5.4.2 *The $(\frac{1}{2}\Delta, C)$-SDE has a solution, unique in law except in the intermediate phase, in which case all solutions are obtained by filtering, on an extension (N, φ) of the noise of the coalescing solution, this coalescing solution φ with respect to a sub-noise of N containing W.*

Remark 5.4.3 *The conditions involving the distance process can be verified using the speed and scale measures of this process which are explicitly determined by the spectral measures of the isotropic fields (cf [18] for $\mathbb{R}^d$ and for $\mathbb{S}^d$).*

5.5 Sobolev flows.

In [18], Sobolev flows $(S_{s,t},\ s \leq t)$ on $\mathbb{R}^d$ and on $\mathbb{S}^d$ are studied. The Sobolev covariances are described with two parameters $\alpha > 0$ and $\eta \in [0,1]$. The associated self-reproducing spaces are Sobolev spaces of vector fields of order $(d+\alpha)/2$. The incompressible and gradient subspaces are orthogonal and respectively weighted by factors η and $1-\eta$.

Let us apply the results obtained in [18]. We will call the stochastic flow associated with $(S_{s,t},\ s \leq t)$ (see section 3.5 and 5.3) Sobolev flow as well. When $\alpha > 2$, we are in case **(a)** and Sobolev flows are flows of diffeomorphisms. More interestingly, when $0 < \alpha < 2$ then

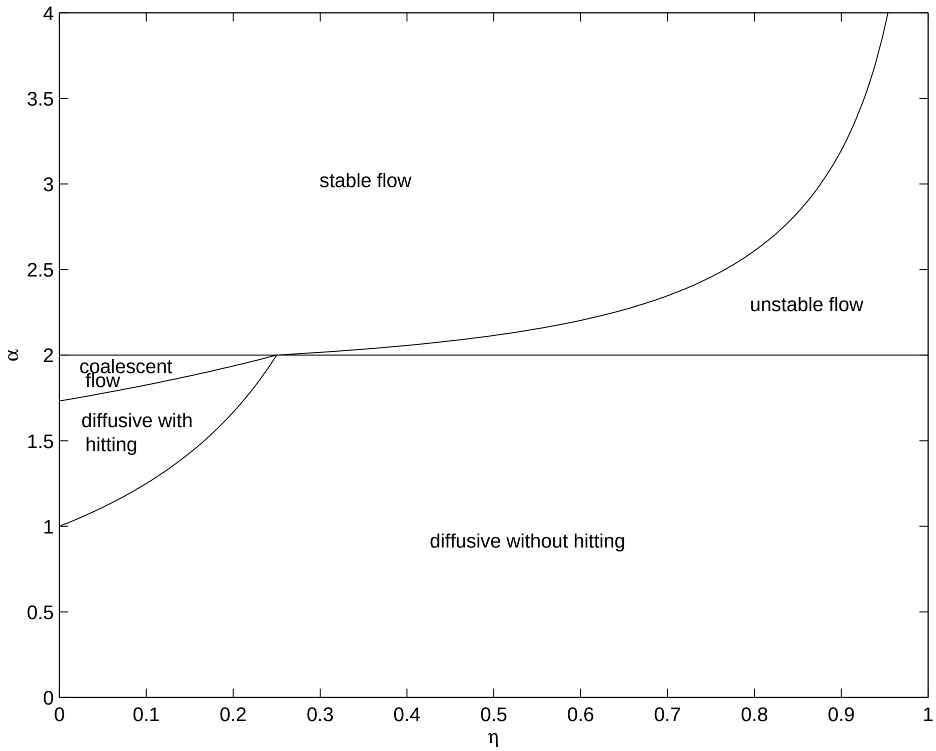
If $d \in \{2,3\}$ and $\eta < 1 - \frac{d}{\alpha^2}$, we are in case **(b)** and the Sobolev flow is a coalescing flow.

If $d \geq 4$ or if $d \in \{2,3\}$ and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we are in case **(c)** and the Sobolev flow is turbulent without hitting.

if $d \in \{2,3\}$ and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we are in case **(d)** (i.e. the intermediate phase) and the Sobolev flow is turbulent with hitting.

By construction, in all these cases, the noise generated by the Sobolev flows are Gaussian noises. And in the intermediate phase, the noise of the associated coalescing flow is predictable but not Gaussian.

These different cases are represented by the phase diagram below, for the homogeneous space S^3.



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