Scaling Limit, Noise, Stability

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Abstract

Linear functions of many independent random variables lead to classical noises (white, Poisson, and their combinations) in the scaling limit. Some singular stochastic flows and some models of oriented percolation involve very nonlinear functions and lead to nonclassical noises. Two examples are examined, Warren's 'noise made by a Poisson snake' and the author's 'Brownian web as a black noise'. Classical noises are stable, nonclassical are not. A new framework for the scaling limit is proposed. Old and new results are presented about noises, stability, and spectral measures.

Contents

Introduction

Functions of n independent random variables and limiting procedures for $n \to \infty$ are a tenor of probability theory.

Classical limit theorems investigate linear functions, such as $f(\xi_1, \dots, \xi_n) = (\xi_1 + \dots + \xi_n)/\sqrt{n}$. The well-known limiting procedure (a classical example of scaling limit) leads to the Brownian motion. Its derivative, the white noise, is not a continuum of independent random variables, but rather an infinitely divisible 'reservoir of independence', a classical example of a continuous product of probability spaces.

Percolation theory investigates some very special nonlinear functions of independent two-valued random variables, either in the limit of an infinite discrete lattice, or in the scaling limit. The latter is now making spectacular progress. The corresponding 'reservoir of independence' is already constructed for oriented percolation (which is much simpler). That is a modern, nonclassical example of a continuous product of probability spaces.

An essential distinction between classical and nonclassical continuous products of probability spaces is revealed by the concept of stability/sensitivity, framed for the discrete case by computer scientists and (in parallel) for the continuous case by probabilists. Everything is stable if and only if the setup is classical.

Some readers prefer discrete models, and treat continuous models as a mean of describing asymptotic behavior. Such readers may skip Sects. 6b, 6c, 8b, 8c, 8d. Other readers are interested only in continuous models. They may restrict themselves to Sects. 3d, 3e, 4i, 5b, 6, 7, 8.

Scaling limit. A new framework for the scaling limit is proposed in Sects. 1b, 2, 3a–3c.

Noise. The idea of a continuous product of probability spaces is formalized by the notions of 'continuous factorization' (Sect. 3d) and 'noise' (Sect. 3e). (Some other types of continuous product are considered in [18], [19].) For two nonclassical examples of noise see Sects. 4, 7.

Stability. Stability (and sensitivity) is studied in Sects. 5, 6a, 6d. For an interplay between discrete and continuous forms of stability/sensitivity, see especially Sects. 5c, 6d.

The spectral theory of noises, presented in Sects. 3c, 3d and used in Sects. 5, 6, generalizes both the Fourier transform on the discrete group $\mathbb{Z}_2^n$ (the Fourier-Walsh transform) and the Itô decomposition into multiple stochastic integrals. For the scaling limit of spectral measures, see Sect. 3c.

Throughout, either by assumption or by construction, all probability spaces will be Lebesgue-Rokhlin spaces; that is, isomorphic mod 0 to an interval with Lebesgue measure, or a discrete (finite or countable) measure space, or a combination of both.

1 A First Look

1a Two toy models

The most interesting thing is a scaling limit as a transition from a lattice model to a continuous model. A transition from a finite sequence to an infinite sequence is much simpler, but still nontrivial, as we'll see on simple toy models.

Classical theorems about independent increments are exhaustive, but a small twist may surprise us. I demonstrate the twist on two models, 'discrete' and 'continuous'. The 'continuous' model is a Brownian motion on the circle. The 'discrete' model takes on two values ± 1 only, and increments are treated multiplicatively: $X(t)/X(s)$ instead of the usual $X(t) - X(s)$. Or equivalently, the 'discrete' process takes on its values in the two-element group $\mathbb{Z}_2$; using additive notation we have $\mathbb{Z}_2 = \{0, 1\}$, $1 + 1 = 0$, increments being $X(t) - X(s)$. In any case, the twist stipulates values in a compact group (the circle, $\mathbb{Z}_2$, etc.), in contrast to the classical theory, where values are in $\mathbb{R}$ (or another linear space). Also, the classical theory assumes continuity (in probability), while our twist does not. The 'continuous' process (in spite of its name) is discontinuous at a single instant $t = 0$. The 'discrete' process is discontinuous at $t = \frac{1}{n}$, $n = 1, 2, \dots$, and also at $t = 0$; it is constant on $[\frac{1}{n+1}, \frac{1}{n})$ for every n.

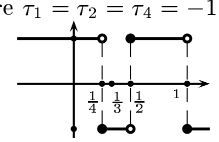
1a1 Example. Introduce an infinite sequence of random signs $\tau_1, \tau_2, \dots$; that is,

$$\mathbb{P}\left(\tau_k = -1 \right) = \mathbb{P}\left(\tau_k = +1 \right) = \frac{1}{2} \quad \text{for each } k,$$
$$\tau_1, \tau_2, \dots \quad \text{are independent.}$$

For each n we define a stochastic process $X_n(\cdot)$, driven by $\tau_1, \dots, \tau_n$, as follows:

$$X_n(t) = \prod_{k: 1/n \le 1/k \le t} \tau_k \,.$$

a sample path of X_4
(here $\tau_1 = \tau_2 = \tau_4 = -1, \tau_3 = +1$)



For $n \to \infty$, finite-dimensional distributions of X_n converge to those of a process $X(\cdot)$. Namely, X consists of countably many random signs, situated on intervals $[\frac{1}{k+1}, \frac{1}{k})$. Almost surely, X has no limit at $0+$. We have

$$\frac{X(t)}{X(s)} = \prod_{k:s<1/k\le t} \tau_k \tag{1a2}$$

whenever $0 < s < t < \infty$. However, (1a2) does not hold when $s < 0 < t$. Here, the product contains infinitely many factors and diverges almost surely; nevertheless, the increment $X(t)/X(s)$ is well-defined. Each X_n satisfies (1a2) for all s,t (including $s < 0 < t$; of course, $k \le n$), but X does not. Still, X is an independent increment process (multiplicatively); that is, $X(t_2)/X(t_1), \dots, X(t_n)/X(t_{n-1})$ are independent whenever $-\infty < t_1 < \dots < t_n < \infty$. However, we cannot describe the whole X by a countable collection of its independent increments. The infinite sequence of $\tau_k = X(\frac{1}{k}+)/X(\frac{1}{k}-)$ does not suffice since, say, $X(1)$ is independent of $(\tau_1, \tau_2, \dots)$. Indeed, the global sign change $x(\cdot) \mapsto -x(\cdot)$ is a measure-preserving transformation that leaves all τ_k invariant. The conditional distribution of $X(\cdot)$ given $\tau_1, \tau_2, \dots$ is concentrated at two functions of opposite global sign. It may seem that we should add to $(\tau_1, \tau_2, \dots)$ one more random sign τ_∞ independent of $(\tau_1, \tau_2, \dots)$ such that $X(\frac{1}{k})$ is a measurable function of $\tau_k, \tau_{k+1}, \dots$ and τ_∞. However, it is impossible. Indeed, $X(1) = \tau_1 \dots \tau_k X(\frac{1}{k})$. Assuming $X(\frac{1}{k}) = f_k(\tau_k, \tau_{k+1}, \dots; \tau_\infty)$ we get $f_1(\tau_1, \tau_2, \dots; \tau_\infty) = \tau_1 \dots \tau_{k-1} f_k(\tau_k, \tau_{k+1}, \dots; \tau_\infty)$ for all k. It follows that $f_1(\tau_1, \tau_2, \dots; \tau_\infty)$ is orthogonal to all functions of the form $g(\tau_1, \dots, \tau_n)h(\tau_\infty)$ for all n, and thus, to a dense (in L_2) set of functions of $\tau_1, \tau_2, \dots; \tau_\infty$; a contradiction.

So, for each n the process X_n is driven by (τ_k), but the limiting process X is not.

1a3 Example. (See also [3].) We turn to the other, the 'continuous' model. For any $\varepsilon \in (0,1)$ we introduce a (complex-valued) stochastic process

$$Y_\varepsilon(t) = \begin{cases} \exp\bigl(\mathrm{i}B(\ln t) - \mathrm{i}B(\ln \varepsilon)\bigr) & \text{for } t \ge \varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

where $B(\cdot)$ is the usual Brownian motion; or rather, $\bigl(B(t)\bigr)_{t\in[0,\infty)}$ and $\bigl(B(-t)\bigr)_{t\in[0,\infty)}$ are two independent copies of the usual Brownian motion. Multiplicative increments $Y_\varepsilon(t_2)/Y_\varepsilon(t_1), \dots, Y_\varepsilon(t_n)/Y_\varepsilon(t_{n-1})$ are independent whenever $-\infty < t_1 < \dots < t_n < \infty$, and the distribution of $Y_\varepsilon(t)/Y_\varepsilon(s)$ does not depend on ε as far as $\varepsilon < s < t$ (in fact, the distribution depends on t/s only). The distribution of $Y_\varepsilon(1)$ converges for $\varepsilon \to 0$ to the uniform distribution on the circle $|z| = 1$. The same for each $Y_\varepsilon(t)$. It follows easily that, when $\varepsilon \to 0$, finite dimensional distributions of Y_ε converge to those of some process Y. For every $t > 0$, $Y(t)$ is distributed uniformly on the circle; Y is an independent increment process (multiplicatively), and $Y(t) = 1$ for $t \le 0$. Almost surely, $Y(\cdot)$ is continuous on $(0, \infty)$, but has no limit at $0+$. We may define $B(\cdot)$ by

$$Y(t) = Y(1) \exp\bigl(\mathrm{i}B(\ln t)\bigr) \quad \text{for } t \in \mathbb{R}\,,$$
$$B(\cdot) \quad \text{is continuous on } \mathbb{R}\,.$$

Then B is the usual Brownian motion, and

$$\frac{Y(t)}{Y(s)} = \frac{\exp(\mathrm{i}B(\ln t))}{\exp(\mathrm{i}B(\ln s))} \quad \text{for } 0 < s < t < \infty\,.$$

However, $Y(1)$ is independent of $B(\cdot)$. Indeed, the global phase change $y(\cdot) \mapsto e^{i\alpha} y(\cdot)$ is a measure preserving transformation that leaves $B(\cdot)$ invariant. The conditional distribution of $Y(\cdot)$ given $B(\cdot)$ is concentrated on a continuum of functions that differ by a global phase (distributed uniformly on the circle). Similarly to the 'discrete' example, we cannot introduce a random variable $B(-\infty)$ independent of $B(\cdot)$, such that $Y(t)$ is a function of $B(-\infty)$ and increments of $B(r)$ for $-\infty < r < \ln t$.

So, for each ε, the process Y_ε is driven by the Brownian motion, but the limiting process Y is not.

Both toy models are singular at a given instant $t = 0$. Interestingly, continuous stationary processes can demonstrate such strange behavior, distributed in time! (See Sects. 4, 7).

1b Our limiting procedures

Imagine a sequence of elementary probabilistic models such that the n-th model is driven by a finite sequence $(\tau_1, \dots, \tau_n)$ of random signs (independent, as before). A limiting procedure may lead to a model driven by an infinite sequence $(\tau_1, \tau_2, \dots)$ of random signs. However, it may also lead to something else, as shown in 1a. This is an opportunity to ask ourselves: what do we mean by a limiting procedure?

The n-th model is naturally described by the finite probability space $\Omega_n = \{-1, +1\}^n$ with the uniform measure. A prerequisite to any limiting procedure is some structure able to join these Ω_n somehow. It may be a sequence of 'observables', that is, functions on the disjoint union,

$$f_k : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R}\,.$$

1b1 Example. Let $f_k(\tau_1,\dots,\tau_n) = \tau_k$ for $n \ge k$. Though f_k is defined only on $\Omega_k \uplus \Omega_{k+1} \uplus \dots$, it is enough. For every k, the joint distribution of $f_1,\dots,f_k$ on Ω_n has a limit for $n \to \infty$ (moreover, the distribution does not depend on n, as far as $n \ge k$). The limiting procedure should extend each f_k to a new probability space Ω such that the joint distribution of $f_1,\dots,f_k$ on Ω_n converges for $n \to \infty$ to their joint distribution on Ω. Clearly, we may take the space of infinite sequences $\Omega = \{-1,+1\}^\infty$ with the product measure, and let f_k be the k-th coordinate function.

1b2 Example. Still $f_k(\tau_1,\dots,\tau_n) = \tau_k$ (for $n \ge k \ge 1$), but in addition, the product $f_0(\tau_1,\dots,\tau_n) = \tau_1 \dots \tau_n$ is included. For every k, the joint distribution of $f_0, f_1,\dots,f_k$ on Ω_n has a limit for $n \to \infty$; in fact, the distribution does not depend on n, as far as $n > k$ (this time, not just $n \ge k$). Thus, in the limit, $f_0, f_1, f_2, \dots$ become independent random signs. The functional dependence $f_0 = f_1 f_2 \dots$ holds for each n, but disappears in the limit. We still may take $\Omega = \{-1,+1\}^\infty$, however, f_0 becomes a new coordinate.

This is instructive; the limiting model depends on the class of 'observables'.

1b3 Example. Let $f_k(\tau_1,\dots,\tau_n) = \tau_k \dots \tau_n$ for $n \ge k \ge 1$. In the limit, f_k become independent random signs. We may define τ_k in the limiting model by $\tau_k = f_k/f_{k+1}$; however, we cannot express f_k in terms of τ_k. Clearly, it is the same as the 'discrete' toy model of 1a.

The second and third examples are isomorphic. Indeed, renaming f_k of the third example as g_k (and retaining f_k of the second example) we have

$$g_k = \frac{f_0}{f_1 \dots f_{k-1}}\,; \qquad f_k = \frac{g_k}{g_{k+1}} \text{ for } k > 0\,, \quad \text{and} \quad f_0 = g_1\,;$$

these relations hold for every n (provided that the same $\Omega_n = \{-1,+1\}^n$ is used for both examples) and naturally, give us an isomorphism between the two limiting models.

That is also instructive; some changes of the class of 'observables' are essential, some are not.

It means that the sequence (f_k) is not really the structure responsible for the limiting procedure. Rather, f_k are generators of the relevant structure. The second and third examples differ only by the choice of generators for the same structure. In contrast, the first example uses a different structure. So, what is the mysterious structure?

I can describe the structure in two equivalent ways. Here is the first description. In the commutative Banach algebra $l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$ of all bounded functions on the disjoint union, we select a subset C (its elements will be called observables) such that

(1b4)

C is a separable closed subalgebra of $l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$ containing the unit.

In other words,

$$\begin{gathered}
C \text{ contains a sequence dense in the uniform topology;} \\
f_n \in C,\, f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C\,; \\
f, g \in C,\, a, b \in \mathbb{R} \quad \Longrightarrow \quad af + bg \in C\,; \\
\mathbf{1} \in C\,; \\
f, g \in C \quad \Longrightarrow \quad fg \in C
\end{gathered} \tag{1b5}$$

(here $\mathbf{1}$ stands for the unity, $\mathbf{1}(\omega) = 1$ for all ω). Or equivalently,

$$\begin{gathered}
C \text{ contains a sequence dense in the uniform topology;} \\
f_n \in C,\, f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C\,; \\
f, g \in C,\, \varphi : \mathbb{R}^2 \to \mathbb{R} \text{ continuous} \quad \Longrightarrow \quad \varphi(f, g) \in C\,.
\end{gathered} \tag{1b6}$$

Indeed, on one hand, both $af+bg$ and fg (and $\mathbf{1}$) are special cases of $\varphi(f, g)$. On the other hand, every continuous function on a bounded subset of $\mathbb{R}^2$ can be uniformly approximated by polynomials. The same holds for $\varphi(f_1, \dots, f_n)$ where $f_1, \dots, f_n \in C$, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Another equivalent set of conditions is also well-known:

$$\begin{gathered}
C \text{ contains a sequence dense in the uniform topology;} \\
f_n \in C,\, f_n \to f \text{ uniformly} \quad \Longrightarrow \quad f \in C\,; \\
f, g \in C,\, a, b \in \mathbb{R} \quad \Longrightarrow \quad af + bg \in C\,; \\
\mathbf{1} \in C\,; \\
f \in C \quad \Longrightarrow \quad |f| \in C\,;
\end{gathered} \tag{1b7}$$

here $|f|$ is the pointwise absolute value, $|f|(\omega) = |f(\omega)|$.

The smallest set C satisfying these (equivalent) conditions (1b4)–(1b7) and containing all given functions f_k is, by definition, generated by these f_k.

Recall that C consists of functions defined on the disjoint union of finite probability spaces Ω_n; a probability measure P_n is given on each Ω_n. The following condition is relevant:

$$\lim_{n\to\infty} \int_{\Omega_n} f \, dP_n \text{ exists for every } f \in C\,. \tag{1b8}$$

Assume that C is generated by given functions f_k. Then the property (1b8) of C is equivalent to such a property of functions f_k:

$$\text{For each } k, \text{ the joint distribution of } f_1, \dots, f_k \text{ on } \Omega_n \text{ weakly converges, when } n \to \infty. \tag{1b9}$$

Proof: (1b9) means convergence of $\int \varphi(f_1, \dots, f_k)\, dP_n$ for every continuous function $\varphi : \mathbb{R}^k \to \mathbb{R}$. However, functions of the form $f = \varphi(f_1, \dots, f_k)$ (for all k, φ) belong to C and are dense in C.

We see that (1b9) does not depend on the choice of generators f_k of a given C.

The second (equivalent) description of our structure is the 'joint compactification' of $\Omega_1, \Omega_2, \dots$ I mean a pair (K, α) such that

$$\begin{gathered} K \text{ is a metrizable compact topological space,} \\ \alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K \text{ is a map,} \\ \text{the image } \alpha(\Omega_1 \uplus \Omega_2 \uplus \dots) \text{ is dense in } K. \end{gathered} \tag{1b10}$$

Every joint compactification (K, α) determines a set C satisfying (1b4). Namely,

$$C = \alpha^{-1}\big(C(K)\big)\,;$$

that is, observables $f \in C$ are, by definition, functions of the form

$$f = g \circ \alpha, \text{ that is, } f(\omega) = g(\alpha(\omega)), \quad g \in C(K)\,.$$

The Banach algebra C is basically the same as the Banach algebra $C(K)$ of all continuous functions on K.

Every C satisfying (1b4) corresponds to some joint compactification. Proof: C is generated by some f_k such that $|f_k(\omega)| \le 1$ for all k, ω. We introduce

$$\begin{gathered} \alpha(\omega) = \big(f_1(\omega), f_2(\omega), \dots\big) \in [-1,1]^\infty\,, \\ K \text{ is the closure of } \alpha(\Omega_1 \uplus \Omega_2 \uplus \dots) \text{ in } [-1,1]^\infty\,; \end{gathered}$$

clearly, (K, α) is a joint compactification. Coordinate functions on K generate $C(K)$, therefore f_k generate $\alpha^{-1}\big(C(K)\big)$, hence $\alpha^{-1}\big(C(K)\big) = C$.

Finiteness of each Ω_n is not essential. The same holds for arbitrary probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$. Of course, instead of $l_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$ we use $L_\infty(\Omega_1 \uplus \Omega_2 \uplus \dots)$, and the map $\alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K$ must be measurable. It sends the given measure P_n on Ω_n into a measure $\alpha(P_n)$ (denoted also by $P_n \circ \alpha^{-1}$) on K. If measures $\alpha(P_n)$ weakly converge, we get the limiting model (Ω, P) by taking $\Omega = K$ and $P = \lim_{n\to\infty} \alpha(P_n)$.

1c Examples of high symmetry

1c1 Example. Let Ω_n be the set of all permutations $\omega : \{1, \dots, n\} \to \{1, \dots, n\}$, each permutation having the same probability ($1/n!$);

$$f : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R} \text{ is defined by}$$
$$f(\omega) = |\{k : \omega(k) = k\}| \,;$$

that is, the number of fixed points of a random permutation. Though f is not bounded, which happens quite often, in order to embed it into the framework of 1b, we make it bounded by some homeomorphism from $\mathbb{R}$ to a bounded interval (say, $\omega \mapsto \arctan f(\omega)$). The distribution of $f(\cdot)$ on Ω_n converges (for $n \to \infty$) to the Poisson distribution $P(1)$. Thus, the limiting model exists; however, it is scanty: just $P(1)$.

We may enrich the model by introducing

$$f_u(\omega) = |\{k < un : \omega(k) = k\}| \,;$$

for instance, $f_{0.5}(\cdot)$ is the number of fixed points among the first half of $\{1, \dots, n\}$. The parameter u could run over $[0, 1]$, but we need a countable set of functions; thus we restrict u to, say, rational points of $[0, 1]$. Now the limiting model is the Poisson process.

Each finite model here is invariant under permutations. Functions f_u seem to break the invariance, but the latter survives in their increments, and turns in the limit into invariance of the Poisson process (or rather, its derivative, the point process) under all measure preserving transformations of $[0, 1]$.

Note also that *independent* increments in the limit emerge from *dependent* increments in finite models.

We feel that all these $f_u(\cdot)$ catch only a small part of the information contained in the permutation. You may think about more information, say, cycles of length $1, 2, \dots$ (and what about length $n/2$?)

1c2 Example. Let Ω_n be the set of all graphs over $\{1, \dots, n\}$. That is, each $\omega \in \Omega_n$ is a subset of the set $\binom{\{1,\dots,n\}}{2}$ of all unordered pairs (treated as edges, while $1, \dots, n$ are vertices); the probability of ω is $p_n^{|\omega|}(1 - p_n)^{n(n-1)/2 - |\omega|}$, where $|\omega|$ is the number of edges. That is, every edge is present with probability p_n, independently of others. Define $f(\omega)$ as the number of isolated vertices. The limiting model exists if (and only if) there exists a limit $\lim_n n(1 - p_n)^{n-1} = \lambda \in [0, \infty)$;[1] the Poisson distribution $P(\lambda)$ exhausts the limiting model.

[1] Formally, the limiting model exists also for $\lambda = \infty$, since the range of f is compactified.

A Poisson process may be obtained in the same way as before.

You may also count small connected components which are more complicated than single points.

Note that the finite model contains a lot of independence (namely, $n(n-1)/2$ independent random variables); the limiting model (Poisson process) also contains a lot of independence (namely, independent increments). However, we feel that independence is not inherited; rather, the independence of finite models is lost in the limiting procedure, and a new independence emerges.

1c3 Example. Let $\Omega_n = \{-1, +1\}^n$ with uniform measure, and $f_n : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R}$ be defined by

$$f_u(\omega) = \frac{1}{\sqrt{n}} \sum_{k<un} \tau_k(\omega)\,;$$

as before, $\tau_1, \dots, \tau_n$ are the coordinates, that is, $\omega = \big(\tau_1(\omega), \dots, \tau_n(\omega)\big)$ and u runs over rational points of $[0,1]$. The limiting model is the Brownian motion, of course.

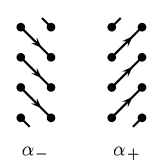
Similarly to 1c1, each finite model is invariant under permutations. The invariance survives in increments of functions f_k, and in the limit, the white noise (the derivative of the Brownian motion) is invariant under all measure preserving transformations of $[0,1]$.

A general argument of 6c will show that a high symmetry model cannot lead to a nonclassical scaling limit.

1d Example of low symmetry

Example 1c3 may be rewritten via the composition of random maps

$$\begin{gathered}\alpha_-, \alpha_+ : \mathbb{Z} \to \mathbb{Z}\,,\\ \alpha_-(k) = k-1\,, \quad \alpha_+(k) = k+1\,;\\ \alpha_\omega = \alpha_{\tau_n(\omega)} \circ \dots \alpha_{\tau_1(\omega)}\,;\end{gathered}$$



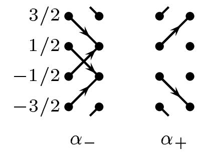
thus, $\alpha_\omega(k) = k + \tau_1(\omega) + \dots + \tau_n(\omega)$, and we may define $f_1(\omega) = \frac{1}{\sqrt{n}}\alpha_\omega(0)$, which conforms to 1c3. Similarly, $f_u(\omega) = \frac{1}{\sqrt{n}}\alpha_{\omega,u}(0)$, where $\alpha_{\omega,u}$ is the composition of $\alpha_{\tau_k(\omega)}$ for $k \le un$. The order does not matter, since α_-, α_+ commute, that is, $\alpha_- \circ \alpha_+ = \alpha_+ \circ \alpha_-$. It is interesting to try a pair of noncommuting maps.

1d1 Example. (See Warren [22].) Define

$$\alpha_-, \alpha_+ : \mathbb{Z} + \frac{1}{2} \to \mathbb{Z} + \frac{1}{2} \,,$$

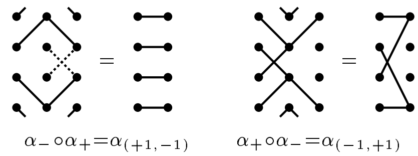
$$\left. \begin{aligned} \alpha_-(x) &= x - 1 \,, \\ \alpha_+(x) &= x + 1 \end{aligned} \right. \quad \text{for } x \in \big(\mathbb{Z} + \tfrac{1}{2}\big) \cap (0, \infty) \,,$$

$$\alpha_-(-x) = -\alpha_-(x) \,, \quad \alpha_+(-x) = -\alpha_+(x) \,.$$



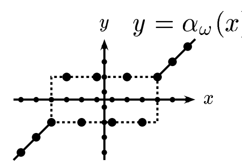
These are not invertible functions; α_- is not injective, α_+ is not surjective. Well, we do not need to invert them, but need their compositions:

$$\alpha_\omega = \alpha_{\tau_n(\omega)} \circ \dots \circ \alpha_{\tau_1(\omega)} \,.$$



All compositions belong to a two-parameter set of functions $h_{a,b}$,

$$\alpha_\omega(x) = h_{a,b}(x) = \begin{cases} x + a & \text{for } x \ge b, \\ x - a & \text{for } x \le -b, \\ (-1)^{b-x}(a+b) & \text{for } -b \le x \le b; \end{cases}$$



$$b, a + b \in \big(\mathbb{Z} + \tfrac{1}{2}\big) \cap (0, \infty) = \{\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \dots\} \,.$$

Indeed, $\alpha_- = h_{-1,1.5}$, $\alpha_+ = h_{1,0.5}$, and $h_{a_2,b_2} \circ h_{a_1,b_1} = h_{a,b}$ where $a = a_1 + a_2$, $b = \max(b_1, b_2 - a_1)$. Thus, $\alpha_\omega = h_{\alpha(\omega),b(\omega)}$, and we define

$$f_1 : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to \mathbb{R}^2 \times \{-1, +1\} \,,$$

$$f_1(\omega) = \left(\frac{a(\omega)}{\sqrt{n}}, \frac{b(\omega)}{\sqrt{n}}, (-1)^{b(\omega)-0.5} \right) .$$

However, the function is neither bounded nor real-valued; in order to fit into the framework of 1b we take, say, $\arctan\big(a(\omega)/\sqrt{n}\big)$, $\arctan\big(b(\omega)/\sqrt{n}\big)$, and $(-1)^{b(\omega)-0.5}$. The latter is essential if, say, $\frac{1}{\sqrt{n}}\alpha_\omega(0.5)$ is treated as an 'observable'; indeed, $\frac{1}{\sqrt{n}}\alpha_\omega(0.5) = (-1)^{b(\omega)-0.5}\frac{1}{\sqrt{n}}\big(a(\omega) + b(\omega)\big)$. The limiting model exists, and is quite interesting. (See also 8c.) As before, a random process appears by considering the composition over $k < un$.

Here, finite models are not invariant under permutations of their independent random variables (since the maps do not commute), and the limiting model appears not to be invariant under measure preserving transformations of $[0, 1]$.

Independence present in finite models survives in the limit, provided that the limit is described by a two-parameter random process; we'll return to this point in 4c.

1e Trees, not cubes

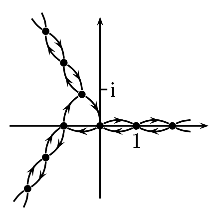
1e1 Example. A particle moves on the sphere S^2. Initially it is at a given point $x_0 \in S^2$. Then it jumps by ε in a random direction. That is, $X_0 = x_0$, while the next random variable X_1 is distributed uniformly on the circle $\{x \in S^2 : |x_0 - x| = \varepsilon\}$. Then it jumps again to X_2 such that $|X_1 - X_2| = \varepsilon$, and so on. We have a Markov chain (X_k) in discrete time (and continuous space). Let Ω_ε be the corresponding probability space; it may be the space of sequences $(x_0, x_1, x_2, \dots)$ satisfying $|x_k - x_{k+1}| = \varepsilon$, or something else, but in any case $X_k : \Omega_\varepsilon \to S^2$. We choose $\varepsilon_n \to 0$ (say, $\varepsilon_n = 1/n$), take $\Omega_n = \Omega_{\varepsilon_n}$ and define $f_u : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to S^2$ by

$$f_u(\omega) = X_k(\omega) \quad \text{for } \varepsilon_n^2 k \le u < \varepsilon_n^2 (k+1)\,, \quad \omega \in \Omega_n\,.$$

Of course, the limiting model is the Brownian motion on the sphere S^2.

In contrast to previous examples, here Ω_n is not a product; the n-th model does not consist of independent random variables. But, though we can parameterize these Markov transitions by independent random variables, there is a lot of freedom in doing so; none of the parameterizations may be called canonical. The same holds for the limiting model. The Brownian motion on S^2 can be driven by the Brownian motion on R^2 according to some stochastic differential equation, but the latter involves a lot of freedom.

1e2 Example. (See [12].) Consider the random walk on such an oriented graph:

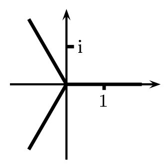


A particle starts at 0 and chooses at random (with probabilities 1/2, 1/2) one of the two outgoing edges, and so on (you see, exactly two edges go out of any vertex). Such $(Z_0, Z_1, \dots)$ is known as the simplest spider walk. It is a complex-valued martingale. The set Ω_n of all n-step trajectories contains 2^n elements and carries its natural structure of a binary tree. (It can be mapped to the binary cube $\{-1, +1\}^n$ in many ways.) We define $f_u : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to$

$\mathbb{C}$ by

$$f_u(\omega) = \frac{1}{\sqrt{n}} Z_k(\omega) \quad \text{for } k \le nu < k+1 \, , \quad \omega \in \Omega_n \, .$$

The limiting model is a continuous complex-valued martingale whose values belong to the union of three rays.



The process is known as Walsh's Brownian motion, a special case of the so-called spider martingale.

1f Sub-σ-fields

Every example considered till now follows the pattern of 1b; a joint compactification of probability spaces Ω_n, and the limiting Ω. Moreover, Ω_n is usually related to a set T_n (a parameter space, interpreted as time or space), and Ω to a joint compactification T of these T_n.

Example	T_n	T
1a1	$\{1, \frac{1}{2}, \dots, \frac{1}{n}\}$	$\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$
1a3	$[\varepsilon_n, 1]$	$[0, 1]$
1c1, 1c2, 1c3, 1d1, 1e1, 1e2	$\{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$	$[0, 1]$

Examples 1a1, 1a3, 1c3 deal (for a finite n) with independent increment processes, taking on their values in a group, namely, 1c3: $\mathbb{R}$ (additive); 1a1: $\{-1, +1\}$ (multiplicative), 1a3: the circle $\{z \in \mathbb{C} : |z| = 1\}$ (multiplicative). Every $t \in T_n$ splits the process into two parts, the past and the future; in order to keep them independent, we define them via increments, not values.[2] In terms of random signs τ_k (for 1a1, 1c3) it means simply $\{-1, +1\}^n = \{-1, +1\}^k \times \{-1, +1\}^{n-k}$; here k depends on t. The same idea (of independent parts) is formalized by sub-σ-fields $\mathcal{F}_{0,t}$ (the past) and $\mathcal{F}_{t,1}$ (the future) on our probability space (Ω_m or Ω). Say, for the Brownian motion 1c3, $\mathcal{F}_{0,t}$ is generated by Brownian increments on $[0, t]$, while $\mathcal{F}_{t,1}$ — on $[t, 1]$. Similarly we may define $\mathcal{F}_{s,t}$ for $s < t$, and we have

$$\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t} \quad \text{whenever } r < s < t \, .$$

[2] In fact, the process of 1a1 has also independent values (not only increments); but that is irrelevant.

It means two things: first, independence,

$$\mathbb{P}\big(A \cap B\big) = \mathbb{P}\big(A\big)\mathbb{P}\big(B\big) \quad \text{whenever } A \in \mathcal{F}_{r,s}, B \in \mathcal{F}_{s,t}\,;$$

and second, $\mathcal{F}_{r,t}$ is generated by $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ (that is, $\mathcal{F}_{r,t}$ is the least sub-σ-field containing both $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$). Such a two-parameter family $(\mathcal{F}_{s,t})$ of sub-σ-fields is called a *factorization* (of the given probability space). Some additional precautions are needed when dealing with semigroups (like 1d1), and also, with discrete time.

Sub-σ-fields $\mathcal{F}_A$ can be defined for some subsets $A \subset T$ more general than intervals, getting

$$\mathcal{F}_A \otimes \mathcal{F}_B = \mathcal{F}_C \quad \text{whenever } A \uplus B = C\,.$$

Models of high symmetry admit arbitrary measurable sets A; models of low symmetry do not. For some examples (such as 1c1, 1c2), a factorization emerges after the limiting procedure.[3]

No factorization at all is given for 1e1, 1e2. Still, the past $\mathcal{F}_{0,t} = \mathcal{F}_t$ is defined naturally. However, the future is not defined, since possible continuations depend on the past. Here we deal with a one-parameter family $(\mathcal{F}_t)$ of sub-σ-fields, satisfying only a monotonicity condition

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \text{whenever } s < t\,;$$

such $(\mathcal{F}_t)$ is called a *filtration.*

2 Abstract Nonsense of the Scaling Limit

2a More on our limiting procedures

The joint compactification K of $\Omega_1 \uplus \Omega_2 \uplus \dots$, used in 1b, is not quite satisfactory. Return to 1c3:

$$f_u(\omega) = \frac{1}{\sqrt{n}} \sum_{k<un} \tau_k(\omega) \quad \text{for } u \in [0,1] \cap \mathbb{Q} \tag{2a1}$$

($\mathbb{Q}$ being the set of rational numbers). The limiting model is the Brownian motion, restricted to $[0,1] \cap \mathbb{Q}$. What about an irrational point, $v \in [0,1] \setminus \mathbb{Q}$? The random variable f_v may be defined on Ω as the limit (say, in L_2) of f_u for $u \to v$, $u \in [0,1] \cap \mathbb{Q}$. On the other hand, f_v is naturally defined on

[3] For 1c2, some factorization is naturally defined for Ω_n, but is lost in the limiting procedure, and a new factorization emerges.

$\Omega_1 \uplus \Omega_2 \uplus \dots$ (by the same formula (2a1)). However, f_v is not a continuous function on the compact space K.[4] Thus, the weak convergence $P_i \to P$ is relevant to f_u but not f_v. Something is wrong!

What is wrong is the uniform topology used in (1b4)–(1b7). A right topology should take measures P_i into account. We have two ways, 'moderate' and 'radical'.

Here is the 'moderate' way. We choose some appropriate subsets $B_m \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, $B_1 \subset B_2 \subset \dots$, such that

$$\inf_i P_i(B_m \cap \Omega_i) \uparrow 1 \quad \text{for } m \to \infty$$

and in (1b5)–(1b7) replace the assumption "$f_n \in C$, $f_n \to f$ uniformly $\Longrightarrow f \in C$" with

$$f_n \in C,\ f_n \to f \text{ uniformly on each } B_m \quad \Longrightarrow \quad f \in C\,. \tag{2a2}$$

2a3 Example. Continuing (2a1) we define B_m by

$$B_m \cap \Omega_i = \left\{ \omega \in \Omega_i : \sup_{0 \le k < l \le i} \frac{\left| \frac{1}{\sqrt{i}} \sum_{j=k}^{l} \tau_j(\omega) \right|}{\left(\frac{l-k}{i} \right)^{1/3}} \le m \right\} ;$$

then[5]

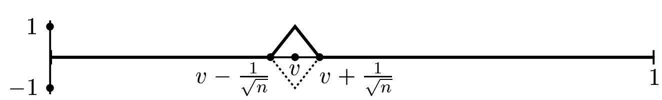
$$|f_u(\omega) - f_v(\omega)| \le m|u-v|^{1/3} \quad \text{for } \omega \in B_m \cap \Omega_i$$

if i is large enough (namely, $2/i < |u-v|$). The set C (satisfying (2a2)) generated by f_u for all rational u, also contains f_v for all irrational v.

Similarly to 1b, we may translate (2a2) into the topological language. For each m, the restriction of C to B_m corresponds to a joint compactification (K_m, α_m) of $B_m \cap \Omega_i$. Clearly, $K_{m_1} \subset K_{m_2}$ for $m_1 < m_2$, and $\alpha_{m_1} = \alpha_{m_2}|_{K_{m_1}}$. Thus, we get a *joint σ-compactification*

$$\alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K_\infty = K_1 \cup K_2 \cup \dots$$

[4] There exist $\omega_n \in \Omega_n$ such that $\lim_n f_u(\omega_n)$ exists for all $u \in [0,1] \cap \mathbb{Q}$, but $\lim_n f_v(\omega_n)$ does not exist.



[5] Of course, $|u-v|^\alpha$ for any $\alpha \in (0, 1/2)$ may be used, not only $|u-v|^{1/3}$.

We do not need a topology on the union K_∞ of metrizable compact spaces $K_1 \subset K_2 \subset \dots$[6] We just define $C(K_\infty)$ as the set of all functions $g : K_\infty \to \mathbb{R}$ such that $g|_{K_m}$ is continuous (on K_m) for each m. We have

$$C = \alpha^{-1}\big(C(K_\infty)\big) \,,$$

that is, observables $f \in C$ are functions of the form

$$f = g \circ \alpha \,, \quad \text{that is, } f(\omega) = g(\alpha(\omega)), \quad g \in C(K_\infty) \,.$$

If measures $\alpha(P_i)$ weakly converge (w.r.t. bounded functions of $C(K_\infty)$, recall (1b8), (1b9)), we get the limiting model (Ω, P) by taking $\Omega = K_\infty$ and $P = \lim_{i\to\infty} \alpha(P_i)$.

2a4 Example. Continuing 2a3 we see that the limiting measure P exists, and the joint distribution of all f_u (extended to K_∞ by continuity) w.r.t. P is the Wiener measure. The 'uniform' metric on K_∞,

$$\operatorname{dist}(x, y) = \sup_{0\le u\le 1} |f_u(x) - f_u(y)| \,,$$

is continuous on each K_m (intersected with the support of P). Therefore, every function continuous in the 'uniform' metric belongs to $C(K_\infty)$. Our joint σ-compactification is another form of the usual weak convergence of random walks to the Brownian motion.

That was the 'moderate way'. It requires special subsets $B_m \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, in contrast to the 'radical way'; basically, the latter allows the sequence of sets B_m to depend on a sequence of functions f_n, see (2a2). In other words, instead of uniform (or 'locally uniform') convergence, we introduce a weaker topology by the metric[7]

(2a5) $$\operatorname{dist}(f, g) = \sup_i \int \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|} \, \mathrm{d}P_i(\omega) \,.$$

[6] But if you want, K_∞ may be equipped with the inductive limit topology; that is, $U \subset K_\infty$ is open if and only if for every m, $U \cap K_m$ is open (in K_m). However, the topology usually is not metrizable.

[7] Alternatively, we may restrict ourselves to bounded functions $\Omega_1 \uplus \Omega_2 \uplus \dots \to [-1, +1]$ (applying a transformation like arctan) and use, say,

$$\operatorname{dist}(f, g) = \sup_i \int |f(\omega) - g(\omega)| \, \mathrm{d}P_i(\omega) \,.$$

If $f_n \in C(K)$ and $\operatorname{dist}(f_n, f) \to 0$ then f_n converge in probability w.r.t. P; thus, f is naturally defined P-almost everywhere.[8]

Let C be the closure of $C(K)$ in the metric (2a5). Then

$$\int \varphi(f_1, \dots, f_d)\, \mathrm{d}P_i \xrightarrow[i\to\infty]{} \int \varphi(f_1, \dots, f_d)\, \mathrm{d}P$$

for every d, every bounded continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and every $f_1, \dots, f_d \in C$. The joint distribution of $f_1, \dots, f_d$ w.r.t. P_i converges (weakly) to that w.r.t. P. So, the weak convergence $P_i \to P$ is relevant for the whole C (not only $C(K)$). That is the idea of the 'radical way', presented systematically in 2b, 2c.

Returning again to 1c3 we see that f_v (for $v \in [0,1]$) is the limit of f_u (for $u \in [0,1] \cap \mathbb{Q}$) in the metric (2a5); thus, $f_v \in C$ for all $v \in [0,1]$.

However, much more can be said. Not only

$$\operatorname{Lim}_{i\to\infty} \biggl(\frac{1}{\sqrt{i}} \sum_{ai<k<bi} \tau_k(\omega) \biggr) = \int_a^b \mathrm{d}B(t)\,,$$

where 'Lim' means the scaling limit (as explained above), but also

$$\begin{gathered}\operatorname{Lim}_{i\to\infty} \biggl(i^{-d/2} \sum_{ai<k_1<\dots<k_d<bi} \tau_{k_1}(\omega) \dots \tau_{k_d}(\omega) \biggr) \\ = \int\limits_{a<t_1<\dots<t_d<b} \!\!\cdots\!\! \int \mathrm{d}B(t_1) \dots \mathrm{d}B(t_d) = \frac{1}{d!} H_d\bigl(B(b)-B(a), b-a\bigr)\end{gathered}$$

where H_d is the Hermite polynomial (see for instance [11, IV.3.8]). Taking finite linear combinations and their closure in the metric (2a5) we get

$$\begin{gathered}\operatorname{Lim}_{i\to\infty} \biggl(\sum_{d=0}^\infty i^{-d/2} \sum_{0<k_1<\dots<k_d<i} \psi_d\bigl(\tfrac{k_1}{i}, \dots, \tfrac{k_d}{i}\bigr) \tau_{k_1}(\omega) \dots \tau_{k_d}(\omega) \biggr) \\ = \sum_{d=0}^\infty \int\limits_{0<t_1<\dots<t_d<1} \!\!\cdots\!\! \int \psi_d(t_1, \dots, t_d)\, \mathrm{d}B(t_1) \dots \mathrm{d}B(t_d)\end{gathered} \tag{2a6}$$

provided that functions ψ_d are Riemann integrable, and vanish for d large enough. The right-hand side is well-defined for all $\psi_d \in L_2$ such that

[8]In fact, every (equivalence class of) P-measurable function can be obtained in that way provided that, for each i, supports of P_i and P do not intersect. It means that every random variable on the limiting probability space is the scaling limit of some function on $\Omega_1 \uplus \Omega_2 \uplus \dots$ (see also 2c8).

$\sum_d \|\psi_d\|_2^2 < \infty$; the scaling limit may be kept by replacing $\psi_d\bigl(\frac{k_1}{i}, \dots, \frac{k_d}{i}\bigr)$ with the mean value of ψ_d on the $1/i$-cube centered at $\bigl(\frac{k_1}{i}, \dots, \frac{k_d}{i}\bigr)$. Now, $(0,1)$ may be replaced with the whole $\mathbb{R}$; ψ_d is defined on $\Delta_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 < \dots < x_d\}$. The right-hand side of (2a6) gives us an isometric linear correspondence between $L_2(\Delta_0 \uplus \Delta_1 \uplus \Delta_2 \uplus \dots)$ and $L_2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the probability space describing the Brownian motion (on the whole $\mathbb{R}$).

2b Coarse probability space: definition and simple example

2b1 Definition. A *coarse probability space* $\bigl((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\bigr)$ consists of a sequence of probability spaces $(\Omega[i], \mathcal{F}[i], P[i])$ and a set $\mathcal{A}$ of subsets of the disjoint union $\Omega[\text{all}] = \Omega(1) \uplus \Omega(2) \uplus \dots$, satisfying the following conditions:

(a) $\forall A \in \mathcal{A}\ \forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$;

(b) $\forall A, B \in \mathcal{A}\ \bigl(A \cap B \in \mathcal{A},\ A \cup B \in \mathcal{A},\ \Omega[\text{all}] \setminus A \in \mathcal{A}\bigr)$;

(c) $\mathcal{A}$ contains every $A \subset \Omega[\text{all}]$ such that $\forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$ and $P[i]\bigl(A \cap \Omega[i]\bigr) \to 0$ for $i \to \infty$;

(d) $\bigl(\cup_{k=1}^\infty A_k\bigr) \in \mathcal{A}$ for every pairwise disjoint $A_1, A_2, \dots \in \mathcal{A}$ such that $\sum_k \sup_i P[i]\bigl(A_k \cap \Omega[i]\bigr) < \infty$;

(e) $\lim_i P[i]\bigl(A \cap \Omega[i]\bigr)$ exists for every $A \in \mathcal{A}$;

(f) there exists a finite or countable subset $\mathcal{A}_1 \subset \mathcal{A}$ that *generates* $\mathcal{A}$ in the sense that the least subset of $\mathcal{A}$ satisfying (b)–(d) and containing $\mathcal{A}_1$ is the whole $\mathcal{A}$.

A set $\mathcal{A}$ satisfying (a)–(f) will be called a *coarse σ-field*[9] (on the *coarse sample space* $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$). Each set A belonging to the coarse σ-field $\mathcal{A}$ will be called *coarsely measurable* (w.r.t. $\mathcal{A}$), or a *coarse event.*

2b2 Remark. Condition 2b1(c) is equivalent to

(c1) $\forall i\ \mathcal{F}[i] \subset \mathcal{A}$. That is, if a set $A \subset \Omega[\text{all}]$ is contained in some $\Omega[i]$, and is $\mathcal{F}[i]$-measurable, then $A \in \mathcal{A}$.

Also, Condition 2b1(d) is equivalent to each of the following conditions (d1)–(d4). There, we assume that $A \subset \Omega[\text{all}]$, $\forall i\ \bigl(A \cap \Omega[i]\bigr) \in \mathcal{F}[i]$, and $\forall k\ A_k \in \mathcal{A}$.

(d1) If $A_k \uparrow A$ (that is, $A_1 \subset A_2 \subset \dots$ and $A = \cup_k A_k$) and $\sup_i P[i]\bigl((A \setminus A_k) \cap \Omega[i]\bigr) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

[9] It is not a σ-field, unless $\mathcal{A}$ contains all sets satisfying 2b1(a).

(d2) If $\sup_i P[i]\big((A \bigtriangleup A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$. (Here $A \bigtriangleup A_k = (A \setminus A_k) \cup (A_k \setminus A)$.)

(d3) If $A_k \uparrow A$ and $\limsup_i P[i]\big((A \setminus A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

(d4) If $\limsup_i P[i]\big((A \bigtriangleup A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

So, we have 10 equivalent combinations: (c)&(d), (c1)&(d), (c)&(d1), (c1)&(d1), (c)&(d2), ..., (c1)&(d4). (I omit the proof.)

However, "$\sup_i$" in (d) cannot be replaced with "$\limsup_i$".

2b3 Lemma. Let $\mathcal{A}_1$ be a finite or countable set satisfying 2b1(a,e) and

(b1) $\forall A, B \in \mathcal{A}_1 \ \big(A \cap B \in \mathcal{A}_1\big)$.

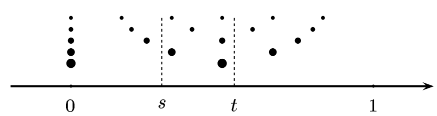
Then the least set $\mathcal{A}$ containing $\mathcal{A}_1$ and satisfying 2b1(b,c,d) is a coarse σ-field.

Proof. The algebra generated by $\mathcal{A}_1$ satisfies (e), since $P[i]\big((A \cup B) \cap \Omega[i]\big) = P[i](A \cap \Omega[i]) + P[i](B \cap \Omega[i]) - P[i]\big((A \cap B) \cap \Omega[i]\big)$. We enlarge the algebra according to (c), which preserves (e), as well as (a), (b). Finally, we enlarge it according to (d), which preserves (a), (b), (e); (c) and (f) hold trivially. $\square$

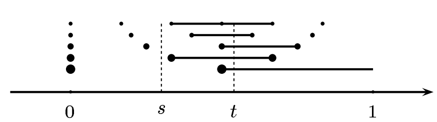
In such a case we say that the coarse σ-field $\mathcal{A}$ is *generated* by the set $\mathcal{A}_1$.

2b4 Example. Let $\Omega[i] = \{0, \frac{1}{i}, \dots, \frac{i-1}{i}\}$, and $P[i]$ be the uniform distribution on $\Omega[i]$. Every interval $(s,t) \subset (0,1)$ gives us a set $A_{s,t} \subset \Omega[\text{all}]$,

$$A_{s,t} \cap \Omega[i] = (s,t) \cap \Omega[i] \,.$$



We take a dense countable set of pairs (s,t) (say, rational s,t) and consider the set $\mathcal{A}_1$ of the corresponding $A_{s,t}$. The set $\mathcal{A}_1$ satisfies the conditions of 2b3, therefore it generates a coarse σ-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $A[i] + (0, 1/i) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; that is, $\operatorname{mes}\big(A[\infty] \bigtriangleup (A[i] + (0,1/i))\big) \to 0$ for $i \to \infty$.



If $A = A_{s,t}$ then, of course, $A[\infty] = (s,t)$.

2b5 Example. Continuing 1b1, we take $\Omega[i] = \{-1, +1\}^i$ with the uniform distribution $P[i]$. Given n and $a = (a_1, \dots, a_n) \in \{-1, +1\}^n$, we consider $A_a \subset \Omega[\text{all}]$,

$$A_a \cap \Omega[i] = \{(\tau_1, \dots, \tau_i) : \tau_1 = a_1, \dots, \tau_n = a_n\} \quad \text{for } i \ge n \,.$$

Such sets A_a (for all a and n) are a countable collection $\mathcal{A}_1$ satisfying the conditions of 2b3, therefore it generates a coarse σ-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $\beta_i^{-1}(A) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; here $\beta_i : (0,1) \to \Omega[i]$ is such a measure preserving map:

$$\beta_i(x) = \big((-1)^{c_1}, \dots, (-1)^{c_i}\big) \quad \text{when } x - \Big(\frac{c_1}{2} + \dots + \frac{c_i}{2^i}\Big) \in \Big(0, \frac{1}{2^i}\Big) \,,$$

for any $c_1, \dots, c_i \in \{0,1\}$.

You may guess that some limiting procedure produces a ('true', not coarse) probability space out of any given coarse probability space. Indeed, such a procedure, called 'refinement', is described in 2c.

2c Good use of joint compactification

Having a coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ and its refinement $(\Omega, \mathcal{F}, P)$ (to be defined later), we may hope that the Hilbert space $L_2[\infty] = L_2(\Omega, \mathcal{F}, P)$ is in some sense the limit of Hilbert spaces $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$. That is indeed the case in the framework of joint compactification, as we'll see. A bad use of the framework, tried in 1b, is a joint compactification of given probability spaces. A good use, considered here, is a joint compactification of metric (Hilbert, ...) spaces built over the given probability spaces.

2c1 Definition. A *coarse Polish space* is $\big((S[i], \rho[i])_{i=1}^\infty, c\big)$, where each $(S[i], \rho[i])$ is a Polish space (that is, a complete separable metric space[10]), and $c \subset S[1] \times S[2] \times \dots$ is a set of sequences $x = \big(x[1], x[2], \dots\big)$ satisfying the following conditions:

(a) if $x_1, x_2 \in S[1] \times S[2] \times \dots$ are such that $\rho[i]\big(x_1[i], x_2[i]\big) \to 0$ (for $i \to \infty$), then $(x_1 \in c) \iff (x_2 \in c)$;

(b) if $x, x_1, x_2, \dots \in S[1] \times S[2] \times \dots$ are such that $\sup_i \rho[i]\big(x_k[i], x[i]\big) \to 0$ (for $k \to \infty$), then $\big(\forall k \;\; x_k \in c\big) \implies \big(x \in c\big)$;

(c) $\lim_i \rho[i]\big(x_1[i], x_2[i]\big)$ exists for every $x_1, x_2 \in c$;

(d) there exists a finite or countable subset $c_1 \subset c$ that *generates* c in the sense that the least subset of c satisfying (a), (b) and containing c_1 is the whole c.

2c2 Remark. Condition 2c1(d) does not change if 'satisfying (a), (b)' is replaced with 'satisfying (b)'. That is, 2c1(d) is just separability of c in the metric $x_1, x_2 \mapsto \sup_i \rho[i]\big(x_1[i], x_2[i]\big)$.

[10] Many authors define a Polish space as a metrizable topological space admitting a complete separable metric. However, I assume that a metric is given.

The refinement of a coarse Polish space $\big((S[i],\rho[i])_{i=1}^\infty, c\big)$ is basically the metric space $\big(c,\tilde\rho\big)$, where

$$\tilde\rho(x_1,x_2) = \lim_i \rho[i]\big(x_1[i],x_2[i]\big)\,.$$

However, $\tilde\rho$ is a pseudometric (semimetric); it may vanish for some $x_1 \ne x_2$. The equivalence class, denoted by $x[\infty]$, of a sequence $x \in c$ consists of all $x_1 \in c$ such that $\rho[i]\big(x_1[i],x[i]\big) \to 0$. On the set $S[\infty]$ of all equivalence classes we introduce a metric $\rho[\infty]$,

$$\rho[\infty]\big(x_1[\infty],x_2[\infty]\big) = \lim_{i\to\infty} \rho[i]\big(x_1[i],x_2[i]\big)\,;$$

thus, $\big(S[\infty],\rho[\infty]\big)$ is a metric space. We write

$$\big(S[\infty],\rho[\infty]\big) = \operatorname{Lim}_{i\to\infty,c}\big(S[i],\rho[i]\big)$$

and call $\big(S[\infty],\rho[\infty]\big)$ the *refinement* of the coarse Polish space $\big((S[i],\rho[i])_{i=1}^\infty, c\big)$. Also, for every $x=(x[1],x[2],\dots)\in c$ we denote its equivalence class $x[\infty]\in S[\infty]$ by

$$x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]\,,$$

and call it the refinement of x.

2c3 Lemma. For every coarse Polish space, its refinement (S,ρ) is a Polish space.

Proof. Separability follows from 2c1(d); completeness is to be proven. Let $x_1,x_2,\dots$ be a Cauchy sequence in (S,ρ); we have to find $x\in S$ such that $\rho(x_k,x)\to 0$. We may assume that $\sum_k \rho(x_k,x_{k+1}) < \infty$. Each x_k is an equivalence class; using (a) we choose for each $k=1,2,3,\dots$ a representative $s_k \in S[1]\times S[2]\times\dots$ of x_k such that $\sup_i \rho[i]\big(s_k[i],s_{k+1}[i]\big) \le 2\rho(x_k,x_{k+1})$. Completeness of $\big(S[i],\rho[i]\big)$ ensures existence of $s_\infty[i]=\lim_k s_k[i]$. Condition (b) ensures $s_\infty\in c$. The equivalence class $x\in S$ of s_∞ satisfies $\rho(x_k,x)\le \sup_i \rho[i]\big(s_k[i],s_\infty[i]\big)\to 0$ for $k\to\infty$. $\square$

Let $(S[i],\rho[i])_{i=1}^\infty, c\big)$ be a coarse Polish space, and (S,ρ) its refinement. On the disjoint union $\big(S[1]\uplus S[2]\uplus\dots\big)\uplus S$ we introduce a topology, namely, the weakest topology making continuous the following functions $f_s : \big(S[1]\uplus S[2]\uplus\dots\big)\uplus S \to [0,\infty)$ for $s\in c$,

$$\begin{aligned} f_s(x) &= \rho[i]\big(x,s[i]\big) \quad \text{for } x\in S[i]\,,\\ f_s(x) &= \rho(x,s[\infty]) \quad \text{for } x\in S\,, \end{aligned}$$

and an additional function $f_0 : \big(S[1] \uplus S[2] \uplus \dots\big) \uplus S \to [0,\infty)$, $f_0(x) = 1/i$ for $x \in S[i]$, $f_0(x) = 0$ for $x \in S$. On every $S[i]$ separately (and also on S), the new topology coincides with the old topology, given by $\rho[i]$ (or ρ).

We may choose a sequence (s_k) dense in c; the topology is generated by functions f_{s_k} (and f_0), therefore it is a metrizable topology. Moreover, the sequence of functions $\big(\frac{f_{s_k}(\cdot)}{1+f_{s_k}(\cdot)}\big)_{k=1}^\infty$ (and f_0) maps the disjoint union into the metrizable compact space $[0,1]^\infty$, and is a homeomorphic embedding. Thus, we have a joint compactification of all $S[i]$ and S; and so, we treat them as subsets of a compact metrizable space K;

$$S[i] \subset K\,, \quad S \subset K\,.$$

2c4 Lemma. Let $s_\infty \in S$, $s_1 \in S[1], s_2 \in S[2], \dots$ Then $s_i \to s_\infty$ in K if and only if $s = (s_1, s_2, \dots) \in c$ and $\operatorname{Lim}_{i\to\infty,c} s_i = s_\infty$.

Proof. The 'if' part. The needed relation, $f_k(s_i) \to f_k(s_\infty)$ for $i \to \infty$, is ensured by 2c1(c).

The 'only if' part. We choose $x \in c$ such that $x[\infty] = s_\infty$; then $\rho[i]\big(s_i, x[i]\big) \to \rho\big(s_\infty, x[\infty]\big) = 0$, thus $s \in c$ by 2c1(a). $\square$

The assumption '$s_\infty \in S$' is essential. Other limiting points (not belonging to S) may exist; corresponding sequences converge in K but do not belong to c. And, of course, sets $S, S[1], S[2], \dots$ are not closed in K, unless they are compact.

2c5 Lemma. A set $c_1 \subset c$ generates c if and only if the set of refinements $\{x[\infty] : x \in c_1\}$ is dense in $S[\infty]$.

Proof. The 'only if' part follows from a simple argument: if S' is a closed subset of S then the set c' of all $x \in c$ such that $x[\infty] \in S'$ satisfies 2c1(a,b).

The 'if' part. Let $\{x[\infty] : x \in c_1\}$ be dense in $S[\infty]$ and $s \in c$. We choose $x_k \in c_1$ such that $x_k[\infty] \to s$. Similarly to the proof of 2c3, we construct $y_k \in c_1$ such that $\rho[i]\big(s_k[i], y_k[i]\big) \to 0$ when $i \to \infty$ for each k, and $\sup_i \rho[i]\big(y_k[i], s[i]\big) \to 0$ when $k \to \infty$. The subset of c generated by c_1 contains all y_k by 2c1(a). Thus, it contains s by 2c1(b). $\square$

Given continuous functions $f[i] : S[i] \to \mathbb{R}$, $f[\infty] : S[\infty] \to \mathbb{R}$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $f[i](x[i]) \to f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. If functions $f[i]$ are equicontinuous (say, $|f[i](x) - f[i](y)| \le \rho[i](x,y)$ for all i and $x, y \in S[i]$), then it is enough to check that $f[i](x_k[i]) \to f[\infty](x_k[\infty])$ for some sequence $(x_k)_{k=1}^\infty$, $x_k \in c$, such that the sequence $(x_k[\infty])_{k=1}^\infty$ is dense in $S[\infty]$.

Given continuous maps $f[i] : S[i] \to S[i]$, $f[\infty] : S \to S$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $\operatorname{Lim}_{i\to\infty,c} f[i](x[i]) = f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. That is, $\operatorname{Lim}\bigl(f[i](x[i])\bigr) = \bigl(\operatorname{Lim} f[i]\bigr)\bigl(\operatorname{Lim} x[i]\bigr)$. If maps $f[i]$ are equicontinuous then, again, convergence may be checked on x_k such that $x_k[\infty]$ are dense.

Given continuous maps $f[i] : S[\infty] \to S[i]$, we may ask whether $\operatorname{Lim}_{i\to\infty,c} f[i](x) = x$ for all $x \in S[\infty]$, or not. If maps $f[i]$ are equicontinuous then, still, convergence may be checked for a dense subset of $S[\infty]$.

If every $S[i]$ is not only a metric space but also a Hilbert (or Banach) space, and c is linear (that is, closed under linear operations), then the refinement S is also a Hilbert (or Banach) space, and linear operations are continuous on $\bigl(S[1] \cup S[2] \cup \dots\bigr) \cup S \subset K$ in the sense that

$$\operatorname{Lim}_{i\to\infty,c}(as_1[i] + bs_2[i]) = a \operatorname{Lim}_{i\to\infty,c} s_1[i] + b \operatorname{Lim}_{i\to\infty,c} s_2[i]$$

for all $s_1, s_2 \in c$.

Consider the case of Hilbert spaces $S[i] = H[i]$, $S = H$. Given linear[11] operators $R[i] : H[i] \to H[i]$, we may ask about $\operatorname{Lim} R[i]$. If it exists, we get

$$\operatorname{Lim}\bigl(R[i]x[i]\bigr) = \bigl(\operatorname{Lim} R[i]\bigr)\bigl(\operatorname{Lim} x[i]\bigr)\,.$$

If $\sup_i \|R[i]\| < \infty$, then $R[i]$ are equicontinuous, and convergence may be checked on a sequence x_k such that vectors $x_k[\infty]$ span H (that is, their linear combinations are dense in H). For example, one-dimensional orthogonal projections; if $x[\infty] = \operatorname{Lim} x[i]$ then $\operatorname{Proj}_{x[\infty]} = \operatorname{Lim} \operatorname{Proj}_{x[i]}$.

Given linear operators $R[i] : H \to H[i]$, we may ask whether $\operatorname{Lim} R[i](x) = x$ for all $x \in H$, or not. If $\sup_i \|R[i]\| < \infty$ then convergence may be checked on a sequence that spans H. Such $R[i]$ always exist; moreover, $\|R[i]\| \le 1$ may be ensured. Proof: we take x_k such that $x_k[\infty]$ are an orthonormal basis of H. After some correction, $x_k[i]$ become orthogonal (for each i), and $\|x_k(i)\| \le 1$.[12] Now we let $R[i]x_k[\infty] = x_k[i]$.

We return to coarse *probability* spaces.

Let $\bigl((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^{\infty}, \mathcal{A}\bigr)$ be a coarse probability space. For each i the pseudometric $A, B \mapsto P[i](A \triangle B)$ on $\mathcal{F}[i]$ gives us the metric space $\operatorname{MALG}[i] = \operatorname{MALG}\bigl(\Omega[i], \mathcal{F}[i], P[i]\bigr)$ of all equivalence classes of measurable sets. It is not only a metric space but also a Boolean algebra, and moreover, a separable measure algebra (as defined in [7, 17.44]). Treating every coarse event $A \in \mathcal{A}$ as a sequence of $A[1] \in \operatorname{MALG}[1], A[2] \in \operatorname{MALG}[2], \dots$ we get a coarse Polish space $\bigl((\operatorname{MALG}[i])_{i=1}^{\infty}, \mathcal{A}\bigr)$. Its refinement is a metric space

[11] Continuous, of course.

[12] Of course, $\|x_k[i]\| \to 1$ for $i \to \infty$, but in general we cannot ensure $\|x_k[i]\| = 1$. It may happen that $\dim H[i] < \infty$ but $\dim H = \infty$.

MALG[∞]. The set $\mathcal{A}$ is closed under Boolean operations (union, intersection, complement). Therefore MALG[∞] is not only a metric space but also a Boolean algebra. Using 2c3 it is easy to check that MALG[∞] is a separable measure algebra. Therefore [7, 17.44] it is (up to isomorphism) of the form

$$\mathrm{MALG}[\infty] = \mathrm{MALG}(\Omega, \mathcal{F}, P)$$

for some probability space $(\Omega, \mathcal{F}, P)$. In the nonatomic case we may take $(\Omega, \mathcal{F}, P) = (0, 1)$ with Lebesgue measure; in general, we may take a shorter (maybe, empty) interval plus a finite (maybe, empty) or countable set of atoms. Such a probability space $(\Omega.\mathcal{F}, P)$ (unique up to isomorphism) will be called the refinement of the coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^{\infty}, \mathcal{A}\big)$, and we write

$$(\Omega, \mathcal{F}, P) = \mathrm{Lim}_{i\to\infty,\mathcal{A}}\big(\Omega[i], \mathcal{F}[i], P[i]\big)$$

(in practice, sometimes I omit "$i \to \infty$" or "$\mathcal{A}$" or both under the "Lim").

Every sequence $A = (A[1], A[2], \dots) \in \mathcal{A}$ has its refinement

$$\mathrm{Lim}_{i\to\infty,\mathcal{A}}\, A[i] = A[\infty] \in \mathrm{MALG}(\Omega, \mathcal{F}, P)\,.$$

2c6 Lemma. A subset $\mathcal{A}_1$ of a coarse σ-field $\mathcal{A}$ generates $\mathcal{A}$ if and only if the refinement $\mathcal{F}$ of $\mathcal{A}$ is generated (mod 0) by refinements $A[\infty]$ of all $A \in \mathcal{A}_1$.

Proof. We apply 2c5 to the algebra generated by $\mathcal{A}_1$. $\square$

In order to define $L_2(\mathcal{A})$ as a set of functions on $\Omega[\text{all}]$, we start with indicators $\mathbf{1}_A$ for $A \in \mathcal{A}$, form their linear combinations, and take their completion in the metric

$$\|f\|_{L_2(\mathcal{A})} = \sup_i \|f[i]\|_{L_2[i]}\,,$$

where $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$; the completion is a Banach (not Hilbert) space $L_2(\mathcal{A})$. Each element f of the completion is evidently identified with a sequence of $f[i] \in L_2[i]$, or a function on $\Omega[\text{all}]$. We have a coarse Polish space $\big((L_2[i])_{i=1}^{\infty}, L_2(\mathcal{A})\big)$. It has its refinement, $L_2[\infty]$.

2c7 Lemma. The refinement $L_2[\infty]$ of $\big((L_2[i])_{i=1}^{\infty}, L_2(\mathcal{A})\big)$ is (canonically isomorphic to) $L_2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the refinement of $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^{\infty}, \mathcal{A}\big)$.

Proof. We define the canonical map $L_2(\mathcal{A}) \to L_2(\Omega, \mathcal{F}, P)$ first on indicators by $\mathbf{1}_A \mapsto \mathbf{1}_{A[\infty]}$, and extend it by linearity and continuity to the whole $L_2(\mathcal{A})$. We note that the image of $f \in L_2(\mathcal{A})$ in $L_2(\Omega, \mathcal{F}, P)$ depends only on the refinement $f[\infty] \in L_2[\infty]$ of f, and their norms are equal (both are equal to

$\lim_i \|f[i]\|$). We have a linear isometric embedding $L_2[\infty] \to L_2(\Omega, \mathcal{F}, P)$. Its image is closed (since $L_2[\infty]$ is complete by 2c3), and contains indicators $\mathbf{1}_B$ for all $B \in \mathrm{MALG}(\Omega, \mathcal{F}, P)$; therefore the image is the whole $L_2(\Omega, \mathcal{F}, P)$. $\square$

2c8 Remark. The same holds for L_p for each $p \in (0, \infty)$, and for the space L_0 of all random variables (equipped with the topology of convergence in probability). Elements of $L_0(\mathcal{A})$ will be called coarsely measurable (w.r.t. $\mathcal{A}$) functions (on $\Omega[\text{all}]$), or *coarse random variables;* elements of $L_2(\mathcal{A})$ — square integrable coarse random variables.

Let f be a coarse random variable. Then (usual) random variables $f[i] : \Omega[i] \to \mathbb{R}$ converge in distribution (for $i \to \infty$) to the refinement $f[\infty] : \Omega \to \mathbb{R}$. The distribution of $f[\infty]$ will be called the *limiting distribution* of f.

It may happen that $f \in L_2(\mathcal{A})$ but $(\mathrm{sgn} f) \notin L_2(\mathcal{A})$. An example: $f(\omega) = \frac{(-1)^i}{i}$ for all $\omega \in \Omega[i]$. Here, the limiting distribution is an atom at 0, and the function 'sgn' is discontinuous at 0.

2c9 Lemma. (a) Let $f : \Omega[\text{all}] \to \mathbb{R}$ be a coarse random variable, and $\varphi : \mathbb{R} \to \mathbb{R}$ a continuous function. Then $\varphi \circ f : \Omega[\text{all}] \to \mathbb{R}$ is a coarse random variable.

(b) The same as (a) but φ may be discontinuous at points of a set $Z \subset \mathbb{R}$, negligible w.r.t. the limiting distribution of f.

Proof. If f is a linear combination of indicators, then $\varphi \circ f$ is another linear combination of the same indicators. A straightforward approximation gives (a) for uniformly continuous φ. In general, for every ε there exists a compact set $K \subset \mathbb{R} \setminus Z$ of probability $\ge 1 - \varepsilon$ w.r.t. the limiting distribution, and also w.r.t. the distribution of $f[i]$ for all i (since all these distributions are a compact set of distributions). The restriction of f to K is uniformly continuous. The limit for $\varepsilon \to 0$ is uniform in i. $\square$

For a given Polish space S we may define a coarse S-valued random variable as a map $f : \Omega[\text{all}] \to S$ such that (usual) random variables $f[i] : \Omega[i] \to S$ converge in distribution (for $i \to \infty$), and $f^{-1}(B) \in \mathcal{A}$ for every $B \subset S$ such that the boundary of B is negligible w.r.t. the limiting distribution of f.

For $S = \mathbb{R}$ the new definition conforms with the old one.

A coarse σ-field generated by a given sequence of sets (coarse events) was defined after 2b3. Often it is convenient to generate a coarse σ-field by a sequence of functions (coarse random variables). A function $f : \Omega[\text{all}] \to \mathbb{R}$ is coarsely $\mathcal{A}$-measurable if and only if $\mathcal{A}$ contains sets $f^{-1}\big((-\infty, x)\big)$ for all

$x \in \mathbb{R}$ except for atoms (if any) of the limiting distribution of f. A dense countable subset of these x is enough. So, a coarse σ-field generated by a finite or countable set of functions f is nothing but the coarse σ-field generated by a countable set of sets of the form $f^{-1}\big((-\infty, x)\big)$. More generally, S-valued (coarse) random variables may be used; they are reduced to the real-valued case by composing with appropriate continuous functions $S \to \mathbb{R}$.

2c10 Lemma. A sequence of functions $f_k : \Omega[\text{all}] \to \mathbb{R}$ generates a coarse σ-field if and only if for every n, n-dimensional random variables $\big(f_1[i], \dots, f_n[i]\big) : \Omega[i] \to \mathbb{R}^n$ converge in distribution (for $i \to \infty$).

Proof. The 'only if' part. Let $f_1, \dots, f_n$ be coarsely measurable (w.r.t. some coarse σ-field), then they have a limiting joint distribution.

The 'if' part. For each n we choose a dense countable set $Q_n \subset \mathbb{R}$ negligible w.r.t. the limiting distribution of f_n. We apply 2b3 to the set $\mathcal{A}_1$ of coarse events of the form $\{f_1(\cdot) \le q_1, \dots, f_n(\cdot) \le q_n\}$ where $q_1 \in Q_1, \dots, q_n \in Q_n$, $n = 1, 2, \dots$ ☐

2c11 Remark. The same holds for an arbitrary Polish space instead of $\mathbb{R}$.

2c12 Remark. Comparing 2c10 and (1b9) we see that every joint compactification of $\Omega_1 \uplus \Omega_2 \uplus \dots$ (in the sense of 1b, assuming (1b8)) may be downgraded to a coarse probability space. Namely, we take a sequence of functions f_k that generates C and consider the coarse σ-field $\mathcal{A}$ generated by (f_k). Every $f \in C$ is a coarse random variable, since $L_0(\mathcal{A})$ is closed under all operations used in (1b5), (1b6), or (1b7).[13] Therefore $\mathcal{A}$ does not depend on the choice of (f_k).

3 Scaling Limit and Independence

3a Product of coarse probability spaces

Having two coarse probability spaces $\big((\Omega_1[i], \mathcal{F}_1[i], P_1[i])_{i=1}^\infty, \mathcal{A}_1\big)$ and $\big((\Omega_2[i], \mathcal{F}_2[i], P_2[i])_{i=1}^\infty, \mathcal{A}_2\big)$, we define their product as the coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ where for each i,

$$(\Omega[i], \mathcal{F}[i], P[i]) = (\Omega_1[i], \mathcal{F}_1[i], P_1[i]) \times (\Omega_2[i], \mathcal{F}_2[i], P_2[i])$$

is the usual product of probability spaces, and $\mathcal{A}$ is the smallest coarse σ-field that contains $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$, where $A_1 \times A_2 \subset \Omega[\text{all}]$ is defined by $\forall i \ \ (A_1 \times A_2)[i] = A_1[i] \times A_2[i]$. Existence of such $\mathcal{A}$ is ensured by 2b3. We write $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$.

[13] Of course, $L_0(\mathcal{A})$ usually contains no sequence dense in the *uniform* topology.

3a1 Lemma. The refinement of the product of two coarse probability spaces is (canonically isomorphic to) the product of their refinements.

Proof. Denote these refinements by $(\Omega_1, \mathcal{F}_1, P_1)$, $(\Omega_2, \mathcal{F}_2, P_2)$ and $(\Omega, \mathcal{F}, P)$. Both $\mathrm{MALG}(\Omega_1, \mathcal{F}_1, P_1)$ and $\mathrm{MALG}(\Omega_2, \mathcal{F}_2, P_2)$ are naturally embedded into $\mathrm{MALG}(\Omega, \mathcal{F}, P)$ as *independent* subalgebras. They generate $\mathrm{MALG}(\Omega, \mathcal{F}, P)$ due to 2c6.

$\square$

Given an arbitrary coarse σ-field $\mathcal{A}$ on the product coarse sample space $\big((\Omega_1[i], \mathcal{F}_1[i], P_1[i]) \times (\Omega_2[i], \mathcal{F}_2[i], P_2[i])\big)_{i=1}^\infty$, we may ask whether $\mathcal{A}$ is a product, that is, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ for some $\mathcal{A}_1, \mathcal{A}_2$, or not. No need to check all $\mathcal{A}_1, \mathcal{A}_2$. Rather, we have to check

$$\mathcal{A}_1 = \{A_1 : A_1 \times \Omega_2 \in \mathcal{A}\}\,, \quad \mathcal{A}_2 = \{A_2 : \Omega_1 \times A_2 \in \mathcal{A}\}\,;$$

of course, $A_1 \times \Omega_2 \subset \Omega[\mathrm{all}]$ is defined by $\forall i \;\; (A_1 \times \Omega_2)[i] = A_1[i] \times \Omega_2[i]$. If $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ generates $\mathcal{A}$, then $\mathcal{A}$ is a product; otherwise, it is not.

The refinement $\mathcal{F}$ of $\mathcal{A}$ contains two sub-σ-fields $\mathcal{F}_1 = \{(A_1 \times \Omega_2)[\infty] : A_1 \in \mathcal{A}_1\}$, $\mathcal{F}_2 = \{(\Omega_1 \times A_2)[\infty] : A_2 \in \mathcal{A}_2\}$. They are independent:

$$P(A \cap B) = P(A)P(B) \quad \text{for } A \in \mathcal{F}_1,\ B \in \mathcal{F}_2\,.$$

3a2 Lemma. $\mathcal{A}$ is a product if and only if $\mathcal{F}_1, \mathcal{F}_2$ generate $\mathcal{F}$.

Proof. We apply 2c6 to $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$. $\square$

3a3 Remark. It is well-known that a generating pair of independent sub-σ-fields means that $(\Omega, \mathcal{F}, P)$ is (isomorphic to) the product of two probability spaces. So, a coarse probability space is a product if and only if its refinement is a product. (Assuming, of course, that the coarse *sample* space is a product.)

Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Consider Hilbert spaces $H_1[i] = L_2(\Omega_1[i], \mathcal{F}_1[i], P_1[i])$, $H_2[i] = L_2(\Omega_2[i], \mathcal{F}_2[i], P_2[i])$, $H[i] = L_2(\Omega[i], \mathcal{F}[i], P[i])$. For each i, the space $H[i]$ is (canonically isomorphic to) $H_1[i] \otimes H_2[i]$. Indeed, for $x_1 \in H_1[i]$, $x_2 \in H_2[i]$ we define $x_1 \otimes x_2 \in H[i]$ by $(x_1 \otimes x_2)(\omega_1, \omega_2) = x_1(\omega_1)x_2(\omega_2)$; then $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$, and factorizable vectors (of the form $x_1 \otimes x_2$) span the space $H[i]$. We know (see 2c7) that the refinement $H[\infty]$ of $\big((H[i])_{i=1}^\infty, L_2(\mathcal{A})\big)$ is $L_2(\Omega, \mathcal{F}, P)$. Also, $H_1[\infty] = L_2(\Omega_1, \mathcal{F}_1, P_1)$ and $H_2[\infty] = L_2(\Omega_2, \mathcal{F}_2, P_2)$. Using 3a1 we get $H[\infty] = H_1[\infty] \otimes H_2[\infty]$. In that sense,

$$\operatorname{Lim}\big(H_1[i] \otimes H_2[i]\big) = \big(\operatorname{Lim} H_1[i]\big) \otimes \big(\operatorname{Lim} H_2[i]\big)\,.$$

If $x \in L_2(\mathcal{A}_1)$, $y \in L_2(\mathcal{A}_2)$, we define $x \otimes y$ by $(x \otimes y)[i] = x[i] \otimes y[i]$ for all i. We get $x \otimes y \in L_2(\mathcal{A})$ and $(x \otimes y)[\infty] = x[\infty] \otimes y[\infty]$, that is,

$$\operatorname{Lim}\big(x[i] \otimes y[i]\big) = \big(\operatorname{Lim} x[i]\big) \otimes \big(\operatorname{Lim} y[i]\big) \,, \tag{3a4}$$

since it holds for (linear combinations of) indicators of coarse events. Note also that linear combinations of factorizable vectors are dense in $L_2(\mathcal{A})$.

Assume that $R_1[i] : H_1[i] \to H_1[i]$, $R_2[i] : H_2[i] \to H_2[i]$ are linear operators, possessing limits $R_1[\infty] = \operatorname{Lim} R_1[i]$, $R_2[\infty] = \operatorname{Lim} R_2[i]$. Consider linear operators $R_1[i] \otimes R_2[i] = R[i] : H[i] \to H[i]$. (It means that $R[i]x[i] = R_1[i]x_1[i] \otimes R_2[i]x_2[i]$ whenever $x[i] = x_1[i] \otimes x_2[i]$.) If $\sup_i \|R_1[i]\| < \infty$, $\sup_i \|R_2[i]\| < \infty$, then $\operatorname{Lim} R[i] = R_1[\infty] \otimes R_2[\infty]$, that is,

$$\operatorname{Lim}\big(R_1[i] \otimes R_2[i]\big) = \big(\operatorname{Lim} R_1[i]\big) \otimes \big(\operatorname{Lim} R_2[i]\big) \,. \tag{3a5}$$

Proof: We have to check that

$$\operatorname{Lim}\big(R_1[i] \otimes R_2[i]\big)x[i] = \big(\operatorname{Lim} R_1[i] \otimes \operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x[i]\big)$$

for all $x \in L_2(\mathcal{A})$. We may assume that x is factorizable, $x = x_1 \otimes x_2$; then

$$\begin{aligned}
&\operatorname{Lim}\big(R_1[i] \otimes R_2[i]\big)\big(x_1[i] \otimes x_2[i]\big) = \\
&\qquad = \operatorname{Lim}\big(R_1[i]x_1[i] \otimes R_2[i]x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i]x_1[i]\big) \otimes \big(\operatorname{Lim} R_2[i]x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i]\big)\big(\operatorname{Lim} x_1[i]\big) \otimes \big(\operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i] \otimes \operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x_1[i] \otimes \operatorname{Lim} x_2[i]\big) \,.
\end{aligned}$$

Especially, let $R_2[i]$ be the orthogonal projection to the one-dimensional subspace of constants (basically, the expectation), and $R_1[i]$ be the unit (identity) operator. Then $\big(R_1[i] \otimes R_2[i]\big)\big(x[i]\big) = \mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_1[i] \right)$, since it holds for factorizable vectors. Further, $R_2[\infty] = \operatorname{Lim} R_2[i]$ is the expectation on $(\Omega_2, \mathcal{F}_2, P_2)$, since convergence of vectors implies convergence of one-dimensional projections, and constant functions on $\Omega_2[\text{all}]$ belong to $L_2(\mathcal{A})$. So,

$$\operatorname{Lim} \mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_1[i] \right) = \mathbb{E}\left(\operatorname{Lim} x[i] \,\middle|\, \mathcal{F}_1 \right) \tag{3a6}$$

for all $x \in L_2(\mathcal{A})$.

All the same holds for the product of any finite number of spaces (not just two).

3b Dyadic case

Let $(\Omega[i], \mathcal{F}[i], P[i])$ be the space of all maps $\frac{1}{i}\mathbb{Z} \to \{-1,+1\}$ with the usual product measure. That is, we have independent random signs $\tau_{k/i}$ for all integers k;[14] each random sign takes on two values ± 1 with probabilities $50\%, 50\%$. The coarse sample space $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$ will be called the *dyadic coarse sample space.*[15] Let $\mathcal{A}$ be a coarse σ-field on the dyadic coarse sample space. What about decomposing it, say, into the past and the future w.r.t. a given instant?

Let us define a *coarse instant* as a sequence $t = \big(t[i]\big)_{i=1}^\infty$ such that $t[i] \in \frac{1}{i}\mathbb{Z}$ (that is, $it[i] \in \mathbb{Z}$) for all i, and there exists $t[\infty] \in \mathbb{R}$ (call it the refinement of the coarse instant) such that $t[i] \to t[\infty]$ for $i \to \infty$. A *coarse time interval* is a pair (s,t) of coarse instants s,t such that $s \le t$ in the sense that $s[i] \le t[i]$ for all i.

For every coarse time interval (s,t) we define the coarse probability space $\big((\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i])_{i=1}^\infty, \mathcal{A}_{s,t}\big)$ as follows. First, $\Omega_{s,t}[i]$ is the space of all maps $\big(\frac{1}{i}\mathbb{Z} \cap [s[i], t[i])\big) \to \{-1,+1\}$.[16] Second, $\mathcal{F}_{s,t}[i]$ and $P_{s,t}[i]$ are defined naturally, and we have the canonical measure preserving map $(\Omega[i], \mathcal{F}[i], P[i]) \to (\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i])$. Third, each $A \subset \Omega_{s,t}[\mathrm{all}]$ has its inverse image in $\Omega[\mathrm{all}]$; if the inverse image of A belongs to $\mathcal{A}$ then (and only then) A belongs to $\mathcal{A}_{s,t}$, which is the definition of $\mathcal{A}_{s,t}$. It is easy to see that $\mathcal{A}_{s,t}$ is a coarse σ-field.

Given coarse time intervals (r,s) and (s,t), we have

$$\big(\Omega_{r,t}[i], \mathcal{F}_{r,t}[i], P_{r,t}[i]\big) = \big(\Omega_{r,s}[i], \mathcal{F}_{r,s}[i], P_{r,s}[i]\big) \times \big(\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i]\big) \,,$$

and we may ask whether $\mathcal{A}_{r,t}$ is a product, that is, $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$, or not.

3b1 Definition. A *dyadic coarse factorization* is a coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ such that $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$ is the dyadic coarse sample space;

$$\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$$

whenever r,s,t are coarse instants such that $r[i] \le s[i] \le t[i]$ for all i; and

$$\mathcal{A} \text{ is generated by } \bigcup_{(s,t)} \mathcal{A}_{s,t} \,,$$

where the union is taken over all coarse time intervals (s,t).

[14] Rigorously, I should denote it by $\tau_k[i]$, but $\tau_{k/i}$ is more expressive. Though $\tau_{2/6}$ is not the same as $\tau_{1/3}$, hopefully, it does not harm.

[15] Sometimes a subsequence is used; say, $i \in \{2,4,8,16,\dots\}$ only; or equivalently, $\Omega[i]$ is the space of maps $2^{-i}\mathbb{Z} \to \{-1,+1\}$; see 3b7, 3b8.

[16] It may happen that $s[i] = t[i]$, then $\Omega_{s,t}[i]$ contains a single point.

3b2 Example. A single function $f : \Omega[\text{all}] \to \mathbb{R}$, defined by $f(\omega) = \tau_{0/i}(\omega)$ for $\omega \in \Omega[i]$, generates a coarse σ-field $\mathcal{A}$. However, the coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A} \big)$ is *not* a dyadic coarse factorization. The equality $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$ is violated when $s[i]$ converges to 0 from both sides; say, $s[i] = (-1)^i/i$. It means that a single point of the time continuum should not carry a random sign. See also 3b9–3b11.

Every family $(\mathcal{A}_{s,t})_{s \le t}$ of coarse σ-fields $\mathcal{A}_{s,t}$ on coarse sample spaces $\big(\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i] \big)_{i=1}^\infty$, indexed by all coarse time intervals (s,t) and satisfying $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$ whenever $r \le s \le t$, corresponds to a dyadic coarse factorization.

3b3 Example. Given a coarse time interval (s,t), we consider $f_{s,t} : \Omega[\text{all}] \to \mathbb{R}$,

$$f_{s,t}(\omega) = \frac{1}{\sqrt{i}} \sum_{k : s[i] \le k/i < t[i]} \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i] \, .$$

Only $s[\infty], t[\infty]$ matter, in the sense that

$$\int_{\Omega[i]} \frac{|\tilde{f}[i] - f[i]|}{1 + |\tilde{f}[i] - f[i]|} \, dP[i] \xrightarrow[i \to \infty]{} 0 \tag{3b4}$$

if $f = f_{s,t}$, and $\tilde{f} = f_{\tilde{s},\tilde{t}}$ is such a function built for a different coarse time interval $(\tilde{s}, \tilde{t})$ satisfying $\tilde{s}[\infty] = s[\infty]$, $\tilde{t}[\infty] = t[\infty]$. Moreover, $\| \tilde{f}[i] - f[i] \|_{L_2[i]} \to 0$ for $i \to \infty$. We choose a sequence of coarse time intervals, $(s_n, t_n)_{n=1}^\infty$, such that the sequence of their refinements, $(s_n[\infty], t_n[\infty])$ is dense among all (usual, not coarse) intervals. The sequence $\big(f_{s_n,t_n} \big)_{n=1}^\infty$ satisfies the condition of 2c10 and therefore it generates a coarse σ-field $\mathcal{A}$. It is easy to see that $\mathcal{A}$ does not depend on the choice of (s_n, t_n). Clearly, the refinement of $f_{s,t}$ is the increment $B(t[\infty]) - B(s[\infty])$ of the usual Brownian motion $B(\cdot)$.

Given three coarse instants $r \le s \le t$, we have

$$f_{r,t} = f_{r,s} + f_{s,t} \, .$$

It shows that $f_{r,t}$ is coarsely measurable w.r.t. the product of two coarse σ-fields $\mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$, which implies $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$. So, we have a dyadic coarse factorization. We may call it the Brownian coarse factorization.

3b5 Example. Let $f_{s,t}(\omega)$ be the same as in 3b3 and in addition,

$$g_{s,t}(\omega) = \frac{1}{\sqrt{i}} \sum_{k : s[i] \le k/i < t[i]} (-1)^k \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i] \, .$$

In the scaling limit we get two independent Brownian motions B_1, B_2; the refinement of $f_{s,t}$ is $B_1(t[\infty]) - B_1(s[\infty])$, the refinement of $g_{s,t}$ is $B_2(t[\infty]) - B_2(s[\infty])$. By the way, $(-1)^k$ cannot be replaced with $(-1)^{k-s[i]}$; it would violate the condition of 2c10.

We may also consider

$$f_{s,t}^{(n)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \exp\Big(2\pi\mathrm{i}\frac{k}{n}\Big)\tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]$$

for $n = 1, 2, 3, \dots$ (here $\mathrm{i} = \sqrt{-1}$, while i is an integer). In the scaling limit we get two real-valued Brownian motions B_1, B_2 and infinitely many complex-valued Brownian motion $B_3, B_4, \dots$ All B_n are independent.

Another construction of that kind:

$$f_{s,t}^{(\lambda)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \exp\Big(2\pi\mathrm{i}\lambda\frac{k}{\sqrt{i}}\Big)\tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

In the scaling limit, each $\lambda \in (0,\infty)$ gives a complex-valued Brownian motion B_λ. Any finite or countable set of numbers λ may be used, and leads to independent Brownian motions. Note that we cannot use more than a countable set of λ, since separability is stipulated by the definition of a coarse probability space.

3b6 Example. For $n = 1, 2, \dots$ we introduce

$$f_{s,t}^{(n)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i\le(k+n)/i<t[i]} \prod_{m=1}^{n} \tau_{(k+m)/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

In the scaling limit we get independent Brownian motions B_n.

Another construction of that kind:

$$f_{s,t}^{(\lambda)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i\le(k+\lambda\sqrt{i})/i<t[i]} \prod_{m=1}^{\mathrm{entier}(\lambda\sqrt{i})} \tau_{(k+m)/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,;$$

any finite or countable set of numbers $\lambda \in (0,\infty)$ may be used, and leads to independent Brownian motions B_λ.

Note that we cannot take the product over $m = 1, \dots, \mathrm{entier}(\lambda i)$; that would destroy factorizability.

3b7 Example. Here we restrict ourselves to $i \in \{2, 4, 8, 16, \dots\}$, thus violating a little of our framework. We let for $\omega \in \Omega[i]$, $i = 2^n$,

$$g_{s,t}(\omega) = \sum_{k:s[i]\le k/i<(k+n-1)/i<t[i]} \frac{1+\tau_{k/i}(\omega)}{2} \prod_{m=1}^{n-1} \frac{1-\tau_{(k+m)/i}(\omega)}{2}\,.$$

That is, $g_{s,t} : \Omega[\text{all}] \to \{0,1,2,\dots\}$ counts combinations '$+ - \dots -$' of one plus sign and $(n-1)$ minus signs in succession. In the scaling limit we get the Poisson process.

3b8 Example. Let $f_{s,t}$ be as in 3b3 (Brownian), while $g_{s,t}$ is as in 3b7 (Poisson). Taken together, they generate a coarse σ-field. The corresponding scaling limit consists of two *independent* processes, Brownian and Poisson.

Let $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^{\infty},\mathcal{A}\big)$ be a dyadic coarse factorization. Being a coarse probability space, it has a refinement $(\Omega,\mathcal{F},P)$. For every coarse time interval (s,t) we have a coarse sub-σ-field $\mathcal{A}_{s,t} \subset \mathcal{A}$ and its refinement, a sub-σ-field $\mathcal{F}_{s,t}[\infty] \subset \mathcal{F}$. By 3a1,

$$\mathcal{F}_{r,t}[\infty] = \mathcal{F}_{r,s}[\infty] \otimes \mathcal{F}_{s,t}[\infty] \quad \text{whenever } r \le s \le t\,.$$

3b9 Lemma. If $s[\infty] = t[\infty]$ then $\mathcal{F}_{s,t}[\infty]$ is degenerate (that is, contains sets of probability 0 or 1 only).

Proof. Consider the coarse instant r,

$$r[i] = \begin{cases} s[i] & \text{for } i \text{ even,} \\ t[i] & \text{for } i \text{ odd.} \end{cases}$$

For every $A \in \mathcal{A}_{s,r}$,

$$P(A[\infty]) = \lim_{i\to\infty} P[i]\big(A[i]\big) = \lim_{i\to\infty} P[2i]\big(A[2i]\big) \in \{0,1\}\,,$$

since $\mathcal{A}_{s,r}[2i]$ is degenerate. So, $\mathcal{F}_{s,r}[\infty]$ is degenerate. Similarly, $\mathcal{F}_{r,t}[\infty]$ is degenerate. However, $\mathcal{F}_{s,t}[\infty] = \mathcal{F}_{s,r}[\infty] \otimes \mathcal{F}_{r,t}[\infty]$. $\square$

3b10 Lemma. $\mathcal{F}_{s,t}[\infty]$ depends only on $s[\infty], t[\infty]$.

Proof. Let (u,v) be another coarse time interval such that $u[\infty] = s[\infty]$ and $v[\infty] = t[\infty]$; we have to prove that $\mathcal{F}_{s,t}[\infty] = \mathcal{F}_{u,v}[\infty]$. Assume that $s[\infty] < t[\infty]$ (otherwise both $\mathcal{F}_{s,t}[\infty]$ and $\mathcal{F}_{u,v}[\infty]$ are degenerate). Assume also that $s[i] \le v[i]$ and $u[i] \le t[i]$ for all i (otherwise we correct them on a finite set of indices i).

Further, we may assume that $s \le u \le v \le t$; otherwise we turn to $s \wedge u \le s \vee u \le t \wedge v \le t \vee v$, where $(s \wedge u)[i] = s[i] \wedge u[i] = \min\big(s[i],u[i]\big)$, etc. Both $\mathcal{F}_{s,t}[\infty]$ and $\mathcal{F}_{u,v}[\infty]$ are sandwiched between $\mathcal{F}_{s\wedge u,t\vee v}[\infty]$ and $\mathcal{F}_{s\vee u,t\wedge v}[\infty]$.

Finally, $\mathcal{F}_{s,t}[\infty] = \mathcal{F}_{s,u}[\infty] \otimes \mathcal{F}_{u,v}[\infty] \otimes \mathcal{F}_{v,t}[\infty] = \mathcal{F}_{u,v}[\infty]$, since $\mathcal{F}_{s,u}[\infty]$ and $\mathcal{F}_{v,t}[\infty]$ are degenerate by 3b9. $\square$

So, a sub-σ-field $\mathcal{F}_{s,t} \subset \mathcal{F}$ is well-defined for every interval $(s,t) \subset \mathbb{R}$ (rather than a coarse time interval), and

$$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} \quad \text{whenever } -\infty < r \le s \le t < +\infty \,.$$

3b11 Lemma. The union of sub-σ-fields $\mathcal{F}_{s+\varepsilon,t-\varepsilon}$ over $\varepsilon > 0$ generates $\mathcal{F}_{s,t}$.

Proof. Consider $\mathcal{F}_{\varepsilon,1}$. We have to prove that $\mathbb{E}\left(x \,\middle|\, \mathcal{F}_{\varepsilon,1} \right)$ converges to x (in $L_2(\Omega)$, for $\varepsilon \to 0+$) for every $x \in L_2(\mathcal{F}_{0,1})$, or for $x[\infty]$ where $x \in L_2(\mathcal{A}_{0,1})$. Assume the contrary. Then

$$\|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{\varepsilon,1} \right)\| < c < \|x[\infty]\|$$

for all ε small enough, and some constant c. We know that

$$\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{\varepsilon,1} \right) = \operatorname{Lim} \mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{\varepsilon,1}[i] \right)$$

for each ε.[17] Therefore

$$\|\mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{\varepsilon,1}[i] \right)\| \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{\varepsilon,1} \right)\| < c \,.$$

We choose a sequence $\varepsilon[i] \xrightarrow[i\to\infty]{} 0$ such that $\|\mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{\varepsilon[i],1}[i] \right)\| < c$ for all i large enough. However, $\operatorname{Lim} \mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{\varepsilon[i],1}[i] \right) = \mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{\varepsilon[\infty],1} \right) = \mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{0,1} \right) = x[\infty]$; a contradiction. $\square$

3c Scaling limit of Fourier-Walsh coefficients

We still consider a dyadic coarse factorization. The Hilbert space $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$ consists of all functions of random signs τ_m, $m \in \frac{1}{i}\mathbb{Z}$. The well-known Fourier-Walsh orthonormal basis of $L_2[i]$ consists of products

$$\tau_M = \prod_{m \in M} \tau_m \,, \quad M \in \mathcal{C}[i] \,, \quad \mathcal{C}[i] = \{ M \subset \tfrac{1}{i}\mathbb{Z} : M \text{ is finite} \} \,.$$

Every $f \in L_2[i]$ is of the form

$$f = \sum_M \hat f_M \tau_M = \hat f_\emptyset + \sum_{m \in \frac{1}{i}\mathbb{Z}} \hat f_{\{m\}} \tau_m + \sum_{m_1,m_2 \in \frac{1}{i}\mathbb{Z}, m_1 < m_2} \hat f_{\{m_1,m_2\}} \tau_{m_1} \tau_{m_2} + \dots \,;$$

coefficients $\hat f_M$ are called Fourier-Walsh coefficients of f. We define the *spectral measure* μ_f on the countable set $\mathcal{C}[i]$ by

$$\mu_f(\mathcal{M}) = \sum_{M \in \mathcal{M}} |\hat f_M|^2 \quad \text{for } \mathcal{M} \subset \mathcal{C}[i] \,;$$

[17] Or rather, an appropriate coarse instant is meant in $\mathcal{F}_{\varepsilon,1}[i]$.

it is a finite positive measure,

$$\mu_f(\mathcal{C}[i]) = \|f\|^2 ; \quad \mu_f(\{\emptyset\}) = (\mathbb{E}\, f)^2 ; \quad \mu_f(\mathcal{C}[i] \setminus \{\emptyset\}) = \operatorname{Var}(f) .$$

Let (s,t) be a coarse time interval. We have

$$\mathbb{E}\left(\tau_M \,\middle|\, \mathcal{F}_{s,t}[i] \right) = \begin{cases} \tau_M & \text{if } M \subset [s[i], t[i]), \\ 0 & \text{otherwise;} \end{cases}$$

$$\|\mathbb{E}\left(f \,\middle|\, \mathcal{F}_{s,t}[i] \right)\|^2 = \mu_f\bigl(\{M \in \mathcal{C}[i] : M \subset [s[i], t[i])\}\bigr) .$$

We apply it to $f = x[i]$ for an arbitrary $x \in L_2(\mathcal{A})$ and arbitrary i; μ_f becomes $\mu_{x[i]}$ or $\mu_x[i]$; by (3a6),

$$\mu_x[i]\bigl(\{M \in \mathcal{C}[i] : M \subset [s[i], t[i])\}\bigr) = \|\mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{s,t}[i] \right)\|^2 \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{s,t}[\infty] \right)\|^2 .$$

For every $\varepsilon > 0$ we can choose s, t so that $\|x[\infty]\|^2 - \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{s,t}[\infty] \right)\|^2 \le \varepsilon$, and moreover,

$$\mu_x[i]\bigl(\{M \in \mathcal{C}[i] : M \not\subset [s[i], t[i])\}\bigr) \le \varepsilon \quad \text{for all } i . \tag{3c1}$$

We consider each $\mu_x[i]$ as a measure on the space $\mathcal{C}[\infty]$ of all compact subsets of $\mathbb{R}$, equipped with the Hausdorff metric; the metric is

$$\operatorname{dist}(M_1, M_2) = \sup_{x\in\mathbb{R}} \Bigl| \min_{y\in M_1} |x-y| - \min_{y\in M_2} |x-y| \Bigr| \tag{3c2}$$

for nonempty M_1, M_2, and $\operatorname{dist}(\emptyset, M) = 1$ for $M \ne \emptyset$. Clearly, $\mathcal{C}[i] \subset \mathcal{C}[\infty]$ for each i; thus, a measure on $\mathcal{C}[i]$ is also a measure on $\mathcal{C}[\infty]$.[18] The set $\{M \in \mathcal{C}[\infty] : M \subset [u,v]\}$ is well-known to be compact, for every $[u,v] \subset \mathbb{R}$. Thus, (3c1) shows that the sequence of measures $\mu_x[i]$ on $\mathcal{C}[\infty]$ is tight.

Let (s_1,t_1) and (s_2,t_2) be two coarse time intervals, $s_1 \le t_1 \le s_2 \le t_2$. Sub-$\sigma$-fields $\mathcal{F}_{s_1,t_1}[i]$ and $\mathcal{F}_{s_2,t_2}[i]$ are independent; they generate a sub-σ-field that may be denoted by

$$\mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] = \mathcal{F}_{s_1,t_1}[i] \otimes \mathcal{F}_{s_2,t_2}[i] .$$

[18]One may turn $(\mathcal{C}[i])_{i=1}^{\infty}$ into a coarse Polish space, and identify its refinement with $\mathcal{C}[\infty]$. It leads to a joint compactification of all $\mathcal{C}[i]$ and $\mathcal{C}[\infty]$, which is a suitable framework for weak convergence of measures on $\mathcal{C}[i]$ to a measure on $\mathcal{C}[\infty]$. However, it is simpler to use natural embeddings, $\mathcal{C}[i] \subset \mathcal{C}[\infty]$.

We have

$$\mathbb{E}\left(\tau_M \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \right) = \begin{cases} \tau_M & \text{if } M \subset [s_1[i],t_1[i]) \cup [s_2[i],t_2[i]), \\ 0 & \text{otherwise;} \end{cases}$$

$$\|\mathbb{E}\left(f \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \right)\|^2 = \mu_f\big(\{ M \in \mathcal{C}[i] : M \subset [s_1[i],t_1[i]) \cup [s_2[i],t_2[i])\}\big) \,;$$

$$\mu_x[i]\big(\{ M \in \mathcal{C}[i] : M \subset [s_1[i],t_1[i]) \cup [s_2[i],t_2[i])\}\big) = \\ = \|\mathbb{E}\left(x[i] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \right)\|^2 \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[\infty] \right)\|^2 \,,$$

where $\mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[\infty] = \mathcal{F}_{s_1,t_1}[\infty] \otimes \mathcal{F}_{s_2,t_2}[\infty] = \mathcal{F}_{s_1[\infty],t_1[\infty]} \otimes \mathcal{F}_{s_2[\infty],t_2[\infty]}$. A generalization of (3a6) to the product of more than two spaces was used here.

The same holds for more than two coarse time intervals:

(3c3) $$\mu_x[i]\big(\{ M \in \mathcal{C}[i] : M \subset [s_1[i],t_1[i]) \cup \ldots \cup [s_n[i],t_n[i])\}\big) \\ \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup\ldots\cup(s_n,t_n)}[\infty] \right)\|^2 \,.$$

We have convergence of spectral measures on a special class of subsets of $\mathcal{C}[\infty]$. Note that the intersection of two such subsets is again such a subset. Therefore, the convergence holds on the algebra of subsets generated by the class. A generic element of the algebra is the union of a finite number of 'cells' of the form

(3c4) $$\{ M \in \mathcal{C}[\infty] : M \subset \cup_{k=1}^n [s_k,t_k) \text{ and } M \cap [s_k,t_k) \ne \emptyset \text{ for } k=1,\ldots,n \} \,;$$

here $[s_k,t_k) \subset \mathbb{R}$ are usual (rather than coarse) time intervals. (Endpoints may be neglected, as we will see soon.) The diameter of the cell (3c4) (w.r.t. the metric (3c2)) does not exceed $\max_k(t_k - s_k)$. Thus, we get weak convergence of measures, which proves the following result.

3c5 Theorem. For every dyadic coarse factorization $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty, \mathcal{A}\big)$ and every $x \in L_2(\mathcal{A})$, the sequence $\big(\mu_x[i]\big)_{i=1}^\infty$ of spectral measures converges weakly to a (finite, positive) measure $\mu_x[\infty]$ on the Polish space $\mathcal{C}[\infty]$.

Convergence of measures $\mu_x[i]$ on a 'cell' of the form (3c3) (or (3c4)) does not ensure that the limit is $\mu_x[\infty]$ on the 'cell'.[19] Rather, the limit lies between $\mu_x[\infty]$-measures of the interior and the closure of the cell,

(3c6) $$\mu_x[\infty]\big(\{ M \in \mathcal{C}[\infty] : M \subset (s_1,t_1) \cup \ldots \cup (s_n,t_n)\}\big) \\ \le \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup\ldots\cup(s_n,t_n)} \right)\|^2 \\ \le \mu_x[\infty]\big(\{ M \in \mathcal{C}[\infty] : M \subset [s_1,t_1] \cup \ldots \cup [s_n,t_n]\}\big) \,.$$

[19] Think for example about an atom at the point $\frac{1}{n}$ of $\mathbb{R}$, and 'cells' of the form $(x,y]$.

3c7 Lemma. For every $t \in \mathbb{R}$,

$$\mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \ni t\}\big) = 0\,.$$

Proof. Lemma 3b11 gives us

$$\|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)} \right)\|^2 \xrightarrow[\varepsilon\to 0]{} \|x[\infty]\|^2\,;$$

therefore

$$\mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \subset (-\infty,\varepsilon] \cup [\varepsilon,+\infty)\}\big) \xrightarrow[\varepsilon\to 0]{} \mu_x[\infty]\big(\mathcal{C}[\infty]\big)\,.$$

□

Applying Fubini's theorem we see that $\mu_x[\infty]$ is concentrated on (the set of all) compact sets M of Lebesgue measure 0 (therefore, nowhere dense).

Due to 3c7 we see that the boundary of a 'cell' is negligible (of measure 0); inequalities (3c6) are, in fact, equalities. So,

$$\mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \subset E\}\big) = \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_E \right)\|^2\,, \tag{3c8}$$

where $E \subset \mathbb{R}$ is an arbitrary elementary set, that is, a finite union of intervals (treated modulo finite sets), $E = (s_1,t_1) \cup \ldots \cup (s_n,t_n)$, and $\mathcal{F}_E = \mathcal{F}_{s_1,t_1} \otimes \cdots \otimes \mathcal{F}_{s_n,t_n}$.

For a finite i, the Fourier-Walsh basis decomposes $L_2[i]$ into one-dimensional subspaces indexed by $M \in \mathcal{C}[i]$, and each subset $\mathcal{M} \subset \mathcal{C}[i]$ leads to a subspace $H_\mathcal{M}$ of $L_2[i]$ spanned by τ_M, $M \in \mathcal{M}$. In particular, for a subset of the form $\mathcal{M}_E = \{M \in \mathcal{C}[i] : M \subset E\}$ we have $H_{\mathcal{M}_E} = L_2(\Omega[i], \mathcal{F}_E[i], P[i])$.

Similarly, for the limiting object, the subspace $H_{\mathcal{M}_E} = L_2(\Omega, \mathcal{F}_E, P)$ of $L_2[\infty]$ corresponds to the set $\mathcal{M}_E = \{M \in \mathcal{C}[\infty] : M \subset E\}$. In 3d a subspace $H_\mathcal{M} \subset L_2[\infty]$ will be defined for every Borel set $\mathcal{M} \subset \mathcal{C}[\infty]$.

3d The limiting object

3d1 Definition. A *continuous factorization* (of probability spaces, over $\mathbb{R}$) consists of a probability space $(\Omega, \mathcal{F}, P)$ and a two-parameter family $(\mathcal{F}_{s,t})_{s\le t}$ of sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ such that[20]

(a) $$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} \quad \text{whenever } r \le s \le t$$

(that is, $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are independent, and together generate $\mathcal{F}_{r,t}$),

(b) $$\bigcup_{\varepsilon>0} \mathcal{F}_{s+\varepsilon,t-\varepsilon} \text{ generates } \mathcal{F}_{s,t} \text{ whenever } s < t,$$

[20] Here r,s,t are real numbers; coarse instants are not used in 3d, 3e.

and

(c) $$\bigcup_{n=1}^{\infty} \mathcal{F}_{-n,n} \text{ generates } \mathcal{F}\,.$$

The refinement of any dyadic coarse factorization is a continuous factorization (as was shown in 3b).

3d2 Definition. Let $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ be a continuous factorization, and $x \in L_2(\Omega,\mathcal{F},P)$. The *spectral measure* μ_x of x is the (finite, positive) measure on the space $\mathcal{C} = \mathcal{C}[\infty]$ of compact subsets of $\mathbb{R}$ such that

$$\mu_x\big(\{M \in \mathcal{C} : M \subset E\}\big) = \|\mathbb{E}\left(x \,\middle|\, \mathcal{F}_E\right)\|^2$$

for all elementary sets $E \subset \mathbb{R}$.

Uniqueness of μ_x is checked easily. Existence of μ_x is proven in 3c by discrete approximation, assuming that the continuous factorization is the refinement of a dyadic coarse factorization. Another proof, without approximation, will be given by 3d9.

The spectral measure is concentrated on (the set of all) nowhere dense compact sets, and

(3d3) $$\mu_x\big(\{M \in \mathcal{C} : M \ni t\}\big) = 0 \quad \text{for each } t \in \mathbb{R}\,,$$

which follows from 3d6 for $s = t$, since $\mathcal{F}_{t,t} = \mathcal{F}_{t,t} \otimes \mathcal{F}_{t,t}$ is degenerate.

3d4 Example. The refinement of the Brownian coarse factorization (see 3b3) is the Brownian continuous factorization,

$$\mathcal{F}_{s,t} \text{ is generated by } \{B(v) - B(u) : s \le u \le v \le t\}\,,$$

where $B(\cdot)$ is the usual Brownian motion. Every $x \in L_2$ admits Itô's decomposition into multiple stochastic integrals,

$$\begin{aligned} x = \hat{x}(\emptyset) + \int \hat{x}(\{t_1\})\,\mathrm{d}B(t_1) + \iint\limits_{t_1<t_2} \hat{x}(\{t_1,t_2\})\,\mathrm{d}B(t_1)\mathrm{d}B(t_2) + \dots \\ = \sum_{n=0}^{\infty} \int\limits_{t_1<\dots<t_n}\!\!\cdots\!\int \hat{x}(\{t_1,\dots,t_n\})\,\mathrm{d}B(t_1)\dots\mathrm{d}B(t_n)\,, \end{aligned}$$

where $\hat{x} \in L_2(\mathcal{C}_{\text{finite}})$, $\mathcal{C}_{\text{finite}}$ being the space of all finite subsets of $\mathbb{R}$, equipped with the natural (Lebesgue) measure, making the transform $x \leftrightarrow \hat{x}$ unitary, according to the formula

$$\mathbb{E}\,|x|^2 = |\hat{x}(\emptyset)|^2 + \int |\hat{x}(\{t_1\})|^2 \,\mathrm{d}t_1 + \iint\limits_{t_1<t_2} |\hat{x}(\{t_1,t_2\})|^2 \,\mathrm{d}t_1\mathrm{d}t_2 + \dots$$
$$= \sum_{n=0}^{\infty} \int\limits_{t_1<\dots<t_n}\!\!\cdots\!\int |\hat{x}(\{t_1,\dots,t_n\})|^2 \,\mathrm{d}t_1\dots\mathrm{d}t_n\,.$$

The spectral measure μ_x of x is

$$\mu_x(A) = \sum_{n=0}^{\infty} \int\limits_{t_1<\dots<t_n,\{t_1,\dots,t_n\}\in A}\!\!\cdots\!\int |\hat{x}(\{t_1,\dots,t_n\})|^2 \,\mathrm{d}t_1\dots\mathrm{d}t_n\,.$$

This is an important property of the Brownian continuous factorization: the spectral measure (of any random variable) is concentrated on the subset $\mathcal{C}_{\text{finite}} \subset \mathcal{C}$, and absolutely continuous w.r.t. the Lebesgue measure on $\mathcal{C}_{\text{finite}}$.

In particular, for $x = \exp\big(\mathrm{i}\sqrt{\lambda}B(t)\big)$ the measure μ_x is just the distribution of the Poisson process of rate λ on $(0,t)$. Indeed,

$$\exp\big(\mathrm{i}\sqrt{\lambda}B(t)\big) = \mathrm{e}^{-\lambda t/2} \sum_{n=0}^{\infty} \lambda^{n/2} \int\limits_{0<t_1<\dots<t_n<t}\!\!\cdots\!\int \mathrm{d}B(t_1)\dots\mathrm{d}B(t_n)\,.$$

3d5 Example. Recall the process Y_ε of 1a3;

$$Y_\varepsilon(t) = \exp\big(\mathrm{i}B(\ln t) - \mathrm{i}B(\ln\varepsilon)\big)\,.$$

We define $\mathcal{F}_{s,t}$ as the σ-field generated by 'multiplicative increments' $Y_\varepsilon(v)/Y_\varepsilon(u)$ for all $(u,v) \subset (s,t)$, that is, by (usual) Brownian increments on $(\ln s, \ln t)$. The spectral measure $\mu_{Y_\varepsilon(t)}$ is the distribution of a non-homogeneous Poisson process on (ε,t), the image of the usual Poisson process (of rate 1) on $(\ln\varepsilon, \ln t)$ under the time change $u \mapsto \mathrm{e}^u$. The rate of the non-homogeneous Poisson process is $\lambda(s) = 1/s$.

The limiting process Y was discussed in 1a3. It may be treated as the refinement of Y_ε for $\varepsilon \to 0$ (I leave the details to the reader). The spectral measure $\mu_{Y(t)}$ should be the distribution of a non-homogeneous Poisson process on $(0,t)$, at the rate $\lambda(s) = 1/s$. Random points accumulate to 0; we add 0 to the random set, making it compact. However, the equality $\mu(\{M : M \ni 0\}) = 1$ does not conform to 3c7! It happens because the limiting object is not a *continuous* factorization. Denote by $\mathcal{F}_{0+,1}$ the σ-field generated by $\cup_{\varepsilon>0}\mathcal{F}_{\varepsilon,1}$. Every $Y(1)/Y(t)$ for $t > 0$ is $\mathcal{F}_{0+,1}$-measurable, but $Y(1)$ is not. The global phase is missing. Of course, for every $t > 0$, there exists an independent complement of $\mathcal{F}_{0+,t}$ in $\mathcal{F}_{-\infty,t}$ (for example, the σ-field generated by $Y(t)$). However, we cannot choose a single complement (to be denoted by $\mathcal{F}_{-\infty,0+}$) for all $t > 0$, since the tail σ-field $\cap_{t>0}\mathcal{F}_{-\infty,t}$ is degenerate.

3d6 Lemma. For every continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ and every $s \le t$,

$$\mathcal{F}_{s,t} = \bigcap_{\varepsilon>0} \mathcal{F}_{s-\varepsilon,t+\varepsilon} \, .$$

Proof. The σ-field $\cap_{\varepsilon>0}\mathcal{F}_{0,\varepsilon}$ is degenerate by Kolmogorov's zero-one law applied to $\mathcal{F}_{1,\infty}, \mathcal{F}_{1/2,1}, \mathcal{F}_{1/3,1/2}, \dots$ Further, $\mathcal{F}_{-\infty,\varepsilon} = \mathcal{F}_{-\infty,0} \otimes \mathcal{F}_{0,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathcal{F}_{-\infty,0}$. Though the equality $\lim(\mathcal{A} \vee \mathcal{B}_n) = \mathcal{A} \vee (\lim \mathcal{B}_n)$ does not hold in general, it does hold for independent $\mathcal{A}$ and $\mathcal{B}_1$ ($\mathcal{B}_1 \supset \mathcal{B}_2 \supset \dots$), which is a rather trivial part of Weizsäcker's criteria [27]. The rest of the proof is left to the reader.

$\square$

The theory of direct integrals of Hilbert spaces may be used on the way to Theorem 3d12. In fact, I did so in [15, Th. 2.3]. Here, however, I choose a self-contained presentation. First, a general result of measure theory, useful for proving the existence of μ_x (without dyadic approximation).

3d7 Lemma. Let X be a compact topological space, $\mathcal{A}$ an algebra of subsets of X, and $\mu : \mathcal{A} \to [0, \infty)$ an additive function satisfying the following regularity condition:

For every $A \in \mathcal{A}$ and $\varepsilon > 0$ there exists $B \in \mathcal{A}$ such that $\overline{B} \subset A$ (here $\overline{B}$ is the closure of B) and $\mu(B) \ge \mu(A) - \varepsilon$.

Then μ has a unique extension to a measure on the σ-field generated by $\mathcal{A}$.

Proof. Due to a well-known theorem, it is enough to prove that μ is σ-additive on $\mathcal{A}$. Let $A_1 \supset A_2 \supset \dots$, $A_1, A_2, \dots \in \mathcal{A}$, $\cap A_k = \emptyset$; we have to prove that $\mu(A_k) \to 0$. Given $\varepsilon > 0$, we can choose $B_k \in \mathcal{A}$ such that $\overline{B}_k \subset A_k$ and $\mu(B_k) \ge \mu(A_k) - 2^{-k}\varepsilon$. Due to compactness, the relation $\cap \overline{B}_k \subset \cap A_k = \emptyset$ implies $\overline{B}_1 \cap \dots \cap \overline{B}_n = \emptyset$ for some n. Thus, $\mu(A_n) = \mu(A_1 \cap \dots \cap A_n) \le \mu(B_1 \cap \dots \cap B_n) + \mu(A_1 \setminus B_1) + \dots + \mu(A_n \setminus B_n) < \varepsilon$. $\square$

3d8 Remark. All $A \in \mathcal{A}$ such that A and $X \setminus A$ both satisfy the regularity condition, are a subalgebra of $\mathcal{A}$. (The proof is left to the reader.) Therefore it is enough to check the condition for A and $X \setminus A$ where A runs over a set that generates the algebra $\mathcal{A}$.

3d9 Lemma. The spectral measure μ_x exists for every $x \in L_2(\Omega, \mathcal{F}, P)$ and every continuous factorization $(\mathcal{F}_{s,t})_{s \le t}$.

Proof. First, compactness. We have $\|\mathbb{E}\left(x \,\middle|\, \mathcal{F}_{-m,m} \right)\|^2 \to \|x\|^2$ for $m \to \infty$ by 3d1(c); thus we may restrict ourselves to x measurable w.r.t. $\mathcal{F}_{-m,m}$ for some m. The corresponding part $\mathcal{C}_m = \{M \in \mathcal{C} : M \subset [-m, m]\}$ of $\mathcal{C}$ is compact.

Second, additivity on an algebra. We have an algebra $\mathcal{A}$ of subsets of $\mathcal{C}_m$, generated by 'cells' of the form (3c4). Such a cell leads to a subspace of $L_2(\Omega, \mathcal{F}_{-m,m}, P)$ spanned by products $f_1 \dots f_n$ where each f_k is measurable w.r.t. $\mathcal{F}_{s_k,t_k}$, square integrable, and $\mathbb{E}\, f_k = 0$. A partition of the interval $[-m, m]$ into n subintervals leads to a partition of $\mathcal{C}_m$ into 2^n parts, and a decomposition of $L_2(\Omega, \mathcal{F}_{-m,m}, P)$ into 2^n orthogonal subspaces. Thus, x decomposes into 2^n orthogonal vectors; their squared norms give us μ_x on a finite subalgebra (of cardinality 2^{2^n}) of $\mathcal{A}$. We see that μ_x is additive on such subalgebras. Their union (over all partitions of $[-m, m]$) is the whole $\mathcal{A}$, and any two of them are contained in some third; therefore, μ_x is additive on $\mathcal{A}$.

Third, regularity (required by 3d7). Due to 3d8, regularity may be checked only for sets $A_E = \{M \in \mathcal{C}_m : M \subset E\}$ and $\mathcal{C}_m \setminus A_E$. It follows easily from 3d1(b) and 3d6. $\square$

3d10 Remark. In the proof of 3d9, an *orthogonal decomposition* of the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ over the algebra $\mathcal{A}$ is constructed; that is, a family $(H_A)_{A \in \mathcal{A}}$ of (closed linear) subspaces $H_A \subset H$ such that $H_{A \cup B} = H_A \oplus H_B$ (it means that H_A and H_B are orthogonal, and their sum is $H_{A \cup B}$) whenever $A \cap B = \emptyset$, and $H_{\mathcal{C}} = H$. The decomposition satisfies

$$H_{\mathcal{M}_E} = L_2(\Omega, \mathcal{F}_E, P) \,,$$

where $\mathcal{M}_E = \{M \in \mathcal{C} : M \subset E\}$, and is uniquely determined by this property.

The following general result will help us construct $H_{\mathcal{M}}$ for all Borel sets $\mathcal{M} \subset \mathcal{C}$.

3d11 Lemma. Let X be a set, $\mathcal{A}$ an algebra of subsets of X, H a Hilbert space, and $(H_A)_{A \in \mathcal{A}}$ an orthogonal decomposition of H over $\mathcal{A}$. Assume that for every $x \in H$ the additive function[21] $A \mapsto \| \operatorname{Proj}_{H_A} x \|^2$ on $\mathcal{A}$ can be extended to a measure on the σ-field $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. Then the orthogonal decomposition can be extended to an orthogonal decomposition $(H_B)_{B \in \sigma(\mathcal{A})}$, σ-additive in the sense that[22] $H_{B_1 \cup B_2 \cup \dots} = H_{B_1} \oplus H_{B_2} \oplus \dots$ whenever $B_1, B_2, \dots \in \sigma(\mathcal{A})$ are pairwise disjoint.

[21] Here $\operatorname{Proj}_{H_A}$ is the orthogonal projection $H \to H_A$.

[22] That is, $H_{B_1 \cup B_2 \cup \dots}$ is the closure of the algebraic sum of H_{B_k}.

Proof. The extension of the additive function $\mu_x : \mathcal{A} \to [0,\infty)$, $\mu_x(A) = \| \operatorname{Proj}_{H_A} x\|^2$, to a measure on $\sigma(\mathcal{A})$ is unique; denote it by μ_x again. Consider the set of all $B \in \sigma(\mathcal{A})$ such that there exists a subspace $H_B \subset H$ satisfying $\| \operatorname{Proj}_{H_B} x\|^2 = \mu_x(B)$ for all $x \in H$. The set contains $\mathcal{A}$, and is a monotone class (that is, closed under the limit of monotone sequences), which is easy to check. Therefore the set is the whole $\sigma(\mathcal{A})$. □

Combining 3d9 and 3d11 we conclude.

3d12 Theorem. For every continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ there exists one and only one σ-additive orthogonal decomposition $(H_{\mathcal{M}})$ of the Hilbert space $L_2(\Omega,\mathcal{F},P)$ over the Borel σ-field of the space $\mathcal{C}$ (of compact subsets of $\mathbb{R}$) such that $H_{\mathcal{M}_E} = L_2(\Omega,\mathcal{F}_E,P)$ for every elementary set $E \subset \mathbb{R}$ (that is, a finite union of intervals); here $\mathcal{M}_E = \{M \in \mathcal{C} : M \subset E\}$. The orthogonal decomposition is related to spectral measures by

$$\| \operatorname{Proj}_{H_{\mathcal{M}}} f\|^2 = \mu_f(\mathcal{M}) \tag{3d13}$$

for all $f \in L_2(\Omega,\mathcal{F},P)$ and all Borel sets $\mathcal{M} \subset \mathcal{C}$.

3e Time shift; noise

Let $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty,\mathcal{A}\big)$ be a dyadic coarse factorization. For each i the lattice $\frac{1}{i}\mathbb{Z}$ acts on $\Omega[i]$ by measure preserving transformations $\alpha_t : \Omega[i] \to \Omega[i]$ (time shift),

$$\alpha_t(\omega)(s) = \omega(s-t) \quad \text{for all } s \in \frac{1}{i}\mathbb{Z}\,.$$

For each coarse instant $t = (t[i])_{i=1}^\infty$ we have a map $\alpha_t : \Omega[\text{all}] \to \Omega[\text{all}]$,

$$\alpha_t(\omega)[i](s) = \omega[i](s - t[i]) \quad \text{for all } s \in \frac{1}{i}\mathbb{Z}\,.$$

Such α_t is an automorphism of the dyadic coarse sample space, but the coarse σ-field $\mathcal{A}$ need not be invariant under α_t. We consider such a condition:

$$\mathcal{A} \text{ is invariant under } \alpha_t \text{ for every coarse instant } t. \tag{3e1}$$

Dyadic coarse factorizations of 3b3, 3b6, 3b7, 3b8 satisfy (3e1), but that of 3b5 does not.

If (3e1) is satisfied, then the refinement $\alpha_t[\infty] = \operatorname{Lim}_{i\to\infty,\mathcal{A}} \alpha_t[i]$ is an automorphism of the refinement $(\Omega,\mathcal{F},P)$ of the dyadic coarse factorization. Existence of the limit for *every* converging sequence $t = (t[i])$ implies that $\alpha_t[\infty]$ depends on $t[\infty]$ only (see 3e4 below), and we get a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms (that is, invertible measure preserving transformations $\bmod 0$) of $(\Omega, \mathcal{F}, P)$. The group is continuous in the sense that $\mathbb{P}\big(A \triangle \alpha_t(A)\big) \xrightarrow[t\to 0]{} 0$ for all $A \in \mathcal{F}$, which is ensured by (3e1) (see 3e4 again).

3e2 Definition. A *noise* $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}, (\alpha_t)_{t\in\mathbb{R}}\big)$ consists of a continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ and a one-parameter group of automorphisms α_t of $(\Omega, \mathcal{F}, P)$ such that

$$\alpha_t^{-1}(\mathcal{F}_{r,s}) = \mathcal{F}_{r-t,s-t} \quad \text{for all } r, s, t \in \mathbb{R},\ r \le s\,,$$
$$P\big(A \triangle \alpha_t^{-1}(A)\big) \xrightarrow[t\to 0]{} 0 \quad \text{for all } A \in \mathcal{F}\,.$$

Unfortunately, the latter assumption (continuity of the group action) is missing in my former publications, which opens the door for pathologies.[23]

3e3 Remark. Continuity of the factorization follows from other assumptions, see [15, Lemma 2.1]. For arbitrary factorizations, continuity is restrictive (recall 3d5); waiving it, we get discontinuity points $t \in \mathbb{R}$ which are a finite or countable set. For a noise, however, the set is invariant under time shifts, and therefore, empty.

3e4 Lemma. For every dyadic coarse factorization satisfying (3e1), its refinement is a noise.

Proof. Our first argument parallels the proof of 3b9. Namely, let s, t be two coarse instants such that $s[\infty] = t[\infty]$. We introduce a coarse event r:

$$r[i] = \begin{cases} s[i] & \text{for } i \text{ even,} \\ t[i] & \text{for } i \text{ odd.} \end{cases}$$

We have

$$\operatorname{Lim} \alpha_s[i] = \operatorname{Lim} \alpha_s[2i] = \operatorname{Lim} \alpha_r[2i] = \operatorname{Lim} \alpha_r[i]\,.$$

Similarly, $\operatorname{Lim} \alpha_t[i] = \operatorname{Lim} \alpha_r[i]$. Thus, $\operatorname{Lim} \alpha_s[i] = \operatorname{Lim} \alpha_t[i]$, and we may define a one-parameter group of automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$ by $\alpha_{t[\infty]} = \operatorname{Lim} \alpha_t[i]$.

Our second argument resembles the proof of 3b11. Namely, assume existence of $A_\infty \in \mathcal{F}$, $\varepsilon > 0$ and $t_n \to 0$ such that $P\big(A_\infty \triangle \alpha_{t_n}^{-1}(A_\infty)\big) \ge \varepsilon$ for all n. We choose a coarse event $A \in \mathcal{A}$ such that $A[\infty] = A_\infty$, and

[23] Most results of these former publications do not depend on the (missing) continuity condition. But anyway, a discontinuous group action is a pathology, no doubt. (In particular, it cannot be Borel measurable.) The proof of Lemma 2.9 of [15], based on Weyl's relation, depends on the continuity condition.

coarse instants s_n such that $s_n[\infty] = t_n$ for all n. Taking into account that $P[i]\big(A[i]\triangle\alpha_{s_n}^{-1}[i]A[i]\big) \to P\big(A_\infty\triangle\alpha_{t_n}^{-1}(A_\infty)\big) \ge \varepsilon$ and $s_n[i] \to t_n$ when $i \to \infty$, we choose integers $i_1 < i_2 < \dots$ such that $P[i]\big(A[i] \triangle \alpha_{s_n}^{-1}[i]A[i]\big) \ge \varepsilon/2$ and $|s_n[i]| \le |t_n| + 1/n$ whenever $i \ge i_n$. We define a coarse instant r by $r[i] = s_n[i]$ whenever $i_n \le i < i_{n+1}$. Clearly, $r[\infty] = 0$; therefore $\operatorname{Lim} \alpha_r^{-1}[i]A[i] = \alpha_0^{-1}A[\infty] = A[\infty]$, and $P[i]\big(A[i] \triangle \alpha_r^{-1}[i]A[i]\big) \to 0$, which is impossible: these probabilities exceed $\varepsilon/2$. The contradiction proves continuity of the group $(\alpha_t)_{t\in\mathbb{R}}$. $\square$

3e5 Question. Is every noise the refinement of some dyadic coarse factorization satisfying (3e1)? I do not know; I guess that the answer is negative. It would be interesting to find some special features of such refinements among all noises. It is also unclear what happens to the class of such refinements, if subsequences are permitted (like in 3b7).

4 Example: The Noise Made by a Poisson Snake

This section is based on a paper by J. Warren entitled "The noise made by a Poisson snake" [23].

4a Three discrete semigroups: algebraic definition

A discrete semigroup (with unit; non-commutative, in general) may be defined by generators and relations.

Two generators f_+, f_- with two relations $f_+f_- = 1$, $f_-f_+ = 1$ generate a semigroup G_1^{discrete} that is in fact a group, just the cyclic group $\mathbb{Z}$. Indeed, every word reduces to some f_+^k or f_-^k (or 1).

Two generators f_+, f_- with a single relation $f_+f_- = 1$ generate a semigroup G_2^{discrete}. Every word reduces to some $f_-^k f_+^l$. The composition is

$$(f_-^{k_1} f_+^{l_1})(f_-^{k_2} f_+^{l_2}) = f_-^k f_+^l\,, \qquad \begin{aligned} k &= k_1 + \max(0, k_2 - l_1)\,, \\ l &= l_2 + \max(0, l_1 - k_2)\,. \end{aligned} \tag{4a1}$$

The canonical homomorphism $G_2^{\text{discrete}} \to G_1^{\text{discrete}}$ maps f_+ to f_+, f_- to f_-, and $f_-^k f_+^l$ into f_-^{k-l} (if $k > l$), f_+^{l-k} (if $k < l$), or 1 (if $k = l$). Accordingly, the composition law (4a1) satisfies

$$l - k = (l_1 - k_1) + (l_2 - k_2)\,.$$

There is a more convenient pair of parameters, $a = l - k$, $b = k$; that is,[24]

(4a2)
$$\begin{gathered} f_{a,b} = f_-^b f_+^{a+b} \quad \text{for } a, b \in \mathbb{Z},\ b \ge 0,\ a + b \ge 0\,; \\ f_{a_1,b_1} f_{a_2,b_2} = f_{a,b}\,, \quad \begin{aligned} a &= a_1 + a_2\,, \\ b &= \max(b_1, b_2 - a_1)\,. \end{aligned} \end{gathered}$$

The canonical homomorphism $G_2^{\text{discrete}} \to G_1^{\text{discrete}}$ maps $f_{a,b}$ to f_a, where $f_a \in G_1^{\text{discrete}}$ is f_+^a for $a > 0$, $f_-^{|a|}$ for $a < 0$, and 1 for $a = 0$.

Three generators f_-, f_+, f_* with three relations

(4a3)
$$f_+ f_- = 1\,, \quad f_* f_- = 1\,, \quad f_* f_+ = f_* f_*$$

generate a semigroup G_3^{discrete}. Every word reduces to some $f_-^k f_+^l f_*^m$. The following homomorphism $G_3^{\text{discrete}} \to G_2^{\text{discrete}}$ will be called canonical: $f_- \mapsto f_-$, $f_+ \mapsto f_+$, $f_* \mapsto f_+$. We have $f_-^k f_+^l f_*^m \mapsto f_-^k f_+^{l+m}$, which suggests such a triple of parameters for G_3^{discrete}: $a = l + m - k$, $b = k$, $c = m$; that is,

(4a4)
$$\begin{gathered} f_{a,b,c} = f_-^b f_+^{a+b-c} f_*^c \quad \text{for } a, b, c \in \mathbb{Z},\ b \ge 0,\ 0 \le c \le a + b\,; \\ f_{a_1,b_1,c_1} f_{a_2,b_2,c_2} = f_{a,b,c}\,, \quad \begin{aligned} a &= a_1 + a_2\,, \\ b &= \max(b_1, b_2 - a_1)\,, \end{aligned} \quad c = \begin{cases} a_2 + c_1 & \text{if } c_1 > b_2, \\ c_2 & \text{otherwise.} \end{cases} \end{gathered}$$

The canonical homomorphism $G_3^{\text{discrete}} \to G_2^{\text{discrete}}$ is just $f_{a,b,c} \mapsto f_{a,b}$.

Note that G_1^{discrete} is commutative, but G_2^{discrete} and G_3^{discrete} are not.

4b The three discrete semigroups: representation

By a representation of a semigroup G on a set S we mean a map $G \times S \ni (g, s) \mapsto g(s) \in S$ such that

$$(g_1 g_2)(s) = g_2\big(g_1(s)\big) \quad \text{and} \quad 1(s) = s$$

for all $g_1, g_2 \in G$, $s \in S$. The representation is called faithful, if

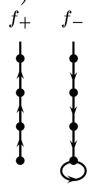
$$g_1 \ne g_2 \quad \Longrightarrow \quad \exists s \in S \ \ \big(g_1(s) \ne g_2(s)\big)\,.$$

Every G has a faithful representation on itself, $S = G$, namely, the regular representation, $g(g_0) = g_0 g$. Fortunately, G_2^{discrete} and G_3^{discrete} have more economical faithful representations on the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Namely,

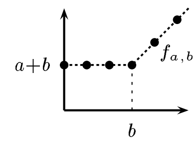
[24]Parameters a, b of (4a2) and a, b, c of (4a4) are suggested by S. Watanabe.

for G_2^{discrete},

(4b1)

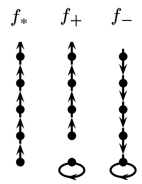


$$f_+(x) = x+1\,, \quad f_-(x) = \max(0, x-1)\,,$$
$$f_{a,b}(x) = a + \max(x,b)\,,$$

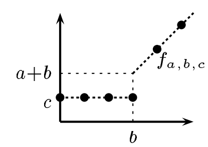


$x \in \mathbb{Z}_+$. For G_3^{discrete},

(4b2)



$$f_*(x) = x+1\,, \quad f_-(x) = \max(0, x-1)\,,$$
$$f_+(x) = \begin{cases} x+1 & \text{for } x > 0, \\ 0 & \text{for } x = 0; \end{cases}$$
$$f_{a,b,c}(x) = \begin{cases} c & \text{for } 0 \le x \le b, \\ x+a & \text{for } x > b. \end{cases}$$



4c Random walks and stochastic flows in discrete semigroups

4c1 Example. The standard random walk on $\mathbb{Z}$ may be described by G_1^{discrete}-valued random variables

$$\begin{gathered} \xi_{s,t} = \xi_{s,s+1}\xi_{s+1,s+2}\dots\xi_{t-1,t} \quad \text{for } s,t \in \mathbb{Z},\ s \le t\,; \\ \xi_{t,t+1} \text{ are independent random variables } (t \in \mathbb{Z})\,; \\ \mathbb{P}\left(\, \xi_{t,t+1} = f_- \,\right) = \frac{1}{2} = \mathbb{P}\left(\, \xi_{t,t+1} = f_+ \,\right) \quad \text{for each } t \in \mathbb{Z}\,. \end{gathered} \tag{4c2}$$

Note that $\xi_{r,s}\xi_{s,t} = \xi_{r,t}$ whenever $r \le s \le t$. Everyone knows that

$$\mathbb{P}\left(\, \xi_{0,t} = f_a \,\right) = \frac{1}{2^t}\binom{t}{\frac{t+a}{2}} \tag{4c3}$$

for $a = -t, -t+2, -t+4, \dots, t$.

In fact, 'the standard random walk' is the random process $t \mapsto \xi_{0,t}$. Taking into account that G_1^{discrete} is a group, $\xi_{s,t}$ may be thought of as an increment, $\xi_{s,t} = \xi_{0,s}^{-1}\xi_{s,t}$.

4c4 Example. Formulas (4c2) work equally well on G_2^{discrete}. Still, $\xi_{r,s}\xi_{s,t} = \xi_{r,t}$. However, G_2^{discrete} is not a group, and $\xi_{s,t}$ is not an increment; moreover, it is not a function of $\xi_{0,s}$ and $\xi_{0,t}$. Indeed, knowing a_1, b_1 and $a_1 + a_2$,

$\max(b_1, b_2 - a_1)$ (recall (4a2)) we can find a_2 but not b_2. Thus, the two-parameter family $(\xi_{s,t})_{s\le t}$ of random variables is more than just a random walk. Let us call such a family an *abstract stochastic flow.* Why 'abstract'? Since G_2^{discrete} is an abstract semigroup rather than a semigroup of transformations (of some set). So, we have the standard abstract flow in G_2^{discrete}. In order to get a (usual, not abstract) stochastic flow, we have to choose a representation of G_2^{discrete}. Of course, the regular representation could be used, but the representation (4b1) is more useful. Introducing integer-valued random variables $a(s,t), b(s,t)$ by

$$\xi_{s,t} = f_{a(s,t),b(s,t)}$$

we express the stochastic flow as

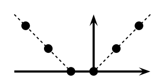
$$\xi_{s,t}(x) = a(s,t) + \max(x, b(s,t))\,.$$

Fixing s and x we get a random process called a single-point motion of the flow. Namely, it is a reflecting random walk. Especially, for $s = 0$ and $x = 0$, the process

$$t \mapsto \xi_{0,t}(0) = a(0,t) + b(0,t)$$

is a reflecting random walk. It is easy to see that two processes

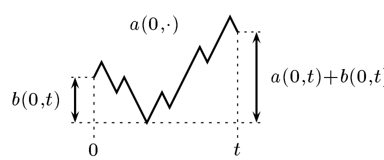
$$\begin{aligned} &t \mapsto \xi_{0,t}(0) = a(0,t) + b(0,t)\,, \\ &t \mapsto \left|a(0,t) + \frac{1}{2}\right| - \frac{1}{2} \end{aligned}$$



are identically distributed. Also,

(4c5)

$$\begin{aligned} b(0,t) &= -\min_{s=0,1,\dots,t} a(0,s)\,, \\ a(0,t) + b(0,t) &= \max_{s=0,1,\dots,t} a(s,t)\,, \end{aligned}$$



and $a(\cdot,\cdot)$ is the standard random walk on $G_1^{\text{discrete}} = \mathbb{Z}$. That is, the canonical homomorphism $G_2^{\text{discrete}} \to G_1^{\text{discrete}}$ transforms the standard flow on G_2^{discrete} into the standard flow (or random walk) on G_1^{discrete}. Using the reflection principle, one gets

(4c6) $$\mathbb{P}\left(\xi_{0,t} = f_{a,b}\right) = \frac{a + 2b + 1}{2^t} \frac{t!}{\left(\frac{t+a}{2} + b + 1\right)!\left(\frac{t-a}{2} - b\right)!}\,.$$

Note that a, b occur only in the combination $a + 2b$.

4c7 Example. On G_3^{discrete}, we have no 'standard' random walk or flow; rather, we introduce a one-parameter family of abstract stochastic flows,

(4c8)
$$
\begin{gathered}
\xi_{s,t} = \xi_{s,s+1}\xi_{s+1,s+2}\dots\xi_{t-1,t} \quad \text{for } s,t \in \mathbb{Z},\ s \le t\,;\\
\xi_{t,t+1} \text{ are independent random variables } (t \in \mathbb{Z})\,;\\
\mathbb{P}\left(\,\xi_{t,t+1} = f_-\,\right) = \frac{1}{2}, \quad \mathbb{P}\left(\,\xi_{t,t+1} = f_+\,\right) = \frac{1-p}{2}, \quad \mathbb{P}\left(\,\xi_{t,t+1} = f_*\,\right) = \frac{p}{2}\,;
\end{gathered}
$$

$p \in (0,1)$ is the parameter. The canonical homomorphism $G_3^{\text{discrete}} \to G_2^{\text{discrete}}$ glues together f_+ and f_*, thus eliminating the parameter p and giving the standard abstract flow on G_2^{discrete}. Defining $a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)$ by

$$\xi_{s,t} = f_{a(s,t),b(s,t),c(s,t)}$$

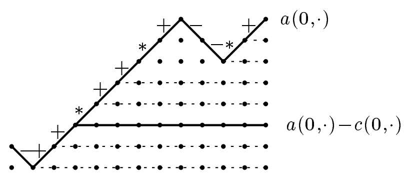
we see that the joint distribution of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ is the same as before.

Representation (4b2) of G_3^{discrete} turns the abstract flow into a stochastic flow on $\mathbb{Z}_+$. Its single-point motion is a sticky random walk,

$$t \mapsto \xi_{0,t}(0) = c(0,t)\,.$$

In order to find the conditional distribution of $c(\cdot,\cdot)$ given $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ we observe that

(4c9) $$a(0,t) - c(0,t) = \min\bigl(a(0,t), \min\{x : \xi_{\sigma(x),\sigma(x)+1} = f_*\}\bigr)$$
$$\text{where } \sigma(x) = \max\{s = 0,\dots,t : a(0,s) = x\}\,, \quad -b(0,t) \le x < a(0,t).$$



Therefore the conditional distribution of $c(0,t)$ is basically the truncated geometric distribution. More exactly, it is the (conditional) distribution of

(4c10) $$\max\bigl(0, a(0,t) + b(0,t) - G + 1\bigr)\,, \qquad G \sim \text{Geom}(p)\,;$$

here G is a random variable, independent of $a(\cdot,\cdot), b(\cdot,\cdot)$, such that $\mathbb{P}\left(\,G = g\,\right) = p(1-p)^{g-1}$ for $g = 1,2,\dots$ This is the discrete counterpart of a well-known result of J. Warren [21]. So,

(4c11) $$\mathbb{P}\left(\,\xi_{0,t} = f_{a,b,c}\,\right) = \frac{a+2b+1}{2^t}\frac{t!}{\left(\frac{t+a}{2}+b+1\right)!\left(\frac{t-a}{2}-b\right)!}\cdot p(1-p)^{a+b-c}$$

for $c > 0$; for $c = 0$ the factor $p(1-p)^{a+b-c}$ turns into $(1-p)^{a+b}$, rather than $p(1-p)^{a+b}$, because of truncation.

4d Three continuous semigroups

The continuous counterpart of the discrete semigroup $G_1^{\text{discrete}} = \mathbb{Z}$ is the semigroup $G_1 = \mathbb{R} = \{f_a : a \in \mathbb{R}\}$, $f_{a_1} f_{a_2} = f_{a_1+a_2}$.

The continuous counterpart of the discrete semigroup $G_2^{\text{discrete}} = \{f_{a,b} : a, b \in \mathbb{Z},\ b \ge 0,\ a + b \ge 0\}$ is the semigroup

(4d1)
$$
\begin{gathered}
G_2 = \{f_{a,b} : a, b \in \mathbb{R},\ b \ge 0,\ a + b \ge 0\}\,, \\
f_{a_1,b_1} f_{a_2,b_2} = f_{a,b}\,, \qquad \begin{aligned} & a = a_1 + a_2\,, \\ & b = \max(b_1, b_2 - a_1) \end{aligned}
\end{gathered}
$$

(recall (4a2)). The canonical homomorphism $G_2 \to G_1$ maps $f_{a,b}$ to f_a.

The continuous counterpart of the discrete semigroup $G_3^{\text{discrete}} = \{f_{a,b,c} : a, b, c \in \mathbb{Z},\ b \ge 0,\ 0 \le c \le a + b\}$ is the semigroup

(4d2)
$$
\begin{gathered}
G_3 = \{f_{a,b,c} : a, b, c \in \mathbb{R},\ b \ge 0,\ 0 \le c \le a + b\}\,, \\
f_{a_1,b_1,c_1} f_{a_2,b_2,c_2} = f_{a,b,c}\,, \quad \begin{aligned} & a = a_1 + a_2\,, \\ & b = \max(b_1, b_2 - a_1)\,, \end{aligned} \quad c = \begin{cases} a_2 + c_1 & \text{if } c_1 > b_2, \\ c_2 & \text{otherwise} \end{cases}
\end{gathered}
$$

(recall (4a4)). The canonical homomorphism $G_3 \to G_2$ maps $f_{a,b,c}$ to $f_{a,b}$.

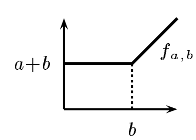
Note that G_1 is commutative but G_2, G_3 are not. Also, G_1 and G_2 are topological semigroups, but G_3 is not (since the composition is discontinuous at $c_1 = b_2$).

There are two one-parameter semigroups in G_2, $\{f_{a,0} : a \in [0,\infty)\}$ and $\{f_{-b,b} : b \in [0,\infty)\}$. They generate G_2 according to the relation $f_{b,0} f_{-b,b} = 1$; namely, $f_{a,b} = f_{-b,b} f_{a+b,0}$.

There are three one-parameter semigroups in G_3, $\{f_{a,0,0} : a \in [0,\infty)\}$, $\{f_{-b,b,0} : b \in [0,\infty)\}$ and $\{f_{c,0,c} : c \in [0,\infty)\}$. They generate G_3 according to relations $f_{b,0,0} f_{-b,b,0} = 1$, $f_{b,0,b} f_{-b,b,0} = 1$, and $f_{c,0,c} f_{a,0,0} = f_{c,0,c} f_{a,0,a}$ for $c > 0$; namely, $f_{a,b,c} = f_{-b,b,0} f_{a+b-c,0,0} f_{c,0,c}$.

Here is a faithful representation of G_2 on $[0,\infty)$ (recall (4b1)):

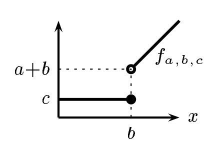
(4d3)
$$
f_{a,b}(x) = a + \max(x, b)\,,
$$



$x \in [0,\infty)$.

Here is a faithful representation of G_3 on $[0,\infty)$ (recall (4b2)):

(4d4)
$$
f_{a,b,c}(x) = \begin{cases} c & \text{for } 0 \le x \le b, \\ x + a & \text{for } x > b. \end{cases}
$$



All functions are increasing, but $f_{a,b}$ are continuous, while $f_{a,b,c}$ are not.

4e Convolution semigroups in these continuous semigroups

4e1 Example. Everyone knows that the binomial distribution (4c3) is asymptotically normal. That is, the distribution of $\sqrt{\varepsilon}a(0,t/\varepsilon)$ converges weakly (for $\varepsilon \to 0$) to the normal distribution $\mu_t^{(1)} = \mathrm{N}(0,t)$. These form a convolution semigroup, $\mu_s^{(1)} * \mu_t^{(1)} = \mu_{s+t}^{(1)}$.

Note however, that $a(s,t)$ and $\xi_{s,t}$ are defined (see (4c2)) only for integers s,t. We may extend them, in one way or another, to real s,t. Or alternatively, we may use coarse instants $t = \big(t[i]\big)_{i=1}^{\infty}$, $t[i] \in \frac{1}{i}\mathbb{Z}$, $t[i] \to t[\infty]$, introduced in 3b. For every coarse instant t, the distribution of $i^{-1/2}a(0,it[i])$ converges weakly (for $i \to \infty$) to $\mu_{t[\infty]}^{(1)} = \mathrm{N}(0,t[\infty])$.

4e2 Example. The two-dimensional distribution (4c6) on G_2^{discrete} has its asymptotics. Namely, the joint distribution of $i^{-1/2}a(0,it[i])$ and $i^{-1/2}b(0,it[i])$ converges weakly (for $i \to \infty$) to the measure $\mu_{t[\infty]}^{(2)}$ with density (on the relevant domain $b > 0$, $a + b > 0$; t means $t[\infty]$):

$$\text{(4e3)} \qquad \frac{\mu_t^{(2)}(\mathrm{d}a\mathrm{d}b)}{\mathrm{d}a\mathrm{d}b} = \frac{2(a+2b)}{\sqrt{2\pi}\,t^{3/2}} \exp\bigg(-\frac{(a+2b)^2}{2t} \bigg) \,.$$

Treating $\mu_t^{(2)}$ (for $t \in [0,\infty)$) as a measure on G_2, we get a convolution semigroup: $\mu_s^{(2)} * \mu_t^{(2)} = \mu_{s+t}^{(2)}$. Of course, the convolution is taken according to the composition (4d1).

4e4 Example. What about the three-dimensional distribution (4c11) on G_3^{discrete}? It has a parameter p. In order to get a non-degenerate asymptotics, we let p depend on i, namely,

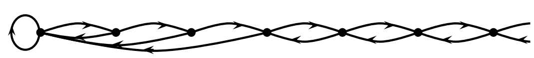
$$p = \frac{1}{\sqrt{i}} \to 0 \,.$$

Then the distribution of $i^{-1/2}G$, where $G \sim \mathrm{Geom}(p)$ (recall (4c10)), converges weakly to the exponential distribution $\mathrm{Exp}(1)$, and the joint distribution of $i^{-1/2}a(0,it[i])$, $i^{-1/2}b(0,it[i])$ and $i^{-1/2}c(0,it[i])$ converges weakly to a measure $\mu_{t[\infty]}^{(3)}$. The measure has an absolutely continuous part and a singular part (at $c = 0$), and may be described (somewhat indirectly) as the joint distribution of three random variables a, b and $(a + b - \eta)^+$, where the pair (a,b) is distributed $\mu_t^{(2)}$ (see (4e3)), η is independent of (a,b), and

$\eta \sim \mathrm{Exp}(1)$. Treating $\mu_t^{(3)}$ (for $t \in [0,\infty)$) as a measure on G_3, we get a convolution semigroup: $\mu_s^{(3)} * \mu_t^{(3)} = \mu_{s+t}^{(3)}$, the convolution being taken according to the composition (4d2). No need to check the relation 'by hand'; it follows from its discrete counterpart. The latter follows from the construction of 4c (since random variables $\xi_{0,1}, \xi_{1,2}, \dots, \xi_{s+t-1,s+t}$ are independent). It may seem that the limiting procedure does not work, since G_3 is not a topological semigroup; the composition (4d2) is discontinuous at $c_1 = b_2$. However, that is not an obstacle, since the equality $c_1 = b_2$ is of zero probability, as far as triples (a_1, b_1, c_1) and (a_2, b_2, c_2) are independent and distributed $\mu_s^{(3)}$, $\mu_t^{(3)}$, respectively ($s, t > 0$). The atom of c_1 at 0 does not matter, since b_2 is nonatomic. The composition is continuous almost everywhere!

4f Getting dyadic

Our flows in G_1^{discrete} and G_2^{discrete} are dyadic (two equiprobable possibilities in each step), which cannot be said about G_3^{discrete}; here, in each step, we have three possibilities f_-, f_+, f_* of probabilities $1/2, (1-p)/2, p/2$. Can a dyadic model produce the same asymptotic behavior? Yes, it can, at the expense of using $i \in \{1, 4, 16, 64, \dots\}$ only (recall 3b7); and, of course, the dyadic model is more complicated.[25] Instead of the trap at 0, we design a trap near 0 as follows:



$$g_+ = f_* = f_{1,0,1}\,; \quad g_- = f_-^m f_+^{m-1} = f_{-1,m,0}\,;$$
$$\mathbb{P}\big(\, \xi_{t,t+1} = g_- \,\big) = \frac{1}{2} = \mathbb{P}\big(\, \xi_{t,t+1} = g_+ \,\big)\,.$$

The old (small) parameter p disappears, and a new (large) parameter m appears. We'll see that the two models are asymptotically equivalent, when $p = 2^{-m}$.

As before, we may denote

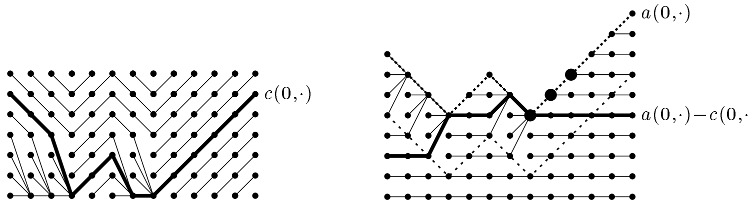
$$\xi_{s,t} = f_{a(s,t),b(s,t),c(s,t)}\,.$$

Note, however, that only $a(s,t)$ is the same as before; $b(s,t)$, $c(s,t)$ and $\xi_{s,t}$ are modified. Formula (4c5) for $b(0,t)$ fails, but still,

(4f1) $$b(0,t) = -\min_{s=0,1,\dots,t} a(0,s) + O(m)\,,$$

[25] Maybe, a still more complicated construction can use all i; I do not know.

which is asymptotically the same. Formula (4c9) for $c(0,t)$ also fails. Instead,



(4f2) $$a(0,t) - c(0,t) = \min\{x : \sigma(x+m-1) - \sigma(x) = m-1\} \,,$$

if such x exists in the set $\mathbb{Z} \cap [\min_{[0,t]} a(0,\cdot), a(0,t) - m + 1]$; otherwise, $c(0,t) = O(m)$. (Here σ is the same as in (4c9).)

The conditional distribution of $c(0,t)$, given the path $a(0,\cdot)$, is not at all geometric (unlike (4c10)), since now $c(0,t)$ is uniquely determined by $a(0,\cdot)$. However, according to (4f2), $a(0,t)-c(0,t)$ is determined by small increments of the process $\sigma(\cdot)$. On the other hand, the large-scale structure of the path $a(0,\cdot)$ is correlated mostly with large increments of $\sigma(\cdot)$; small increments are numerous, but contribute little to the sum. Using this argument, one can show that $c(0,t)$ is asymptotically independent of $a(0,t)$ (and $b(0,t)$, due to (4f1)).

The unconditional distribution of $c(0,t)$ can be found from (4f2), taking into account that increments $\sigma(x+1) - \sigma(x)$ are independent, and each increment is equal to 1 with probability 1/2. We have Bernoulli trials, and we wait for the first block of $m-1$ 'successes'. For large m, the waiting time is approximately exponential, with the mean 2^m.[26] Thus, $2^{-m}\big(a(0,t) - c(0,t) - \min_{[0,t]} a(0,\cdot)\big)$ is asymptotically Exp(1), truncated (at $c = 0$) as in 4e.

Taking the limit $i = 2^{2m} \to \infty$, we get for $i^{-1/2}a(0,it[i])$, $i^{-1/2}b(0,it[i])$, $i^{-1/2}c(0,it[i])$ the limiting distribution $\mu^{(3)}_{t[\infty]}$, the same as in 4e.

4g Scaling limit

For any coarse instants s,t such that $s \le t$, the distribution $\mu^{(n)}_{s,t}[i]$ of $i^{-1/2}\xi^{(n)}_{is[i],it[i]}$ converges weakly (for $i \to \infty$) to the measure $\mu^{(n)}_{s,t}[\infty] = \mu^{(n)}_{t[\infty]-s[\infty]}$ on G_n, for our three models, $n = 1,2,3$. Of course, multiplication of ξ by $i^{-1/2}$ is understood as multiplication of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ by $i^{-1/2}$, which is a homomorphic embedding of G_n^{discrete} into G_n.

Let r,s,t be coarse instants, $r \le s \le t$. Due to independence, the joint distribution $\mu^{(n)}_{r,s}[i] \otimes \mu^{(n)}_{s,t}[i]$ of random variables $i^{-1/2}\xi^{(n)}_{ir[i],is[i]}$ and $i^{-1/2}\xi^{(n)}_{is[i],it[i]}$

[26] Such a block appears, in the mean, after 2^{m-1} shorter blocks, of mean length ≈ 2 each.

converges weakly to $\mu^{(n)}_{r,s}[\infty] \otimes \mu^{(n)}_{s,t}[\infty]$. However, we need the joint distribution of three random variables,

$$i^{-1/2}\xi^{(n)}_{ir[i],is[i]}\,, \quad i^{-1/2}\xi^{(n)}_{is[i],it[i]}\,, \quad i^{-1/2}\xi^{(n)}_{ir[i],it[i]}\,,$$

the third being the product of the first and the second in the semigroup G_n. For $n = 1, 2$ weak convergence for the triple follows immediately from weak convergence for the pair, since the composition is continuous. For $n = 3$, discontinuity of the composition in G_3 does not invalidate the argument, since the composition is continuous almost everywhere w.r.t. the relevant measure (recall 4e).

Similarly, for every k and all coarse instants $t_1 \le \dots \le t_k$, the joint distribution of $k(k-1)/2$ random variables $i^{-1/2}\xi^{(n)}_{it_l[i],it_m[i]}$, $1 \le l < m \le k$, converges weakly (for $i \to \infty$). We choose a sequence $(t_k)_{k=1}^\infty$ of coarse instants such that the sequence of numbers $(t_k[\infty])_{k=1}^\infty$ is dense in $\mathbb{R}$, and use 2c10, getting a coarse probability space.

The Hölder condition, the same as in 2a3, holds for all three models. I mean Hölder continuity of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$. Indeed, $a(\cdot,\cdot)$ is the same as in 2a3; $b(\cdot,\cdot)$ is related to $a(\cdot,\cdot)$ via (4c5) or (4f1), and $c(\cdot,\cdot)$ satisfies (on any interval)

$$\max_{|s-t|\le x} |c(0,s) - c(0,t)| \le \max_{|s-t|\le x} |a(0,s) - a(0,t)|\,,$$

though, for the model of 4f, $O(m)$ must be added.

Thus, a joint σ-compactification is constructed for all three models (the third model — in two versions, 4c7 and 4f).

4h Noises

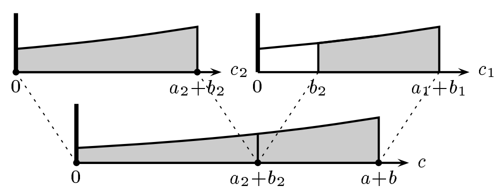
4h1 Example. The standard flow in G_1^{discrete}, rescaled by $i^{-1/2}$, gives us a coarse probability space, identical to that of 3b3. It is a dyadic coarse factorization. Its refinement is the Brownian continuous factorization. Equipped with the natural time shift, it is a noise.

4h2 Example. The standard flow in G_2^{discrete}, rescaled by $i^{-1/2}$, gives us another coarse probability space. It is also a dyadic coarse factorization (the proof is similar to the previous case). Its 'two-dimensional nature' is a delusion; the dyadic coarse factorization is identical to that of 4h1. The second dimension $b(\cdot,\cdot)$ reduces to the first dimension, $a(\cdot,\cdot)$, by (4c5).

4h3 Example. The flow in G_3, introduced in 4c7, rescaled by $i^{-1/2}$ with $p = i^{-1/2}$ (recall 4e4), gives us a coarse probability space. It is not a dyadic coarse factorization, since it is not dyadic. However, it satisfies a natural generalization of 3b1 to the non-dyadic case (the proof is as before). Its refinement is a continuous factorization, and (with natural time shift), a noise; it may be called the noise of stickiness.

Once again, the second dimension, $b(\cdot,\cdot)$, reduces to the first dimension, $a(\cdot,\cdot)$. Indeed, the joint distribution of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ is the same as in 4h2. What about the third dimension, $c(\cdot,\cdot)$?

The conditional distribution of $c(s,t)$, given $a(s,t)$ and $b(s,t)$, is basically truncated exponential. Namely, it is the distribution of $\big(a(s,t)+b(s,t)-\eta\big)^+$ where $\eta \sim \mathrm{Exp}(1)$; see 4e4. Moreover, for any $r < s < t$, the conditional distribution of $c(r,t)$ given $a(r,s), b(r,s)$ and $a(s,t), b(s,t)$, is still the distribution of $\big(a(r,t)+b(r,t)-\eta\big)^+$. In other words, $c(r,t)$ is conditionally independent of $a(r,s), b(r,s), a(s,t), b(s,t)$, given $a(r,t), b(r,t)$. That is a property of the composition (4d2); if $c_1 \sim \big(a_1+b_1-\eta_1\big)^+$ and $c_2 \sim \big(a_2+b_2-\eta_2\big)^+$ then $c \sim \big(a+b-\eta\big)^+$.



It follows by induction that the conditional distribution of $c(t_1,t_n)$, given all $a(t_i,t_j)$ and $b(t_i,t_j)$, is given by the same formula $\big(a(t_1,t_n)+b(t_1,t_n)-\eta\big)^+$, $\eta \sim \mathrm{Exp}(1)$, for every n and $t_1 < \dots < t_n$. Therefore, the same holds for the conditional distribution of $c(s,t)$ given all $a(u,v)$ and $b(u,v)$ for u,v such that $s \le u \le v \le t$ (a well-known result of J. Warren [21]). We see that $c(\cdot,\cdot)$ is not a function of $a(\cdot,\cdot)$ (and $b(\cdot,\cdot)$).

4h4 Example. Another flow in G_3^{discrete}, introduced in 4f, being rescaled by $i^{-1/2}$ with $i = 2^{2m}$, gives us a dyadic coarse factorization. Its refinement is the same continuous factorization (and noise) as in 4h3.

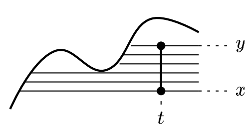
4i The Poisson snake

Formula (4c9) suggests a description of the sticky flow in G_3^{discrete} by a combination of a simple random walk $a(\cdot,\cdot)$ and a random subset of the set of its 'chords'. A chord may be defined as an interval $[s,t]$, $s,t \in \mathbb{Z}$, $s < t$, such that $a(s,t) = 0$ and $a(s,u) > 0$ for all $u \in (s,t) \cap \mathbb{Z}$. Or equivalently, a chord is a horizontal straight segment on the plane that connects points $\big(s, a(0,s)\big)$ and $\big(t, a(0,t)\big)$ and goes below the graph of $a(0,\cdot)$. The random subset of chords is very simple: every chord belongs to the subset with probability p, independently of others. Note that $p = i^{-1/2}$ is equal to the vertical pitch (after rescaling $a(\cdot,\cdot)$ by $i^{-1/2}$). The scaling limit suggests itself: a Poisson random subset of the set of all chords of the Brownian sample path.

4i1 Definition. A *finite chord* of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is a set of the form $[s,t] \times \{x\} \subset \mathbb{R}^2$ where $s < t$, $x = f(s)$ and $t = \inf\{u \in (s,\infty) : f(u) \le x\}$. An *infinite chord* of f is a set of the form $[s,\infty) \times \{x\} \subset \mathbb{R}^2$ where $x = f(s)$ and $f(t) > x$ for all $t \in (s,\infty)$. A *chord* of f is either a finite chord of f, or an infinite chord of f.

If f decreases, it has no chords. Otherwise it has a continuum of chords. The set of chords is, naturally, a standard Borel space,[27] due to the one-one correspondence between a chord and its initial point $(s,x) \in \mathbb{R}^2$.

4i2 Lemma. For every continuous function $f : \mathbb{R} \to \mathbb{R}$ there exists one and only one σ-finite positive Borel measure[28] on the space of all chords of f, such that the set of chords that intersect a vertical segment $\{t\} \times [x,y]$ is of measure $y - x$, whenever t, x, y are such that $\inf_{s\in(-\infty,t)} f(s) \le x < y \le f(t)$.



The proof is left to the reader. Hint: for every $\varepsilon > 0$, the set of chords longer than ε is elementary; on this set, the measure is locally finite.

The map $[s,t] \times \{x\} \mapsto s$ (also $[s,\infty) \times \{x\} \mapsto s$, of course) sends the measure on the set of chords (described in 4i2) into a measure on $\mathbb{R}$. If f is of locally finite variation, then the measure on $\mathbb{R}$ is just $(df)^+$, the positive part of the Lebesgue-Stieltjes measure. However, we need the opposite case: f is of infinite variation on every interval, and the measure is also infinite on every interval. Nevertheless, it is σ-finite (but not locally finite). We denote it $(df)^+$ anyway.

The measure $(df)^+$ is concentrated on the set of points of 'local minimum from the right'. If f is a Brownian sample path then such points are a set of Lebesgue measure 0.

[27] For a definition, see [7, Sect. 12.B].
[28] For a definition, see [7, Sect. 17.A].

So, the set of all chords is a measure space; it carries a natural σ-finite (sometimes, finite) measure. The latter is the intensity measure of a unique Poisson random measure.[29] This way, (the distribution of) a random set of chords is well-defined.

Or equivalently, we may consider a Poisson random subset of $\mathbb{R}$, whose intensity measure is $(df)^+$.

However, it is not so easy to substitute a Brownian sample path $B(\cdot)$ for $f(\cdot)$. In order to get a (Poisson) random variable, we may ask how many random points belong to a given Borel set $A \subset \mathbb{R}$ such that $(dB)^+(A) < \infty$. Note that for any *interval* A, $(dB)^+(A) = \infty$ a.s. We cannot choose an appropriate A without knowing the path $B(\cdot)$. The set of all countable dense subsets of $\mathbb{R}$ does not carry a natural (non-pathological) Borel structure.

In this aspect, chords are better than points. Chords are parameterized by three (or two) numbers, and thus, carry a natural Borel structure, irrespective of $B(\cdot)$. The random countable set of chords is not dense; rather, it accumulates toward short chords.

A point (t, x) belongs to a random chord of $B(\cdot)$ if and only if

$$x \in \sigma_t^{-1}(\Pi) \,, \quad \text{that is,} \quad \sigma_t(x) \in \Pi \,,$$
$$\text{where } \sigma_t(x) = \sup\{s \in (-\infty, t] : B(s) \le x\} \text{ for } x \in (-\infty, B(t))$$

(recall (4c9)), and Π is the Poisson random subset of $\mathbb{R}$, whose intensity measure is $(dB)^+$. Do not confuse the inverse image $\sigma_t^{-1}(\Pi)$ with the image $B(\Pi)$. True, $B(\sigma_t(x)) = x$, but $\sigma_t(B(s)) \ne s$. Sets Π and $B(\Pi)$ are dense, but the set $\sigma_t^{-1}(\Pi)$ is locally finite. Moreover, $\sigma_t^{-1}(\Pi)$ is a Poisson random subset of $(-\infty, B(t)]$, its intensity being just 1.

The random countable dense set Π itself is bad; we have no measurable functions of it. However, the pair $\big(B(\cdot), \Pi\big)$ of the Brownian path and the set is good; we have measurable functions of the pair. In particular, we may use measurable functions of the locally finite set $\sigma_t^{-1}(\Pi)$. Especially,

$$a(0,t) - c(0,t) = \min\big(a(0,t), \min\{x : \sigma_t(x) \in \Pi \cap (0,\infty)\}\big) \,.$$

4i3 Lemma. The σ-field $\mathcal{F}_{s,t}$ of the noise of stickiness (see 4h3) is generated by Brownian increments $B(u) - B(s)$ for $u \in (s,t)$ and random sets $\sigma_u^{-1}\big(\Pi \cap (s,t)\big)$ for $u \in (s,t)$ (treated as random variables whose values are finite subsets of $\mathbb{R}$).

The proof is left to the reader.

[29] See for instance [11, XII.1.18].

5 Stability

5a Discrete case

Fourier-Walsh coefficients, introduced in 3c for an arbitrary dyadic coarse factorization,

$$f = \sum_{M \in \mathcal{C}[i]} \hat{f}_M \tau_M = \hat{f}_\emptyset + \sum_{m \in \frac{1}{i}\mathbb{Z}} \hat{f}_{\{m\}} \tau_m + \sum_{m_1, m_2 \in \frac{1}{i}\mathbb{Z}, m_1 < m_2} \hat{f}_{\{m_1, m_2\}} \tau_{m_1} \tau_{m_2} + \dots$$

help us to examine the stability of a function f, as explained below. Imagine another array of random signs $(\tau'_m)_{m \in \frac{1}{i}\mathbb{Z}}$ (also independent equiprobable ± 1) correlated with the array $(\tau_m)_{m \in \frac{1}{i}\mathbb{Z}}$,

$$\mathbb{E}\, \tau_m \tau'_m = \rho \quad \text{for each } m \in \frac{1}{i}\mathbb{Z} \,;$$

$\rho \in [-1, +1]$ is a parameter. Other correlations vanish. That is, the joint distribution of all τ_m and τ'_m is the product (over $m \in \frac{1}{i}\mathbb{Z}$) of (copies of) such a four-atom distribution:

τ'_m \ τ_m	-1	$+1$
-1	$\frac{1+\rho}{4}$	$\frac{1-\rho}{4}$
$+1$	$\frac{1-\rho}{4}$	$\frac{1+\rho}{4}$

Denoting by $\tilde{\Omega}[i]$ the product of these four-point probability spaces, we have a natural measure preserving map $\alpha : \tilde{\Omega}[i] \to \Omega[i]$; as before, $\Omega[i]$ is the product of two-point probability spaces. In addition, we have another measure preserving map $\alpha' : \tilde{\Omega}[i] \to \Omega[i]$,

$$\tau_m \circ \alpha = \tau_m \,, \quad \tau_m \circ \alpha' = \tau'_m \,;$$

we use the same 'τ_m' for denoting a coordinate function on $\Omega[i]$ and $\tilde{\Omega}[i]$.

For products

$$\tau_M = \prod_{m \in M} \tau_m \,, \quad M \in \mathcal{C}[i] \,, \quad \mathcal{C}[i] = \{ M \subset \tfrac{1}{i}\mathbb{Z} : |M| < \infty \}$$

we have

$$\mathbb{E}\, \tau_M \tau'_M = \rho^{|M|} \,, \quad \tau_M \circ \alpha = \tau_M \,, \quad \tau_M \circ \alpha' = \tau'_M \,,$$

where $|M|$ is the number of elements of M. Therefore

$$\mathbb{E}\,(f \circ \alpha)(g \circ \alpha') = \sum_M \rho^{|M|} \hat f_M \hat g_M = \langle g, \rho^{\mathbf{N}[i]} f \rangle\,,$$

$$\rho^{\mathbf{N}[i]} : L_2[i] \to L_2[i]\,, \quad \rho^{\mathbf{N}[i]} \tau_M = \rho^{|M|} \tau_M\,, \quad \rho^{\mathbf{N}[i]} f = \sum_M \rho^{|M|} \hat f_M \tau_M\,.$$

The Hermite operator $\rho^{\mathbf{N}[i]}$ is a function of a self-adjoint operator $\mathbf{N}[i]$ defined by $\mathbf{N}[i]\tau_M = |M|\tau_M$ for $M \in \mathcal{C}[i]$.

Every bounded function $\varphi : \mathcal{C}[i] \to \mathbb{R}$ acts on $L_2[i]$ by the operator $f \mapsto \sum_{M \in \mathcal{C}[i]} \varphi(M) \hat f_M \tau_M$. A commutative operator algebra is isomorphic to the algebra of functions. The operator $\rho^{\mathbf{N}[i]}$ corresponds to the function $M \mapsto \rho^{|M|}$. (In some sense, the unbounded operator $\mathbf{N}$ corresponds to the unbounded function $M \mapsto |M|$.)

A function $\varphi : \mathcal{C}[i] \to \{0,1\}$, the indicator of a subset of $\mathcal{C}[i]$, corresponds to a projection operator. Say, for the (indicator of) the set $\{\emptyset\}$, the operator projects to the one-dimensional space of constants (the expectation). For the set $\{M : M \subset (0,\infty)\}$, the operator is the conditional expectation, $\mathbb{E}\,\big(\cdot \,\big|\, \mathcal{F}_{0,\infty}[i]\,\big)$.

The function $M \mapsto |M|$ is the sum (over $m \in \frac1i \mathbb{Z}$) of localized functions $M \mapsto |M \cap \{m\}|$. The latter is the indicator of the set $\{M : M \ni m\}$, corresponding to the projection operator $1 - \mathbb{E}\,\big(\cdot \,\big|\, \mathcal{F}_{\frac1i\mathbb{Z}\setminus\{m\}}\,\big)$. Thus,

$$\mathbf{N} f = \sum_m \big(f - \mathbb{E}\,\big(\, f \,\big|\, \mathcal{F}_{\frac1i\mathbb{Z}\setminus\{m\}}\,\big)\,.$$

The operator $\rho^{\mathbf{N}[i]}$ may be interpreted as the conditional expectation w.r.t. the sub-σ-field $\alpha^{-1}(\mathcal{F})$ generated by $\tau_m \circ \alpha$, $m \in \frac1i\mathbb{Z}$:

$$\mathbb{E}\,\big(\, f \circ \alpha' \,\big|\, \alpha^{-1}(\mathcal{F})\,\big) = (\rho^{\mathbf{N}[i]} f) \circ \alpha \quad \text{for } f \in L_2[i]\,.$$

We may imagine that our data τ_m are an unreliable copy of the true data τ'_m; each sign τ_m is either correct (with probability $(1+\rho)/2$) or inverted (with probability $(1-\rho)/2$). If ρ is close to 1, our knowledge of τ'_M is satisfactory for moderate $|M|$ (when $\rho^{|M|} \approx 1$) but very bad for large $|M|$ (when $\rho^{|M|} \approx 0$). The position of a given function f between the two extremes is indicated by the number $\|f - \rho^{\mathbf{N}} f\|$.

5a1 Example. In the Brownian coarse factorization (recall 3b3),

$$\sup_i \|f[i] - \rho^{\mathbf{N}[i]} f[i]\| \to 0 \quad \text{for } \rho \to 1$$

for all $f \in L_2(\mathcal{A})$. This follows easily from convergence of operators (recall 2c and 3d4):

$$\operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = \rho^{\mathbf{N}[\infty]} \,,$$
$$\rho^{\mathbf{N}[\infty]} f = \sum_{n=0}^\infty \rho^n \int\limits_{t_1<\dots<t_n} \!\!\cdots\!\! \int \hat f(\{t_1,\dots,t_n\}) \, \mathrm{d}B(t_1) \dots \mathrm{d}B(t_n) \,.$$

Convergence of operators follows from (2a6). The same holds for 3b5.

5a2 Example. A very different situation appears in 3b6. The second Brownian motion B_2 (or rather, its discrete approximation) is not linear but quadratic in random signs τ_m, $m \in \frac1i \mathbb{Z}$. It is two times less stable:

$$\mathbf{N}[i] f^{(2)}_{s,t}[i] = 2 f^{(2)}_{s,t}[i] \,; \qquad \operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = \rho^{2\mathbf{N}[\infty]} \,,$$

if $\mathbf{N}[\infty]$ is defined in the same way as in 5a1. For B_3 it is $\rho^{3\mathbf{N}[\infty]}$, and so on. Still, $\sup_i \| f[i] - \rho^{\mathbf{N}[i]} f[i] \| \to 0$ for $\rho \to 1$. For B_λ, however, the change is dramatic. Namely,

$$\mathbf{N}[i] f^{(\lambda)}_{s,t}[i] = \operatorname{entier}(\lambda \sqrt i) f^{(\lambda)}_{s,t}[i] \,; \qquad \operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = 0^{\mathbf{N}[\infty]}$$

for all $\rho \in (-1,+1)$; here $0^{\mathbf{N}[\infty]} = \lim_{\rho\to0} \rho^{\mathbf{N}[\infty]}$ is the orthogonal projection to the one-dimensional subspace of constants (just the expectation). The same holds for 3b7.

Notions of stability and sensitivity are introduced in [2, Sects. 1.1, 1.4] for a sequence of two-valued functions of $1,2,3,\dots$ two-valued variables. For arbitrary (not just two-valued) functions, a number of equivalent definitions can be found in [12, Sect. 1]. They may be adapted to our framework as follows. We consider a function $f : \Omega[\mathrm{all}] \to \mathbb{R}$ such that $0 < \liminf_i \| f[i] \| \le \limsup_i \| f[i] \| < \infty$. We say that f is stable, if $\sup_i \| f[i] - \rho^{\mathbf{N}[i]} f[i] \| \to 0$ when $\rho \to 1$. We say that f is sensitive, if $\| \rho^{\mathbf{N}[i]} f[i] - 0^{\mathbf{N}[i]} f[i] \| \to 0$ when $i \to \infty$, for some (therefore, every) $\rho \in (0,1)$. These definitions conform to [12] when $f[i]$ depends only on i signs $\tau_{1/i},\dots,\tau_{i/i}$. In terms of the two ρ-correlated arrays (τ_m), (τ'_m), stability means that $\mathbb{E}\,\big((f[i]\circ\alpha')(f[i]\circ\alpha) \big) \to \|f[i]\|^2$ for $\rho \to 1$, uniformly in i. Or, equivalently, $\mathbb{E}\,\big(\operatorname{Var}\big(f[i] \circ \alpha' \,\big|\, \alpha^{-1}(\mathcal{F}) \big) \big) \to 0$ when $\rho \to 1$, uniformly in i. Sensitivity means that $\mathbb{E}\,\big((f[i]\circ\alpha')(f[i]\circ\alpha) \big) \to \big(\mathbb{E}\, f[i] \big)^2$ when $n \to \infty$, for some (therefore, every) $\rho \in (0,1)$. Or, equivalently, $\mathbb{E}\, \big| \mathbb{E}\,\big(f[i] \circ \alpha' \,\big|\, \alpha^{-1}(\mathcal{F}) \big) - \mathbb{E}\, f[i] \big|^2 \to 0$ when $n \to \infty$, for some (therefore, every) $\rho \in (0,1)$.

In particular, those definitions can be applied to any $f \in L_2(\mathcal{A})$ such that $\| f[\infty] \| \ne 0$.

Example 5a1 shows that everything is stable in the Brownian coarse factorization. In contrast, everything is sensitive in the coarse factorization generated by B_λ in 5a2. In 5c we will find a reason to rename this 'stability' and 'sensitivity' as 'micro-stability' and 'micro-sensitivity'.

A sufficient condition for sensitivity is found by Benjamini, Kalai and Schramm in terms of the influence of a (two-valued) variable on a function, see [2, Sect. 1.2]. In our framework, the influence of the variable τ_m on a function $f[i] : \Omega[i] \to \mathbb{R}$ may be defined as the expectation of the square root of the conditional variance,

$$\mathbb{E} \sqrt{\operatorname{Var}\big(f[i] \,\big|\, \mathcal{F}_{\frac1i \mathbb{Z}\setminus\{m\}} \big)}\,;$$

here $\mathcal{F}_{\frac1i \mathbb{Z}\setminus\{m\}}$ is the sub-σ-field of $\mathcal{F}[i]$ generated by all random signs except for τ_m. The root of the conditional variance is simply one half of the difference between two values of the function $f[i]$, one value for $\tau_m = +1$, the other for $\tau_m = -1$. Thus, our formula gives two times less than [2, (1.3)], but the coefficient does not matter. Similarly, for any set $M \subset \frac1i \mathbb{Z}$, the influence of M (that is, of all variables τ_m, $m \in M$) on $f[i]$ may be defined as

$$\mathbb{E} \sqrt{\operatorname{Var}\big(f[i] \,\big|\, \mathcal{F}_{\frac1i \mathbb{Z}\setminus M} \big)}\,.$$

By the way, for a *linear* function, the *squared* influence is additive (in M); indeed, if $f[i] = \sum_m c_m \tau_m$, then $\operatorname{Var}\big(f[i] \,\big|\, \mathcal{F}_{\frac1i \mathbb{Z}\setminus M} \big) = \mathbb{E}\,\big(\sum_{m\in M} c_m \tau_m \big)^2 = \sum_{m \in M} c_m^2$. The sum of squared influences appears in the following remarkable result (adapted to our framework).

5a3 Theorem (Benjamini, Kalai, Schramm). Let a function $f : \Omega[\text{all}] \to \{0,1\}$ be such that each $f[i]$ depends on i variables $\tau_{1/i}, \dots, \tau_{i/i}$ only. If

$$\sum_{k=1}^{i} \Big(\mathbb{E} \sqrt{\operatorname{Var}\big(f[i] \,\big|\, \mathcal{F}_{\frac1i \mathbb{Z}\setminus\{k/i\}} \big)} \Big)^2 \xrightarrow[i\to\infty]{} 0\,,$$

then f is sensitive.

See [2, Th. 1.3]. We will return to the point in 6d.

5b Continuous case

We start with the *Brownian* continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$. Using the Wiener-Itô decomposition of $L_2(\Omega,\mathcal{F},P)$,

$$f = \sum_{n=0}^{\infty} \underbrace{\int \cdots \int_{t_1<\dots<t_n} \hat f(\{t_1,\dots,t_n\})\, \mathrm{d}B(t_1)\dots\mathrm{d}B(t_n)}_{\text{belongs to } n\text{-th Wiener chaos}}\,, \quad \hat f \in L_2(\mathcal{C}_{\text{finite}})\,,$$

we can define a self-adjoint operator $\mathbf{N} : L_2 \to L_2$ such that for each n, $\mathbf{N}f = nf$ for all f of n-th Wiener chaos. Accordingly, $\rho^{\mathbf{N}} f = \rho^n f$ for these f. Informally, $\mathbf{N}(\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)) = n\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)$.

Every bounded Borel function φ on $\mathcal{C}_{\text{finite}}$ acts on $L_2(\Omega, \mathcal{F}, P)$ by the operator R_φ,

$$R_\varphi f = \sum_{n=0}^{\infty} \int \cdots \int_{t_1 < \dots < t_n} \varphi\big(\{t_1, \dots, t_n\}\big) \hat{f}\big(\{t_1, \dots, t_n\}\big) \, \mathrm{d}B(t_1) \dots \mathrm{d}B(t_n) \, . \tag{5b1}$$

The operator $\rho^{\mathbf{N}}$ corresponds to the function $M \mapsto \rho^{|M|}$. (In some sense, the unbounded operator $\mathbf{N}$ corresponds to the unbounded function $M \mapsto |M|$.) The decomposition $|M| = |M \cap (-\infty, t)| + |M \cap (t, \infty)$ (it holds for μ_f-almost all M) leads to the operator decomposition $\mathbf{N} = \mathbf{N}_{-\infty,t} + \mathbf{N}_{t,\infty}$. Informally, $\mathbf{N}_{-\infty,t}\big(\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)\big) = k\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)$ and $\mathbf{N}_{t,\infty}\big(\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)\big) = (n-k)\mathrm{d}B(t_1) \dots \mathrm{d}B(t_n)$ whenever $t_1 < \dots < t_k < t < t_{k+1} < \dots < t_n$. Accordingly, $\rho^{\mathbf{N}} = \rho^{\mathbf{N}_{-\infty,t}} \otimes \rho^{\mathbf{N}_{t,\infty}}$.

A function $\varphi : \mathcal{C}_{\text{finite}} \to \{0, 1\}$, the indicator of a Borel subset $\mathcal{M}$ of $\mathcal{C}_{\text{finite}}$, corresponds to the orthogonal projection operator onto the corresponding (recall Theorem 3d12) subspace $H_{\mathcal{M}}$. Say, for the (indicator of the) set $\{\emptyset\}$, the operator projects onto the one-dimensional space of constants (the expectation). For the set $\{M : M \subset (0, \infty)\}$ the operator is the conditional expectation, $\mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{0,\infty} \right)$.

The function

$$\varphi_{s,t}(M) = \begin{cases} 1 & \text{if } M \cap (s,t) \ne \emptyset, \\ 0 & \text{if } M \cap (s,t) = \emptyset \end{cases}$$

acts by the operator $\mathbf{1} - \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{(-\infty,s)\cup(t,\infty)} \right)$.

For a finite set $L = \{s_1, \dots, s_n\} \subset \mathbb{R}$, $s_1 < \dots < s_n$, the function $\varphi_L(M) = \varphi_{s_1,s_2}(M) + \dots + \varphi_{s_{n-1},s_n}(M)$ counts intervals (s_j, s_{j+1}) that intersect M. Clearly, $\varphi_L(M) \le |M|$, and

$$\varphi_{L_n}(M) \uparrow |M| \quad \text{for } \mu_f\text{-almost all } M$$

if $L_1 \subset L_2 \subset \dots$ are chosen so that their union is dense in $\mathbb{R}$. Accordingly,

$$\mathbf{N}_{L_n} \uparrow \mathbf{N} \, ,$$

$$\mathbf{N}_{\{s_1,\dots,s_n\}} = \sum_{j=1}^{n-1} \Big(\mathbf{1} - \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{(-\infty,s_j)\cup(s_{j+1},\infty)} \right) \Big) \, . \tag{5b2}$$

The operator $\mathbf{N}$ is thus expressed in terms of the factorization only, irrespective of the Wiener-Itô decomposition, which gives us a bridge to *arbitrary* continuous factorizations. Operators R_φ described in the next lemma generalize (5b1).

5b3 Lemma. For every continuous factorization $\bigl((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\bigr)$ there exists one and only one map $\varphi \mapsto R_\varphi$ from the set of all bounded Borel functions $\varphi : \mathcal{C} \to \mathbb{R}$ to the set of (bounded linear) operators on $L_2(\Omega,\mathcal{F},P)$ such that

(a) the map is a homomorphism of algebras; that is, $R_{a\varphi} = aR_\varphi$, $R_{\varphi+\psi} = R_\varphi + R_\psi$, $R_{\varphi\psi} = R_\varphi R_\psi$;

(b) $\|R_\varphi\| \le \sup_{M\in\mathcal{C}} |\varphi(M)|$;

(c) $R_{\mathbf{1}_\mathcal{M}} = \operatorname{Proj}_{H_\mathcal{M}}$ for every Borel set $\mathcal{M} \subset \mathcal{C}$; here $\mathbf{1}_\mathcal{M}$ is the indicator of $\mathcal{M}$, and $(H_\mathcal{M})$ is the orthogonal decomposition provided by Theorem 3d12.

The map also satisfies the condition

(d) let $\varphi,\varphi_1,\varphi_2,\dots : \mathcal{C} \to [0,1]$ be Borel functions such that $\varphi_k \to \varphi$ pointwise (that is, $\varphi_k(M) \xrightarrow[k\to\infty]{} \varphi(M)$ for each $M \in \mathcal{C}$); then $R_{\varphi_k} \to R_\varphi$ strongly (that is, $\|R_{\varphi_k}x - R_\varphi x\| \xrightarrow[k\to\infty]{} 0$ for every $x \in L_2(\Omega,\mathcal{F},P)$).

Proof. Uniqueness and existence are easy: Condition (c) and linearity determine the map on the algebra of Borel functions $\varphi : \mathcal{C} \to \mathbb{R}$ having finite sets of values; it remains to extend the map by continuity.

For proving Condition (d) we note the equality

$$\langle R_\varphi x, x\rangle = \int \varphi \, d\mu_x \,,$$

where μ_x is the spectral measure of x; it holds for φ having finite sets of values, and therefore, for all φ. The bounded convergence theorem gives us not only $\langle R_{\varphi_k}x, x\rangle \to \langle R_\varphi x, x\rangle$, but also $\langle R_{(\varphi_k-\varphi)^2}x, x\rangle \to 0$. However, $\|R_{\varphi_k}x - R_\varphi x\|^2 = \langle R_{\varphi_k-\varphi}x, R_{\varphi_k-\varphi}x\rangle = \langle R_{(\varphi_k-\varphi)^2}x, x\rangle$. $\square$

5b4 Lemma. For every continuous factorization $\bigl((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\bigr)$, all finite sets $L_1 \subset L_2 \subset \dots$ whose union is dense in $\mathbb{R}$, and every $\lambda \in [0,\infty)$, the limit

$$U_\lambda = \lim_n \exp(-\lambda \mathbf{N}_{L_n}) \,,$$

where $\mathbf{N}_L$ is defined by (5b2), exists in the strong operator topology, and does not depend on the choice of $L_1, L_2, \dots$ Also,

$$U_\lambda U_\mu = U_{\lambda+\mu} \quad \text{for all } \lambda,\mu \in [0,\infty) \,.$$

Proof. We have $\varphi_L = \sum \varphi_{s_k,s_{k+1}}$ and $R_{\varphi_{s,t}} = \mathbf{1} - \mathbb{E}\bigl(\cdot \,\big|\, \mathcal{F}_{(-\infty,s)\cup(t,\infty)}\bigr)$; thus $R_{\varphi_L} = \mathbf{N}_L$. It follows that $R_{\exp(-\lambda\varphi_L)} = \exp(-\lambda\mathbf{N}_L)$. However, $\exp(-\lambda\varphi_{L_n}) \to \varphi_\lambda$, where $\varphi_\lambda(M) = \exp(-\lambda|M|)$ (and $e^{-\infty} = 0$, of course). By 5b3(d), $\exp(-\lambda\mathbf{N}_{L_n}) \to R_{\varphi_\lambda} = U_\lambda$. The semigroup relation $U_\lambda U_\mu = U_{\lambda+\mu}$ for operators follows from the corresponding relation $\varphi_\lambda\varphi_\mu = \varphi_{\lambda+\mu}$ for functions. $\square$

In the Brownian factorization we know that $U_\lambda = \exp(-\lambda \mathbf{N})$, $\mathbf{N} = \lim_n \mathbf{N}_{L_n}$. In general, however, the semigroup $(U_\lambda)_{\lambda \ge 0}$ is discontinuous at $\lambda = 0$ (and $\mathbf{N}$ is ill-defined).

5b5 Definition. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t}\big)$ be a continuous factorization, and $f \in L_2(\Omega, \mathcal{F}, P)$.

(a) f is called *stable,* if $\| f - U_\lambda f \| \to 0$ for $\lambda \to 0$, or equivalently, if μ_f is concentrated on $\mathcal{C}_{\text{finite}} = \{ M \in \mathcal{C} : |M| < \infty \}$.

(b) f is called *sensitive,* if $U_\lambda f = 0$ for all $\lambda > 0$, or equivalently, if μ_f is concentrated on $\mathcal{C} \setminus \mathcal{C}_{\text{finite}} = \{ M \in \mathcal{C} : |M| = \infty \}$.

Of course, $U_0 f = f$ anyway. For proving equivalence, apply 5b3(d) to $U_\lambda = R_{\varphi_\lambda}$, $\varphi_\lambda(M) = \mathrm{e}^{-\lambda |M|}$.

The space $L_2(\Omega, \mathcal{F}, P)$ decomposes into the direct sum of two subspaces, stable and sensitive, according to the decomposition of $\mathcal{C}$ into the union of two disjoint subsets, $\mathcal{C}_{\text{finite}}$ and $\mathcal{C} \setminus \mathcal{C}_{\text{finite}}$.

A continuous factorization is called *classical* (or *stable*), if the stable subspace is the whole $L_2(\Omega, \mathcal{F}, P)$.

A noise is called classical, if its continuous factorization is classical.

In order to understand probabilistic meaning of U_λ, consider first $\rho^{\mathbf{N}_L}$, $L = \{s_1, \dots, s_n\}$, $s_1 < \dots < s_n$. We have

$$\Omega = \Omega_{-\infty, s_1} \times \Omega_{s_1, s_2} \times \dots \times \Omega_{s_{n-1}, s_n} \times \Omega_{s_n, \infty}$$

or rather, $(\Omega, \mathcal{F}, P) = (\Omega_{-\infty,s_1}, \mathcal{F}_{-\infty,s_1}, P_{-\infty,s_1}) \times \dots$, but let me use the shorter notation. Each $\omega \in \Omega$ may be thought of as a sequence $(\omega_{-\infty,s_1}, \omega_{s_1,s_2}, \dots \omega_{s_{n-1},s_n}, \omega_{s_n,\infty})$ of local portions of data. Imagine another portion of data $\omega'_{s_1,s_2} \in \Omega_{s_1,s_2}$, either equal to ω_{s_1,s_2} (with probability ρ), or independent of it (with probability $1-\rho$). The joint distribution of ω_{s_1,s_2} and ω'_{s_1,s_2} is a convex combination of two probability measures on $\tilde\Omega_{s_1,s_2} = \Omega_{s_1,s_2} \times \Omega_{s_1,s_2}$. One measure is concentrated on the diagonal and is the image of P_{s_1,s_2} under the map $\Omega_{s_1,s_2} \ni \omega_{s_1,s_2} \mapsto (\omega_{s_1,s_2}, \omega_{s_1,s_2}) \in \tilde\Omega_{s_1,s_2}$; this measure occurs with the coefficient ρ. The other measure is the product measure $P_{s_1,s_2} \otimes P_{s_1,s_2}$; it occurs with the coefficient $1-\rho$.

Similarly we introduce $\tilde\Omega_{s_2,s_3}, \dots, \tilde\Omega_{s_{n-1},s_n}$ and construct $\tilde\Omega = \Omega_{-\infty,s_1} \times \tilde\Omega_{s_1,s_2} \times \dots \times \tilde\Omega_{s_{n-1},s_n} \times \Omega_{s_n,\infty}$ (the factors being equipped with corresponding measures). It is the same idea as in 5a. Again, we have two measure preserving maps $\alpha, \alpha' : \tilde\Omega \to \Omega$. It appears that

$$\mathbb{E}\,\big(f \circ \alpha' \,\big|\, \alpha^{-1}(\mathcal{F}) \big) = (\rho^{\mathbf{N}_L} f) \circ \alpha \quad \text{for } f \in L_2(\Omega, \mathcal{F}, P)\,.$$

This is the probabilistic interpretation of $\rho^{\mathbf{N}_L}$; each portion of data is either correct (with probability ρ), or wrong (with probability $1-\rho$).[30] However, the portions are not small yet. The limit $n \to \infty$ makes them infinitesimal, and turns $\rho^{\mathbf{N}_L}$ into U_λ, where ρ and λ are related by $\rho = \mathrm{e}^{-\lambda}$.

The interpretation above motivates the terms 'stable' and 'sensitive'.

Constant functions on Ω are stable; sensitive functions are of zero mean. This is a terminological deviation from the discrete case; according to 5a, constant functions are both stable and sensitive.

Two limiting cases of U_λ are projections. Namely, $U_\infty = \lim_{\lambda\to\infty} U_\lambda$ is the expectation, and $U_{0+} = \lim_{\lambda\to 0+} U_\lambda$ is the projection onto the stable subspace. Restricting the 'perturbation of local data' to a given interval (s,t) we get operators $U_\lambda^{(s,t)}$. These correspond to functions $\mathcal{C} \ni M \mapsto \exp(-\lambda|M \cap (s,t)|)$ and satisfy

$$
\begin{gathered}
U_\lambda^{(s,t)} U_\mu^{(s,t)} = U_{\lambda+\mu}^{(s,t)}\,; \quad U_\lambda^{(r,s)} U_\lambda^{(s,t)} = U_\lambda^{(r,t)}\,; \\
U_\infty^{(s,t)} = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s} \otimes \mathcal{F}_{t,\infty} \right); \\
U_{0+}^{(s,t)} = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s} \otimes \mathcal{F}_{s,t}^{\text{stable}} \otimes \mathcal{F}_{t,\infty} \right).
\end{gathered} \tag{5b6}
$$

Note that (5b2) may be written as

$$
\mathbf{N}_{\{s_1,\dots,s_n\}} = \bigl(\mathbf{1} - U_\infty^{(s_1,s_2)}\bigr) + \dots + \bigl(\mathbf{1} - U_\infty^{(s_{n-1},s_n)}\bigr)\,. \tag{5b7}
$$

5b8 Lemma. Let $\bigl((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\bigr)$ be a continuous factorization, $f \in L_2(\Omega,\mathcal{F},P)$, and $g = \eta \circ f$ where $\eta : \mathbb{R} \to \mathbb{R}$ satisfies $|\eta(x) - \eta(y)| \le |x-y|$ for all $x,y \in \mathbb{R}$. Then

$$
\mu_g(\mathcal{C} \setminus \mathcal{M}_E) \le \mu_f(\mathcal{C} \setminus \mathcal{M}_E)
$$

for all elementary sets $E \subset \mathbb{R}$; here $\mathcal{M}_E = \{M \in \mathcal{C} : M \subset E\}$.

Proof. We have (up to isomorphism) $\Omega = \Omega_E \times \Omega_{\mathbb{R}\setminus E}$ (the product of probability spaces is meant). We introduce $\tilde\Omega = \Omega \times \Omega = (\Omega_E \times \Omega_E) \times (\Omega_{\mathbb{R}\setminus E} \times \Omega_{\mathbb{R}\setminus E})$ and equip the second factor $\Omega_{\mathbb{R}\setminus E} \times \Omega_{\mathbb{R}\setminus E}$ with the product measure, while the first factor $\Omega_E \times \Omega_E$ is equipped with the measure concentrated on the

[30] This time, $\rho \in [0,1]$ rather than $[-1,1]$. The relation to the approach of 5a is expressed by the equality

$$
\begin{aligned}
\frac{1+\rho}{2} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \frac{1-\rho}{2} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} &= \begin{pmatrix} (1+\rho)/4 & (1-\rho)/4 \\ (1-\rho)/4 & (1+\rho)/4 \end{pmatrix} \\
&= \rho \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + (1-\rho) \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}.
\end{aligned}
$$

diagonal, such that (equipping $\tilde{\Omega}$ with the product of these two measures), the measure preserving 'coordinate' maps $\alpha, \alpha' : \tilde{\Omega} \to \Omega$ satisfy

$$
\begin{gathered}
f \circ \alpha = f \circ \alpha' \quad \text{for all } \mathcal{F}_E\text{-measurable } f, \\
f \circ \alpha \text{ and } g \circ \alpha' \text{ are independent, for all } \mathcal{F}_{\mathbb{R}\setminus E}\text{-measurable } f, g.
\end{gathered}
$$

Then

$$
\mathbb{E}\left(f \circ \alpha' \,\middle|\, \alpha^{-1}(\mathcal{F}) \right) = \mathbb{E}\left(f \,\middle|\, \mathcal{F}_E \right) \circ \alpha \quad \text{for all } f \in L_2(\Omega, \mathcal{F}, P) \, .
$$

Therefore (recall Theorem 3d12),

$$
\begin{gathered}
\mathbb{E}\big((f \circ \alpha')(g \circ \alpha) \big) = \mathbb{E}\left(g \mathbb{E}\left(f \,\middle|\, \mathcal{F}_E \right) \right) ; \\
\mathbb{E}\big((f \circ \alpha')(f \circ \alpha) \big) = \langle \operatorname{Proj}_{H_{\mathcal{M}_E}} f, f \rangle = \mu_f(\mathcal{M}_E) ; \\
\frac{1}{2} \mathbb{E} \left(f \circ \alpha' - f \circ \alpha \right)^2 = \mu_f(\mathcal{C}) - \mu_f(\mathcal{M}_E) = \mu_f(\mathcal{C} \setminus \mathcal{M}_E) \, .
\end{gathered}
$$

The same holds for g. It remains to note that $|g \circ \alpha' - g \circ \alpha| = |\eta \circ f \circ \alpha' - \eta \circ f \circ \alpha| \le |f \circ \alpha' - f \circ \alpha|$ everywhere on $\tilde{\Omega}$. $\square$

We introduce a special set S of Borel functions $\varphi : \mathcal{C} \to [0,1]$ in three steps. First, we take all functions of the form $\mathbf{1}_{\mathcal{M}_E}$,

$$
\mathbf{1}_{\mathcal{M}_E}(M) = \begin{cases} 1 & \text{if } M \subset E, \\ 0 & \text{otherwise,} \end{cases}
$$

where $E \subset \mathbb{R}$ runs over all elementary sets. Second, we consider all (finite) convex combinations of these $\mathbf{1}_{\mathcal{M}_E}$. Third, we consider the least set S containing these convex combinations and closed under pointwise convergence (that is, if $\varphi_k \in S$ and $\varphi_k(M) \to \varphi(M)$ for each $M \in \mathcal{C}$ then $\varphi \in S$).

The set S is convex (since the third step preserves convexity). It is also closed under multiplication: $\varphi\psi \in S$ for all $\varphi, \psi \in S$. Indeed, multiplicativity holds in the first step, and is preserved in the second and third steps.

5b9 Lemma. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t} \big)$ be a continuous factorization, $f \in L_2(\Omega, \mathcal{F}, P)$, and $g = \eta \circ f$ where $\eta : \mathbb{R} \to \mathbb{R}$ satisfies $|\eta(x) - \eta(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$. Then

$$
\int (1 - \varphi) \, d\mu_g \le \int (1 - \varphi) \, d\mu_f
$$

for all $\varphi \in S$.

Proof. In the first step, for $\varphi = \mathbf{1}_{\mathcal{M}_E}$, the inequality is stated by 5b8. The second step evidently preserves the inequality. And the third step preserves it due to the bounded convergence theorem. $\square$

5b10 Lemma. Let a Borel set $\mathcal{M} \subset \mathcal{C}$ be such that its indicator function $\mathbf{1}_\mathcal{M}$ belongs to the set S. Then for every continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$, the subspace $H_\mathcal{M} = \{f : \mu_f(\mathcal{C} \setminus \mathcal{M}) = 0\}$ of $L_2(\Omega, \mathcal{F}, P)$ is of the form

$$H_\mathcal{M} = L_2(\Omega, \mathcal{F}_\mathcal{M}, P)$$

where $\mathcal{F}_\mathcal{M}$ is a sub-σ-field of $\mathcal{F}$.

Proof. The subspace satisfies

$$f \in H_\mathcal{M} \quad \text{implies} \quad |f| \in H_\mathcal{M}$$

(here $|f|(M) = |f(M)|$ for $M \in \mathcal{C}$). Indeed,

$$\int (1 - \mathbf{1}_\mathcal{M}) \, d\mu_{|f|} \le \int (1 - \mathbf{1}_\mathcal{M}) \, d\mu_f$$

by 5b9; that is, $\mu_{|f|}(\mathcal{C} \setminus \mathcal{M}) \le \mu_f(\mathcal{C} \setminus \mathcal{M})$. A subspace satisfying such a condition is necessarily of the form $L_2(\Omega, \mathcal{F}_\mathcal{M}, P)$. $\square$

Recall the decomposition of $L_2(\Omega, \mathcal{F}, P)$ into the sum of two orthogonal subspaces, stable and sensitive, according to the decomposition of $\mathcal{C}$ into the union of two disjoint subsets, $\mathcal{C}_{\text{finite}}$ and $\mathcal{C} \setminus \mathcal{C}_{\text{finite}}$.

5b11 Theorem. For every continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ there exists a sub-σ-field $\mathcal{F}_{\text{stable}}$ of $\mathcal{F}$ such that for all $f \in L_2(\Omega, \mathcal{F}, P)$

$$\begin{gathered} f \text{ is stable if and only if } f \text{ is } \mathcal{F}_{\text{stable}}\text{-measurable;} \\ f \text{ is sensitive if and only if } \mathbb{E}\left(f \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0 \,. \end{gathered}$$

Proof. The second statement (about sensitive functions) follows from the first (about stable functions). By 5b10 it is enough to prove that the indicator of $\mathcal{C}_{\text{finite}}$ belongs to S.

For every $\lambda \in (0, \infty)$ the function $\varphi_\lambda : \mathcal{C} \to [0,1]$ defined by $\varphi_\lambda(M) = \exp(-\lambda |M|)$ belongs to S due to the limiting procedure $\varphi_\lambda = \lim \exp(-\lambda \varphi_{L_n})$ used in the proof of 5b4. For each n the function $\exp(-\lambda \varphi_{L_n}) = \prod \exp(-\lambda \varphi_{s_k, s_{k+1}})$ belongs to S, since each $\exp(-\lambda \varphi_{s,t})$ is a convex combination of two indicators, of $\mathcal{M}_{(-\infty,s)\cup(t,\infty)}$ and of the whole $\mathcal{M}$.

It remains to note that φ_λ converges for $\lambda \to 0$ to the indicator of $\mathcal{C}_{\text{finite}}$. $\square$

So, a continuous factorization (or a noise) is classical if and only if $\mathcal{F}_{\text{stable}} = \mathcal{F}$.

5c Back to discrete: two kinds of stability

The operator equality $\operatorname{Lim} \rho^{\mathbf{N}[i]} = \rho^{\mathbf{N}[\infty]}$ holds for some dyadic coarse factorizations (recall 5a1) but fails for some others (recall 5a2). Nothing like that happens for spectral measures; $\mu_f[i] \to \mu_f[\infty]$ always (see Theorem 3c5 and 3d). However, the operator $\rho^{\mathbf{N}[i]}$ corresponds to the function $\mathcal{C}[i] \ni M \mapsto \rho^{|M|}$ treated as an element of $L_\infty(\mu_f[i])$, and the operator $\rho^{\mathbf{N}[\infty]}$ corresponds to the function $\mathcal{C}[\infty] \ni M \mapsto \rho^{|M|}$ treated as an element of $L_\infty(\mu_f[\infty])$. How is it possible? Where is the origin of the clash between discrete and continuous?

The origin is discontinuity of functions $M \mapsto \rho^{|M|}$ and $M \mapsto |M|$ w.r.t. the Hausdorff topology on $\mathcal{C}$.

5c1 Example. Return to the equality $\mathbf{N}[i] f_{s,t}^{(2)}[i] = 2 f_{s,t}^{(2)}[i]$ for $f_{s,t}^{(2)}[i] = i^{-1/2} \sum \tau_m \tau_{m+(1/i)}$ (see 5a2 and 3b6). The spectral measure of $f_{s,t}^{(2)}[i]$ is concentrated on two-point sets $M \subset \frac{1}{i}\mathbb{Z}$, namely, on pairs of two adjacent points $\{m, m+(1/i)\}$. However, $f_{s,t}^{(2)}[\infty]$ is just a Brownian increment; its spectral measure is concentrated on single-point sets. Now we see what happens; two close points merge in the limit! Multiplicity of spectral points eludes the continuous model.

The effect becomes dramatic for $f_{s,t}^{(\lambda)}[i]$; everything is stable in the continuous model ($i = \infty$), while everything is sensitive (for $i \to \infty$) in the discrete model. A finite spectral set on the continuum hides the infinite multiplicity of each point.

Conformity between discrete and continuous can be restored by modifying the idea of stability introduced in 5a. Instead of inverting each τ_m (with probability $(1-\rho)/2$) independently of others, we may invert blocks $\tau_{s[i]}, \tau_{s[i]+(1/i)}, \dots, \tau_{t[i]}$ where coarse instants s, t satisfy $t[\infty] - s[\infty] = \varepsilon$. Each block is inverted with probability $(1-\rho)/2$, independently of other blocks. Ultimately we let $\varepsilon \to 0$, but the order of limits is crucial: $\lim_{\varepsilon\to 0} \lim_{i\to\infty}(\dots)$. This way, we can define (in discrete time setup) *block stability* and *block sensitivity*, equivalent to stability and sensitivity (resp.) of the refinement. In contrast, the approach of 5a leads to what may be called *micro-stability* and *micro-sensitivity* (for discrete time only).

The function $\mathcal{C} \ni M \mapsto \rho^{|M|}$ is not continuous, but it is upper semicontinuous. Therefore, every micro-stable function is block stable, and every block sensitive function is micro-sensitive.

5c2 Example. The function $g_{s,t}$ of 3b7 is micro-sensitive but block stable. The same holds for all coarse random variables in that dyadic coarse factorization. It holds also for the second construction of 3b6 (I mean $f_{s,t}^{(\lambda)}$).

6 Generalizing Wiener Chaos

6a First chaos, decomposable processes, stability

We consider an arbitrary continuous factorization. As was shown in Theorem 3d12 and 5b3, Borel functions $\varphi : \mathcal{C} \to \mathbb{R}$ act on $L_2(\Omega, \mathcal{F}, P)$ by linear operators R_φ, and (indicators of) Borel subsets $\mathcal{M} \subset \mathcal{C}$ act by orthogonal projections to subspaces $H_{\mathcal{M}}$.

In particular, for the Brownian factorization, only $\mathcal{C}_{\text{finite}}$ is relevant. The set $\{M \in \mathcal{C}_{\text{finite}} : |M| = n\}$ corresponds to the subspace called n-th Wiener chaos.

In general, we may define n-th chaos as the subspace of $L_2(\Omega, \mathcal{F}, P)$ that corresponds to $\{M \in \mathcal{C} : |M| = n\}$. These subspaces are orthogonal, and span the stable subspace — not the whole $L_2(\Omega, \mathcal{F}, P)$, unless the noise is classical.

For each $t \in \mathbb{R}$ the set $\mathcal{M}_t = \{M : M \ni t\}$ is negligible in the sense that $H_{\mathcal{M}_t} = \{0\}$ (recall 3c7 and (3d3)). Neglecting $\mathcal{M}_t$ we may treat $\mathcal{C}$ as the product,[31]

(6a1) $$\mathcal{C} = \mathcal{C}_{-\infty,t} \times \mathcal{C}_{t,\infty} \,,$$

where $\mathcal{C}_{a,b}$ is the space of all compact subsets of (a,b); namely, we treat a set $M \in \mathcal{C}$ as the pair of sets $M \cap (-\infty, t)$ and $M \cap (t, \infty)$, assuming $t \notin M$.

On the other hand, the Hilbert space $H = H_{\mathcal{C}} = L_2(\Omega, \mathcal{F}, P)$ may be treated as the tensor product,

$$H = H_{-\infty,t} \otimes H_{t,\infty} \,,$$

of two Hilbert spaces $H_{-\infty,t} = H_{\mathcal{C}_{-\infty,t}} = L_2(\Omega, \mathcal{F}_{-\infty,t}, P)$ and $H_{t,\infty} = H_{\mathcal{C}_{t,\infty}} = L_2(\Omega, \mathcal{F}_{t,\infty}, P)$. Namely, $f \otimes g$ is just the usual product fg of random variables $f \in L_2(\Omega, \mathcal{F}_{-\infty,t}, P)$ and $g \in L_2(\Omega, \mathcal{F}_{t,\infty}, P)$; note that f and g are necessarily independent, therefore $\mathbb{E}\,|fg|^2 = \big(\mathbb{E}\,|f|^2\big)\big(\mathbb{E}\,|g|^2\big)$.

Subspaces $H_{\mathcal{M}} \subset H_{-\infty,t}$ for Borel subsets $\mathcal{M} \subset \mathcal{C}_{-\infty,t}$ are a σ-additive orthogonal decomposition of $H_{-\infty,t}$. The same holds for (t, ∞).

6a2 Lemma. $H_{\mathcal{M}_1 \times \mathcal{M}_2} = H_{\mathcal{M}_1} \otimes H_{\mathcal{M}_2}$ for all Borel sets $\mathcal{M}_1 \subset \mathcal{C}_{-\infty,t}$ and $\mathcal{M}_2 \subset \mathcal{C}_{t,\infty}$.

Proof. The equality holds for the special case $\mathcal{M}_1 = \{M : M \subset E_1\}$, $\mathcal{M}_2 = \{M : M \subset E_2\}$ where $E_1 \subset (-\infty, t)$ and $E_2 \subset (t, \infty)$ are elementary sets;

[31] Sorry, the formula '$\mathcal{C} = \mathcal{C}_{-\infty,t} \times \mathcal{C}_{t,\infty}$' may be confusing since, on the other hand, $\mathcal{C}_{-\infty,t} \subset \mathcal{C}$ and $\mathcal{C}_{t,\infty} \subset \mathcal{C}$. The same can be said about the next formula, $H = H_{-\infty,t} \otimes H_{t,\infty}$.

indeed, $L_2(\Omega,\mathcal{F}_{E_1},P)\otimes L_2(\Omega,\mathcal{F}_{E_2},P) = L_2(\Omega,\mathcal{F}_{E_1\cup E_2},P)$ since $\mathcal{F}_{E_1\cup E_2} = \mathcal{F}_{E_1}\otimes\mathcal{F}_{E_2}$. The general case follows by the monotone class theorem. $\square$

6a3 Theorem. The sub-σ-field generated by the first chaos is equal to $\mathcal{F}_{\text{stable}}$.

Proof. The σ-field is evidently included in $\mathcal{F}_{\text{stable}}$. Given a finite set $L = \{s_1,\dots,s_n\}\subset\mathbb{R}$, $s_1<\dots<s_n$, we consider the set $\mathcal{M}_L$ of all $M\in\mathcal{C}$ such that $M\subset(s_1,s_n)$ and each $[s_k,s_{k+1}]$ contains at most one point of M. The set $\mathcal{M}_L$ being the product (over k), 6a2 shows that $H_{\mathcal{M}_L}$ is the tensor product (over k) of subspaces of $L_2(\Omega,\mathcal{F}_{s_k,s_{k+1}},P)$; each factor is the first chaos on (s_k,s_{k+1}) plus constants. Therefore each function of $H_{\mathcal{M}_L}$ is measurable w.r.t. the σ-field generated by the first chaos. We choose $L_1\subset L_2\subset\dots$ whose union is dense in $\mathbb{R}$; then $\mathcal{M}_{L_n}\uparrow\mathcal{C}_{\text{finite}}$, and corresponding subspaces span the stable subspace. $\square$

A random variable $X\in L_2(\Omega,\mathcal{F},P)$ belongs to the first chaos if and only if

$$X = \mathbb{E}\left(X\,\middle|\,\mathcal{F}_{-\infty,t}\right)+\mathbb{E}\left(X\,\middle|\,\mathcal{F}_{t,\infty}\right)\quad\text{for all } t\in\mathbb{R}\,.$$

For such X, letting $X_{s,t}=\mathbb{E}\left(X\,\middle|\,\mathcal{F}_{s,t}\right)$ we get a *decomposable process*, that is, a family $(X_{s,t})_{s\le t}$ of random variables such that $X_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and $X_{r,s}+X_{s,t}=X_{r,t}$ whenever $r\le s\le t$. This way we get decomposable processes satisfying $\mathbb{E}\,|X_{s,t}|^2<\infty$ and $\mathbb{E}\,X_{s,t}=0$. Waiving these additional conditions we get a larger set of processes, but the sub-σ-field generated by these processes is still $\mathcal{F}_{\text{stable}}$. We may also consider complex-valued *multiplicative decomposable processes;* it means that $X_{s,t}:\Omega\to\mathbb{C}$ is $\mathcal{F}_{s,t}$-measurable and $X_{r,s}X_{s,t}=X_{r,t}$. The generated sub-σ-field is $\mathcal{F}_{\text{stable}}$, again. The same holds under the restriction $|X_{s,t}|=1$ a.s. See [20, Th. 1.7].

Dealing with a noise (rather than factorization) we may restrict ourselves to stationary Brownian and Poisson decomposable processes. 'Stationary' means $X_{r,s}\circ\alpha_t = X_{r-t,s-t}$. 'Brownian' means $X_{s,t}\sim\mathrm{N}(0,t-s)$. 'Poisson' means $X_{s,t}\sim\mathrm{Poisson}(\lambda(t-s))$ for some $\lambda\in(0,\infty)$. The generated sub-σ-field is still $\mathcal{F}_{\text{stable}}$. See [15, Lemma 2.9]. (It was written for the Brownian component, but works also for the Poisson component.)

For a finite set $L=\{s_1,\dots,s_n\}\subset\mathbb{R}$, $s_1<\dots<s_n$, we introduce an operator Q_L on the space $L_2^0=\{X\in L_2(\Omega,\mathcal{F},P):\mathbb{E}\,X=0\}$ by

$$Q_L=\mathbb{E}\left(\cdot\,\middle|\,\mathcal{F}_{-\infty,s_1}\right)+\mathbb{E}\left(\cdot\,\middle|\,\mathcal{F}_{s_1,s_2}\right)+\dots+\mathbb{E}\left(\cdot\,\middle|\,\mathcal{F}_{s_{n-1},s_n}\right)+\mathbb{E}\left(\cdot\,\middle|\,\mathcal{F}_{s_n,\infty}\right).$$

6a4 Theorem. If finite sets $L_1\subset L_2\subset\dots$ are such that their union is dense in $\mathbb{R}$, then operators Q_{L_n} converge in the strong operator topology to the orthogonal projection from L_2^0 onto the first chaos.

Proof. Q_L is the projection onto $H_{\mathcal{M}_L}$, where $\mathcal{M}_L$ is the set of all nonempty $M \in \mathcal{C}$ contained in one of the $n+1$ intervals. The intersection of subspaces corresponds to the intersection of subsets. $\square$

Stochastic analysis gives us another useful tool for calculating the first chaos, pioneered by Jon Warren [23, Th. 12]. Let $(B_{s,t})_{s \le t}$ be a decomposable Brownian motion, that is, a decomposable process such that $B_{s,t} \sim \mathrm{N}(0, t-s)$. One says that B has the *representation property,* if every $X \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}\, X = 0$ is equal to a stochastic integral,

$$X = \int_{-\infty}^{+\infty} H(t)\, \mathrm{d}B_{0,t}\,,$$

where H is a predictable *process* w.r.t. the filtration $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$.

6a5 Lemma. If B has the representation property then the first chaos is equal to the set of all *linear* stochastic integrals

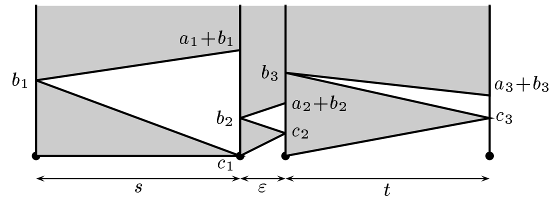
$$\int_{-\infty}^{+\infty} \varphi(t)\, \mathrm{d}B_{0,t}\,, \qquad \varphi \in L_2(\mathbb{R})\,.$$

Proof. Linear stochastic integrals evidently belong to the first chaos. Let X belong to the first chaos. Consider martingales $B(t) = B_{0,t}$, $X(t) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_{-\infty,t} \big) = \int_{-\infty}^t H(s)\, \mathrm{d}B(s)$ and their bracket process $\langle X, B \rangle_t = \int_{-\infty}^t H(s)\, \mathrm{d}s$. The two-dimensional process $(B(\cdot), X(\cdot))$ has independent increments; therefore the bracket process has independent increments as well. On the other hand, the bracket process is a continuous process of finite variation. Therefore it is degenerate (non-random), and $H(\cdot)$ is also non-random. $\square$

It follows that $\mathcal{F}_{\text{stable}}$ is generated by B.

6a6 Example. For the noise of stickiness (see Sect. 4), the process $\big(a(s,t)\big)_{s\le t}$ is a decomposable Brownian motion having the representation property. Therefore it generates $\mathcal{F}_{\text{stable}}$. On the other hand we know (recall 4h3) that $a(\cdot,\cdot)$ does not generate the whole σ-field. So, the sticky noise is not classical (Warren [23]).

The approach of Theorem 6a4 is also applicable. Let $\varphi : G_3 \to [-1,+1]$ be a Borel function, and $0 < t - \varepsilon < t < 1$. We consider $\varphi(\xi_{0,1}) = \varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1})$ (you know, $\xi_{t-\varepsilon,t} = f_{a(t-\varepsilon,t),b(t-\varepsilon,t),c(t-\varepsilon,t)}$), and compare it with $\varphi(\xi_{0,t-\varepsilon}\tilde{\xi}_{t-\varepsilon,t}\xi_{t,1})$, where $\tilde{\xi}_{t-\varepsilon,t} = f_{a(t-\varepsilon,t),b(t-\varepsilon,t),0}$.



It appears that

$$\|\varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1}) - \varphi(\xi_{0,t-\varepsilon}\tilde{\xi}_{t-\varepsilon,t}\xi_{t,1})\|_{L_2} = O(\varepsilon^{3/4}) = o(\sqrt{\varepsilon})\,,$$

provided that t is bounded away from 1 (otherwise we get $O(\varepsilon^{3/4}(1-t)^{-1/2})$ with an absolute constant). Taking into account that $\tilde{\xi}_{t-\varepsilon,t}$ is measurable w.r.t. the σ-field generated by $a(\cdot,\cdot)$ we conclude that the projection of $\varphi(\xi_{0,1})$ onto the first chaos is measurable w.r.t. the σ-field generated by $a(\cdot,\cdot)$. See 7b for the rest.

6b Higher levels of chaos

We still consider an arbitrary continuous factorization. Any Borel subset $\mathcal{M} \subset \mathcal{C}$ determines a subspace $H_{\mathcal{M}} \subset L_2(\Omega,\mathcal{F},P)$. However, the subset $\mathcal{C}_{\text{finite}} \subset \mathcal{C}$ is special; the corresponding subspace, being equal to $L_2(\mathcal{F}_{\text{stable}})$ by Theorem 5b11, is of the form $L_2(\mathcal{F}_1)$ for a sub-σ-field $\mathcal{F}_1 \subset \mathcal{F}$.

Another interesting subset is $\mathcal{C}_{\text{countable}}$, the set of all at most countable compact subsets of $\mathbb{R}$. It is not a Borel subset of $\mathcal{C}$ [7, Th. 27.5] but still, it is universally measurable [7, Th. 21.10] (that is, measurable w.r.t. every Borel measure), since its complement is analytic [7, Th. 27.5]. The Cantor-Bendixson derivative M' of $M \in \mathcal{C}$ is, by definition, the set of all limit points of M. Clearly, $M' \in \mathcal{C}$, $M' \subset M$, and $M' = \emptyset$ if and only if M is finite. The iterated Cantor-Bendixson derivative $M^{(\alpha)}$ is defined for every ordinal α by transfinite recursion: $M^{(0)} = M$; $M^{(\alpha+1)} = (M^{(\alpha)})'$; and $M^{(\alpha)} = \cap_{\beta<\alpha} M^{(\beta)}$ if α is a limit ordinal; see [7, Sect. 6.C]. If $M \notin \mathcal{C}_{\text{countable}}$ then $M^{(\alpha)} \neq \emptyset$ for all α. If $M \in \mathcal{C}_{\text{countable}}$ then $M^{(\alpha)} = \emptyset$ for some finite or countable ordinal α; the least α such that $M^{(\alpha)} = \emptyset$ is called the Cantor-Bendixson rank of $M \in \mathcal{C}_{\text{countable}}$. It is always of the form $\beta + 1$, and $M^{(\beta)}$ is a finite set.

Recall the proof of Theorem 5b11: the indicator of $\mathcal{C}_{\text{finite}}$ belongs to the set S introduced in 5b. Here is a more general fact.

6b1 Lemma. Let α be an at most countable ordinal, and $\mathcal{M}_\alpha$ the set of all $M \in \mathcal{C}$ such that $M^{(\alpha)} = \emptyset$. Then the indicator function of $\mathcal{M}_\alpha$ belongs to the set S.

Proof. Transfinite induction in α. For $\alpha = 0$ the claim is trivial. Let α be a limit ordinal. We take $\alpha_k \uparrow \alpha$, $\alpha_k < \alpha$, and note that $\mathcal{M}_\alpha = \mathcal{M}_{\alpha_1} \cup \mathcal{M}_{\alpha_2} \cup \dots$ (indeed, $M^{(\alpha_k)} \downarrow M^{(\alpha)}$, and $M^{(\alpha_k)}$ are compact). Thus, indicators of $\mathcal{M}_{\alpha_k}$ converge to the indicator of $\mathcal{M}_\alpha$.

The transition from α to $\alpha + 1$ needs the following property of S: for every $\varphi \in S$ and a closed elementary set E, the function $M \mapsto \varphi(M \cap E)$ belongs to S. Proof: In the first step of constructing S, φ is the indicator of some $\{M : M \subset E_1\}$; thus $M \mapsto \varphi(M \cap E)$ is the indicator of $\{M : M \subset E_1 \cup (\mathbb{R} \setminus E)\}$. The second and third steps preserve the property.

Assume that the indicator function of $\mathcal{M}_\alpha$ belongs to S; we have to prove the same for $\alpha + 1$. The indicator of $\mathcal{M}_{\alpha+1}$ is $M \mapsto \varphi(M^{(\alpha)})$, where φ is the indicator of $\mathcal{C}_{\text{finite}}$. Taking into account that $\varphi \in S$ (see the proof of Theorem 5b11), we will prove a more general fact: the function $M \mapsto \varphi(M^{(\alpha)})$ belongs to S for every $\varphi \in S$ (not just the indicator of $\mathcal{C}_{\text{finite}}$). The property is evidently preserved by the second and third steps of constructing S; it remains to prove it in the first step. Here φ is the indicator of $\{M : M \subset E\}$ for an elementary E. We have to express the set $\{M : M^{(\alpha)} \subset E\}$ as a limit of sets of the form $\{M : (M \cap E_1)^{(\alpha)} = \emptyset\}$ where E_1 is a closed elementary set. The indicator of $\{M : (M \cap E_1)^{(\alpha)} = \emptyset\}$ belongs to S, since it is $\mathbf{1}_{\mathcal{M}_\alpha}(M \cap E_1)$. We note that, for $\varepsilon \to 0$,

$$\{M : (M \cap (-\infty, \varepsilon])^{(\alpha)} = \emptyset\} \uparrow \{M : M^{(\alpha)} \subset (0, \infty)\},$$
$$\{M : (M \cap (-\infty, -\varepsilon])^{(\alpha)} = \emptyset\} \downarrow \{M : M^{(\alpha)} \subset [0, \infty)\},$$

which does the job for two special cases, $E = (0, \infty)$ and $E = [0, \infty)$, and shows how to deal with a boundary point, belonging to E or not. The general case is left to the reader. □

6b2 Theorem. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t}\big)$ be a continuous factorization.

(a) There exists a sub-σ-field $\mathcal{E}$ of $\mathcal{F}$ such that for all $f \in L_2(\Omega, \mathcal{F}, P)$, f is $\mathcal{E}$-measurable if and only if μ_f is concentrated on $\mathcal{C}_{\text{countable}}$.

(b) For every at most countable ordinal α there exists a sub-σ-field $\mathcal{E}_\alpha$ of $\mathcal{F}$ such that for all $f \in L_2(\Omega, \mathcal{F}, P)$, f is $\mathcal{E}_\alpha$-measurable if and only if μ_f is concentrated on the set of $M \in \mathcal{C}$ such that $M^{(\alpha)} = \emptyset$ (that is, of Cantor-Bendixson rank less than or equal to α).

Proof. Item (a) follows from (b), since $\mathcal{E}_\alpha = \mathcal{E}_{\alpha+1}$ for countable α large enough (see [7, Th. 6.9]), and $\mu_f(\mathcal{C}_{\text{countable}}) = \sup_\alpha \mu_f\{M : M^{(\alpha)} = \emptyset\}$ (see [7], the proof of Th. 21.10, and Th. 35.23).

Item (b) follows from 6b1, 5b10. □

Let us concentrate on Item (b) for $\alpha = 0, 1, 2$. The case $\alpha = 0$ is trivial: only the empty set M, and only constant functions f. The case $\alpha = 1$ was discussed before: finite sets M and stable functions f. The case $\alpha = 2$ means that M' is finite.

We define the n-th *superchaos* as the subspace $H_{\mathcal{M}} \subset L_2(\Omega, \mathcal{F}, P)$ corresponding to $\{M \in \mathcal{C} : |M'| = n\}$. These subspaces are orthogonal. The 0-th superchaos is the stable subspace, while for $n = 1, 2, \dots$ the n-th superchaos consists of (some) sensitive functions. By Theorem 6b2(b), the subspace spanned by n-th superchaos spaces for all $n = 0, 1, 2, \dots$ is of the form $L_2(\Omega, \mathcal{E}_2, P)$ where $\mathcal{E}_2$ is a sub-σ-field of $\mathcal{F}$. Similarly to Theorem 6a3, the sub-σ-field generated by the first superchaos and $\mathcal{F}_{\text{stable}}$ is equal to $\mathcal{E}_2$.

Similarly to (5b2) and (5b7) we may 'count' points of M' by the operator

$$\begin{aligned}\mathbf{N}'_{\{s_1,\dots,s_n\}} &= \sum_{j=1}^{n-1} \big(\mathbf{1} - \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s_j} \otimes \mathcal{F}^{\text{stable}}_{s_j,s_{j+1}} \otimes \mathcal{F}_{s_{j+1},\infty} \right) \big) \\ &= \big(\mathbf{1} - U_{0+}^{(s_1,s_2)} \big) + \dots + \big(\mathbf{1} - U_{0+}^{(s_{n-1},s_n)} \big) \,,\end{aligned}$$

or rather its limit $\mathbf{N}' = \lim_n \mathbf{N}'_{L_n}$. Further, similarly to 5b4, we may define

$$V_\lambda = \lim_n \exp(-\lambda \mathbf{N}'_{L_n}) \,.$$

This way, an ordinal hierarchy of operators may be constructed. It corresponds to the Cantor-Bendixson hierarchy of countable compact sets.

Introducing

$$\begin{aligned}Q'_{\{s_1,\dots,s_n\}} X &= \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{-\infty,s_1} \otimes \mathcal{F}^{\text{stable}}_{s_1,\infty} \right) + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_1} \otimes \mathcal{F}_{s_1,s_2} \otimes \mathcal{F}^{\text{stable}}_{s_2,\infty} \right) \\ &+ \dots + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_{n-1}} \otimes \mathcal{F}_{s_{n-1},s_n} \otimes \mathcal{F}^{\text{stable}}_{s_n,\infty} \right) + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_n} \otimes \mathcal{F}_{s_n,\infty} \right)\end{aligned}$$

for $X \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0$, we get such a counterpart of Theorem 6a4.

6b3 Theorem. If finite sets $L_1 \subset L_2 \subset \dots$ are such that their union is dense in $\mathbb{R}$, then operators Q'_{L_n} converge in the strong operator topology to the orthogonal projection from the sensitive subspace onto the first superchaos.

Proof. Q'_L is the projection onto $H_{\mathcal{M}_L}$, where $\mathcal{M}_L$ is the set of all nonempty $M \in \mathcal{C}$ such that M' is contained in one of the $n+1$ intervals. The intersection of subspaces corresponds to the intersection of subsets. $\square$

6b4 Example. For the sticky noise, consider such a random variable X: the number of random chords $[s,t] \times \{x\}$ such that $s > 0$ and $t > 1$. In other words (see 4i),

$$X = |\{x : \sigma_1(x) \in \Pi \cap (0, \infty)\}| \,.$$

The conditional distribution of X given the Brownian path $B(\cdot) = a(0, \cdot)$ is Poisson(λ) with $\lambda = a(0,1) + b(0,1) = B(1) - \min_{[0,1]} B(\cdot)$, which is easy to guess from the discrete counterpart (see (4c10)). That is a generalization of a claim from 4h3. In fact, the conditional distribution of the set $\{x : \sigma_1(x) \in \Pi \cap (0,\infty)\}$, given the Brownian path, is the Poisson point process of intensity 1 on $[-b(0,1), a(0,1)]$, which is a result of Warren [23]. Taking into account that the σ-field generated by $B(\cdot)$ is $\mathcal{F}_{\text{stable}}$ (recall 6a6), we get $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = a(0,1) + b(0,1)$. The random variable

$$Y = X - \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = X - a(0,1) - b(0,1)$$

is sensitive, that is, $\mathbb{E}\left(Y \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0$. I claim that Y belongs to the first superchaos.

The proof is based on Theorem 6b3. Given $0 < s_1 < \dots < s_n < 1$, we have to check that Y can be decomposed into a sum $Y_0 + \dots + Y_n$ such that each Y_j is measurable w.r.t. $\mathcal{F}^{\text{stable}}_{0,s_j} \otimes \mathcal{F}_{s_j,s_{j+1}} \otimes \mathcal{F}^{\text{stable}}_{s_{j+1},1}$. Here is the needed decomposition:

$$\begin{gathered} X_j = \left| \{ x : \sigma_1(x) \in \Pi \cap (s_j, s_{j+1}) \} \right| , \\ Y_j = X_j - \mathbb{E}\left(X_j \,\middle|\, \mathcal{F}_{\text{stable}} \right) . \end{gathered}$$

We apply a small perturbation on $(0, s_j)$ and $(s_{j+1}, 1)$ but not on (s_j, s_{j+1}). The set $\Pi \cap (s_j, s_{j+1})$ remains unperturbed. The function σ_1 is perturbed, but only a little; being a function of $B(\cdot)$, it is stable.

So, Y belongs to the first superchaos, and X belongs to the first superchaos plus $L_2(\mathcal{F}_{\text{stable}})$. It means that μ_X is concentrated on sets M such that $|M'| \le 1$.

The same holds for random variables $X_u = |\{ x : x \le u, \sigma_1(x) \in \Pi \cap (0,\infty) \}|$, for any u. They all are measurable w.r.t. the σ-field generated by the first superchaos and $\mathcal{F}_{\text{stable}}$. The random variable $c(0,1)$ is a (nonlinear!) function of these X_u (recall 4i). We see that the first superchaos and $\mathcal{F}_{\text{stable}}$ generate the whole σ-field $\mathcal{F}$. Every spectral set (of every random variable) has only a finite number of limit points.

6b5 Example. Another nonclassical noise, discovered and investigated by Warren [22], see also Watanabe [25], may be called the noise of splitting. It is the scaling limit of the model of 1d1; see also 8c. Spectral measures of the most interesting random variables are described explicitly! A spectral set contains a single limit point, and two sequences converging to the point from the left and from the right.

Again, every spectral set (of every random variable) has only a finite number of limit points.

6b6 Question. We have no example of a noise whose spectral sets M are at most countable, and M' is not always finite. Can it happen at all? Can it happen for the refinement of a dyadic coarse factorization satisfying (3e1)?

Beyond $\mathcal{C}_{\text{countable}}$ it is natural to use the Hausdorff dimension, $\dim M$, of compact sets $M \in \mathcal{C}$. The set S used in Theorems 5b11 and 6b2 helps again. First, a general lemma.

6b7 Lemma. For every probability measure μ on $\mathcal{C}$ the function $\varphi : \mathcal{C} \to [0,1]$ defined by $\varphi(M) = \mu\{M_1 \in \mathcal{C} : M \cap M_1 = \emptyset\}$, belongs to the set S.

Proof. We may restrict ourselves to compact subsets of a bounded interval; let it be just $[0,1]$. For any such set M let $M^{(n)}$ denote the union of intervals $[\frac{k}{n}, \frac{k+1}{n}]$ $(k = 0, \dots, n-1)$ that intersect M. The sequence $(M^{(n)})_{n=1}^{\infty}$ decreases and converges to M (in the Hausdorff metric). For every n, the function $\varphi_n(M) = \mu\{M_1 : M \cap M_1^{(n)} = \emptyset\}$ belongs to S, since it is the convex combination of indicators of $\{M : M \subset E\}$ with coefficients $\mu\{M_1 : M_1^{(n)} = [0,1] \setminus E\}$, where E runs over 2^n elementary sets. It remains to note that $\varphi_n(M) \uparrow \varphi(M)$, since $M \cap M_1 = \emptyset$ if and only if $M \cap M_1^{(n)} = \emptyset$ for some n. □

6b8 Lemma. For every $\alpha \in (0,1)$ there exists a function $\varphi \in S$ such that $\varphi(M) = 1$ for all M satisfying $\dim M < \alpha$, and $\varphi(M) = 0$ for all M satisfying $\dim M > \alpha$.

Proof. We may restrict ourselves to the space $\mathcal{C}_{0,1}$ of all compact subsets of $(0,1)$. There exists a probability measure μ on $\mathcal{C}_{0,1}$ such that the function $\varphi(M) = \mu\{M_1 : M_1 \cap M = \emptyset\}$ satisfies two conditions: $\varphi(M) = 1$ for all M such that $\dim M < \alpha$, and $\varphi(M) < 1$ for all M such that $\dim M > \alpha$. That is a result of J. Hawkes, see [6, Th. 6], [10, Lemma 5.1]. By 6b7, $\varphi \in S$. By multiplicativity (of S), also $\varphi^n \in S$ for all n. The function $\lim_n \varphi^n$ satisfies the required conditions. □

As a by-product we see that the Hausdorff dimension is a *Borel* function $\mathcal{C} \to \mathbb{R}$. (To this end we use an additional limiting procedure, as in the proof of Theorem 6b9.)

6b9 Theorem. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t}\big)$ be a continuous factorization, and $\alpha \in (0,1)$ a number. Then there exist sub-σ-fields $\mathcal{E}_{\alpha-}, \mathcal{E}_{\alpha+}$ of $\mathcal{F}$ such that for all $f \in L_2(\Omega, \mathcal{F}, P)$,

(a) f is measurable w.r.t. $\mathcal{E}_{\alpha-}$ if and only if μ_f is concentrated on the set of $M \in \mathcal{C}$ such that $\dim M < \alpha$;

(b) f is measurable w.r.t. $\mathcal{E}_{\alpha+}$ if and only if μ_f is concentrated on the set of $M \in \mathcal{C}$ such that $\dim M \le \alpha$.

Proof. We choose $\alpha_k \to \alpha$, apply 6b8 for each k, consider the limit φ of corresponding functions φ_k, and use 5b10. The case $\alpha_k < \alpha$ leads to (a), the case $\alpha_k > \alpha$ leads to (b). $\square$

A more general notion behind Theorems 5b11, 6b2 and 6b9 is an ideal. Recall that a subset I of $\mathcal{C}$ is called an ideal, if

$$\begin{gathered} M_1 \subset M_2,\ M_2 \in I \quad \Longrightarrow \quad M_1 \in I\,, \\ M_1, M_2 \in I \quad \Longrightarrow \quad (M_1 \cup M_2) \in I\,. \end{gathered}$$

In particular, $\mathcal{C}_{\text{finite}}$ and $\mathcal{C}_{\text{countable}}$ are ideals. For every finite or countable ordinal α, all $M \in \mathcal{C}$ such that $M^{(\alpha)} = \emptyset$ are an ideal. For every $\alpha \in (0,1)$, all $M \in \mathcal{C}$ such that $\dim M < \alpha$ are an ideal. The same holds for '$\dim M \le \alpha$'. All these ideals are shift-invariant:

$$\begin{gathered} M \in I \quad \Longrightarrow \quad (M+t) \in I \quad \text{for all } t\,, \\ M + t = \{ m + t : m \in M \}\,, \end{gathered}$$

but in general, an ideal need not be shift-invariant. Also, all ideals mentioned above are Borel subsets of $\mathcal{C}$, except for $\mathcal{C}_{\text{countable}}$; the latter is universally measurable, but not Borel. The following theorem is formulated for Borel ideals, but holds also for universally measurable ideals. Conditions 6b10 (a,b,c) parallel 3d1 (a,b,c), which means that sub-σ-fields $\mathcal{E}_{s,t}$ form a continuous factorization of the quotient probability space $(\Omega, \mathcal{F}, P)/\mathcal{E}$.

6b10 Theorem. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t} \big)$ be a continuous factorization, $I \subset \mathcal{C}$ a Borel ideal, $\mathcal{E} \subset \mathcal{F}$ a sub-σ-field, and for every $f \in L_2(\Omega, \mathcal{F}, P)$, f be $\mathcal{E}$-measurable if and only if μ_f is concentrated on I. Then sub-σ-fields $\mathcal{E}_{s,t} = \mathcal{E} \cap \mathcal{F}_{s,t}$ satisfy the conditions

(a) $$\mathcal{E}_{r,t} = \mathcal{E}_{r,s} \otimes \mathcal{E}_{s,t} \quad \text{whenever } r \le s \le t\,,$$

(b) $$\bigcup_{\varepsilon > 0} \mathcal{E}_{s+\varepsilon, t-\varepsilon} \text{ generates } \mathcal{E}_{s,t} \text{ whenever } s < t,$$

(c) $$\bigcup_{n=1}^{\infty} \mathcal{E}_{-n,n} \text{ generates } \mathcal{E}\,.$$

Proof. (a) We introduce Borel subsets $I_{s,t} = \{ M \in I : M \subset (s,t) \}$ of $\mathcal{C}$ and the corresponding subspaces $H_{s,t} = H_{I_{s,t}}$ of $L_2(\Omega, \mathcal{F}, P)$. The equality $I_{r,t} = I_{r,s} \times I_{s,t}$ (treated according to (6a1)) follows easily from the fact that I is an ideal. Lemma 6a2 (or rather, its evident generalization) states that $H_{r,t} = H_{r,s} \otimes H_{s,t}$. On the other hand,

$$L_2(\mathcal{E}_{s,t}) = L_2(\mathcal{E} \cap \mathcal{F}_{s,t}) = L_2(\mathcal{E}) \cap L_2(\mathcal{F}_{s,t}) = H_I \cap H_{\mathcal{C}_{s,t}} = H_{I \cap \mathcal{C}_{s,t}} = H_{s,t}\,.$$

So, $L_2(\mathcal{E}_{r,t}) = L_2(\mathcal{E}_{r,s}) \otimes L_2(\mathcal{E}_{s,t})$, therefore $\mathcal{E}_{r,t} = \mathcal{E}_{r,s} \otimes \mathcal{E}_{s,t}$.

(c) $\cup_n I_{-n,n} = I$, therefore $\cup_n H_{I_{-n,n}}$ is dense in H_I; that is, $\cup_n L_2(\mathcal{E}_{-n,n})$ is dense in $L_2(\mathcal{E})$, therefore $\cup_n \mathcal{E}_{-n,n}$ generates $\mathcal{E}$.

(b): similarly to (c). $\square$

6b11 Remark. If the ideal I is shift-invariant and the given object is a noise (not only a factorization), then the sub-factorization $(\mathcal{E}_{s,t})$ becomes a sub-noise. In particular, every nonclassical noise has its classical (in other words, stable) sub-noise.

6b12 Question. Does every Borel ideal correspond to a sub-σ-field? (For an arbitrary continuous factorization, I mean. Though, the question is also open for noises and shift-invariant ideals.)

6c An old question of Jacob Feldman

Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ be a continuous factorization. Sub-σ-fields $\mathcal{F}_E$ correspond to elementary sets $E \subset \mathbb{R}$ (recall 3d) and satisfy

$$\mathcal{F}_{E_1\cup E_2} = \mathcal{F}_{E_1} \otimes \mathcal{F}_{E_2} \quad \text{whenever } E_1 \cap E_2 = \emptyset \,. \tag{6c1}$$

It is natural to ask whether or not the map $E \mapsto \mathcal{F}_E$ can be extended to all Borel sets $E \subset \mathbb{R}$ in such a way that (6c1) is still satisfied and in addition,

$$\mathcal{F}_{E_n} \uparrow \mathcal{F}_E \quad \text{whenever } E_n \uparrow E \,. \tag{6c2}$$

The answer is positive if and only if the given continuous factorization is classical (Theorem 6c7 below, see also [18]), which solves a question of Feldman [4].

Note that (6c2) implies

$$\mathcal{F}_{E_n} \downarrow \mathcal{F}_E \quad \text{whenever } E_n \downarrow E \,. \tag{6c3}$$

Proof: Let $E_n \downarrow E$, then $\mathcal{F}_{\mathbb{R}\setminus E_n} \uparrow \mathcal{F}_{\mathbb{R}\setminus E}$ by (6c2), and so $\mathcal{F}_{\mathbb{R}\setminus E}$ is independent of $\cap \mathcal{F}_{E_n}$. If $\mathcal{F}_E$ is strictly less than $\cap \mathcal{F}_{E_n}$, then $\mathcal{F}_E \otimes \mathcal{F}_{\mathbb{R}\setminus E}$ is strictly less than $(\cap \mathcal{F}_{E_n}) \otimes \mathcal{F}_{\mathbb{R}\setminus E}$, which cannot happen, since $\mathcal{F}_E \otimes \mathcal{F}_{\mathbb{R}\setminus E} = \mathcal{F}$ by (6c1).

An extension satisfying (6c2), (6c3) is unique (if it exists) by the monotone class theorem. Therefore an extension (of $(\mathcal{F}_E)$ to the Borel σ-field) satisfying (6c1), (6c2) is unique (if it exists).

6c4 Lemma. If the factorization is classical then an extension satisfying (6c1), (6c2) exists.

Proof. By (slightly generalized) Theorem 6a3, for every elementary E, the σ-field $\mathcal{F}_E = \mathcal{F}_E^{\text{stable}}$ is generated by the corresponding portion $H_E^{(1)} = L_2(\mathcal{F}_E) \cap H^{(1)}$ of the first chaos $H^{(1)}$. The space $H_E^{(1)}$ corresponds (in the sense of Theorem 3d12) to the subset $\mathcal{M}_E^{(1)} \subset \mathcal{C}$ of all single-point subsets of E.

Given an arbitrary Borel set $E \subset \mathbb{R}$, we define the subset $\mathcal{M}_E^{(1)} \subset \mathcal{C}$ as above (that is, all single-point subsets of E), consider the corresponding subspace $H_E^{(1)} \subset H^{(1)}$, and introduce the sub-$\sigma$-field $\mathcal{F}_E \subset \mathcal{F}$ generated by $H_E^{(1)}$.

Given $f \in H^{(1)}$, we denote by f_E the orthogonal projection of f to $H_E^{(1)}$; here E is an arbitrary Borel set. If $E_n \uparrow E$ (or $E_n \downarrow E$) then $f_{E_n} \to f$ in L_2. If E is elementary then

$$\mathbb{E}\, \mathrm{e}^{\mathrm{i}f} = \big(\mathbb{E}\, \mathrm{e}^{\mathrm{i}f_E}\big)\big(\mathbb{E}\, \mathrm{e}^{\mathrm{i}f_{\mathbb{R}\setminus E}}\big)$$

due to independence. The monotone class theorem extends the equality to all Borel sets E. We conclude that f_E and $f_{\mathbb{R}\setminus E}$ are independent. Therefore σ-fields $\mathcal{F}_E$ and $\mathcal{F}_{\mathbb{R}\setminus E}$ are independent for every Borel set E. Taking into account that $H_{E_1 \cup E_2}^{(1)} = H_{E_1}^{(1)} \oplus H_{E_2}^{(1)}$ whenever $E_1 \cap E_2 = \emptyset$ we get (6c1).

If $E_n \uparrow E$ then $H_{E_n}^{(1)} \uparrow H_E^{(1)}$, which ensures (6c2). $\square$

Condition (a) of the next lemma is evidently necessary for the extension to exist. In more topological language, for every open set $G \subset \mathbb{R}$ the corresponding σ-field $\mathcal{F}_G$ is naturally defined by approximation (of G by elementary sets) from within, while a closed set is approximated from the outside. The necessary condition, $\mathcal{F}_G \otimes \mathcal{F}_{\mathbb{R}\setminus G} = \mathcal{F}$, appears to be equivalent to the following (see 6c5(b)): the set $M \cap G$ is compact, for almost all $M \in \mathcal{C}$.

6c5 Lemma. For all elementary sets $E_1 \subset E_2 \subset \dots$ the following two conditions are equivalent:

(a) $\Big(\bigvee\limits_n \mathcal{F}_{E_n}\Big) \otimes \Big(\bigwedge\limits_n \mathcal{F}_{\mathbb{R}\setminus E_n}\Big) = \mathcal{F}$;

(b) the set $\{M \in \mathcal{C} : \forall n \; M \cap \big((\cup E_k) \setminus E_n\big) \ne \emptyset\}$ is negligible w.r.t. the spectral measure μ_f for every $f \in L_2(\Omega, \mathcal{F}, P)$.

Proof. Denote $F_n = \mathbb{R} \setminus E_n$, $\mathcal{E}_n = \mathcal{F}_{E_n}$, $\mathcal{F}_n = \mathcal{F}_{\mathbb{R}\setminus E_n}$, $\mathcal{E}_\infty = \vee_n \mathcal{E}_n$, $\mathcal{F}_\infty = \wedge_n \mathcal{F}_n$. Clearly, $\mathcal{E}_\infty$ and $\mathcal{F}_\infty$ are independent, and (a) becomes $\mathcal{E}_\infty \vee \mathcal{F}_\infty = \mathcal{F}$. Denote also $\mathcal{M}_n = \{M \in \mathcal{C} : M \subset E_n\}$, $\mathcal{N}_n = \{M \in \mathcal{C} : M \subset F_n\}$, $\mathcal{M}_\infty = \cup_n \mathcal{M}_n = \{M \in \mathcal{C} : \exists n \; M \subset E_n\}$, $\mathcal{N}_\infty = \cap_n \mathcal{N}_n = \{M \in \mathcal{C} : M \subset \cap F_n\}$; then $H_{\mathcal{M}_n} = L_2(\mathcal{E}_n)$, $H_{\mathcal{N}_n} = L_2(\mathcal{F}_n)$. We have $\mathcal{M}_n \uparrow \mathcal{M}_\infty$ and $\mathcal{N}_n \downarrow \mathcal{N}_\infty$; therefore $L_2(\mathcal{E}_n) = H_{\mathcal{M}_n} \uparrow H_{\mathcal{M}_\infty}$ and $L_2(\mathcal{F}_n) = H_{\mathcal{N}_n} \downarrow H_{\mathcal{N}_\infty}$. On the other hand, $\mathcal{E}_n \uparrow \mathcal{E}_\infty$ and $\mathcal{F}_n \downarrow \mathcal{F}_\infty$; therefore $L_2(\mathcal{E}_n) \uparrow L_2(\mathcal{E}_\infty)$ and $L_2(\mathcal{F}_n) \downarrow L_2(\mathcal{F}_\infty)$. So,

$$H_{\mathcal{M}_\infty} = L_2(\mathcal{E}_\infty)\,, \quad H_{\mathcal{N}_\infty} = L_2(\mathcal{F}_\infty)\,.$$

Denote $\mathcal{M}_\infty \vee \mathcal{N}_\infty = \{M_1 \cup M_2 : M_1 \in \mathcal{M}_\infty, M_2 \in \mathcal{N}_\infty\}$; the same for $\mathcal{M}_1 \vee \mathcal{N}_\infty$ etc. We have $H_{\mathcal{M}_1 \vee \mathcal{N}_n} = H_{\mathcal{M}_1} \otimes H_{\mathcal{N}_n}$ and $\mathcal{M}_1 \vee \mathcal{N}_n \downarrow \mathcal{M}_1 \vee \mathcal{N}_\infty$; thus $H_{\mathcal{M}_1 \vee \mathcal{N}_\infty} = H_{\mathcal{M}_1} \otimes H_{\mathcal{N}_\infty}$ (note a relation to 6a2). Similarly, $H_{\mathcal{M}_n \vee \mathcal{N}_\infty} = H_{\mathcal{M}_n} \otimes H_{\mathcal{N}_\infty}$. However, $\mathcal{M}_n \vee \mathcal{N}_\infty \uparrow \mathcal{M}_\infty \vee \mathcal{N}_\infty$, and we get $H_{\mathcal{M}_\infty \vee \mathcal{N}_\infty} = H_{\mathcal{M}_\infty} \otimes H_{\mathcal{N}_\infty}$, that is,

$$H_{\mathcal{M}_\infty \vee \mathcal{N}_\infty} = L_2(\mathcal{E}_\infty) \otimes L_2(\mathcal{F}_\infty)\,.$$

Now (a) becomes $H_{\mathcal{M}_\infty \vee \mathcal{N}_\infty} = H$, which means negligibility of the set $\mathcal{C} \setminus (\mathcal{M}_\infty \vee \mathcal{N}_\infty) = \{M : \forall n \; M \cap \big((\cup E_k) \setminus E_n\big) \ne \emptyset\}$, that is, (b). $\square$

Every classical factorization satisfies 6c5(b), since a finite set M cannot intersect $(\cup E_k) \setminus E_n$ for all n.

6c6 Lemma. If Condition 6c5(b) is satisfied for every (E_n) then the factorization is classical.

Proof. Let the factorization be not classical. Then we can choose a sensitive $f \in L_2(\Omega, \mathcal{F}, P)$, $\|f\| = 1$. Assume for convenience that $f \in L_2(\mathcal{F}_{0,1})$, and consider the spectral measure μ_f; μ_f-almost all M are infinite subsets of $(0,1)$. We choose $p_1, p_2, \dots \in (0,1)$ such that $\sum p_k \le 1/3$ (say, $p_k = 2^{-k}/3$). Integer parameters $n_1 < n_2 < \dots$ will be chosen later. We introduce independent random elementary sets $B_1, B_2, \dots \subset [0,1]$ as follows:

$$\mathbb{P}\Big\{ B_k = \Big(\frac{l_1 - 1}{n_k}, \frac{l_1}{n_k}\Big) \cup \dots \cup \Big(\frac{l_m - 1}{n_k}, \frac{l_m}{n_k}\Big) \Big\} = p_k^m (1 - p_k)^{n_k - m}$$

whenever $1 \le l_1 < \dots < l_m \le n_k$, $m \in \{0, \dots, n_k\}$. That is, we have a two-parameter family of independent events, $\big(\frac{l-1}{n_k}, \frac{l}{n_k}\big) \subset B_k$, where $l \in \{1, \dots, n_k\}$, $k \in \{1, 2, \dots\}$. The probability of such an event is equal to p_k. We define $E_k = B_1 \cup \dots \cup B_k$; thus $E_1 \subset E_2 \subset \dots$ is a (random) increasing sequence of elementary subsets of $[0,1]$.

We treat M as a random compact subset of $(0,1)$, distributed μ_f and independent of $B_1, B_2, \dots$ Let $\tilde{P}$ be the corresponding probability measure (in fact, product measure) on the space $\tilde{\Omega}$ of sequences (of sets) $(M, B_1, B_2, \dots)$. For each $k = 0, 1, 2, \dots$ we define an event A_k, that is, a measurable subset of $\tilde{\Omega}$, by the following condition on $(M, B_1, B_2, \dots)$:

$$M \setminus E_k \text{ is infinite and does not intersect } B_{k+1}\,;$$

of course, $E_0 = \emptyset$.

We can choose $n_1, n_2, \dots$ such that $\sum_k \tilde{P}(A_k) \le 1/3$. Proof: $\tilde{P}(A_k)$ is a function of $n_1, \dots, n_k, n_{k+1}$ that converges to 0 when $n_{k+1} \to \infty$ (while $n_1, \dots, n_k$ are fixed).

The probability of the event

$$M \setminus E_k \text{ is infinite for all } k$$

is no less than $1 - \sum p_k \ge 2/3$. Proof: Each M has a limit point (at least one), and the point is covered by (the closure of) $B_1 \cup B_2 \cup \dots$ with probability $\le \sum p_k$.

So, there is a positive probability ($\ge 1/3$) to such an event:

$$\text{for each } k, \text{ the set } M \setminus E_k \text{ is infinite and intersects } B_{k+1}\,.$$

However, the conditional probability, given $B_1, B_2, \dots$ (but not M) of the event

$$\text{for each } k, \text{ the set } M \setminus E_k \text{ intersects } B_{k+1}$$

must vanish according to 6c5(b). $\square$

6c7 Theorem. A continuous factorization is classical if and only if the map $E \mapsto \mathcal{F}_E$ can be extended from the algebra of elementary sets to the Borel σ-field, satisfying (6c1) and (6c2).

Proof. If the factorization is classical then the extension exists by 6c4. Let the extension exist; then 6c5(a) is satisfied for all (E_k), therefore 6c5(b) is also satisfied, and the factorization is classical by 6c6. $\square$

6d Black noise

6d1 Definition. A noise is *black,* if its stable σ-field $\mathcal{F}_{\text{stable}}$ is degenerate. In other words: its first chaos contains only 0.

Why 'black'? Well, the white noise is called 'white' since its spectral density is constant. It excites harmonic oscillators of all frequencies to the same extent. For a black noise, however, the response of any linear sensor is zero!

What could be a physically reasonable nonlinear sensor able to sense a black noise? Maybe a fluid could do it, which is hinted at by the following words of Shnirelman [13, p. 1263] about the paradoxical motion of an ideal incompressible fluid: '... very strong external forces are present, but they are infinitely fast oscillating in space and therefore are indistinguishable from zero in the sense of distributions. The smooth test functions are not "sensitive" enough to "feel" these forces.'

The very idea of black noises, nonclassical factorizations, etc. was suggested to me by Anatoly Vershik in 1994.

6d2 Lemma. Let $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ be a continuous factorization, $a<b$, $\mathcal{M}$ a Borel subset of $\mathcal{C}_{a,b}=\{M\in\mathcal{C}:M\subset(a,b)\}$, and $\tilde{\mathcal{M}}=\{M\in\mathcal{C}:M\cap(a,b)\in\mathcal{M}\}$. If $\mu_f(\mathcal{M})=0$ for all $f\in L_2(\Omega,\mathcal{F},P)$ then $\mu_f(\tilde{\mathcal{M}})=0$ for all $f\in L_2(\Omega,\mathcal{F},P)$.

Proof. I prove it for $(a,b)=(0,\infty)$, leaving the general case to the reader. We have $\mathcal{C}=\mathcal{C}_{-\infty,0}\times\mathcal{C}_{0,\infty}$, $\mathcal{M}\subset\mathcal{C}_{0,\infty}$ and $\tilde{\mathcal{M}}=\mathcal{C}_{-\infty,0}\times\mathcal{M}$ (in the sense of (6a1)). By 6a2, $H_{\tilde{\mathcal{M}}}=H_{\mathcal{C}_{-\infty,0}\times\mathcal{M}}=H_{\mathcal{C}_{-\infty,0}}\otimes H_{\mathcal{M}}$. By (3d13), the space $H_{\mathcal{M}}$ is trivial (that is, $\{0\}$). Therefore $H_{\tilde{\mathcal{M}}}$ is also trivial; it remains to use (3d13) again. $\square$

Recall that a compact set M is called perfect, if it has no isolated points. (The empty set is also perfect.) The set $\mathcal{C}_{\text{perfect}}$ of all perfect compact subsets of $\mathbb{R}$ is a Borel set in $\mathcal{C}$, see [7, proof of Th. 27.5].

6d3 Theorem. For every continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ the following two conditions are equivalent:

(a) the first chaos space is trivial (contains only 0);

(b) for every $f\in L_2(\Omega,\mathcal{F},P)$ the spectral measure μ_f is concentrated on $\mathcal{C}_{\text{perfect}}$.

Proof. (b) implies (a) evidently (a single-point set cannot be perfect). Assume (a). Applying 6d2 to the set $\mathcal{M}$ of all single-point subsets of (a,b) we see that μ_f-almost all $M\in\mathcal{C}$ are such that $M\cap(a,b)$ is not a single-point set, for all rational $a<b$. It means that M is perfect. $\square$

So, a noise is black if and only if spectral measures are concentrated on (the set of all) perfect sets.

Existence of black noises was proven first by Tsirelson and Vershik [20, Sect. 5]. A simpler and more natural example is described in the next section. Another example is found by Watanabe [26].

If all spectral sets are finite or countable (as in 6b4, 6b5), such a noise cannot contain a black sub-noise.

6d4 Question. If a noise contains no black sub-noise, does it follow that all spectral sets are at most countable?

Perfect sets may be classified, say, by Hausdorff dimension. For any $\alpha\in(0,1)$, sets $M\in\mathcal{C}$ of Hausdorff dimension $\le\alpha$ are a shift invariant ideal, corresponding to a sub-noise. Also, all $M\in\mathcal{C}$ of Hausdorff dimension α correspond to a 'chaos subspace number α'. A continuum of such chaos subspaces (not in a single noise, of course) could occur, describing different 'levels of sensitivity'. For now, however, I know of perfect spectral sets of Hausdorff dimension 1/2 only.

6d5 Question. Can a noise have perfect spectral sets of Hausdorff dimension other than $1/2$? (See also the end of 8c.)

6d6 Question. Can a black noise emerge as the refinement of a dyadic coarse factorization satisfying (3e1)?

The following results (especially 6d14) may be treated as continuous-time counterparts of Theorem 5a3 (of Benjamini, Kalai and Schramm). Given a continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ and a function $f\in L_2(\Omega,\mathcal{F},P)$, we define

$$\mathbf{H}(f)=\limsup_{\{t_1,\dots,t_n\}\uparrow}\sum_{k=1}^{n+1}\Big(\mathbb{E}\sqrt{\operatorname{Var}\big(f\,\big|\,\mathcal{F}_{\mathbb{R}\setminus(t_{k-1},t_k)}\big)}\Big)^2\,;$$

here $t_0=-\infty$, $t_{n+1}=+\infty$, and the '$\limsup$' is taken over all finite sets $L=\{t_1,\dots,t_n\}\subset\mathbb{R}$, $t_1<\dots<t_n$, ordered by inclusion. That is, 'for every ε there exists L_ε such that for all $L\supset L_\varepsilon$ …' and so on. We also introduce

$$\mathbf{H}_1(f)=\lim_{\{t_1,\dots,t_n\}\uparrow}\sum_{k=1}^{n+1}\operatorname{Var}\big(\mathbb{E}\big(f\,\big|\,\mathcal{F}_{t_{k-1},t_k}\big)\big)\,.$$

This time we may write '$\lim$' (or '$\inf$') instead of '$\limsup$' due to monotonicity (w.r.t. inclusion); the more $L=\{t_1,\dots,t_n\}$ the less the sum.

6d7 Lemma. $\sqrt{\operatorname{Var}\big(\mathbb{E}\big(f\,\big|\,\mathcal{F}_{s,t}\big)\big)}\le\mathbb{E}\sqrt{\operatorname{Var}\big(f\,\big|\,\mathcal{F}_{\mathbb{R}\setminus(s,t)}\big)}$ for all $f\in L_2(\Omega,\mathcal{F},P)$ and $s<t$.

Proof. The space $L_2(\Omega,\mathcal{F},P)=L_2(\mathcal{F})=L_2(\mathcal{F}_{s,t}\otimes\mathcal{F}_{\mathbb{R}\setminus(s,t)})=L_2(\mathcal{F}_{s,t})\otimes L_2(\mathcal{F}_{\mathbb{R}\setminus(s,t)})$ may also be thought of as the space $L_2\big(\mathcal{F}_{\mathbb{R}\setminus(s,t)},L_2(\mathcal{F}_{s,t})\big)$ consisting of $\mathcal{F}_{\mathbb{R}\setminus(s,t)}$-measurable square integrable vector-functions, taking on values in $L_2(\mathcal{F}_{s,t})$. We consider the element $\tilde f\in L_2\big(\mathcal{F}_{\mathbb{R}\setminus(s,t)},L_2(\mathcal{F}_{s,t})\big)$ corresponding to $f\in L_2(\mathcal{F})$ (according to the canonical isomorphism of these two spaces). The mean value of the vector-function is $\mathbb{E}\tilde f=\mathbb{E}\big(f\,\big|\,\mathcal{F}_{s,t}\big)$ (these two '$\mathbb{E}$' act on different spaces). Convexity of the seminorm $\sqrt{\operatorname{Var}(\cdot)}$ on $L_2(\mathcal{F}_{s,t})$ gives $\sqrt{\operatorname{Var}(\mathbb{E}\tilde f)}\le\mathbb{E}\sqrt{\operatorname{Var}(\tilde f)}$, where $\operatorname{Var}(\tilde f)$ means the pointwise variance (each value of $\tilde f$ is a random variable; the latter has its variance), basically the same as $\operatorname{Var}\big(f\,\big|\,\mathcal{F}_{\mathbb{R}\setminus(s,t)}\big)$. □

6d8 Corollary. $\mathbf{H}_1(f)\le\mathbf{H}(f)$.

6d9 Lemma. $\mathbf{H}_1(f)=\|Q_1f\|$ for all $f\in L_2(\Omega,\mathcal{F},P)$; here Q_1 is the orthogonal projection onto the first chaos.

Proof. Follows immediately from Theorem 6a4. $\square$

6d10 Corollary. Every $f \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbf{H}(f) = 0$ is orthogonal to the first chaos.

6d11 Corollary. If a noise is such that $\mathbf{H}(f) = 0$ for all $f \in L_2(\Omega, \mathcal{F}, P)$, then the noise is black.

6d12 Lemma. Let $g \in L_2(\mathcal{F})$, $h \in L_\infty(\mathcal{F}_{0,\infty})$, and $f = \mathbb{E}\left(gh \,\middle|\, \mathcal{F}_{-\infty,0} \right)$. Then $\mathbf{H}(f) \le \|h\|_\infty^2 \mathbf{H}(g)$.

Proof. It is sufficient to prove the inequality for the influence, $\mathbb{E}\sqrt{\operatorname{Var}\left(f \,\middle|\, \mathcal{F}_{\mathbb{R}\setminus(s,t)} \right)} \le \|h\|_\infty \mathbb{E}\sqrt{\operatorname{Var}\left(g \,\middle|\, \mathcal{F}_{\mathbb{R}\setminus(s,t)} \right)}$ for any $(s,t) \subset (-\infty, 0)$. Similarly to the proof of 6d7, we consider $\tilde g \in L_2\big(\mathcal{F}_{0,\infty}, L_2(\mathcal{F}_{-\infty,0})\big)$ corresponding to $g \in L_2(\mathcal{F}_{-\infty,0} \otimes \mathcal{F}_{0,\infty})$. We have $\tilde g h \in L_2\big(\mathcal{F}_{0,\infty}, L_2(\mathcal{F}_{-\infty,0}\big)$, $\mathbb{E}\,(\tilde g h) = f$. Convexity of the seminorm $\mathbb{E}\sqrt{\operatorname{Var}\left(\cdot \,\middle|\, \mathcal{F}_{(-\infty,0)\setminus(s,t)} \right)}$ on $L_2(\mathcal{F}_{-\infty,0})$ gives $\mathbb{E}\sqrt{\operatorname{Var}\left(f \,\middle|\, \mathcal{F}_{(-\infty,0)\setminus(s,t)} \right)} \le \mathbb{E}\,\mathbb{E}\sqrt{\operatorname{Var}\left(\tilde g h \,\middle|\, \mathcal{F}_{(-\infty,0)\setminus(s,t)} \right)}$, where 'Var' and the internal '$\mathbb{E}$' act on $L_2(\mathcal{F}_{-\infty,0})$, while the outer '$\mathbb{E}$' acts on $L_2(\mathcal{F}_{0,\infty})$. The right-hand side is equal to $\mathbb{E}\left(|h| \mathbb{E}\sqrt{\operatorname{Var}\left(\tilde g \,\middle|\, \mathcal{F}_{(-\infty,0)\setminus(s,t)} \right)}\right)$ and so, cannot exceed $\|h\|_\infty \mathbb{E}\,\mathbb{E}\sqrt{\operatorname{Var}\left(\tilde g \,\middle|\, \mathcal{F}_{(-\infty,0)\setminus(s,t)} \right)} = \|h\|_\infty \mathbb{E}\sqrt{\operatorname{Var}\left(g \,\middle|\, \mathcal{F}_{\mathbb{R}\setminus(s,t)} \right)}$.

$\square$

6d13 Lemma. If $f \in L_2(\Omega, \mathcal{F}, P)$ is such that $\mathbf{H}(f) = 0$, then μ_f is concentrated on $\mathcal{C}_{\text{perfect}}$.

Proof. Similarly to the proof of Theorem 6d3, it is sufficient to prove, for every $(a,b) \subset \mathbb{R}$, that μ_f-almost all $M \in \mathcal{C}$ are such that $M \cap (a,b)$ is not a single-point set. Lemma 6a2 shows that the subspace corresponding to $\{M \in \mathcal{C} : |M \cap (a,b)| = 1\}$ is $H_{-\infty,a} \otimes H_{a,b}^{(1)} \otimes H_{b,\infty}$, where $H_{a,b}^{(1)}$ is the first chaos intersected with $H_{a,b}$. We have to prove that f is orthogonal to $H_{-\infty,a} \otimes H_{a,b}^{(1)} \otimes H_{b,\infty}$, that is, to gh for every $g \in H_{a,b}^{(1)}$, $h \in H_{-\infty,a} \otimes H_{b,\infty} = L_2(\mathcal{F}_{\mathbb{R}\setminus(a,b)})$, and we may assume that $h \in L_\infty(\mathcal{F}_{\mathbb{R}\setminus(a,b)})$.

We have $\mathbb{E}\,(fgh) = \mathbb{E}\left(g \mathbb{E}\left(fh \,\middle|\, \mathcal{F}_{a,b} \right)\right)$. Lemma 6d12 (slightly generalized) shows that $\mathbf{H}\big(\mathbb{E}\left(fh \,\middle|\, \mathcal{F}_{a,b} \right)\big) \le \|h\|_\infty^2 \mathbf{H}(f)$. Thus, $\mathbf{H}\big(\mathbb{E}\left(fh \,\middle|\, \mathcal{F}_{a,b} \right)\big) = 0$; by 6d10, $\mathbb{E}\left(g \mathbb{E}\left(fh \,\middle|\, \mathcal{F}_{a,b} \right)\right) = 0$. $\square$

6d14 Corollary. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s \le t}\big)$ be a continuous factorization. If $f \in L_2(\Omega, \mathcal{F}, P)$ satisfies $\mathbf{H}(f) = 0$ and $\mathbb{E}\, f = 0$, then f is sensitive.

Here are counterparts of 5b8 and Theorem 5b11 inspired by the work [9] of Le Jan and Raimond.

6d15 Lemma. Let $f \in L_2(\Omega, \mathcal{F}, P)$, and $g = \eta \circ f$ where $\eta : \mathbb{R} \to \mathbb{R}$ satisfies $|\eta(x) - \eta(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$. Then

$$\mathbf{H}(g) \le \mathbf{H}(f)\,.$$

Proof. It is sufficient to prove the inequality for the influence, $\mathbb{E}\sqrt{\operatorname{Var}\big(\,g\,\big|\,\mathcal{F}_{\mathbb{R}\setminus(s,t)}\big)} \le \mathbb{E}\sqrt{\operatorname{Var}\big(\,f\,\big|\,\mathcal{F}_{\mathbb{R}\setminus(s,t)}\big)}$, or a stronger inequality $\operatorname{Var}\big(\,g\,\big|\,\mathcal{F}_E\big) \le \operatorname{Var}\big(\,f\,\big|\,\mathcal{F}_E\big)$ a.s., for an arbitrary elementary set E. It is a conditional counterpart of the inequality $\operatorname{Var}(\eta \circ X) \le \operatorname{Var}(X)$ for any random variable X. A proof of the latter: $\operatorname{Var}(\eta\circ X) = \frac12 \mathbb{E}\,(\eta\circ X_1 - \eta\circ X_2)^2 \le \frac12 \mathbb{E}\,(X_1 - X_2)^2 = \operatorname{Var}(X)$, where X_1, X_2 are independent copies of X.

□

6d16 Theorem. For every continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ there exists a sub-σ-field $\mathcal{F}_{\text{jetblack}}$ of $\mathcal{F}$ such that $L_2(\Omega, \mathcal{F}_{\text{jetblack}}, P)$ is the closure (in $L_2(\Omega, \mathcal{F}, P)$) of $\{f \in L_2(\Omega, \mathcal{F}, P) : \mathbf{H}(f) = 0\}$.

Proof. The set $\{f : \mathbf{H}(f) = 0\}$ is closed under linear operations, and also under the nonlinear operation $f \mapsto |f|$, therefore its closure is of the form $L_2(\mathcal{F}_{\text{jetblack}})$. □

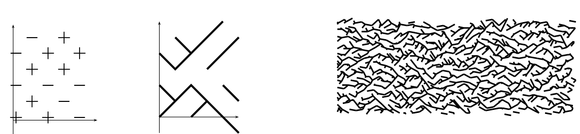
6d17 Corollary. $L_2(\mathcal{F}_{\text{jetblack}}) \subset H_{\mathcal{C}_{\text{perfect}}}$.

6d18 Question. Whether $\mathcal{F}_{\text{jetblack}}$ is nontrivial for every black noise, or not?

7 Example: The Brownian Web as a Black Noise

7a Convolution semigroup of the Brownian web

A one-dimensional array of random signs can produce some classical and nonclassical noises in the scaling limit, but I still do not know whether it can produce a black noise, or not (see 6d6).



(a) (b) (c)

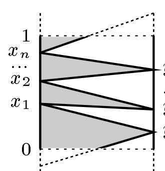
This is why I turn to a two-dimensional array of random signs (a). It produces a system of coalescing random walks (b) that converges to the so-called

Brownian web (c), consisting of infinitely many coalescing Brownian motions (independent before coalescence).

The Brownian web was investigated by Arratia, Toth, Werner, Soucaliuc, and recently by Fontes, Isopi, Newman and Ravishankar [5] (other references may be found therein). The scaling limit may be interpreted in several ways, depending on the choice of 'observables', and may involve delicate points, because of complicated topological properties of the Brownian web as a random geometric configuration on the plane. However, we avoid these delicate points by treating the Brownian web as a stochastic flow in the sense of Sect. 4, that is, a two-parameter family of random variables in a semigroup.

In order to keep finite everything that can be kept finite, we consider Brownian motions in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ rather than the line $\mathbb{R}$.

It is well-known that a countable dense set of coalescing 'particles', given at the initial instant, becomes finite, due to coalescence, after any positive time. Moreover, the finite number is of finite expectation. Thus, for any given $t > 0$, the Brownian web on the time interval $(0, t)$ gives us a random map $\mathbb{T} \to \mathbb{T}$ of the following elementary form (a step function):



$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n} : \mathbb{T} \to \mathbb{T}\,,$$
$$x_1 < \dots < x_n < x_1,\ y_1 < \dots < y_n < y_1 \text{ (cyclically)},$$
$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n}(x) = y_{k+1} \text{ for } x \in (x_k, x_{k+1}]\,.$$

Of course, n is random, as well as $x_1, \dots, x_n$ and $y_1, \dots, y_n$. The value at x_k does not matter; we let it be y_k for convenience, but it could equally well be y_{k+1}, or remain undefined. Points $x_1, \dots, x_n$ will be called left critical points of the map, while $y_1, \dots, y_n$ are right critical points.

We introduce the set G_∞ consisting of all step functions $\mathbb{T} \to \mathbb{T}$ and, in addition, the identity function. If $f, g \in G_\infty$ then their composition fg belongs to G_∞; thus G_∞ is a semigroup. It consists of pieces of dimensions $2, 4, 6, \dots$ and the identity. Similarly to G_3 (recall (4d2)), G_∞ is not a topological semigroup, since the composition is discontinuous.

The distribution of the random map is a probability measure μ_t on G_∞. These maps form a convolution semigroup, $\mu_s * \mu_t = \mu_{s+t}$. Similarly to 4e, discontinuity of composition does not harm, since the composition is continuous almost everywhere (w.r.t. $\mu_s \otimes \mu_t$). Left and right critical points do not meet.[32]

[32] They meet with probability 0, as long as s and t are fixed. Otherwise, delicate points are involved...

Having the convolution semigroup, we can construct the stochastic flow, that is, a family of G_∞-valued random variables $(\xi_{s,t})_{s\le t}$ such that

$$\begin{gathered}\xi_{s,t} \sim \mu_{t-s}\,,\\ \xi_{r,s}\xi_{s,t} = \xi_{r,t} \quad \text{a.s.}\end{gathered}$$

whenever $-\infty < r < s < t < \infty$, and

$$\xi_{t_1,t_2}, \dots, \xi_{t_{n-1},t_n} \quad \text{are independent}$$

whenever $-\infty < t_1 < \dots < t_n < \infty$.

Indeed, for each i, we can take independent $\xi_{k/i,(k+1)/i} : \Omega[i] \to G_\infty$ for $k \in \mathbb{Z}$ according to the discrete model, and define $\xi_{k/i,l/i} = \xi_{k/i,(k+1)/i} \dots \xi_{(l-1)/i,l/i}$. For any two coarse instants $s \le t$, the distribution of $\xi_{s[i],t[i]}$ converges weakly (for $i \to \infty$) to $\mu_{t[\infty]-s[\infty]}$. The refinement gives us

$$\xi_{s,t} : \Omega \to G_\infty\,, \qquad \xi_{s,t} = f_{x_1(s,t),\dots,x_{n(s,t)}(s,t)}^{y_1(s,t),\dots,y_{n(s,t)}(s,t)}\,;$$

$x_k(\cdot,\cdot)$ and $y_k(\cdot,\cdot)$ are continuous a.s. Also,

$$\text{(7a1)} \qquad \mathbb{E}\, n(s,t) < \infty\,.$$

We consider the sub-σ-field $\mathcal{F}_{s,t}$ generated by all $\xi_{u,v}$ for $(u,v) \subset (s,t)$ and get a continuous factorization. Time shifts are evidently introduced, and so, we get a noise — the *noise of coalescence.*

7b Some general arguments

Probably we could use **H** and Theorem 6d16 in order to prove that the noise of coalescence is black (see also [9]). However, I choose another way (via $\mathbf{H}_1$ rather than **H**).

Random variables of the form $\varphi(\xi_{s,t})$ for arbitrary $s < t$ and arbitrary bounded Borel function $\varphi : G_\infty \to \mathbb{R}$ generate the whole σ-field $\mathcal{F}$. Products of the form $\varphi_1(\xi_{t_0,t_1}) \dots \varphi_n(\xi_{t_{n-1},t_n})$ for $t_0 < \dots < t_n$ span L_2 (as a closed subspace); however, we cannot expect that *linear* combinations of such $\varphi(\xi_{s,t})$ are dense in L_2.

Denote by Q_1 the orthogonal projection of $L_2(\Omega,\mathcal{F},P)$ onto the first chaos.

7b1 Lemma. Linear combinations of all $Q_1\varphi(\xi_{s,t})$ are dense in the first chaos.

Proof: Follows easily from the next (quite general) result, or rather, its evident generalization to n factors.

7b2 Lemma. Let $r \le s \le t$, $X \in L_2(\mathcal{F}_{r,s})$, $Y \in L_2(\mathcal{F}_{s,t})$. Then $Q_1(XY) = Q_1(X)\mathbb{E}\,(Y) + \mathbb{E}\,(X)Q_1(Y)$.

Proof. In terms of operators R_φ given by 5b3 we have $Q_1(XY) = R_{\varphi_{r,t}}(XY)$, where $\varphi_{r,t} : \mathcal{C}_{r,t} \to \mathbb{R}$ is the indicator of $\{M \in \mathcal{C} : |M \cap (r,t)| = 1\}$. Similarly, $Q_1(X) = R_{\varphi_{r,s}}(X)$, and $\mathbb{E}\,(X) = R_{\psi_{r,s}}(X)$, where $\psi_{r,s}$ is the indicator of $\{M \in \mathcal{C} : |M \cap (r,s)| = 0\}$. However, $\varphi_{r,t} = \varphi_{r,s}\psi_{s,t} + \psi_{r,s}\varphi_{s,t}$ almost everywhere on $\mathcal{C}_{r,t}$ (w.r.t. every spectral measure). $\square$

In order to prove that the noise (of coalescence) is black, it suffices to prove that $Q\varphi(\xi_{s,t}) = 0$ for all s, t, φ. We'll prove that $Q\varphi(\xi_{0,1}) = 0$; the general case is similar. According to 6d9 we have to prove that $\mathbf{H}_1(\varphi(\xi_{0,1})) = 0$. Assuming that $\mathbb{E}\,\varphi(\xi_{0,1}) = 0$ we will check the sufficient condition:

$$\|\mathbb{E}\,\big(\,\varphi(\xi_{0,1})\,\big|\,\mathcal{F}_{t-\varepsilon,t}\,\big)\| = o(\sqrt{\varepsilon}) \quad \text{for } \varepsilon \to 0\,,$$

uniformly in t. When doing so, we may assume that t is bounded away from 0 and 1. Indeed, $\|\mathbb{E}\,\big(\,\varphi(\xi_{0,1})\,\big|\,\mathcal{F}_{t,1}\,\big)\| \to 0$ for $t \to 1-$, due to continuity of the factorization (recall 3d1(b)).

7b3 Lemma. $\mathbb{E}\,\big(\,\varphi(\xi_{0,1})\,\big|\,\mathcal{F}_{t-\varepsilon,t}\,\big) = \mathbb{E}\,\big(\,\varphi(\xi_{0,1})\,\big|\,\xi_{t-\varepsilon,t}\,\big)$.

The proof is left to the reader; a hint:

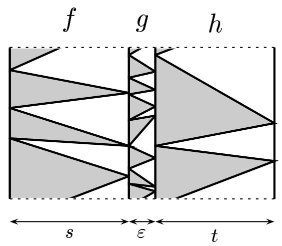
$$\mathbb{E}\,\big(\,\varphi(\xi_{t_1,t_5})\,\big|\,\xi_{t_2,t_3}, \xi_{t_3,t_4}\,\big) = \iint \varphi(\xi_{12}\xi_{23}\xi_{34}\xi_{45})\,\mathrm{d}\mu_{t_2-t_1}(\xi_{12})\mathrm{d}\mu_{t_5-t_4}(\xi_{45}) \\ = \mathbb{E}\,\big(\,\varphi(\xi_{t_1,t_5})\,\big|\,\xi_{t_2,t_4}\,\big)\,.$$

7c The key argument

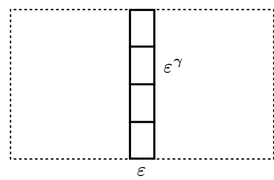
Similarly to 6a6, we consider $X = \varphi(\xi_{0,1}) = \varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1})$, $\mathbb{E}\,X = 0$, $|X| \le 1$ a.s. We have to prove that $\|\mathbb{E}\,\big(\,X\,\big|\,\xi_{t-\varepsilon,t}\,\big)\| = o(\sqrt{\varepsilon})$ for $\varepsilon \to 0$, uniformly in t, when t is bounded away from 0 and 1. Clearly,

$$\mathbb{E}\,\big(\,X\,\big|\,\xi_{t-\varepsilon,t}\,\big) = \iint \varphi(fgh)\,\mathrm{d}\mu_{t-\varepsilon}(f)\mathrm{d}\mu_{1-t}(h)\,,$$

where $g = \xi_{t-\varepsilon,t}$.



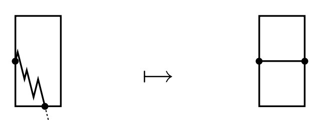
We choose $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and divide the strip $(t-\varepsilon, t) \times \mathbb{T}$ into $\sim \varepsilon^{-\gamma}$ 'cells' $(t-\varepsilon, t) \times (z_k, z_{k+1})$ of height $z_{k+1} - z_k \sim \varepsilon^\gamma$.



We want to think of g as consisting of independent cells. Probably it can be done in continuous time, but we have no such technique for now. Instead, we retreat to the discrete-time model. The needed inequality for continuous time results in the scaling limit $i \to \infty$ provided that in discrete time our estimations are uniform in i (for i large enough).

So, random signs that produce g are divided into cells. Cells are independent and, taken together, they determine g uniquely.

However, a path may cross many cells. This is rather improbable, since $\gamma < 1/2$, but it may happen. We enforce locality by a forgery! Namely, if the path starting at the middle of a cell reaches the bottom or the top edge of the cell, we replace the whole cell with some other cell (it may be chosen once and for all) where it does not happen.



Now cells are 'local'; a path cannot cross more than two cells, but of course, the stochastic flow is changed. Namely, g is changed with an exponentially small (for $\varepsilon \to 0$) probability, which changes $\mathbb{E}\left(X \,\middle|\, \xi_{t-\varepsilon,t}\right)$ by $o(\sqrt{\varepsilon})$ (much less, in fact). Still, cells are independent.

Does a cell (of g) influence the composition, fgh? It depends on f and h. If the left edge $\{t-\varepsilon\} \times [z_k, z_{k+1}]$ of the cell contains no right critical point of f, the cell can influence, since a path starting in an adjacent cell can cross the boundary between cells. However, if the enlarged left edge $\{t-\varepsilon\} \times [z_k - \varepsilon^\gamma, z_{k+1} + \varepsilon^\gamma]$ contains no right critical point of f (in which case we say 'the cell is blocked by f'), then the cell cannot influence, because of the enforced locality. Similarly, if the enlarged right edge $\{t\} \times [z_k - \varepsilon^\gamma, z_{k+1} + \varepsilon^\gamma]$ contains no left critical point of h (in which case we say 'the cell is blocked by h'), the cell cannot influence.

The probability of being not blocked by f is the same for all cells, since the distribution of f is invariant under rotations of $\mathbb{T}$ (discretized as needed).

The sum of these probabilities does not exceed $3\mathbb{E}\, n(0, t-\varepsilon)$ (recall (7a1)), which is $O(1)$ when $\varepsilon \to 0$. (Here we need t to be bounded away from 0.) Thus,

$$\begin{gathered}\mathbb{P}\left(\text{a given cell is not blocked by } f\right) = O(\varepsilon^\gamma)\,;\\ \mathbb{P}\left(\text{a given cell is not blocked by } h\right) = O(\varepsilon^\gamma)\,;\\ \mathbb{P}\left(\text{a given cell is not blocked}\right) = O(\varepsilon^{2\gamma})\,;\\ \mathbb{P}\left(\text{at least one cell is not blocked}\right) = O(\varepsilon^\gamma)\,.\end{gathered}$$

In the latter case we may say that g is not blocked (by f, h).

Denote by A the event "g is not blocked by f, h" (it is determined by f and h, not g); $\mathbb{P}\left(A\right) = O(\varepsilon^\gamma)$. Taking into account that

$$\begin{gathered}X = X - \mathbb{E}\, X = \bigl(X \cdot \mathbf{1}_A - \mathbb{E}\,(X \cdot \mathbf{1}_A)\bigr) + \bigl(X \cdot (\mathbf{1} - \mathbf{1}_A) - \mathbb{E}\,(X \cdot (\mathbf{1} - \mathbf{1}_A))\bigr)\,,\\ \mathbb{E}\left(X \cdot (\mathbf{1} - \mathbf{1}_A) \,\middle|\, g\right) = \mathbb{E}\,(X \cdot (\mathbf{1} - \mathbf{1}_A))\,,\\ \mathbb{E}\left(X \,\middle|\, g\right) = \mathbb{E}\left(X \cdot \mathbf{1}_A \,\middle|\, g\right) - \mathbb{E}\,(X \cdot \mathbf{1}_A)\,,\end{gathered}$$

we have to prove that $\|\mathbb{E}\left(X \cdot \mathbf{1}_A \,\middle|\, g\right) - \mathbb{E}\,(X \cdot \mathbf{1}_A)\| = o(\sqrt{\varepsilon})$. Note that it does not result from the trivial estimation $\|X \cdot \mathbf{1}_A\| \le \|\mathbf{1}_A\| = \sqrt{\mathbb{P}\left(A\right)} = O(\varepsilon^{\gamma/2})$, $\gamma \in \bigl(\frac13, \frac12\bigr)$. Note also that, when g influences X, its influence is usually not small (irrespective of ε) because of the stepwise nature of f and h.

We express the norm in terms of covariance,

$$\|\mathbb{E}\left(X \cdot \mathbf{1}_A \,\middle|\, g\right) - \mathbb{E}\,(X \cdot \mathbf{1}_A)\| = \sup_\psi \operatorname{Cov}\bigl(X \cdot \mathbf{1}_A, \psi(g)\bigr)\,,$$

where the supremum is taken over all Borel functions $\psi : G_\infty \to \mathbb{R}$ such that $\operatorname{Var}\bigl(\psi(g)\bigr) \le 1$. In terms of the correlation coefficient

$$\operatorname{Corr}\bigl(X \cdot \mathbf{1}_A, \psi(g)\bigr) = \frac{\operatorname{Cov}\bigl(X \cdot \mathbf{1}_A, \psi(g)\bigr)}{\sqrt{\operatorname{Var}(X \cdot \mathbf{1}_A)}\sqrt{\operatorname{Var}(\psi(g))}}\,,$$

it is enough to prove that

$$\operatorname{Corr}\bigl(X \cdot \mathbf{1}_A, \psi(g)\bigr) = o(\varepsilon^{(1-\gamma)/2})\,,$$

since it implies $\operatorname{Cov}(\dots) = o(\varepsilon^{(1-\gamma)/2}) \cdot \|X \cdot \mathbf{1}_A\| = o(\varepsilon^{(1-\gamma)/2}\varepsilon^{\gamma/2}) = o(\sqrt{\varepsilon})$. Instead of $o(\varepsilon^{(1-\gamma)/2})$ we will get $O(\varepsilon^\gamma)$, which is also enough since $\gamma > 1/3$.

It remains to apply the quite general lemma given below, interpreting its Y_k as the whole k-th cell (of g), X_k as the indicator of the event "the k-th cell is not blocked" ($k = 1, \dots, n$), X_0 as the pair (f, h), and $\varphi(\dots)$ as $X \cdot \mathbf{1}_A$.

The lemma is formulated for real-valued random variables Y_k, but this does not matter; the same clearly holds for arbitrary spaces, and in fact, we need only finite spaces. The product $X_k Y_k$ is a trick for 'blocking' Y_k when $X_k = 0$. Note that dependence between $X_0, X_1, \dots, X_n$ is allowed.

7c1 Lemma. Let $(X_0, X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be two independent random vectors, $Y_k : \Omega \to \mathbb{R}$, $X_k : \Omega \to \{0,1\}$ for $k = 1, \dots, n$, $X_0 : \Omega \to \mathbb{R}$, and random variables $Y_1, \dots, Y_n$ be independent. Then

$$\operatorname{Corr}\big(\varphi(X_0, X_1Y_1, \dots, X_nY_n),\, \psi(Y_1, \dots, Y_n)\big) \le \sqrt{\max_{k=1,\dots,n} \mathbb{P}\left(X_k = 1 \right)}$$

for all Borel functions $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$, $\psi : \mathbb{R}^n \to \mathbb{R}$ such that the correlation is well-defined (that is, $0 < \operatorname{Var} \varphi(\dots) < \infty$, $0 < \operatorname{Var} \psi(\dots) < \infty$).

Proof. We may assume that $X_1, \dots, X_n$ are functions of X_0. Consider the orthogonal (in $L_2(\Omega)$) projection Q from the space of all random variables of the form $\psi(Y_1, \dots, Y_n)$ to the space of all random variables of the form $\varphi(X_0, X_1Y_1, \dots, X_nY_n)$, that is, $Q\psi(Y_1, \dots, Y_n) = \mathbb{E}\left(\psi(Y_1, \dots, Y_n) \,\middle|\, X_0, X_1Y_1, \dots, X_nY_n \right)$. We have to prove that $\|Q\psi(Y_1, \dots, Y_n)\|^2 \le \big(\max_k \mathbb{P}\left(X_k = 1 \right)\big) \|\psi(Y_1, \dots, Y_n)\|^2$ whenever $\mathbb{E}\, \psi(Y_1, \dots, Y_n) = 0$. The space of all $\psi(Y_1, \dots, Y_n)$ is spanned by *factorizable* random variables $\psi(Y_1, \dots, Y_n) = \psi_1(Y_1) \dots \psi_n(Y_n)$. For such a ψ we have

$$\begin{aligned} Q\psi(Y_1, \dots, Y_n) &= \mathbb{E}\left(\psi_1(Y_1) \dots \psi_n(Y_n) \,\middle|\, X_0, X_1Y_1, \dots, X_nY_n \right) \\ &= \Bigg(\prod_{k:X_k=0} \mathbb{E}\, \psi_k(Y_k) \Bigg) \Bigg(\prod_{k:X_k=1} \psi_k(Y_k) \Bigg) ; \end{aligned}$$

$$\begin{aligned} \|Q\psi(Y_1, \dots, Y_n)\|^2 &= \mathbb{E}\Big(\mathbb{E}\left(|Q\psi(Y_1, \dots, Y_n)|^2 \,\middle|\, X_0 \right) \Big) \\ &= \mathbb{E}\left(\Bigg(\prod_{k:X_k=0} |\mathbb{E}\, \psi_k(Y_k)|^2 \Bigg) \Bigg(\prod_{k:X_k=1} \mathbb{E}\, |\psi_k(Y_k)|^2 \Bigg) \right) . \end{aligned}$$

If, in addition, $\mathbb{E}\, \psi_1(Y_1) = 0$ then $\|Q\psi(Y_1, \dots, Y_n)\|^2 \le \mathbb{P}\left(X_1 = 1 \right) \|\psi(Y_1, \dots, Y_n)\|^2$. Similarly,

$$\|Q\psi(Y_1, \dots, Y_n)\|^2 \le \Big(\max_k \mathbb{P}\left(X_k = 1 \right) \Big) \|\psi(Y_1, \dots, Y_n)\|^2$$

if $\mathbb{E}\, \psi(Y_1, \dots, Y_n) = 0$ and, of course, ψ is factorizable, that is, $\psi(Y_1, \dots, Y_n) = \psi_1(Y_1) \dots \psi_n(Y_n)$. The latter assumption cannot be eliminated just by saying that factorizable random variables of zero mean span all random variables of zero mean. Instead, we use two facts.

The first fact. The space of all random variables $\psi(\dots)$ has an *orthogonal* basis consisting of factorizable random variables satisfying an additional condition: each factor $\psi_k(Y_k)$ is either of zero mean, or equal to 1. (For a proof, start with an orthogonal basis for functions of Y_1 only, the first basis function being constant; do the same for Y_2; take all products; and so on.)

The second fact. The operator Q maps *orthogonal* factorizable random variables, satisfying the additional condition, into *orthogonal* random variables. Indeed, let $\psi(Y_1,\dots,Y_n) = \psi_1(Y_1)\dots\psi_n(Y_n)$, $\psi'(Y_1,\dots,Y_n) = \psi'_1(Y_1)\dots\psi'_n(Y_n)$, and each $\psi_k(Y_k)$ be either of zero mean, or equal to 1; the same for each $\psi'_k(Y_k)$. If $\mathbb{E}\,\bigl(\psi(Y_1,\dots,Y_n)\psi'(Y_1,\dots,Y_n)\bigr) = 0$ then $\mathbb{E}\,\bigl(\psi_k(Y_k)\psi'_k(Y_k)\bigr) = 0$ for at least one k; let it happen for $k=1$. We have not only $\mathbb{E}\,\bigl(\psi_1(Y_1)\psi'_1(Y_1)\bigr) = 0$ but also $\bigl(\mathbb{E}\,\psi_1(Y_1)\bigr)\bigl(\mathbb{E}\,\psi'_1(Y_1)\bigr) = 0$, since ψ_1 and ψ'_1 cannot both be equal to 1. Therefore

$$
\begin{aligned}
&\mathbb{E}\,\bigl(Q\psi(Y_1,\dots,Y_n)\bigr)\bigl(Q\psi'(Y_1,\dots,Y_n)\bigr) = \\
&\quad = \mathbb{E}\,\Biggl(\biggl(\prod_{k:X_k=0}\bigl(\mathbb{E}\,\psi_k(Y_k)\bigr)\bigl(\mathbb{E}\,\psi'_k(Y_k)\bigr)\biggr)\biggl(\prod_{k:X_k=1}\psi_k(Y_k)\psi'_k(Y_k)\biggr)\Biggr) = 0\,,
\end{aligned}
$$

since the first term vanishes whenever $X_1 = 0$, and the second term vanishes whenever $X_1 = 1$.

$\square$

Combining all together, we get the conclusion.

7c2 Theorem. The noise of coalescence is black.

7d Remarks

Another proof of Theorem 7c2 should be possible, by showing that all (zero mean) random variables are sensitive. To this end, we divide the time axis $\mathbb{R}$ into intervals of small length ε, and choose a random subset of intervals such that each interval is chosen with a small probability $1-\rho = 1-\mathrm{e}^{-\lambda} \sim \lambda$, independently of others. On each chosen interval we replace local random data with fresh (independent) data.

Consider the path $X(\cdot)$ of the Brownian web, starting at the origin, $X(t) = \xi_{0,t}(0)$ for $t \in [0,\infty)$; it behaves like a Brownian motion. After the replacement we get another path $Y(\cdot)$. Their difference, $\bigl(X(t)-Y(t)\bigr)/\sqrt{2}$, behaves like another Brownian motion when outside 0, but is somewhat sticky at 0. Namely, during each chosen (to the random set) time interval, the point 0 has nothing special; however, outside these time intervals, the point 0 is absorbing. In this sense, chosen time intervals act like factors f_* in the random product of factors f_-, f_+, f_* studied in Sect. 4. There, f_* occurs with a small probability $1/(2\sqrt{i}) \to 0$ (recall 4e4), which produces a non-degenerate stickiness in the scaling limit. Here, in contrast, a time interval is chosen with probability $1-\rho \sim \lambda$ that does not tend to 0 when the interval length ε tends to 0. Naturally, stickiness disappears in the limit $\varepsilon \to 0$ (a proof uses the idea of (4c9)). That is, interaction between $X(\cdot)$ and $Y(\cdot)$ disappears in the limit $\varepsilon \to 0$. They become independent, no matter how small $1-\rho$ is.

Probably, the same argument works for any finite number of paths $X_k(t) = \xi_{0,t}(x_k)$; they should be asymptotically independent of $Y_k(\cdot)$ for $\varepsilon \to 0$, but I did not prove it.

The spectral measure μ_X of the random variable $X = \xi_{0,1}(0)$ is written down explicitly in [16]. Or rather, its discrete counterpart is found; the scaling limit follows by (a generalization of) Theorem 3c5 (see also [17]). The measure μ_X is a probability measure (since $\|X\| = 1$), it may be thought of as the distribution of a random perfect subset of $(0,1)$. Note that the random subset is not at all a function on the probability space $(\Omega, \mathcal{F}, P)$ that carries the Brownian web. There is no sense in speaking about 'the joint distribution of the random set and the Brownian web'. In fact, they may be treated as incompatible (non-commuting) measurements in the framework of quantum probability, see [15].

A wonder: μ_X is the distribution of $(\theta - M) \cap (0,1)$, where M is the set of zeros of the usual Brownian motion, and θ is independent of M and distributed uniformly on $(0,1)$.

Moreover, the corresponding equality holds exactly (not only asymptotically) in the discrete-time model. Strangely enough, the Brownian motion (or rather, random walk) does not appear in the calculation of the spectral measure. The relation to Brownian motion is observed at the end, as a surprise!

7d1 Question. Can μ_X (for $X = \xi_{0,1}(0)$) be found via some natural construction of a Brownian motion whose zeros form the spectral set (after the transformation $x \mapsto \theta - x$)? (See [16, Problem 1.5].)

We see that μ_X (for $X = \xi_{0,1}(0)$) is concentrated on sets of Hausdorff dimension 1/2.

7d2 Question. Is μ_X concentrated on sets of Hausdorff dimension 1/2 for an arbitrary random variable X such that $\mathbb{E}\, X = 0$ (over the noise of coalescence)?

An affirmative answer would probably give us another proof that the noise is black. A stronger conjecture may be made.

7d3 Question. Is μ_X for an arbitrary $\mathcal{F}_{0,1}$-measurable X (over the noise of coalescence), satisfying $\mathbb{E}\,X = 0$, absolutely continuous w.r.t. $\mu_{\xi_{0,1}(0)}$?

7e A combinatorial by-product

Consider a Markov chain $X = (X_k)_{k=0}^{\infty}$ (a half-difference of two independent simple random walks, or a double-speed simple random walk divided by two): $X_0 = 0$ and

$$\mathbb{P}\left(X_{k+1} = X_k + \Delta x \,\middle|\, X_k \right) = \begin{cases} 1/4 & \text{for } \Delta x = -1, \\ 1/2 & \text{for } \Delta x = 0, \\ 1/4 & \text{for } \Delta x = +1 \end{cases}$$

for each $k = 0, 1, 2, \dots$

Let Z be the (random) set of zeros of X, that is,

$$Z = \{ k = 0, 1, \dots \, : X_k = 0 \} \, .$$

Given a set $S \subset \{0, 1, 2, \dots\}$ and a number $k = 0, 1, 2, \dots$, we consider the event $Z \cap [0, k] \subset k - S$, that is, $\forall l = 0, \dots, k \, \big(l \in Z \implies k - l \in S \big)$, and its probability. We define

$$p_{n,S} = \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}\left(Z \cap [0, k] \subset k - S \right);$$

of course, only $k \in S$ can contribute (since $0 \in Z$).

On the other hand, we may trap X at 0 on S; that is, given a set $S \subset \{0, 1, 2, \dots\}$, we introduce another Markov chain $X^{(S)} = \big(X_k^{(S)} \big)_{k=0}^{\infty}$ such that $X_0^{(S)} = 0$ and for each $k = 0, 1, 2, \dots$

$$\mathbb{P}\left(X_{k+1}^{(S)} = x + \Delta x \,\middle|\, X_k^{(S)} = x \right) = \begin{cases} 1/4 & \text{for } \Delta x = -1, \\ 1/2 & \text{for } \Delta x = 0, \\ 1/4 & \text{for } \Delta x = +1 \end{cases}$$

except for the case $k \in S$, $x = 0$,

$$\mathbb{P}\left(X_{k+1}^{(S)} = 0 \,\middle|\, X_k^{(S)} = 0 \right) = 1 \quad \text{if } k \in S \, .$$

7e1 Theorem. $p_{n,S} = \frac{1}{n} \sum_{k \in S} \mathbb{P}\left(X_k^{(S)} = 0 \right)$ for every $n = 1, 2, \dots$ and $S \subset \{0, 1, \dots, n-1\}$.

7e2 Example. Before proving the theorem, consider a special case; namely, let S consist of just a single number s. Then $\mathbb{P}\big(Z \cap [0,k] \subset k - S \big) = \mathbb{P}\big(Z \cap [0,k] \subset \{k-s\} \big)$ vanishes for $k \ne s$. For $k = s$ it becomes $\mathbb{P}\big(Z \cap [0,s] = \{0\} \big) = 2^{-(2s-1)} \big(\binom{2s-2}{s-1} + \binom{2s-2}{s} \big)$. Therefore $p_{n,\{s\}} = \frac1n 2^{-(2s-1)} \big(\binom{2s-2}{s-1} + \binom{2s-2}{s} \big)$, assuming $s \ge 2$; also, $p_{n,\{0\}} = \frac1n$ and $p_{n,\{1\}} = \frac1{2n}$. On the other hand, $\frac1n \sum_{k\in S} \mathbb{P}\big(X_k^{(S)} = 0 \big) = \frac1n \mathbb{P}\big(X_s = 0 \big) = \frac1n \cdot 2^{-2s} \binom{2s}{s}$. The equality becomes $\binom{2s-2}{s-1} + \binom{2s-2}{s} = \frac12 \binom{2s}{s}$ (for $s \ge 2$).

Proof (sketch). We use the discrete-time counterpart of the Brownian web (see 7a and [16, Sect. 1]) and consider $\xi_{0,n}(0)$, the value at time n of the path starting at the origin. At every instant $k \notin S$ we replace the corresponding random signs with fresh (independent) copies, which leads to another random variable $\xi'_{0,n}(0)$. We calculate the covariance $\mathbb{E}\big(\xi_{0,n}(0)\xi'_{0,n}(0)\big)$ in two ways, and compare the results.

The first way. The difference process $\xi_{0,\cdot}(0) - \xi'_{0,\cdot}(0)$ is distributed like the process $2X^{(S)}$ (similarly to 7d). Thus

$$4\mathbb{E}\big(X_n^{(S)}\big)^2 = \mathbb{E}\big(\xi_{0,n}(0) - \xi'_{0,n}(0)\big)^2 = 2n - 2\mathbb{E}\big(\xi_{0,n}(0)\xi'_{0,n}(0)\big) \,.$$

On the other hand, $\frac12 - \mathbb{E}\big(X_{k+1}^{(S)}\big)^2 + \mathbb{E}\big(X_k^{(S)}\big)^2 = \frac12 \mathbb{P}\big(X_k^{(S)} = 0 \big)$ if $k \in S$, otherwise 0. Therefore $n - 2\mathbb{E}\big(X_n^{(S)}\big)^2 = \sum_{k\in S} \mathbb{P}\big(X_k^{(S)} = 0 \big)$. So,

$$\mathbb{E}\big(\xi_{0,n}(0)\xi'_{0,n}(0)\big) = \sum_{k\in S} \mathbb{P}\big(X_k^{(S)} = 0 \big) \,.$$

The second way. In terms of the spectral measure μ of the random variable $\xi_{0,n}(0)$ we have $\mathbb{E}\big(\xi_{0,n}(0)\xi'_{0,n}(0)\big) = \mu\{M : M \subset S\}$. However, the probability measure $\frac1n \mu$ is equal to the distribution of $(\theta - Z) \cap [0,\infty)$; here Z is (as before) the set of zeros of X, and θ is a random variable independent of Z and distributed uniformly on $\{0,1,\dots,n-1\}$. (See [16, Prop. 1.3], see also [24].) Therefore $\frac1n \mu\{M : M \subset S\} = \mathbb{P}\big((\theta - Z) \cap [0,\infty) \subset S \big) = \mathbb{P}\big(Z \cap [0,\theta] \subset \theta - S \big) = p_{n,S}$. So,

$$\mathbb{E}\big(\xi_{0,n}(0)\xi'_{0,n}(0)\big) = n p_{n,S} \,.$$

$\square$

7e3 Question. Is there a simpler proof of Theorem 7e1? Namely, can we avoid the spectral measure and its relation to the set of zeros?

A continuous-time counterpart of Theorem 7e1 is left to the reader.

8 Miscellany

8a Beyond the one-dimensional time

Scaling limits of models driven by *two-dimensional* arrays of random signs are evidently important. The best examples appear in percolation theory. Also the Brownian web is an example and, after all, it may be treated as an oriented percolation.

In such cases, independent sub-σ-fields should correspond to disjoint regions of $\mathbb{R}^2$, not only of the form $(s,t)\times\mathbb{R}$. In fact, a rudimentary use of these can be found in Sect. 7 (recall 'cells' in 7c). In general it is unclear what kind of regions can be used; probably, regions with piecewise smooth boundaries always fit, while arbitrary open sets do not fit unless the two-dimensional noise is classical (recall 6c).

In spite of the great and spectacular progress of the percolation theory (see for instance [14] and references therein), 'the noise of percolation' is still a dream.

8a1 Question. For the critical site percolation on the triangular lattice, invent an appropriate coarse σ-field, and check two-dimensional counterparts of the two conditions of 3b1 for an appropriate class of two-dimensional domains. Is it possible?

8a2 Remark. Hopefully, the answer is affirmative, that is, the two-dimensional noise of percolation will be defined. Then it should appear to be a (two-dimensional) black noise, due to (appropriately adapted) 6d11, 7b1 and (most important) the critical exponent for a small cell of size $\varepsilon\times\varepsilon$ being pivotal [14, Sect. 5, Item 2]. The probability is $O(\varepsilon^{5/4})$, therefore $o(\varepsilon)$. The sum for $\mathbf{H}(f)$ contains $O(1/\varepsilon^2)$ terms, $o(\varepsilon^2)$ each.[33]

Sensitivity of percolation events, disclosed in [2], is micro-sensitivity (recall 5c). Existence of the black noise of percolation would mean a stronger property: block sensitivity. (See also [2, Problem 5.4].)

It would be the most important example of a black noise!

For the *general* theory of stability, spectral measures, decomposable processes etc., the dimension of the underlying space is of little importance. Basically, regions must form a Boolean algebra. Such a general approach is used in [20], [18].

Nonclassical factorizations appear already in zero-dimensional 'time', be it a Cantor set, or even a convergent sequence with limit point. For Cantor

[33]Different arguments (especially, 7c1) are used in Sect. 7, since an infinite two-dimensional spectral set could have a finite one-dimensional projection.

sets, see [20, Sect. 4]; some interesting models of combinatorial nature, with large symmetry groups (instead of 'time shifts' of a noise) are examined there. For a convergent sequence with limit point, see Chapter 1 here (namely, 1a1), and [18, Appendix].

8b The 'wave noise' approach

A completely different way of constructing noises is sketched here.

Consider the linear wave equation in dimension $1+1$,

(8b1)
$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \Big) u(x,t) = 0 \, ,$$

with initial conditions $u(x,0)=0$, $u_t(x,0)=f(x)$. Its solution is well-known:

$$u(x,t) = \frac12 \int_{x-t}^{x+t} f(y) \, dy = \frac12 F(x+t) - \frac12 F(x-t) \, ,$$

where F is defined by $F'(x)=f(x)$. The formula holds in a generalized sense for nonsmooth F, which covers the following case: $F(x)=B(x)=$ Brownian motion (combined out of two independent branches, on $[0,+\infty)$ and on $(-\infty,0]$); $f(x)=B'(x)$ is the white noise. The random field on $(-\infty,\infty)\times[0,\infty)$,

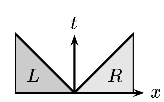
$$u(x,t) = \frac12 B(x+t) - \frac12 B(x-t) \, , \qquad B = \text{ Brownian motion,}$$

is continuous, stationary in x, scaling invariant (for any c the random field $u(cx,ct)/\sqrt{c}$ has the same distribution as $u(x,t)$), satisfies the wave equation (8b1) and the following independence condition:

(8b2)

$u\big|_L$ and $u\big|_R$ are independent,

where $L=\{(x,t) : x < -t < 0\}$, $R=\{(x,t) : x > t > 0\}$.



The independence is a manifestation of: (1) the independence inherent to the white noise (its integrals over disjoint segments are independent), and (2) the hyperbolicity of the wave equation (propagation speed does not exceed 1).

A solution with such properties is essentially unique. That is, if $u(x,t)$ is a continuous random field on $(-\infty,\infty)\times(0,\infty)$, stationary in x, satisfying the wave equation (8b1) and the independence condition (8b2), then necessarily $u(x,t) = \mu_0 + \mu_1 t + \sigma\big(B(x+t) - B(x-t)\big)$ for a Brownian motion B. Scaling invariance forces $\mu_0 = \mu_1 = 0$.

It is instructive that a wave equation may be used in a non-traditional way. Traditionally, a solution is determined by its initial values. In contrast, the independence condition (8b2), combined with some more conditions, determines a random solution with no help of initial conditions! Not an individual sample function is determined, of course, but its distribution (a probability measure on the space of solutions of the wave equation).

Somebody with no preexisting idea of white noise or Brownian motion can, in principle, use the above approach. Observing that $u(x,0) = 0$ but $u_t(x,0)$ does not exist (in the classical sense), he may investigate $u(x,t)/t$ for $t \to 0$ as a way toward the white noise.

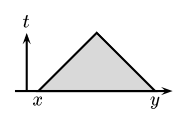
8b3 Question. Can we construct a nonclassical (especially, black) noise, using a nonlinear hyperbolic equation?

I once tried the nonlinear wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u(x,t) = \varepsilon t^{-(3-\varepsilon)/2} \sin\bigl(t^{-(1+\varepsilon)/2} u(x,t) \bigr) , \tag{8b4}$$

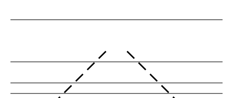
ε being a small positive parameter. The equation is scaling-invariant: if $u(x,t)$ is a solution, then $u(cx,ct)/c^{(1+\varepsilon)/2}$ is also a solution. We search for a random field $u(t,x)$, continuous, stationary in x, scaling invariant, satisfying (8b4) and the independence condition (8b2). Its behavior for $t \to 0$ should give us a new noise. Does such a random field exist? Is it unique (in distribution)? If the answers are affirmative, then we get a noise,

$\mathcal{F}_{x,y}$ is the σ-field generated by $\{ u(z,t) : x+t < z < y-t \}$,



and maybe it is black. However, I did not succeed with it.

A modified 'waive noise' approach was used successfully in [20, Sect. 5], proving, for the first time, the existence of a black noise. The modification is to keep the auxiliary dimension, but make it discrete rather than continuous:



More specifically, consider a sequence of stationary random processes $u_k(\cdot)$ on $\mathbb{R}$ such that

- u_k is $2\varepsilon_k$-dependent (for some $\varepsilon_k \to 0$); it means that $u_k\big|_{(-\infty,-\varepsilon_k]}$ and $u_k\big|_{[\varepsilon_k,+\infty)}$ are independent;

- $u_{k-1}(x)$ is uniquely determined by $u_k\big|_{[x-(\varepsilon_{k-1}-\varepsilon_k),\,x+(\varepsilon_{k-1}-\varepsilon_k)]}$.

Such a sequence (u_k) determines a noise; namely, $\mathcal{F}_{x,y}$ is generated by all $u_k(z)$ such that $x+\varepsilon_k \le z \le y-\varepsilon_k$. White noise can be obtained by a linear system of Gaussian processes:

$$u_{k-1}(x) = \int_{x-(\varepsilon_{k-1}-\varepsilon_k)}^{x+(\varepsilon_{k-1}-\varepsilon_k)} V_k(y-x)u_k(y)\,dy\,,$$

where kernels V_k, concentrated on $[-(\varepsilon_{k-1}-\varepsilon_k),(\varepsilon_{k-1}-\varepsilon_k)]$, are chosen appropriately. A nonlinear system (of quite non-Gaussian processes) of the form

$$u_{k-1}(x) = \varphi\biggl(\frac{\text{const}}{\varepsilon_{k-1}-\varepsilon_k}\int_{x-(\varepsilon_{k-1}-\varepsilon_k)}^{x+(\varepsilon_{k-1}-\varepsilon_k)} u_k(y)\,dy\biggr)$$

was used for constructing a black noise. But, it is not really a *construction* of a specific noise. Existence of (u_k) is proven, but uniqueness (in distribution) is not. True, every such (u_k) determines a black noise. However, none of them is singled out.

8c Groups, semigroups, kernels

A Brownian motion X in a topological group G is defined as a continuous G-valued random process with stationary independent increments, starting from the unit of G. For example, if G is the additive group of reals, then the general form of a Brownian motion in G is $X(t)=\sigma B(t)+vt$, where $B(\cdot)$ is the standard Brownian motion, $\sigma\in[0,\infty)$ and $v\in\mathbb{R}$ are parameters. If G is a Lie group, then Brownian motions X in G correspond to Brownian motions Y in the tangent space of G (at the unit) via the stochastic differential equation $(dX)\cdot X^{-1}=dY$ (in the sense of Stratonovich).

A noise corresponds to every Brownian motion in a topological group, just as the white noise corresponds to $B(\cdot)$. If the noise is classical, it is the white noise of some dimension ($0,1,2,\dots$ or ∞). If this is the case for all Brownian motions in G, we call G a *white group.* Thus, $\mathbb{R}$ is white, and every Lie group is white. Every commutative topological group is white (see [15, Th. 1.8]). The group of all unitary operators in l_2 (equipped with the strong operator topology) is white (see [15, Th. 1.6]). Many other groups are white since they are embeddable into a group known to be white; for example, the group of diffeomorphisms is white (an old result of Baxendale).

8c1 Question. Is the group of all homeomorphisms of (say) $[0,1]$ white?

In a topological group, Brownian motions X and continuous abstract stochastic flows ξ are basically the same:

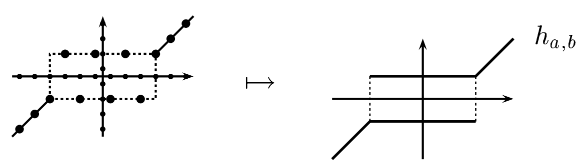
$$X(t) = \xi_{0,t}\,; \qquad \xi_{s,t} = X^{-1}(s)X(t)\,.$$

In a semigroup, however, a noise corresponds to a flow, not to a Brownian motion (see also 4c4).

A nonclassical noise (of stickiness) was constructed in Sect. 4 out of an abstract flow in a 3-dimensional semigroup G_3; however, G_3 is not a topological semigroup, since composition is discontinuous.

8c2 Question. Can a nonclassical noise arise from an abstract stochastic flow in a finite-dimensional topological semigroup?

The continuous (but not topological) semigroup G_3 emerged in Sect. 4 from the discrete semigroup G_3^{discrete} via the scaling limit. Or rather, a flow in G_3 emerged from a flow in G_3^{discrete} via the scaling limit. A similar approach to the discrete model of 1d1 gives something unexpected. The continuous semigroup that emerges is G_2, the two-dimensional topological semigroup described in (4d1). However, its representation is not single-valued:



Namely, $h_{a,b}(x)$ for $x \in (-b, b)$ is $\pm(a+b)$, that is, either $a+b$ or $-(a+b)$ with probabilities $0.5, 0.5$. Such h is not a function, of course. Rather, it is a *kernel,* that is, a measurable map from $\mathbb{R}$ into the space of probability measures on $\mathbb{R}$. Composition of kernels is well-defined, thus, a representation (of a semigroup) by kernels (rather than functions) is also well-defined.

The stochastic flow in G_2, resulting from 1d1 via the scaling limit, is identical to the flow $(\xi_{s,t}^{(2)})$ of 4g. Its noise is the usual (one-dimensional) white noise. The representation of G_2 by kernels turns the abstract flow into a *stochastic flow of kernels* as defined by Le Jan and Raimond [8, Def. 1.1.3]. However, a kernel (unlike a function) introduces an additional level of randomness. When the kernel says that $h_{a,b}(x) = \pm(a+b)$, someone has to choose at random one of the two possibilities. Who makes the decision?

One may treat a point as a macroscopically small collection of many microscopic atoms, and $\omega \in \Omega$ as a macroscopic flow (on the whole space-time); given ω, atoms are (conditionally) independent, “which means that two points[34] thrown initially at the same place separate" [8, p. 4]. No need to deal explicitly with a continuum of independent choices. "Turbulent evolutions [are represented] by flows of probability kernels obtained by dividing infinitely the initial point" [8, p. 4].

Alternatively, one can postulate that if two atoms meet at a (macroscopic!) point, they must coalesce. In one-dimensional space (and *sometimes* in higher dimensions) such a postulate itself prevents a continuum of independent choices and leads to a flow of maps (the Brownian web is an example). A countable dense set of atoms makes decisions; others must obey. A flow of maps is a (degenerate) special case of a flow of kernels. However, coalescence can produce a flow of maps out of a non-degenerate flow of kernels, as explained in [8, Sect. 2.3].

Conversely, a coalescent flow can produce a non-degenerate flow of kernels via "filtering by a sub-noise" [8, Sect. 2.3]. In the simplest case (filtering by a trivial sub-noise), we just retain the one-particle motion of the given coalescent flow, forget the rest of the flow, and let atoms perform the motion independently.

A large class of flows on $\mathbb{R}^n$ (and other homogeneous spaces) is investigated in [8]. Some of these flows are shown to be coalescent and to generate nonclassical noises (neither white nor black). Flows are homogeneous in space (and isotropic). Thus, we have a hierarchy of nonclassical models. First, toy models (recall 1a1, 1a3) having a singular time point. Second, 'simple' models (1d, 4i) homogeneous in time but having a singular spatial point. Third, 'serious' models (the Brownian web, and Le Jan-Raimond's isotropic Brownian flows), homogeneous in space and time.

Noises generated by one-dimensional flows (also homogeneous in space and time) are investigated by Warren and Watanabe [24]. Spectral sets of Hausdorff dimension other than 0 and 1/2 are found! Roughly, it answers Question 6d5; however, these spectral sets are not perfect — they have isolated points.

8d Abstract nonsense of Le Jan-Raimond's theory

A new semigroup, introduced recently by Le Jan and Raimond [8], is quite interesting for the theory of stochastic flows and noises. Its definition involves some technicalities considered here.

A kernel is defined in [8] as a measurable mapping from a compact metric space $\mathcal{M}$ to the (also compact) space $\mathcal{P}(\mathcal{M})$ of all probability measures on $\mathcal{M}$. The space E of all kernels is equipped with the σ-field $\mathcal{E}$ generated

[34] Or rather, atoms.

by evaluations, $E \ni K \mapsto K(x) \in \mathcal{P}(\mathcal{M})$, at points $x \in \mathcal{M}$. Note that every $\mathcal{E}$-measurable function uses the values of $K(x)$ only for a countable set of points x, which is scanty, since $K(x)$ is just measurable (rather than continuous) in x. Thus, $(E, \mathcal{E})$ is not a standard Borel space,[35] and the composition of kernels is not a measurable operation, which obscures the technique and makes proofs more difficult (as noted on page 11 of [8]).

Fortunately, the theory can be reformulated equivalently in terms of Borel operations on standard Borel spaces, as outlined below. Additional simplification comes from disentangling space and time (entangled in Theorem 1.1.4 of [8]) and explicit use of the de Finetti theorem.

The hassle about measurability is another manifestation of the well-known clash between finite-dimensional distributions and modifications of a random process. Say, for the usual Poisson process on $[0, \infty)$, its finite-dimensional distributions do not tell us whether sample paths are continuous from the left (right), or not. A process $X = X(t, \omega)$ has a lot of modifications $Y(t, \omega)$; these satisfy $\forall t \; \mathbb{P}\big(\{\omega : X(t, \omega) = Y(t, \omega)\} \big) = 1$, which does not imply $\mathbb{P}\big(\{\omega : \forall t \; X(t, \omega) = Y(t, \omega)\} \big) = 1$. If a process admits continuous sample paths (like the Brownian motion), the continuous modification is preferable. If a process is just continuous in probability (like the Poisson process, but also, say, some stationary Gaussian processes, unbounded on every interval), we are unable to prefer one modification to others, in general.

In order to describe the class of all modifications of a random process, we have two well-known tools: first, a compatible family of finite-dimensional distributions, and second, a probability measure on the (non-standard!) Borel space of all (or only measurable; but definitely, not only continuous) sample paths, whose σ-field is generated by evaluations. Assuming the process to be continuous in probability, we find the first tool much better; joint distributions depend on points continuously, and everything is standard.

The same for kernels. These may be thought of as sample paths of a random process whose 'time' runs over $\mathcal{M}$, and 'values' belong to $\mathcal{P}(\mathcal{M})$. However, the process will appear (implicitly) only in Theorem 8d3; its finite-dimensional distributions are $\nu_n(x_1, \dots, x_n)$ there.

8d1 Definition. A *multikernel* from a compact metric space $\mathcal{M}_1$ to a compact metric space $\mathcal{M}_2$ is a sequence $(P_n)_{n=1}^{\infty}$ of continuous maps $P_n : \mathcal{M}_1^n \to$

[35]For a definition, see [7, Sect. 12.B] or [1, Def. 7.1].

$\mathcal{P}(\mathcal{M}_2^n)$, compatible in the sense that[36]

$$\int_{\mathcal{M}_2^n} g \, \mathrm{d}P_n(x_1,\dots,x_n) = \int_{\mathcal{M}_2^m} f \, \mathrm{d}P_m(x_{i_1},\dots,x_{i_m})$$

for all n and $x_1,\dots,x_n \in \mathcal{M}_1$, whenever $i_1,\dots i_m$ are pairwise distinct elements of $\{1,\dots,n\}$, $f : \mathcal{M}_2^m \to \mathbb{R}$ is a continuous function, and $g : \mathcal{M}_2^n \to \mathbb{R}$ is defined by $g(y_1,\dots,y_n) = f(y_{i_1},\dots,y_{i_m})$ for $y_1,\dots,y_n \in \mathcal{M}_2$.

We do not assume $i_1 < \dots < i_m$. For example:

$$\begin{aligned}
g(y_1,y_2) = f(y_1) \quad &\Longrightarrow \quad \int g \, \mathrm{d}P_2(x_1,x_2) = \int f \, \mathrm{d}P_1(x_1)\,;\\
g(y_1,y_2) = f(y_2) \quad &\Longrightarrow \quad \int g \, \mathrm{d}P_2(x_1,x_2) = \int f \, \mathrm{d}P_1(x_2)\,;\\
g(y_1,y_2) = f(y_2,y_1) \quad &\Longrightarrow \quad \int g \, \mathrm{d}P_2(x_1,x_2) = \int f \, \mathrm{d}P_2(x_2,x_1)\,.
\end{aligned}$$

Note also that $x_1,x_2,\dots$ need not be distinct.

8d2 Definition. A multikernel $(P_n)_{n=1}^\infty$ is *single-valued,* if

$$\int_{\mathcal{M}_2^2} g \, \mathrm{d}P_2(x,x) = \int_{\mathcal{M}_2} f \, \mathrm{d}P_1(x) \quad \text{for all } x \in \mathcal{M}_1\,,$$

whenever $g : \mathcal{M}_2^2 \to \mathbb{R}$ is a continuous function, and $f : \mathcal{M}_2 \to \mathbb{R}$ is defined by $f(y) = g(y,y)$ for $y \in \mathcal{M}_2$.

An equivalent definition: $(P_n)_{n=1}^\infty$ is single-valued, if

$$\int_{\mathcal{M}_2^2} \rho \, \mathrm{d}P_2(x,x) = 0 \quad \text{for all } x \in \mathcal{M}_1\,,$$

where $\rho : \mathcal{M}_2^2 \to \mathbb{R}$ is the metric, $\rho(y_1,y_2) = \operatorname{dist}(y_1,y_2)$.

Another equivalent definition:

$$\sup_{\rho(x_1,x_2)\le\varepsilon} \int_{\mathcal{M}_2^2} \rho \, \mathrm{d}P_2(x_1,x_2) \to 0 \quad \text{for } \varepsilon \to 0\,.$$

(Compare it with continuity in probability.)

[36]Here $\int g \, \mathrm{d}P_n(x_1,\dots,x_n)$ is not an integral in $x_1,\dots,x_n$. Rather, $x_1,\dots,x_n$ are parameters. The integral is taken in other variables (say, $y_1,\dots,y_n$), suppressed in the notation and running over $\mathcal{M}_2$.

My 'multikernel' is a time-free counterpart of a 'compatible family of Feller semigroups' of [8]. My 'single-valued' corresponds to their (1.7). What could correspond to their 'stochastic convolution semigroup'? It is a single-valued multikernel from $\mathcal{M}_1$ to $\mathcal{P}(\mathcal{M}_2)$. Yes, I mean it: maps from $\mathcal{M}_1^n$ to $\mathcal{P}\big((\mathcal{P}(\mathcal{M}_2))^n\big)$. It may look frightening, but think what happens if $\mathcal{M}_1$ contains only one point, and $\mathcal{M}_2$ — only two points, say, 0 and 1. Then a multikernel from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a law of an exchangeable sequence of events. A single-valued multikernel from $\mathcal{M}_1$ to $\mathcal{M}_2$ would mean that all events coincide, but we need rather a single-valued multikernel from $\mathcal{M}_1$ to $\mathcal{P}(\mathcal{M}_2) = [0,1]$; nothing but a probability measure on $[0,1]$. The De Finetti theorem (see [1], for instance) tells us that every exchangeable sequence of events arises from a probability measure on $[0,1]$. Here is a more general result.

8d3 Theorem. For every multikernel $(P_n)_{n=1}^\infty$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ there exists a single-valued multikernel $(\nu_n)_{n=1}^\infty$ from $\mathcal{M}_1$ to $\mathcal{P}(\mathcal{M}_2)$ such that

$$\int_{\mathcal{M}_2^n} f \,\mathrm{d}P_n(x_1,\dots,x_n) = \int_{(\mathcal{P}(\mathcal{M}_2))^n} F \,\mathrm{d}\nu_n(x_1,\dots,x_n)$$

for all n and $x_1,\dots,x_n \in \mathcal{M}_1$, whenever $f : \mathcal{M}_2^n \to \mathbb{R}$ is a continuous function, and $F : (\mathcal{P}(\mathcal{M}_2))^n \to \mathbb{R}$ is defined by $F(\mu_1,\dots,\mu_n) = \int f \, d(\mu_1 \otimes \dots \otimes \mu_n)$ for $\mu_1,\dots,\mu_n \in \mathcal{P}(\mathcal{M}_2)$.

Proof. We choose a discrete probability measure μ_0 on $\mathcal{M}_1$ whose support is the whole $\mathcal{M}_1$. That is, we choose a countable (or finite) dense set $A \subset \mathcal{M}_1$, and give a positive probability to each point of A. For every n we consider the following measure Q_n on $(\mathcal{M}_1 \times \mathcal{M}_2)^n$:

$$\begin{aligned}&\int f_1 \otimes g_1 \otimes \dots \otimes f_n \otimes g_n \,\mathrm{d}Q_n \\ &= \int \Big(\int g_1 \otimes \dots \otimes g_n \,\mathrm{d}P_n(x_1,\dots,x_n) \Big) f_1(x_1) \dots f_n(x_n) \,\mathrm{d}\mu_0(x_1) \dots \mathrm{d}\mu_0(x_n) \,.\end{aligned}$$

In other words, if Q_n is the distribution of $(X_1,Y_1;\dots;X_n,Y_n)$, then $X_1,\dots,X_n$ are i.i.d. distributed μ_0 each, and the conditional distribution of $(Y_1,\dots,Y_n)$ given $(X_1,\dots,X_n)$ is $P_n(X_1,\dots,X_n)$. The measure Q_n is invariant under the group of $n!$ permutations of n pairs, due to compatibility of the multikernel $(P_n)_{n=1}^\infty$. For the same reason, Q_n is the marginal of Q_{n+1}. Thus, $(Q_n)_{n=1}^\infty$ is the distribution of an exchangeable infinite sequence of $\mathcal{M}_1 \times \mathcal{M}_2$-valued random variables (X_n, Y_n).

The De Finetti theorem [1, Th. 3.1 and Prop. 7.4] states that the joint distribution of all (X_n, Y_n) is a mixture of products, in the sense that there exists a probability measure ν on $\mathcal{P}(\mathcal{M}_1 \times \mathcal{M}_2)$ such that for every n, the joint distribution of n pairs $(X_1, Y_1), \dots, (X_n, Y_n)$ is the mixture of products $Q^{\otimes n} = Q \otimes \dots \otimes Q$, where $Q \in \mathcal{P}(\mathcal{M}_1 \times \mathcal{M}_2)$ is distributed ν. The first marginal of Q is equal to μ_0 (for ν-almost every Q), since X_n are i.i.d. (μ_0).

Let $x_1, \dots, x_n \in A$. The event $X_1 = x_1, \dots, X_n = x_n$ is of positive probability. Given the event, the conditional distribution $P_n(x_1, \dots, x_n)$ of $Y_1, \dots, Y_n$ is the mixture of products $Q_{x_1} \otimes \dots \otimes Q_{x_n}$, where Q_x is the conditional measure on $\mathcal{M}_2$, that corresponds to Q, and $Q \in \mathcal{P}(\mathcal{M}_1 \times \mathcal{M}_2)$ is distributed ν; indeed, ν-almost all Q ascribe the same probability to the event $X_1 = x_1, \dots, X_n = x_n$.

We define $\nu_n(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in A$ as the joint distribution of $\mathcal{P}(\mathcal{M}_2)$-valued random variables $Q_{x_1}, \dots, Q_{x_n}$, where Q is distributed ν; then

$$\begin{aligned}(8\text{d}4)\quad & \int_{(\mathcal{P}(\mathcal{M}_2))^n} F \, \mathrm{d}\nu_n(x_1, \dots, x_n) \\ &= \int_{\mathcal{P}(\mathcal{M}_1 \times \mathcal{M}_2)} \left(\int_{\mathcal{M}_2^n} f \, \mathrm{d}(Q_{x_1} \otimes \dots \otimes Q_{x_n}) \right) \mathrm{d}\nu(Q) \\ &= \int_{\mathcal{M}_2^n} f \, \mathrm{d}P_n(x_1, \dots, x_n)\end{aligned}$$

whenever $f : \mathcal{M}_2^n \to \mathbb{R}$ is a continuous function, and $F : (\mathcal{P}(\mathcal{M}_2))^n \to \mathbb{R}$ is defined by $F(\mu_1, \dots, \mu_n) = \int f \, d(\mu_1 \otimes \dots \otimes \mu_n)$ for $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathcal{M}_2)$.

Till now, $\nu_n(x_1, \dots, x_n)$ is defined for $x_1, \dots, x_n \in A$ (rather than $\mathcal{M}_1$). We want to check that $\int \tilde\rho_2 \, \mathrm{d}\nu_2(x_1, x_2) \to 0$ for $\rho_1(x_1, x_2) \to 0$; here ρ_1 is a metric on $\mathcal{M}_1$ conforming to its topology, and $\tilde\rho_2$ is a metric on $\mathcal{P}(\mathcal{M}_2)$ conforming to its weak topology. Due to compactness of $\mathcal{P}(\mathcal{M}_2)$, it is enough to check that $\int h^2 \, \mathrm{d}\nu_2(x_1, x_2) \to 0$ for $\rho_1(x_1, x_2) \to 0$ whenever $h : \mathcal{P}(\mathcal{M}_2) \times \mathcal{P}(\mathcal{M}_2) \to \mathbb{R}$ is of the form $h(Q_1, Q_2) = \int f \, \mathrm{d}Q_1 - \int f \, \mathrm{d}Q_2$ for a continuous function $f : \mathcal{M}_2 \to \mathbb{R}$. Consider $\tilde f : \mathcal{P}(\mathcal{M}_2) \to \mathbb{R}$, $\tilde f(Q) = \int f \, \mathrm{d}Q$ for $Q \in \mathcal{P}(\mathcal{M}_2)$. We have

$$\int_{(\mathcal{P}(\mathcal{M}_2))^2} \tilde f \otimes \tilde f \, \mathrm{d}\nu_2(x_1, x_2) = \int_{\mathcal{M}_2^2} f \otimes f \, \mathrm{d}P_2(x_1, x_2) \, ,$$

which is a special case of (8d4). It may also be written as

$$\mathbb{E} \, \tilde f(Q_{x_1}) \tilde f(Q_{x_2}) = \mathbb{E} \left(f(Y_1) f(Y_2) \,\middle|\, X_1 = x_1, X_2 = x_2 \right) ;$$

here Q_{x_1} and Q_{x_2} are treated as random variables on the probability space $\big(\mathcal{P}(\mathcal{M}_1 \times \mathcal{M}_2), \nu\big)$ (thus, the two expectations are taken on different probability spaces). The right-hand side is a continuous function of x_1, x_2; denote it $\varphi(x_1, x_2)$. We have

$$\int h^2 \, \mathrm{d}\nu_2(x_1, x_2) = \mathbb{E} \big(\tilde{f}(Q_{x_1}) - \tilde{f}(Q_{x_2}) \big)^2 \\ = \varphi(x_1, x_1) - \varphi(x_1, x_2) - \varphi(x_2, x_1) + \varphi(x_2, x_2) \, ,$$

which tends to 0 for $\rho_1(x_1, x_2) \to 0$. So,

$$\int_{(\mathcal{P}(\mathcal{M}_2))^2} \tilde{\rho}_2 \, \mathrm{d}\nu_2(x_1, x_2) \to 0 \quad \text{for } \rho_1(x_1, x_2) \to 0 \, .$$

It follows easily that each ν_n is uniformly continuous on A^n and, extending it by continuity to $\mathcal{M}_1^n$, we get a single-valued multikernel. $\square$

Definition 8d1 may be reformulated as follows.

8d5 Definition. A multikernel from a compact metric space $\mathcal{M}_1$ to a compact metric space $\mathcal{M}_2$ is a continuous map $P_\infty : \mathcal{M}_1^\infty \to \mathcal{P}(\mathcal{M}_2^\infty)$, satisfying conditions (1) and (2) below. Here $\mathcal{M}^\infty = \mathcal{M} \times \mathcal{M} \times \dots$ is the product of an infinite sequence of copies of $\mathcal{M}$ (still a metrizable compact space).

(1) P_∞ intertwines the natural actions of the permutation group of the index set $\{1, 2, 3, \dots\}$ on $\mathcal{M}_1^\infty$ and $\mathcal{P}(\mathcal{M}_2^\infty)$ (via $\mathcal{M}_2^\infty$).

(2) For every n, the projection of the measure $P_\infty(m)$ to the product $\mathcal{M}_1^n$ of the first n factors depends only on the first n coordinates $m_1, \dots, m_n$ of the point $(m_1, m_2, \dots) = m \in \mathcal{M}_1^\infty$.

Proof of equivalence between definitions 8d1 and 8d5 is left to the reader.

It is well-known that a continuous map $\mathcal{M}_1 \to \mathcal{P}(\mathcal{M}_2)$ is basically the same as a linear operator $C(\mathcal{M}_2) \to C(\mathcal{M}_1)$, positive and preserving the unit. Thus, a multikernel from $\mathcal{M}_1$ to $\mathcal{M}_2$ may be thought of as a positive unit-preserving linear operator $C(\mathcal{M}_2^\infty) \to C(\mathcal{M}_1^\infty)$ satisfying two conditions parallel to 8d5(1,2).

Given three compact metric spaces $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, a multikernel from $\mathcal{M}_1$ to $\mathcal{M}_2$ and a multikernel from $\mathcal{M}_2$ to $\mathcal{M}_3$, we may define their composition, a multikernel from $\mathcal{M}_1$ to $\mathcal{M}_3$. In terms of operators it is just the product of two operators, $C(\mathcal{M}_3^\infty) \to C(\mathcal{M}_2^\infty) \to C(\mathcal{M}_1^\infty)$.

The set of all multikernels from $\mathcal{M}_1$ to $\mathcal{M}_2$, treated as operators $C(\mathcal{M}_2^\infty) \to C(\mathcal{M}_1^\infty)$, is a closed (and bounded, but not compact) subset of the operator space equipped with the strong operator topology. Thus, the set of multikernels becomes a Polish space (that is, a topological space underlying a complete separable metric space).

Composition of multikernels, $C(\mathcal{M}_3^\infty) \to C(\mathcal{M}_2^\infty) \to C(\mathcal{M}_1^\infty)$, is a (jointly) continuous operation. (Indeed, the product of operators is continuous in the strong operator topology, as far as all operators are of norm ≤ 1.)

So, multikernels from $\mathcal{M}$ to $\mathcal{M}$ are a Polish semigroup (that is, a topological semigroup whose topological space is Polish).

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