Nonclassical stochastic flows and continuous products

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Abstract

Contrary to the classical wisdom, processes with independent values (defined properly) are much more diverse than white noise combined with Poisson point processes, and product systems are much more diverse than Fock spaces.

This text is a survey of recent progress in constructing and investigating nonclassical stochastic flows and continuous products of probability spaces and Hilbert spaces.

Contents

Introduction

The famous Brownian motion $(B_t)_{t\in[0,\infty)}$ may be thought of as an especially remarkable bizarre random function. For instance, one may use it in a probabilistic proof of existence of a nowhere differentiable continuous function. Various fine properties of Brownian sample paths are investigated, but are beyond the scope of this survey.

Stochastic differential equations are a different (and maybe more important) way of using B_t. An example (simple and widely known):

$$\mathrm{d}X_t = X_t \,\mathrm{d}B_t\,, \qquad X_0 = 1\,. \tag{0.1}$$

It may be thought of as the scaling limit (for $n \to \infty$) of a discrete-time equation

$$X^{(n)}_{(k+1)/n} - X^{(n)}_{k/n} = X^{(n)}_{k/n} \frac{\tau_k}{\sqrt{n}}\,, \qquad X^{(n)}_0 = 1 \tag{0.2}$$

(after rescaling); here $\tau_0, \tau_1, \dots$ are random signs, that is, i.i.d. random variables taking on two values ± 1 with probabilities $1/2, 1/2$. Accordingly, $(B_t)_t$ is the scaling limit of the random walk

$$B^{(n)}_{k/n} = \frac{\tau_0 + \dots + \tau_{k-1}}{\sqrt{n}}\,. \tag{0.3}$$

We can rewrite (0.2) as

$$X^{(n)}_{(k+1)/n} - X^{(n)}_{k/n} = X^{(n)}_{k/n} \big(B^{(n)}_{(k+1)/n} - B^{(n)}_{k/n} \big)\,, \tag{0.4}$$

which is closer to (0.1) but less simple than (0.2). The random signs are more relevant than the random walk. Similarly, in (0.1) the stochastic differentials $\mathrm{d}B_t$ are more relevant than the Brownian motion. The latter is rather an infinitely divisible reservoir of independent random variables. (One could object that B_t occurs in the solution $X_t = \exp\bigl(B_t - \frac12 t\bigr)$. However, this is a feature of (0.1); in general X_t involves B_s for all $s \in [0,t]$.)

It is tempting to by-pass B_t by treating $\mathrm{d}B_t$ as the scaling limit of τ_k. Of course, a Brownian sample path is not a differentiable function, but may be differentiated as a generalized function (Schwarz distribution). Accordingly, we may consider the (random) locally integrable function

$$W^{(n)}(t) = \sqrt{n}\,\tau_k \qquad \text{for } t \in \bigl(\tfrac{k}{n}, \tfrac{k+1}{n}\bigr)$$

as a (random) Schwarz distribution. Then $W^{(n)}$ converges in distribution (as $n \to \infty$) to the so-called white noise $\frac{\mathrm{d}}{\mathrm{d}t}B_t$. This is a classical wisdom: the scaling limit of random signs is the white noise. Similarly, the scaling limit of a two-dimensional array of random signs is a white noise over the plane $\mathbb{R}^2$. These are examples of processes with independent values. Conceptually, nothing is simpler than independent values; but technically, they cannot be treated as random functions.

A spectacular achievement of percolation theory (S. Smirnov, 2001) is existence (and conformal invariance) of a scaling limit of critical site percolation on the triangular lattice (see for instance [32] and references therein). The model is based on a two-dimensional array of random signs (colors of vertices). Does it mean that the scaling limit is driven by the white noise over the plane? No, it does not. The percolation model uses the random signs in a nonclassical way (see Sect. 11b).

Contrary to the classical wisdom, processes with independent values (defined properly) are much more diverse than white noises, Poisson point processes and their combinations, time derivatives of Lévy processes. The Lévy-Itô theorem does not lie; processes with independent increments are indeed exhausted by Brownian motions, Poisson processes and their combinations. The scope of the classical theory is limited by its treatment of independent values via independent increments belonging to $\mathbb{R}$ or another linear space (or commutative group).

Nowadays, independent increments are investigated also in noncommutative groups and semigroups, consisting of homeomorphisms or more general maps (say, $\mathbb{R} \to \mathbb{R}$), kernels (that is, maps from points to measures), bounded linear operators in a Hilbert space, etc. These are relevant to stochastic flows. Some flows, being smooth enough, are strong solutions of stochastic differential equations; these flows are classical. Other flows contain some singularities (turbulence, coalescence, stickiness, splitting etc.); these flows tend to be nonclassical. They still are scaling limits of discrete models driven by random signs, but these signs are used in a nonclassical way.

The intuitive idea of a process with independent values appeared to be deeper than its classical treatment. A general formalization of the idea is the heart of this survey. Sect. 1 is a preliminary presentation via simplest examples, it may be thought of as an extended introduction. The main definitions appear in Sects. 2 and 3. More interesting (and less simple) examples are introduced in Sect. 4. But only in Sect. 5 the distinction between classical and nonclassical is defined, and the examples (of Sects. 1, 4) are shown to be nonclassical.

In discrete time, independence corresponds to the product of probability spaces. In continuous time, a process with independent values corresponds to a continuous product of probability spaces. The corresponding Hilbert spaces L_2 of square integrable random variables form a continuous tensor product of Hilbert spaces. Such products are a notion well-known in analysis (the theory of operator algebras) and relevant to noncommutative probability (and quantum theory). A part of the (noncommutative) theory of such products got recently in close contact with the (commutative) probability theory and is surveyed here (Sections 3, 6(d–g), 9d, 10). The classical case is well-known as Fock spaces, or type I Arveson systems. The nonclassical case consists of Arveson systems of types II and III. Their existence was revealed (in different terms) in 1987 by R. Powers [25] (see also [26] and Chapter 13 of recent monograph [4] by W. Arveson) while the corresponding probabilistic theory was in latency; the idea was discussed repeatedly by A. Vershik and J. Feldman, but the only publication [13] was far from revealing existence of nonclassical systems. The first nonclassical continuous product of probability spaces was published in 1998 [41] by A. Vershik and the author. (I was introduced to the topic by A. Vershik in 1994.)

The classical part is well understood in both setups (commutative and noncommutative), see Sect. 6. Sometimes the classical part is trivial, which is called 'black noise' in the commutative setup and 'type III' in the noncommutative setup. Nonsmooth stochastic flows, homogeneous in space and time, can lead to black noises, see Sect. 7. The interaction between noises theory and turbulence theory enriches both sides.

Stochastic flows of unitary operators in a Hilbert space belong to commutative probability, but are closely connected with noncommutative probability. The connection is used in Sect. 8 for proving that such flows are classical.

The 'classical/nonclassical' dichotomy is the starting point of a more detailed classification, see Sect. 9.

Rich sources of examples for the noncommutative theory are found by the commutative theory, see Sect. 10.

Hopefully this survey contributes to the interaction between the commutative and the noncommutative.

1 Singularity concentrated in time (toy models)

1a Terminology: flows

Independent increments are more interesting for us than values of a process (X_t) with independent increments. Thus we denote

$$X_{s,t} = X_t - X_s \quad \text{for } s < t \,,$$

use two properties

$$X_{r,t} = X_{r,s} + X_{s,t} \quad \text{for } r < s < t \,, \tag{1a1}$$

$$X_{t_1,t_2}, X_{t_2,t_3}, \dots, X_{t_{n-1},t_n} \text{ are independent for } t_1 < t_2 < \dots < t_n \,, \tag{1a2}$$

and discard X_t. A two-parameter family $(X_{s,t})_{s<t}$ of random variables $X_{s,t} : \Omega \to \mathbb{R}$ satisfying (1a1), (1a2) will be called an $\mathbb{R}$-flow (or, more formally, a stochastic $(\mathbb{R},+)$-flow; here $(\mathbb{R},+)$ is the additive group of real numbers). Why not just 'a flow in $\mathbb{R}$? Since the latter is widely used for a family of random diffeomorphisms of $\mathbb{R}$ (or more general maps, kernels etc). On the other hand, $\mathbb{R}$ acts on itself by shifts ($y \mapsto x + y$), which justifies calling $(X_{s,t})_{s<t}$ a stochastic flow. Similarly we may consider, say, the multiplicative semigroup of complex numbers; a $(\mathbb{C},\cdot)$-flow satisfies $X_{s,t} : \Omega \to \mathbb{C}$ and $X_{r,t} = X_{r,s}X_{s,t}$ instead of (1a1). Often, indices s,t of $X_{s,t}$ run over $[0,\infty)$ each, but any other linearly ordered set may be specified, if needed. Full generality is postponed to Sect. 2.

1b Two examples

Tracing nonclassical behavior to bare bones we get very simple models shown here. They may be discrete or continuous, this is a matter of taste.

A stationary (not just 'with stationary increments') random walk (or Brownian motion) is impossible on such groups as $\mathbb{Z}$ or $\mathbb{R}$, but possible on compact groups such as the finite cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ or the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Discrete example

We choose $m \in \{2,3,\dots\}$ and take the group $\mathbb{Z}_m$. For every $n = 1,2,\dots$ we construct a $\mathbb{Z}_m$-flow $X^{(n)} = (X^{(n)}_{s,t})_{s<t;s,t\in T}$ over the time set $T = \{0,1,2,\dots\} \cup \{\infty\}$ (with the natural linear order) as follows:

$$\mathbb{P}\big(X^{(n)}_{s,s+1} = 0\big) = \frac{1}{2} = \mathbb{P}\big(X^{(n)}_{s,s+1} = 1\big) \quad \text{for } s = 0,\dots,n-1\,;$$
$$\mathbb{P}\big(X^{(n)}_{s,\infty} = 0\big) = 1 \quad \text{for } s = n, n+1, \dots$$

These conditions (together with (1a1), (1a2)) determine uniquely the joint distribution of all $X_{s,t}$ for $s,t \in T$, $s < t$. Namely, $X^{(n)}_{s,t}$ may be thought of as increments of a random walk (in $\mathbb{Z}_m$) stopped at the instant n. (Till now, $\mathbb{Z}$ could be used instead of $\mathbb{Z}_m$, but the next claim would be violated; indeed, $X^{(n)}_{0,\infty}$ would not be tight.) Random processes $X^{(n)}$ converge in distribution (for $n \to \infty$) to a random process X. (It means weak convergence of finite-dimensional distributions, or equivalently, probability measures on the compact space $(\mathbb{Z}_m)^{\{(s,t)\in T\times T : s<t\}}$, a product of countably many finite topological spaces.) The limiting process $X = (X_{s,t})_{s<t;s,t\in T}$ is again a $\mathbb{Z}_m$-flow.

Here is a remarkable feature of X (in contrast to $X^{(n)}$): the random variable $X_{0,\infty}$, distributed uniformly on $\mathbb{Z}_m$, is independent of the whole finite-time part of the process, $(X_{s,t})_{s<t<\infty}$. (The same holds for each $X_{t,\infty}$ separately, but surely not for $X_{0,\infty} - X_{1,\infty} = X_{0,1}$.) Therefore $X_{0,\infty}$ is not a function of the i.i.d. sequence $(X_{t,t+1})_{t<\infty}$. A paradox! You may guess that an additional random variable $X_{\infty-,\infty}$, independent of all $X_{t,t+1}$, squeezes somehow through a gap between finite numbers and infinity. However, such an explanation does not work. It cannot happen that $X_{s,\infty} = f_s(X_{s,s+1}, X_{s+1,s+2}, \dots; X_{\infty-,\infty})$ for all s. Here is a proof (sketch). Assume that it happens. The conditional distribution of $X_{0,\infty}$ given $X_{0,1},\dots,X_{s-1,s}$ and $X_{\infty-,\infty}$ is uniform on $\mathbb{Z}_m$, since

$$X_{0,\infty} = X_{0,s} + X_{s,t} + f_t(X_{t,t+1}, X_{t+1,t+2}, \dots; X_{\infty-,\infty})\,,$$

and the conditional distribution of $X_{s,t}$ is nearly uniform for large t. Thus, $X_{0,\infty}$ is independent of $X_{0,1}, X_{1,2}, \dots$ and $X_{\infty-,\infty}$; a contradiction.

We see that some flows cannot be locally parameterized by independent random variables. The group $\mathbb{Z}_m$ is essential; every $\mathbb{Z}$-flow (or $\mathbb{R}$-flow, or $\mathbb{R}^n$-flow) $(X_{s,t})_{s<t;s,t\in T}$ can be locally parameterized by $X_{s,s+1}$ ($s = 0,1,\dots$) and $X_{\infty-,\infty} = X_{0,\infty} - \lim_{k\to\infty}(X_{0,k} - c_k)$ for appropriate centering constants $c_1, c_2, \dots$ (see also Corollary 6a3). It is also essential that the time set $T = \{0,1,2,\dots\} \cup \{\infty\}$ contains a limit point (∞). Otherwise, say, for $T = \mathbb{Z}$, every flow $(X_{s,t})_{s<t}$ reduces to independent random variables $X_{s,s+1}$. Time

reversal does not matter; the same phenomenon manifests itself for $T = \{-k : k = 0, 1, \dots\} \cup \{-\infty\}$, as well as $T = \{2^{-k} : k = 1, 2, \dots\} \cup \{0\}$, or just $t = [0, \infty)$, the latter being used below.

Continuous example

We take the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the circle). For every $\varepsilon > 0$ we construct a $\mathbb{T}$-flow $Y^{(\varepsilon)} = (Y^{(\varepsilon)}_{s,t})_{s<t;s,t\in[0,\infty)}$ as follows:

$$Y^{(\varepsilon)}_{s,t} = B\Big(\ln\frac{t}{\varepsilon}\Big) - B\Big(\ln\frac{s}{\varepsilon}\Big) \pmod 1 \qquad \text{for } \varepsilon \le s < t < \infty\,,$$
$$Y^{(\varepsilon)}_{0,t} = 0 \quad \text{for } t \in [0, \varepsilon]\,;$$

here $\big(B(t)\big)_{t\in[0,\infty)}$ is the usual Brownian motion. These conditions determine uniquely the joint distribution of all $Y^{(\varepsilon)}_{s,t}$ for $s, t \in [0, \infty)$, $s < t$. Namely, $Y^{(\varepsilon)}_{s,t}$ may be thought of as increments of a process, Brownian (on the circle) in logarithmic time after the instant ε, but constant before ε. Random processes $Y^{(\varepsilon)}$ converge in distribution (as $\varepsilon \to 0$) to a random process Y (weak convergence of finite-dimensional distributions is meant), and the limiting process $Y = (Y_{s,t})_{s<t;s,t\in[0,\infty)}$ is again a $\mathbb{T}$-flow. The random variable $Y_{0,1}$, distributed uniformly on $\mathbb{T}$, is independent of all $Y_{s,t}$ for $0 < s < t$. We cannot locally parametrize Y by increments of a Brownian motion (and possibly an additional random variable $Y_{0,0+}$ independent of the Brownian motion).

The one-parameter random process $(Y_{0,\mathrm{e}^t})_{t\in\mathbb{R}}$ is a stationary Brownian motion in $\mathbb{T}$. The complex-valued random process $(Z_t)_{t\in[0,\infty)}$,

$$Z_t = \sqrt{t}\,\mathrm{e}^{2\pi\mathrm{i}Y_{0,t}} \quad \text{for } t \in [0, \infty)$$

is a continuous martingale, and satisfies the stochastic differential equation

$$\mathrm{d}Z_t = \frac{\mathrm{i}}{\sqrt{t}} Z_t \,\mathrm{d}B_t\,, \quad Z_0 = 0\,,$$

where $(B_t)_{t\in[0,\infty)}$ is the usual Brownian motion. However, the random variable Z_1 is independent of the whole Brownian motion (B_t). The weak solution of the stochastic differential equation is not a strong solution. See also [48], [12], and [40, Sect. 1a].

1c Stability and sensitivity

Stability and sensitivity of Boolean functions of many Boolean variables were introduced in 1999 by Benjamini, Kalai and Schramm [8] and applied to percolation, random graphs etc. They introduce errors (perturbation) into a given Boolean array by flipping each Boolean variable with a small probability (independently of others), and observe the effect of these errors by comparing the new (perturbed) value of a given Boolean function with its original (unperturbed) value. They prove that percolation is sensitive! Surprisingly, their 'stability' is basically the same as our 'classicality'. See also [29].

The $\mathbb{Z}_m$-flow X of Sect. 1b (denote it here by X^{1b}) contains i.i.d. random variables $X_{s,s+1}$ that are Boolean in the sense that each one takes on two values 0 and 1, with probabilities $1/2, 1/2$. However, $X_{0,\infty}$ is not a function of these (Boolean) variables. Here is a proper formalization of the idea. A pair of two correlated G-flows (one 'unperturbed', the other 'perturbed') is a $(G \times G)$-flow; here $G \times G$ is the direct product, that is, the set of all pairs (g_1, g_2) for $g_1, g_2 \in G$ with the group operation $(g_1, g_2)(g_3, g_4) = (g_1 g_3, g_2 g_4)$. (For $G = \mathbb{Z}_m$ we prefer additive notation: $(g_1, g_2) + (g_3, g_4) = (g_1 + g_3, g_2 + g_4) \in \mathbb{Z}_m \oplus \mathbb{Z}_m$.) Let $X = (X_{s,t})_{s<t;s,t\in T}$ be a $(G \times G)$-flow; we have $X_{s,t} = (X'_{s,t}, X''_{s,t})$, that is, $X = (X', X'')$, where X', X'' are G-flows on the same probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}', \mathcal{F}''$ be sub-σ-fields of $\mathcal{F}$ generated by X', X'' respectively. We introduce the maximal correlation

$$\rho_{\max}(X) = \rho_{\max}(X', X'') = \rho_{\max}(\mathcal{F}', \mathcal{F}'') = \sup |\mathbb{E}\,(fg)|\,,$$

where the supremum is taken over all $f \in L_2(\Omega, \mathcal{F}', P)$, $g \in L_2(\Omega, \mathcal{F}'', P)$ such that $\mathbb{E}\,f = 0$, $\mathbb{E}\,g = 0$, $\operatorname{Var} f \le 1$, $\operatorname{Var} g \le 1$. The idea of a (non-degenerate) perturbation of a flow may be formalized by the condition $\rho_{\max}(X) < 1$.

1c1 Proposition. Let $X = (X', X'')$ be a $(\mathbb{Z}_m \oplus \mathbb{Z}_m)$-flow such that X', X'' both are distributed like X^{1b} (the discrete $\mathbb{Z}_m$-flow of Sect. 1b). If $\rho_{\max}(X) < 1$ then random variables $X'_{0,\infty}$ and $X''_{0,\infty}$ are independent.

A small perturbation has a dramatic effect on the random variable $X^{1b}_{0,\infty}$; this is instability (and moreover, sensitivity). All flows in Proposition 1c1 use the time set $T = \{0, 1, 2, \dots\} \cup \{\infty\}$. Nothing like that happens on $T = \{0, 1, 2, \dots\}$ or $T = \mathbb{Z}$. Also, the group $\mathbb{Z}_m$ is essential; nothing like that happens for $\mathbb{Z}$-flows (or $\mathbb{R}$-flows, or $\mathbb{R}^n$-flows; see also Corollary 6a3). *Sketch of the proof of Proposition* 1c1 *for the special case* $m = 2$*:*

$$|\mathbb{E}\,(-1)^{X'_{0,\infty}}(-1)^{X''_{0,\infty}}| =$$

$$= |\mathbb{E}\,(-1)^{X'_{t,\infty}}(-1)^{X''_{t,\infty}}| \cdot \prod_{s=0}^{t-1} |\mathbb{E}\,(-1)^{X'_{s,s+1}}(-1)^{X''_{s,s+1}}| \le 1 \cdot (\rho_{\max}(X))^t \xrightarrow[t\to\infty]{} 0\,.$$

The same can be said about the other (continuous) example Y^{1b} of Sect. 1b. The random variable $Y^{1b}_{0,1}$ is sensitive.

See also Sect. 5, especially 5d.

1d Hilbert spaces, quantum bits (spins)

Return to $X = X^{1\mathrm{b}}$ (the discrete $\mathbb{Z}_m$-flow of Sect. 1b) and consider the Hilbert space H of all square integrable complex-valued measurable functions of random variables $X_{s,t}$, with the norm

$$\|f(X_{0,1}, X_{1,2}, \dots ; X_{0,\infty})\|^2 = \mathbb{E}\,|f(X_{0,1}, X_{1,2}, \dots ; X_{0,\infty})|^2$$

(other $X_{s,t}$ are redundant). Equivalently, $H = L_2(\Omega, \mathcal{F}, P)$ provided that $\mathcal{F}$ is the σ-field generated by X.

We may split X at the instant 1 in two independent components: the past, — just a single random variable $X_{0,1}$; and the future, — all $X_{s,t}$ for $1 \le s < t$, $s, t \in T$. Accordingly, H splits into the tensor product, $H = H_{0,1} \otimes H_{1,\infty}$. The Hilbert space $H_{0,1}$ is two-dimensional (since $X_{0,1}$ takes on two values, 0 and 1), spanned by two orthonormal vectors $2^{-1/2}(1 - X_{0,1})$ and $2^{-1/2} X_{0,1}$. Using this basis, we may treat the famous Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as operators on $H_{0,1}$, and also on H (identifying σ_k with $\sigma_k \otimes \mathbf{1}$). Thus, the 2×2 matrix algebra $\mathrm{M}_2(\mathbb{C})$ acts on H.

Similarly,

$$H = H_{0,1} \otimes \dots \otimes H_{t-1,t} \otimes H_{t,\infty}$$

for any $t = 1, 2, \dots$; each $H_{s-1,s}$ is two-dimensional, and $\{2^{-1/2}(1 - X_{s-1,s}), 2^{-1/2} X_{s-1,s}\}$ is its orthonormal basis. We get commuting copies of $\mathrm{M}_2(\mathbb{C})$;

$$\begin{gathered} \sigma_k^{t-1,t} : H \to H \quad \text{for } k = 1,2,3 \text{ and } t = 1,2,\dots \,; \\ [\sigma_k^{s-1,s}, \sigma_k^{t-1,t}] = 0 \quad \text{for } s \ne t \,. \end{gathered}$$

The random variable $\exp\bigl(\frac{2\pi\mathrm{i}}{m} X_{0,\infty}\bigr)$ is a factorizing vector,

$$\exp\Bigl(\frac{2\pi\mathrm{i}}{m} X_{0,\infty}\Bigr) = \exp\Bigl(\frac{2\pi\mathrm{i}}{m} X_{0,1}\Bigr) \otimes \dots \otimes \exp\Bigl(\frac{2\pi\mathrm{i}}{m} X_{t-1,t}\Bigr) \otimes \exp\Bigl(\frac{2\pi\mathrm{i}}{m} X_{t,\infty}\Bigr) \,.$$

The first factor is not a basis vector but a linear combination,

$$\exp\Bigl(\frac{2\pi\mathrm{i}}{m} X_{0,1}\Bigr) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \exp \frac{2\pi\mathrm{i}}{m} \end{pmatrix} ;$$

each factor $\exp\bigl(\frac{2\pi\mathrm{i}}{m} X_{t-1,t}\bigr)$ is a copy of this vector. The quantum state on a local algebra,

$$M \mapsto \langle M\psi, \psi \rangle \,, \quad M \in M_{2^t}(\mathbb{C}) = \underbrace{M_2(\mathbb{C}) \otimes \dots \otimes M_2(\mathbb{C})}_{t}$$

corresponding to the vector $\psi = \exp\bigl(\frac{2\pi \mathrm{i}}{m} X_{0,\infty}\bigr) \in H$ is equal to the quantum state corresponding to the vector

$$\begin{pmatrix} 2^{-1/2} \\ 2^{-1/2} \exp \frac{2\pi \mathrm{i}}{m} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 2^{-1/2} \\ 2^{-1/2} \exp \frac{2\pi \mathrm{i}}{m} \end{pmatrix} = \begin{pmatrix} 2^{-1/2} \\ 2^{-1/2} \exp \frac{2\pi \mathrm{i}}{m} \end{pmatrix}^{\otimes t} \in \mathbb{C}^{2^t} .$$

The vector $\exp\bigl(\frac{2\pi \mathrm{i}}{m} X_{0,\infty}\bigr) \in H$ may be interpreted as a factorizing state $\bigl({2^{-1/2} \atop 2^{-1/2} \exp \frac{2\pi \mathrm{i}}{m}} \bigr)^{\otimes\infty}$ of an infinite collection of spins. Similarly, for each $k = 0, 1, \dots, m-1$ the vector $\exp\bigl(\frac{2\pi \mathrm{i} k}{m} X_{0,\infty}\bigr) \in H$ may be interpreted as $\bigl({2^{-1/2} \atop 2^{-1/2} \exp \frac{2\pi \mathrm{i} k}{m}} \bigr)^{\otimes\infty}$.

Vectors of H of the form

$$f(X_{0,1}, X_{1,2}, \dots) \exp\Bigl(\frac{2\pi \mathrm{i} k}{m} X_{0,\infty} \Bigr)$$

are a subspace $H_k \subset H$ invariant under all local operators; and $H = H_0 \oplus \dots \oplus H_{m-1}$. Each H_k is irreducible in the sense that it contains no nontrivial subspace invariant under all local operators.

The operator R defined by

$$Rf(X_{0,1}, X_{1,2}, \dots; X_{0,\infty}) = f(X_{0,1}, X_{1,2}, \dots; X_{0,\infty} + 1)$$

(where $X_{0,\infty} + 1$ is treated $\bmod m$) commutes with all local operators. Every operator commuting with all local operators is one of R^k, $k = 0, 1, \dots, m-1$. The subspaces H_k are eigenspaces of R, their eigenvalues being $\exp\bigl(\frac{2\pi \mathrm{i} k}{m}\bigr)$. In a more physical language, the group $\{R^k : k = 0, 1, \dots, m-1\}$ is the gauge group, and H_k are superselection sectors. Each sector has its own asymptotic behavior of remote spins.

See also Sect. 3, [36, Appendix], [19, Sect. 8.4].

2 From convolution semigroups to continuous products of probability spaces

2a Terminology: probability spaces, morphisms etc.

Throughout, either by assumption or by construction, all probability spaces are standard. All claims and constructions are invariant under mod 0 *isomorphisms.*

Recall that a *standard probability space* (known also as a Lebesgue-Rokhlin space) is a probability space isomorphic (mod 0) to an interval with the

Lebesgue measure, a finite or countable collection of atoms, or a combination of both (see [17, (17.41)]). Nonseparable L_2 spaces of random variables are thus disallowed!

A σ-field $\mathcal{F}$ is sometimes shown in the notation $(\Omega, \mathcal{F}, P)$, sometimes suppressed in the shorter notation (Ω, P).

Every function on any probability space is treated mod 0. That is, I write $f : \Omega \to \mathbb{R}$ for convenience, but I mean that f is an equivalence class. The same for maps $\Omega_1 \to \Omega_2$ etc. A *morphism* $\Omega_1 \to \Omega_2$ is a measure preserving (not just non-singular) measurable map (P_1, P_2 are suppressed in the notation). An *isomorphism* (known also as 'mod 0 isomorphism') is an invertible morphism whose inverse is also a morphism. An *automorphism* is an isomorphism to itself. Every sub-σ-field is assumed to contain all negligible sets. Every morphism $\alpha : \Omega \to \Omega'$ generates a sub-σ-field $\mathcal{E} \subset \mathcal{F}$, and every sub-$\sigma$-field $\mathcal{E} \subset \mathcal{F}$ is generated by a morphism $\alpha : \Omega \to \Omega'$, determined by $\mathcal{E}$ uniquely up to isomorphism ($\Omega' \leftrightarrow \Omega''$, making the diagram commutative...); it is the *quotient space* $(\Omega', P') = (\Omega, P)/\mathcal{E}$.

A *standard measurable space* (or 'standard Borel space') is a set E equipped with a σ-field $\mathcal{B}$ such that the measurable space $(E, \mathcal{B})$ is isomorphic either to $\mathbb{R}$ (with the Borel σ-field) or to its finite or countable subset. See [17, 12.B and 15.B].

Equivalence classes of all measurable functions $\Omega \to \mathbb{R}$ are a topological linear space $L_0(\Omega) = L_0(\Omega, \mathcal{F}, P)$; its metrizable topology corresponds to convergence in probability. Any Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ leads to a (nonlinear) map $L_0(\Omega) \to L_0(\Omega)$, $X \mapsto \varphi \circ X$, discontinuous (in general), but Borel measurable. (Hint: if $\varphi_n(x) \to \varphi(x)$ for all $x \in \mathbb{R}$ then $\varphi_n \circ X \to \varphi \circ X$ for all $X \in L_0(\Omega)$.) Given a standard measurable space E, the set $L_0(\Omega \to E)$ of equivalence classes of all measurable maps $\Omega \to E$ carries a natural Borel σ-field and is a standard measurable space (neither linear nor topological, in general).

A stochastic flow (and any random process) is generally treated as a family of equivalence classes (rather than functions). The distinction is essential when dealing with uncountable families of random variables. The phrase (say)

$$f_t = g_t \quad \text{a.s. for all } t$$

is interpreted as

$$\inf_t \mathbb{P}\{\omega : f_t(\omega) = g_t(\omega)\} = 1$$

rather than $\mathbb{P}\bigl(\cap_t \{\omega : f_t(\omega) = g_t(\omega)\}\bigr) = 1$. In spite of that, when dealing with (say) a Brownian motion $(B_t)_t$ and writing (say) $\max_{t\in[0,1]} B_t$, we rely on path continuity. Here the Brownian motion is treated as a random continuous function rather than a family of random variables.

I stop writing 'standard' (probability space), 'mod 0' and 'measure preserving', but I still mean it!

2b From convolution systems to flow systems

A weakly continuous (one-parameter) convolution semigroup in $\mathbb{R}$ is a family $(\mu_t)_{t\in(0,\infty)}$ of probability measures μ_t on $\mathbb{R}$ such that $\mu_s * \mu_t = \mu_{s+t}$ for all $s,t \in (0,\infty)$, and $\lim_{t\to 0} \mu_t\big((-\varepsilon,\varepsilon)\big) = 1$ for all $\varepsilon > 0$. Two basic cases are normal distributions $\mathrm{N}(0,t)$ and Poisson distributions $\mathrm{P}(t)$. They correspond to the Brownian motion and the Poisson process, respectively. Every convolution semigroup decomposes into a combination of these two basic cases, and corresponds to a process with independent increments; the process decomposes into Brownian and Poisson processes. That is the classical theory (Lévy-Khinchin-Itô).

The convolution relation $\mu_s * \mu_t = \mu_{s+t}$ means that the map $\mathbb{R}^2 \to \mathbb{R}$, $(s,t) \mapsto s+t$ sends the product measure $\mu_s \times \mu_t$ into μ_{s+t}. More generally, each μ_t may sit on its own space G_t, in which case some measure preserving maps $G_s \times G_t \to G_{s+t}$ should be given (instead of the group operation). Another generalization is, abandoning stationarity (that is, homogeneity in time).

2b1 Definition. A *convolution system* consists of probability spaces $(G_{s,t}, \mu_{s,t})$ given for all $s,t \in \mathbb{R}$, $s<t$, and morphisms $G_{r,s} \times G_{s,t} \to G_{r,t}$ given for all $r,s,t \in \mathbb{R}$, $r<s<t$, satisfying the associativity condition:

$$(xy)z = x(yz) \quad \text{for almost all } x \in G_{r,s},\ y \in G_{s,t},\ z \in G_{t,u}$$

whenever $r,s,t,u \in \mathbb{R}$, $r<s<t<u$.

Here and henceforth the given map $G_{r,s} \times G_{s,t} \to G_{r,t}$ is denoted simply $(x,y) \mapsto xy$. Any linearly ordered set (not just $\mathbb{R}$) may be used as the time set.

Every convolution semigroup (μ_t) in $\mathbb{R}$ leads to a convolution system; namely, $(G_{s,t}, \mu_{s,t}) = (\mathbb{R}, \mu_{t-s})$, and the map $G_{r,s} \times G_{s,t} \to G_{r,t}$ is $(x,y) \mapsto x+y$. Another example: $(G_{s,t}, \mu_{s,t}) = (\mathbb{Z}_m, \mu)$ for all s,t; here $m \in \{2,3,\dots\}$ is a parameter, and μ is the uniform distribution on the finite cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$; the map $G_{r,s} \times G_{s,t} \to G_{r,t}$ is $(x,y) \mapsto x+y \pmod m$. The latter example is much worse than the former; indeed, the former is separable (see Definition 2b4 below), and the latter is not.

Here is a generalization of the classical transition from convolution semigroups to independent increments.

2b2 Definition. Let $(G_{s,t},\mu_{s,t})_{s<t;s,t\in T}$ be a convolution system over a linearly ordered set T. A *flow system* (corresponding to the given convolution system) consists of a probability space $(\Omega,\mathcal{F},P)$ and morphisms $X_{s,t}:\Omega\to G_{s,t}$ (for $s<t$; $s,t\in T$) such that $\mathcal{F}$ is generated by all $X_{s,t}$ ('non-redundancy'), and

(a) $\quad X_{t_1,t_2},X_{t_2,t_3},\dots,X_{t_{n-1},t_n}$ are independent for $t_1<t_2<\dots<t_n$;

(b) $$X_{r,t}=X_{r,s}X_{s,t}\quad \text{(a.s.) for } r<s<t\,.$$

The non-redundancy can be enforced by taking the quotient space $(\Omega,P)/\mathcal{F}_{-\infty,\infty}$, where $\mathcal{F}_{-\infty,\infty}$ is the σ-field generated by all $X_{s,t}$.

2b3 Proposition. For every convolution system over a finite or countable T, the corresponding flow system exists and is unique up to isomorphism.

By an isomorphism between flow systems $(X_{s,t})_{s<t}$, $X_{s,t}:\Omega\to G_{s,t}$ and $(X'_{s,t})_{s<t}$, $X'_{s,t}:\Omega'\to G_{s,t}$ we mean an isomorphism $\alpha:\Omega\to\Omega'$ such that $X_{s,t}=X'_{s,t}\circ\alpha$ for $s<t$.

Proof (sketch). Existence: if T is finite, $T=\{t_1,\dots,t_n\}$, $t_1<\dots<t_n$, we just take $\Omega=G_{t_1,t_2}\times\dots\times G_{t_{n-1},t_n}$ with the product measure. If T is countable, we have a consistent family of finite-dimensional distributions on the product $\prod_{s<t;s,t\in T}G_{s,t}$ of countably many probability spaces.

Uniqueness follows from the fact that the joint distribution of all $X_{s,t}$ is uniquely determined by the measures $\mu_{s,t}$. $\square$

Given a convolution system $(G_{s,t},\mu_{s,t})_{s<t;s,t\in T}$ and a subset $T_0\subset T$, the restriction $(G_{s,t},\mu_{s,t})_{s<t;s,t\in T_0}$ is also a convolution system. If T is countable and $T_0\subset T$, we get two flow systems, $(X_{s,t})_{s<t;s,t\in T}$ on (Ω,P) and $(X^0_{s,t})_{s<t;s,t\in T_0}$ on (Ω_0,P_0) related via a morphism $\alpha:\Omega\to\Omega_0$ such that $X_{s,t}=X^0_{s,t}\circ\alpha$ (a.s.) for $s,t\in T_0$, $s<t$. It may happen that α is an isomorphism, in which case we say that T_0 is *total* in T (with respect to the given convolution system).

2b4 Definition. A convolution system $(G_{s,t},\mu_{s,t})_{s<t;s,t\in\mathbb{R}}$ is *separable,* if there exists a countable set $T_0\subset\mathbb{R}$ such that for every countable $T\subset\mathbb{R}$ satisfying $T_0\subset T$, the subset T_0 is total in T with respect to the restriction $(G_{s,t},\mu_{s,t})_{s<t;s,t\in T}$ of the given convolution system.

When checking separability, one may restrict himself to the case when the difference $T\setminus T_0$ is a single point.

Here is a counterpart of Proposition 2b3 for the uncountable time set $\mathbb{R}$.

2b5 Proposition. The following two conditions on a convolution system $(G_{s,t}, \mu_{s,t})_{s<t;s,t\in\mathbb{R}}$ are equivalent.

(a) There exists a flow system corresponding to the given convolution system.

(b) The given convolution system is separable.

Proof (sketch). (a) $\Longrightarrow$ (b): a sub-σ-field generated by an uncountable set is also generated by some countable subset.

(b) $\Longrightarrow$ (a): we take the flow system for T_0; separability implies that each $X_{s,t}$ (for $s, t \in \mathbb{R}$) is equal (a.s.) to a function of $(X_{s,t})_{s<t;s,t\in T_0}$. $\square$

2c From flow systems to continuous products, and back

2c1 Definition. A *continuous product of probability spaces* consists of a probability space $(\Omega, \mathcal{F}, P)$ and sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ (given for all $s, t \in \mathbb{R}$, $s < t$) such that $\mathcal{F}$ is generated by the union of all $\mathcal{F}_{s,t}$ ('non-redundancy'), and

(2c2) $$\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t} \quad \text{whenever } r < s < t\,.$$

The latter means that $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are independent and generate $\mathcal{F}_{r,t}$. See also Def. 2c6 below. The non-redundancy can be enforced by taking the quotient space $(\Omega, P)/\mathcal{F}_{-\infty,\infty}$. Any linearly ordered set (not just $\mathbb{R}$) may be used as the time set. It is convenient to enlarge the time set from $\mathbb{R}$ to $[-\infty, \infty]$ defining $\mathcal{F}_{-\infty,t}$ as the σ-field generated by the union of all $\mathcal{F}_{s,t}$ for $s \in (-\infty, t)$; the same for $\mathcal{F}_{s,\infty}$ and $\mathcal{F}_{-\infty,\infty}$.

2c3 Proposition. Let $(X_{s,t})_{s<t}$ be a flow system, and $\mathcal{F}_{s,t}$ be defined (for $s < t$) as the sub-σ-field generated by $\{X_{u,v} : s \le u < v \le t\}$. Then sub-$\sigma$-fields $\mathcal{F}_{s,t}$ form a continuous product of probability spaces.

Proof (sketch). $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ generate $\mathcal{F}_{r,t}$ by 2b2(b) and are independent by (2b2)(a) (and (b)). $\square$

Having a continuous product of probability spaces $(\mathcal{F}_{s,t})_{s<t}$ we may introduce quotient spaces

(2c4) $$(\Omega_{s,t}, P_{s,t}) = (\Omega, P)/\mathcal{F}_{s,t}\,.$$

The relation (2c2) becomes

(2c5) $$(\Omega_{r,s}, P_{r,s}) \times (\Omega_{s,t}, P_{s,t}) = (\Omega_{r,t}, P_{r,t})\,;$$

the equality is treated here via a canonical isomorphism. It is not unusual; for example, the evident equality $(A\times B)\times C = A\times(B\times C)$ for Cartesian products of (abstract) sets is also treated not literally but via a canonical bijection $((a,b),c)\mapsto(a,(b,c))$ between the two sets. The canonical isomorphisms implicit in (2c5) satisfy associativity (stipulated by Definition 2b1); indeed, for $r<s<t<u$ we have $(\Omega_{r,s},P_{r,s})\times(\Omega_{s,t},P_{s,t})\times(\Omega_{t,u},P_{t,u}) = (\Omega_{r,u},P_{r,u})$. Thus, $(\Omega_{s,t},P_{s,t})$ form a convolution system (as defined by Definition 2b1) satisfying an additional condition: the morphisms $G_{r,s}\times G_{s,t}\to G_{r,t}$ become isomorphisms. This is another approach to continuous products of probability spaces.

2c6 Definition. A *continuous product of probability spaces* consists of probability spaces $(\Omega_{s,t},P_{s,t})$ (given for all $s,t\in\mathbb{R}$, $s<t$), and isomorphisms $\Omega_{r,s}\times\Omega_{s,t}\to\Omega_{r,t}$ given for all $r,s,t\in\mathbb{R}$, $r<s<t$, satisfying the associativity condition:

$$(\omega_1\omega_2)\omega_3=\omega_1(\omega_2\omega_3)\quad\text{for almost all }\omega_1\in\Omega_{r,s},\ \omega_2\in\Omega_{s,t},\ \omega_3\in\Omega_{t,u}$$

whenever $r,s,t,u\in\mathbb{R}$, $r<s<t<u$.

(As before, the given map $\Omega_{r,s}\times\Omega_{s,t}\to\Omega_{r,t}$ is denoted simply $(\omega_1,\omega_2)\mapsto\omega_1\omega_2$.) Having $(\mathcal{F}_{s,t})_{s<t}$ as in Definition 2c1 we get the corresponding $(\Omega_{s,t},P_{s,t})$ as in Definition 2c6 by means of (2c4). And conversely, each $(\Omega_{s,t},P_{s,t})$ as in Definition 2c6 leads to the corresponding $(\mathcal{F}_{s,t})_{s<t}$ of Definition 2c1. Namely, we may take $(\Omega,P)=\prod_{k\in\mathbb{Z}}(\Omega_{k,k+1},P_{k,k+1})$, define $X_{k,k+1}:\Omega\to\Omega_{k,k+1}$ as coordinate projections, use the relation $\Omega_{k,k+1}=\Omega_{k,k+\theta}\times\Omega_{k+\theta,k+1}$ for constructing $X_{k,k+\theta}:\Omega\to\Omega_{k,k+\theta}$ and so forth. Alternatively, we may treat $(\Omega_{s,t},P_{s,t})_{s<t}$ as a (special) convolution system and use relations discussed below.

A separable convolution system leads to a flow system by 2b5; a flow system leads to a continuous product of probability spaces by 2c3; and a continuous product of probability spaces is a special case of a separable convolution system.

$$\begin{tikzcd} \text{separable convolution systems} \arrow[r] & \text{flow systems} \arrow[dl] \\ \text{continuous products of probability spaces} \arrow[u, hook] & \end{tikzcd} \tag{2c7}$$

For example, a weakly continuous convolution semigroup $(\mu_t)_{t\in(0,\infty)}$ in $\mathbb{R}$ is a separable convolution system. (Any dense countable subset of $\mathbb{R}$ may be used as T_0 in Def. 2b4.) The corresponding flow system consists of the increments of the Lévy process corresponding to $(\mu_t)_t$. It leads to a continuous product of probability spaces $(\Omega_{s,t}, P_{s,t})$. Namely, $(\Omega_{0,t}, P_{0,t})$ may be treated as the space of sample paths of the Lévy process on $[0,t]$; $(\Omega_{s,t}, P_{s,t})$ is a copy of $(\Omega_{0,t-s}, P_{0,t-s})$; and the composition $\Omega_{r,s} \times \Omega_{s,t} \ni (\omega_1, \omega_2) \mapsto \omega_3 \in \Omega_{r,t}$ is

$$\omega_3(u) = \begin{cases} \omega_1(u) & \text{for } u \in [r,s], \\ \omega_1(s) + \omega_2(u-s) & \text{for } u \in [s,t]. \end{cases}$$

Note that $(\Omega_{s,t}, P_{s,t})$ is much larger than $(G_{s,t}, \mu_{s,t})$. We may treat $(\Omega_{s,t}, P_{s,t})_{s<t}$ as another convolution system; the two convolution systems, $(\mathbb{R}, \mu_{t-s})_{s<t}$ and $(\Omega_{s,t}, P_{s,t})$, lead to the same (up to isomorphism) continuous product of probability spaces. The same holds in general:

$$\begin{tikzcd}
\text{separable convolution system} \arrow[r, mapsto] & \text{flow system} \arrow[d, mapsto] \\
\textit{another}\text{ separable convolution system} & \text{continuous product of probability spaces} \arrow[l, mapsto]
\end{tikzcd}$$

but

$$\begin{tikzcd}
\text{continuous product of probability spaces} \arrow[r, mapsto] \arrow[d, equal] & \text{separable convolution system} \arrow[d, mapsto] \\
\textit{the same}\text{ continuous product of probability spaces} & \text{flow system} \arrow[l, mapsto]
\end{tikzcd}$$

2d Stationary case: noise

Returning to stationarity (that is, homogeneity in time), abandoned in Sects. 2b, 2c, we add time shifts to Def. 2c1, after a general discussion of one-parameter groups of automorphisms.

DIGRESSION: MEASURABLE ACTION

In the spirit of our conventions (Sect. 2a), an action of $\mathbb{R}$ on a probability space $\Omega = (\Omega, \mathcal{F}, P)$ is treated as a homomorphism of $(\mathbb{R}, +)$ to the group of automorphisms $\mathrm{Aut}(\Omega)$, each automorphism being an equivalence class rather than a map $\Omega \to \Omega$. Thus, an action is not quite a map $\mathbb{R} \times \Omega \to \Omega$. The group $\mathrm{Aut}(\Omega)$ is topological (in fact, Polish) [17, 17.46], and a homomorphism $\mathbb{R} \to \mathrm{Aut}(\Omega)$ is Borel measurable if and only if it is continuous (which is a special case of a well-known general theorem [17, (9.10)]). Such a homomorphism will be called a *measurable action* of $\mathbb{R}$ on Ω. Every such action $T : \mathbb{R} \to \mathrm{Aut}(\Omega)$

corresponds to some (non-unique) measurable map $\mathbb{R} \times \Omega \to \Omega$. Moreover, the map can be chosen to satisfy *everywhere* the relation $T_r(T_s(\omega)) = T_{r+s}(\omega)$ (Mackey, Varadarajan and Ramsy). More detailed discussion can be found in [15, Introduction].

END OF DIGRESSION

2d1 Definition. A *noise,* or a *homogeneous continuous product of probability spaces,* consists of a probability space $(\Omega, \mathcal{F}, P)$, sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ given for all $s, t \in \mathbb{R}$, $s < t$, and a measurable action $(T_h)_h$ of $\mathbb{R}$ on Ω, having the following properties:

(a) $$\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t} \quad \text{whenever } r < s < t\,,$$

(b) $$T_h \text{ sends } \mathcal{F}_{s,t} \text{ to } \mathcal{F}_{s+h,t+h} \quad \text{whenever } s < t \text{ and } h \in \mathbb{R}\,,$$

(c) $$\mathcal{F} \text{ is generated by the union of all } \mathcal{F}_{s,t}\,.$$

The time set $\mathbb{R}$ may be enlarged to $[-\infty, \infty]$, as noted after Def. 2c1. Of course, the index h of T_h runs over $\mathbb{R}$ only, and $-\infty + h = -\infty$, $\infty + h = \infty$.

Weakly continuous convolution semigroups in $\mathbb{R}$ (unlike convolution systems in general) lead to noises.

Measurability of the action does not follow from other conditions; a counterexample is similar to the 'pathologic example' of Sect. 3c.

A probabilist might feel that noises are too abstract; σ-fields do not catch distributions. (Similarly a geometer might complain that topological invariants do not catch volumes.) However, they do! The delusion is suggested by the discrete-time counterpart. Indeed, the product of countably many copies of a probability space does not distinguish any specific random variable (or distribution). Continuous time is quite different. Consider for example the white noise $(\mathcal{F}_{s,t})$, (T_h), corresponding to the $\mathbb{R}$-flow $X_{s,t} = B_t - B_s$ of Brownian increments. At first sight, $X_{s,t}$ cannot be reconstructed from $(\mathcal{F}_{s,t})$ and (T_h), but in fact they can! The conditions

$$\begin{gathered}
X_{s,t} \text{ is } \mathcal{F}_{s,t}\text{-measurable}\,, \\
X_{r,s} + X_{s,t} = X_{r,t}\,, \\
X_{s,t} \circ T_h = X_{s+h,t+h}\,, \\
\mathbb{E}\, X_{s,t} = 0\,, \quad \mathbb{E}\, X_{s,t}^2 = t - s
\end{gathered}$$

determine them uniquely up to a sign; $X_{s,t} = \pm(B_t - B_s)$. For the Poisson noise the situation is similar. However, for a Lévy process with different jump sizes, only their rates are encoded in $(\mathcal{F}_{s,t})$, (T_h); the sizes are lost.

2d2 Proposition. Every noise satisfies the 'upward continuity' condition

$$\mathcal{F}_{s,t} \text{ is generated by } \bigcup_{\varepsilon>0} \mathcal{F}_{s+\varepsilon,t-\varepsilon} \quad \text{for all } s,t \in \mathbb{R},\ s<t\,. \tag{2d3}$$

Using the enlarged time set $[-\infty,\infty]$ we interprete $-\infty+\varepsilon$ as $-1/\varepsilon$ and $\infty-\varepsilon$ as $1/\varepsilon$.

Proof (sketch). In the Hilbert space $H = L_2(\Omega,\mathcal{F},P)$ we consider projections $Q_{s,t} : f \mapsto \mathbb{E}\big(f \,\big|\, \mathcal{F}_{s,t}\big)$. They commute, and $Q_{s,t} = Q_{-\infty,t}Q_{s,\infty}$. The *monotone* operator-valued function $t \mapsto Q_{-\infty,t}$ must be continuous (in the strong operator topology) at every $t \in \mathbb{R}$ except for an at most countable set, since H is separable. By shift invariance, continuity at a single t implies continuity at all t. Thus, $\|Q_{-\infty,t}f - Q_{-\infty,t-\varepsilon}f\| \to 0$ when $\varepsilon \to 0$ for every $f \in H$. Similarly, $\|Q_{s,\infty}f - Q_{s+\varepsilon,\infty}f\| \to 0$. For $f \in Q_{s,t}H$ we have $\|f - Q_{s+\varepsilon,t-\varepsilon}f\| \le \|f - Q_{-\infty,t-\varepsilon}f\| + \|Q_{-\infty,t-\varepsilon}(f - Q_{s+\varepsilon,\infty}f)\| \to 0$. $\square$

2d4 Corollary. Every noise satisfies the 'downward continuity' condition

$$\mathcal{F}_{s,t} = \bigcap_{\varepsilon>0} \mathcal{F}_{s-\varepsilon,t+\varepsilon} \quad \text{for all } s,t \in \mathbb{R},\ s \le t\,; \tag{2d5}$$

here $\mathcal{F}_{t,t}$ is the trivial σ-field.

Using the enlarged time set $[-\infty,\infty]$ we interprete $-\infty-\varepsilon$ as $-\infty$ and $\infty+\varepsilon$ as ∞.

Proof (sketch). The σ-field $\mathcal{F}_{s-,t+} = \cap_{\varepsilon>0}\mathcal{F}_{s-\varepsilon,t+\varepsilon}$ is independent of $\mathcal{F}_{-\infty,s-\varepsilon} \vee \mathcal{F}_{t+\varepsilon,\infty}$ for every ε, therefore (using the proposition), also of $\mathcal{F}_{-\infty,s} \vee \mathcal{F}_{t,\infty}$. We have $\mathcal{F}_{s,t} \subset \mathcal{F}_{s-,t+}$ and $\mathcal{F}_{-\infty,s} \vee \mathcal{F}_{s,t} \vee \mathcal{F}_{t,\infty} = \mathcal{F}_{-\infty,s} \vee \mathcal{F}_{s-,t+} \vee \mathcal{F}_{t,\infty}$; in combination with the independence it implies $\mathcal{F}_{s,t} = \mathcal{F}_{s-,t+}$. $\square$

The two continuity conditions ('upward' and 'downward') make sense also for (non-homogeneous) continuous products of probability spaces. The time set may be $\mathbb{R}$, or $[-\infty,\infty]$, or (say) $[0,1]$, etc. Of course, using $[0,1]$ we interprete $0-\varepsilon$ as 0 and $1+\varepsilon$ as 1. Still, the upward continuity implies the downward continuity. (Indeed, the proof of 2d4 does not use the homogeneity. See also 6d18.) The converse does not hold. For example, the $\mathbb{T}$-flow $Y^{1\text{b}}$ (the $\mathbb{T}$-flow Y of Sect. 1b) leads to a continuous product of probability spaces, continuous downwards but not upwards. Namely, $Y^{1\text{b}}_{0,1}$ is $\mathcal{F}_{0,1}$-measurable but independent of the σ-field $\mathcal{F}_{0+,1}$ generated by $\cup_{\varepsilon>0}\mathcal{F}_{\varepsilon,1}$. On the other hand, the σ-field $\mathcal{F}_{-\infty,0+} = \mathcal{F}_{0-,0+}$ is trivial. See also [40, 3d6 and 3e3], [35, 2.1], [41, 1.5(2)].

Similarly to Proposition 2d2, upward continuity holds for (non-homogeneous) continuous products of probability spaces, provided however that r, s do not belong to a finite or countable set of discontinuity points.

Adding stationarity to Def. 2c1 we get Def. 2d1. It is more difficult to add stationarity to Def. 2c6. It should mean isomorphisms $\Omega_s \times \Omega_t \to \Omega_{s+t}$. However, what is the counterpart of the measurability of $T_h(\omega)$ in (ω, h) stipulated in 2d1? Can we treat the disjoint union $\cup_{t\in(0,\infty)}\Omega_t$ as a measurable space such that the map $\big((s,\omega_1),(t,\omega_2)\big) \mapsto (s+t, \omega_1\omega_2)$ is measurable?

By a *Borel semigroup* we mean a semigroup G equipped with a σ-field $\mathcal{B}$ such that the measurable space $(G, \mathcal{B})$ is standard, and the binary operation $(x, y) \mapsto xy$ is a measurable map $(G \times G, \mathcal{B} \otimes \mathcal{B}) \to (G, \mathcal{B})$.

Given a Borel semigroup G, by a G-flow we mean a family $(X_{s,t})_{s<t}$ of G-valued random variables $X_{s,t}$ (given for all $s, t \in \mathbb{R}$, $s < t$ on some probability space), satisfying 2b2(a,b).

2d6 Question. (a) Does every noise correspond to a G-flow (for some Borel semigroup G)?

(b) More specifically, is the following statement true? For every noise $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}, (T_h)_h\big)$ there exists a Borel semigroup G and a G-flow $(X_{s,t})_{s<t}$ such that for all $s, t, h \in \mathbb{R}$, $s < t$,

$$\mathcal{F}_{s,t} \text{ is the } \sigma\text{-field generated by } X_{s,t}\,,$$
$$X_{s,t} \circ T_h = X_{s+h,t+h} \quad \text{a.s,}$$

and in addition, there exists a Borel map $L : G \to (0, \infty)$ ('graduation') such that

$$L(X_{s,t}) = t - s \quad \text{a.s. whenever } s < t\,,$$
$$L(g_1 g_2) = L(g_1) + L(g_2) \quad \text{for all } g_1, g_2 \in G\,.$$

The convolution $\mu * \nu$ of two probability measures μ, ν on a Borel semigroup G is defined evidently (as the image of $\mu \times \nu$ under $(x, y) \mapsto xy$). A (one-parameter) convolution semigroup $(\mu_t)_{t\in(0,\infty)}$ in G is defined accordingly and may be treated as a special (stationary) case of a convolution system (as defined by 2b1). If it is separable, we get a G-valued flow system $(X_{s,t})_{s<t}$ (recall 2b4, 2b5), and in addition, automorphisms T_h of the corresponding probability space, satisfying $X_{s,t} \circ T_h = X_{s+h,t+h}$.

2d7 Question. Is $T_h(\omega)$ jointly measurable in ω and h? In other words: does every separable convolution semigroup (in every Borel semigroup) lead to a noise, or not?

It was said that weakly continuous convolution semigroups in $\mathbb{R}$ lead to noises. This fact can be generalized to topological semigroups, which is however not enough for applications. In Sect. 4 we deal with semigroups that are finite-dimensional topological spaces but not topological semigroups, since the binary operation is not continuous.

2d8 Definition. A *topo-semigroup* is a semigroup G equipped with a topology such that

(a) G is a separable metrizable topological space;

(b) G is a Borel semigroup (w.r.t. the σ-field generated by the topology);

(c) the semigroup G contains a unit 1, and

$$x_n \to 1 \qquad \text{implies} \qquad x_n y \to y \quad \text{and} \quad y x_n \to y$$

for all $y, x_1, x_2, \dots \in G$.

By a weakly continuous (one-parameter) convolution semigroup $(\mu_t)_{t>0}$ in a topo-semigroup G we mean a convolution semigroup in the Borel semigroup G such that

$$\mu_t(U) \to 1 \quad \text{as } t \to 0$$

for every neighborhood U of the unit 1 of G. It follows easily that $\int f \, d\mu_t$ is continuous in $t \in (0, \infty)$ (and tends to $f(1)$ as $t \to 0$) for every bounded continuous function $f : G \to \mathbb{R}$.

2d9 Proposition. Let G be a topo-semigroup and $(\mu_t)_{t>0}$ a weakly continuous convolution semigroup in G. Then the convolution system $(G, \mu_{t-s})_{s<t}$ is separable, and leads to a noise.

Proof (sketch). The flow system on a countable $T \subset \mathbb{R}$ satisfies $X_{s,t_n} \to X_{s,t}$ in probability whenever $t_n \downarrow t > s$. It implies separability of the convolution system. Thus, $X_{s,t}$ are defined for all s, t, and for every s the function $t \mapsto X_{s,t}$ from (s, ∞) to $L_0(\Omega \to G)$ is right-continuous, therefore Borel measurable. Similarly, for every t the function $s \mapsto X_{s,t}$ from $(-\infty, t)$ to $L_0(\Omega \to G)$ is Borel measurable. It follows that $X_{r,t} = X_{r,s} X_{s,t}$ is a Borel measurable function of r, t. Joint measurability of $T_h(\omega)$ follows. $\square$

3 Continuous products: from probability spaces to Hilbert spaces

3a Continuous products of spaces L_2

If (Ω_1, P_1), (Ω_2, P_2) are probability spaces and $(\Omega, P) = (\Omega_1, P_1) \times (\Omega_2, P_2)$ is their product, then Hilbert spaces $H_1 = L_2(\Omega_1, P_1)$, $H_2 = L_2(\Omega_2, P_2)$,

$H = L_2(\Omega, P)$ are related via tensor product,

$$H = H_1 \otimes H_2 \,.$$

In terms of bases it means that, having orthonormal bases $(f_i)_{i \in I}$ in H_1 and $(g_j)_{j \in J}$ in H_2, we get an orthonormal basis $(f_i \otimes g_j)_{(i,j) \in I \times J}$ in H; here

$$(f \otimes g)(\omega_1, \omega_2) = f(\omega_1) g(\omega_2) \quad \text{for } \omega_1 \in \Omega_1, \, \omega_2 \in \Omega_2 \,.$$

Complex spaces $L_2\big((\Omega, P), \mathbb{C}\big)$ and real spaces $L_2\big((\Omega, P), \mathbb{R}\big)$ may be used equally well.

In other words: having a probability space $(\Omega, \mathcal{F}, P)$ and two sub-σ-fields $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$ such that $\mathcal{F}_1 \otimes \mathcal{F}_2 = \mathcal{F}$ (recall (2c2)), we introduce Hilbert spaces $H_1 = L_2(\mathcal{F}_1)$ (that is, $H_1 = L_2(\Omega, \mathcal{F}_1, P)$), $H_2 = L_2(\mathcal{F}_2)$, $H = L_2(\mathcal{F})$ and get $H = H_1 \otimes H_2$. This time, $f \otimes g$ is just the (pointwise) product of the two functions f, g on Ω; note that these are *independent* random variables. In addition we have $H_1 \subset H_1 \otimes H_2$ and $H_2 \subset H_1 \otimes H_2$, which does not happen in general. Here it happens because of a special vector $\mathbf{1}$ (the constant function on Ω) of H_1 (and H_2); H_2 is identified with $\mathbf{1} \otimes H_2 \subset H_1 \otimes H_2$.

Given a continuous product of probability spaces (Ω, P), $(\mathcal{F}_{s,t})_{s<t}$ (as defined by 2c1), we introduce Hilbert spaces

$$\begin{aligned} H_{s,t} &= L_2(\mathcal{F}_{s,t}) \quad \text{for } s < t \,; \\ H_{r,t} &= H_{r,s} \otimes H_{s,t} \quad \text{for } r < s < t \,. \end{aligned}$$

A *unitary operator* is a linear isometric invertible operator between Hilbert spaces (over $\mathbb{R}$ or $\mathbb{C}$). The group of all unitary operators $H \to H$ will be denoted $\mathrm{U}(H)$. Here is a counterpart of Def. 2c6.

3a1 Definition. A *continuous product of Hilbert spaces* consists of separable Hilbert spaces $H_{s,t}$ (given for all $s, t \in [-\infty, \infty]$, $s < t$; possibly finite-dimensional, but not zero-dimensional), and unitary operators $H_{r,s} \otimes H_{s,t} \to H_{r,t}$ (given for all $r, s, t \in [-\infty, \infty]$, $r < s < t$), satisfying the associativity condition:

$$(fg)h = f(gh) \quad \text{for all } f \in H_{r,s}, \, g \in H_{s,t}, \, h \in H_{t,u}$$

whenever $r, s, t, u \in [-\infty, \infty]$, $r < s < t < u$. Here fg stands for the image of $f \otimes g$ under the given operator $H_{r,s} \otimes H_{s,t} \to H_{r,t}$.

Note the time set $[-\infty, \infty]$ rather than $\mathbb{R}$. Enlarging $\mathbb{R}$ to $[-\infty, \infty]$ is easy when dealing with probability spaces (as noted after Def. 2c1) but not Hilbert spaces. Any linearly ordered set could be used as the time set in

Def. 3a1; however, existence of the least and greatest elements ($\pm\infty$) will be used in Sect. 3b. The time set $\mathbb{R}$ will be treated in Sects. 3c, 3d in the stationary setup. Homeomorphic time sets $[-\infty, \infty]$ and $[0, 1]$ are the same in the general setup (3a, 3b) but quite different in the stationary setup (3c, 3d).

Every continuous product of probability spaces leads to a continuous product of Hilbert spaces.

Given a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, we may consider the disjoint union $\mathcal{E}$ of all $H_{s,t}$,

$$\mathcal{E} = \biguplus_{s<t} H_{s,t} = \{(s, t, f) : -\infty \le s < t \le \infty,\ f \in H_{s,t}\}$$

and a partial binary operation

$$\big((r, s, f), (s, t, g)\big) \mapsto (r, t, fg)$$

from a subset of $\mathcal{E} \times \mathcal{E}$ to $\mathcal{E}$; namely, a pair $\big((s_1, t_1, f_1), (s_2, t_2, f_2)\big)$ belongs to the subset iff $t_1 = s_2$. The operation is associative.

If the continuous product of Hilbert spaces corresponds to a continuous product of probability spaces, then all $H_{s,t}$ are embedded into $H = H_{-\infty,\infty}$, therefore $\mathcal{E}$ is a subset of $\mathbb{R} \times \mathbb{R} \times H$. It is a Borel subset. *Sketch of the proof:* the function $(s, t, f) \mapsto \operatorname{dist}(f, H_{s,t})$ is Borel measurable, since it is continuous unless s or t belong to a finite or countable set of discontinuity points (recall 2d).

The set $\mathcal{E}$ inherits from $\mathbb{R} \times \mathbb{R} \times H$ the structure of a standard measurable space. The domain of the binary operation is evidently Borel measurable. And the binary operation is (jointly) Borel measurable. *Sketch of the proof:* the (pointwise) product $(f, g) \mapsto fg$ is a continuous map $L_2(\Omega, P) \times L_2(\Omega, P) \to L_1(\Omega, P)$.

DIGRESSION: MEASURABLE FAMILY OF HILBERT SPACES

Dealing with a Hilbert space that depends on a (non-discrete) parameter, one should bother about measurability in the parameter. To this end we choose a single model of an infinite-dimensional separable Hilbert space, say, the space l_2 of sequences; and for each n, a single model of an n-dimensional Hilbert space, say, the space $l_2^{(n)}$ of n-element sequences. These are our favourites. Given a standard measurable space $(X, \mathcal{X})$, we have a favourite model $(l_2)_{x\in X}$ of a family $(H_x)_{x\in X}$ of infinite-dimensional separable Hilbert spaces. The disjoint union $\biguplus_{x\in X} l_2$, being just $X \times l_2$, is a standard measurable space. More generally, given a measurable function $n : X \to \{0, 1, 2, \dots\} \cup \{\infty\}$, we consider $\big(l_2^{(n(x))}\big)_{x\in X}$; here $l_2^{(\infty)} = l_2$. Still, $\biguplus_{x\in X} l_2^{(n(x))}$

is a standard measurable space; indeed, it is $\cup_k \big(\{x : n(x) = k\} \times l_2^{(k)} \big)$. The general case, defined below, is the same up to measurable, fiberwise unitary maps.

3a2 Definition. A *standard measurable family of Hilbert spaces* (over a standard measurable space $(X, \mathcal{X})$) consists of separable Hilbert spaces H_x, given for all $x \in X$, and a σ-field on the disjoint union $\biguplus_{x \in X} H_x = \{(x, h) : x \in X, h \in H_x\}$ satisfying the condition:

There exist a measurable function $n : X \to \{0, 1, 2, \dots\} \cup \{\infty\}$ and unitary operators $U_x : l_2^{(n(x))} \to H_x$ (for all $x \in X$) such that the map $(x, h) \mapsto (x, U_x h)$ is a Borel isomorphism of $\biguplus_{x \in X} l_2^{(n(x))}$ onto $\biguplus_{x \in X} H_x$.

Such a σ-field on $\biguplus_{x \in X} H_x$ will be called a *measurable structure* on the family $(H_x)_{x \in X}$ of Hilbert spaces.

Instead of unitary operators U_x one may use vectors $e_k(x) = U_x e_k$ where $e_1, e_2, \dots$ are the basis vectors of l_2. For each x vectors $e_k(x)$ are an orthonormal basis of H_x provided that $\dim H_x = \infty$; otherwise the first $n = \dim H_x$ vectors are such a basis, and other vectors vanish. Also, $x \mapsto \big(x, e_k(x)\big)$ is a measurable map $X \to \biguplus_{x \in X} H_x$ (for each k). These properties ensure that the map $(x, h) \mapsto (x, U_x h)$ is a Borel measurable bijective map $\biguplus_{x \in X} l_2^{(n(x))} \to \biguplus_{x \in X} H_x$. The map is a Borel isomorphism if and only if $\biguplus_{x \in X} H_x$ is a *standard* measurable space.

Given two standard measurable families of Hilbert spaces $(H'_x)_{x \in X}$, $(H''_x)_{x \in X}$ over the same base $(X, \mathcal{X})$, the family of tensor products $(H'_x \otimes H''_x)_{x \in X}$ is also a standard measurable family of Hilbert spaces (according to $U'_x \otimes U''_x : l_2^{(n'(x) n''(x))} = l_2^{(n'(x))} \otimes l_2^{(n''(x))} \to H'_x \otimes H''_x$).

3a3 Lemma. Let $h'_x \in H'_x$ and $h''_x \in H''_x$ be such that $h'_x \otimes h''_x$ is measurable in x (that is, the map $x \mapsto (x, h'_x \otimes h''_x)$ from X to $\biguplus_{x \in H} H'_x \otimes H''_x$ is measurable). Then there exists a function $c : X \to \mathbb{C} \setminus \{0\}$ such that both $c(x) h'_x$ and $(1/c(x)) h''_x$ are measurable in x.

Proof (sketch). We may assume that $\|h'_x\| = 1$ and $\|h''_x\| = 1$ (since the norm is a measurable function of a vector). Also we may assume that $H'_x = l_2$ and $H''_x = l_2$ (finite dimensions are left to the reader). Consider the sphere $S(l_2) = \{h \in l_2 : \|h\| = 1\}$, and the map $(h_1, h_2) \mapsto h_1 \otimes h_2$ from $S(l_2) \times S(l_2)$ to $S(l_2 \otimes l_2)$. Inverse image of each point of $S(l_2 \otimes l_2)$ is either empty or a *compact* subset of $S(l_2) \times S(l_2)$ of the form $\{(c h_1, (1/c) h_2) : c \in \mathbb{C}, |c| = 1\}$. There exists a Borel function ('selector') on the set of factorizing vectors of $S(l_2 \otimes l_2)$ that chooses a point from each inverse image. Applying the selector to $h'_x \otimes h''_x$ we get $c(x) h'_x$ and $(1/c(x)) h''_x$. $\square$

Assume now that $(H'_x)_{x\in X}$, $(H''_x)_{x\in X}$ are just families (not 'measurable'!) of Hilbert spaces, $H_x = H'_x \otimes H''_x$, and a measurable structure $\mathcal{B}$ is given on $(H_x)_{x\in X}$. We say that $\mathcal{B}$ is *factorizing,* if it results from some measurable structures $\mathcal{B}'$, $\mathcal{B}''$ on $(H'_x)_{x\in X}$, $(H''_x)_{x\in X}$. Generally, this is not the case. Indeed, a family $(U_x)_{x\in X}$ of unitary operators $U_x \in \mathrm{U}(l_2 \otimes l_2)$ in general is not of the form $U_x = V_x(U'_x \otimes U''_x)$ where $U'_x, U''_x \in \mathrm{U}(l_2)$ are arbitrary, but $V_x \in \mathrm{U}(l_2 \otimes l_2)$ is a measurable function of x.

Assume that $\mathcal{B}$ is factorizing, that is, $\mathcal{B}$ results from some $\mathcal{B}', \mathcal{B}''$. Does $\mathcal{B}$ determine $\mathcal{B}', \mathcal{B}''$ uniquely? No, it does not. Indeed, let $c : X \to \mathbb{C}$, $|c(\cdot)| = 1$, be a non-measurable function. The transformation $(x,h) \mapsto (x, c(x)h)$ of $\biguplus_{x\in X} H'_x$ sends $\mathcal{B}'$ to another σ-field. Combining it with the transformation $(x,h) \mapsto (x, (1/c(x))h)$ of $\biguplus_{x\in X} H''_x$ we get the trivial transformation of of $\biguplus_{x\in X} H'_x \otimes H''_x$, since $(c(x)h'_x) \otimes ((1/c(x))h''_x) = h'_x \otimes h''_x$.

3a4 Lemma. Let $\mathcal{B}'_1, \mathcal{B}'_2$ be two measurable structures on $(H'_x)_{x\in X}$ and $\mathcal{B}''_1, \mathcal{B}''_2$ — on $(H''_x)_{x\in X}$. Assume that the corresponding structures $\mathcal{B}_1, \mathcal{B}_2$ on $(H'_x \otimes H''_x)_{x\in X}$ coincide, $\mathcal{B}_1 = \mathcal{B}_2$. (Here $\mathcal{B}_1$ results from $\mathcal{B}'_1, \mathcal{B}''_1$ and $\mathcal{B}_2$ — from $\mathcal{B}'_2, \mathcal{B}''_2$.) Then there exists a function $c : X \to \mathbb{C}$ such that $|c(\cdot)| = 1$, the map $(x,h) \mapsto (x, c(x)h)$ sends $\mathcal{B}'_1$ to $\mathcal{B}'_2$, and the map $(x,h) \mapsto (x, (1/c(x))h)$ sends $\mathcal{B}''_1$ to $\mathcal{B}''_2$.

Proof (sketch). If vectors $\psi_x, \xi_x \in l_2$ are such that $\psi_x \otimes \xi_x$ is a measurable function of x, then $c(x)\psi_x$ and $(1/c(x))\xi_x$ are measurable functions of x for some choice of $c(\cdot)$. Thus, if $U'_x \otimes U''_x$ is a measurable function of x then $c(x)U'_x$ and $(1/c(x))U''_x$ are measurable functions of x for some choice of $c(\cdot)$. □

END OF DIGRESSION

Returning to the continuous product of Hilbert spaces that corresponds to a continuous product of probability spaces we claim that $\mathcal{E} = \biguplus_{s<t} H_{s,t}$ (equipped with the σ-field inherited from $\mathbb{R}\times\mathbb{R}\times H$) is a standard measurable family of Hilbert spaces. *Sketch of the proof:*[1] let $e_1, e_2, \dots$ span H and $e^{(k)}_{s,t}$ be the projection of e_k to $H_{s,t} \subset H$, then $e^{(1)}_{s,t}, e^{(2)}_{s,t}, \dots$ span $H_{s,t}$, and $e^{(k)}_{s,t}$ is measurable in s,t (being continuous outside a countable set). Using orthogonalization (for each (s,t) separately; zero vectors, if any, are skipped) we turn $e^{(k)}_{s,t}$ into an orthonormal basis of $H_{s,t}$.

In terms of basis vectors $e^{(k)}_{s,t}$, measurability of the partial binary operation means that its matrix element

$$\langle e^{(k)}_{r,s} e^{(l)}_{s,t}, e^{(m)}_{r,t} \rangle$$

is a Borel measurable function of $r,s,t \in [-\infty,\infty]$, $r<s<t$, for any k,l,m.

[1] Similarly one can prove a more general fact: the family of *all* (closed linear) subspaces of a separable Hilbert space is a standard measurable family of Hilbert spaces.

3b Continuous product of Hilbert spaces

The measurable structure, introduced in Sect. 3a on $\biguplus_{s<t} L_2(\mathcal{F}_{s,t})$, exists also on $\biguplus_{s<t} H_{s,t}$ in general.

3b1 Theorem. For every continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ (as defined by 3a1) there exists a measurable structure on the family $(H_{s,t})_{s<t}$ of Hilbert spaces that makes the given map $H_{r,s} \otimes H_{s,t} \to H_{r,t}$ Borel measurable in r, s, t.

In other words, there exist orthonormal bases $\big(e_{s,t}^{(k)}\big)_k$ in the spaces $H_{s,t}$ such that $\langle e_{r,s}^{(k)} e_{s,t}^{(l)}, e_{r,t}^{(m)} \rangle$ is Borel measurable in r, s, t. Recall that $e_{r,s}^{(k)} e_{s,t}^{(l)}$ is the image of $e_{r,s}^{(k)} \otimes e_{s,t}^{(l)}$ inder the given map $H_{r,s} \otimes H_{s,t} \to H_{r,t}$.

In Sect. 3a, spaces $H_{s,t}$ are both subspaces and factors of $H = H_{-\infty,\infty}$; now they are only factors (in the sense that H may be treated as $H_{-\infty,s} \otimes H_{s,t} \otimes H_{t,\infty}$), which means that a different technique is needed.

DIGRESSION: FACTORS

The algebra $\mathcal{B}(l_2 \otimes l_2)$ of all (bounded linear) operators on the Hilbert space $l_2 \otimes l_2$ contains two special subalgebras, $\mathcal{B}(l_2) \otimes \mathbf{1} = \{A \otimes \mathbf{1} : A \in \mathcal{B}(l_2)\}$ and $\mathbf{1} \otimes \mathcal{B}(l_2) = \{\mathbf{1} \otimes A : A \in \mathcal{B}(l_2)\}$. Recall that $(A \otimes B)(x \otimes y) = Ax \otimes By$, thus, $(A \otimes \mathbf{1})(x \otimes y) = Ax \otimes y$ and $(\mathbf{1} \otimes A)(x \otimes y) = x \otimes Ay$. The two subalgebras are commutants to each other: $\mathbf{1} \otimes \mathcal{B}(l_2) = \{A \in \mathcal{B}(l_2 \otimes l_2) : \forall B \in \mathcal{B}(l_2) \otimes \mathbf{1} \;\; AB = BA\}$.

A unitary operator $U \in \mathrm{U}(l_2 \otimes l_2)$ transforms the two subalgebras in two other subalgebras, $U(\mathcal{B}(l_2) \otimes \mathbf{1})U^{-1}$ and $U(\mathbf{1} \otimes \mathcal{B}(l_2))U^{-1}$; still, they are commutants to each other. Of course, $U(\mathcal{B}(l_2) \otimes \mathbf{1})U^{-1} = \{UAU^{-1} : A \in \mathcal{B}(l_2) \otimes \mathbf{1}\}$. If U is factorizing, that is, $U = U_1 U_2$ for some unitary $U_1, U_2 \in \mathcal{B}(l_2)$ then $U(\mathcal{B}(l_2) \otimes \mathbf{1})U^{-1} = \mathcal{B}(l_2) \otimes \mathbf{1}$ and $U(\mathbf{1} \otimes \mathcal{B}(l_2))U^{-1} = \mathbf{1} \otimes \mathcal{B}(l_2)$. And conversely, these two (mutually equivalent) relations imply factorizability of U.

The set of all subalgebras $\mathcal{A}$ of the form $U(\mathcal{B}(l_2) \otimes \mathbf{1})U^{-1}$ may be turned into a measurable space as follows. The ball $\{A \in \mathcal{B}(l_2 \otimes l_2) : \|A\| \le 1\}$ equipped with the weak operator topology is a metrizable compact topological space, and $\{A \in \mathcal{A} : \|A\| \le 1\}$ is its closed subset. The set of all closed subsets of a metrizable compact space is a standard measurable space, known as Effros space, see [17, Sect. 12.C]. Thus, each algebra $\mathcal{A} = U(\mathcal{B}(l_2) \otimes \mathbf{1})U^{-1}$ may be treated as a point of the Effros space.

The set $\mathrm{U}(l_2 \otimes l_2)$ of all unitary operators, being a subset of the ball, is also a measurable space. It is well-known to be a standard measurable space (and in fact, a non-closed G_δ-subset of the ball), see [17, 9.B.6].

3b2 Lemma. (a) The set A of all subalgebras $\mathcal{A}$ of the form $U(\mathcal{B}(l_2)\otimes\mathbf{1})U^{-1}$ is a standard measurable space.

(b) There exists a Borel map $\mathcal{A}\mapsto U_{\mathcal{A}}$ from A to the space of unitary operators on $l_2\otimes l_2$ such that

$$\mathcal{A}=U_{\mathcal{A}}(\mathcal{B}(l_2)\otimes\mathbf{1})U_{\mathcal{A}}^{-1}\quad\text{for all }\mathcal{A}\in\mathsf{A}\,.$$

Proof (sketch). The group $G=\mathrm{U}(l_2\otimes l_2)$ is a Polish group, and factorizing operators are its closed subgroup $G_0=\mathrm{U}(l_2)\times\mathrm{U}(l_2)$. Left-cosets $gG_0=\{gh:h\in G_0\}$ (for $g\in G$) are a Polish space G/G_0 [7, 1.2.3], and by a theorem of Dixmier (see [17, (12.17)] or [7, 1.2.4]) there exists a Borel function ('selector') $s:G/G_0\to G$ such that $s(gG_0)\in gG_0$ for all g.

The map $U\mapsto U(\mathcal{B}(l_2)\otimes\mathbf{1})U^{-1}$ is a Borel map $G\to\mathsf{A}$; indeed, for each $A\in\mathcal{B}(l_2)$ the map $U\mapsto U(A\otimes\mathbf{1})U^{-1}$ is Borel, and $U(\mathcal{B}(l_2)\otimes\mathbf{1})U^{-1}$ is the closure of the sequence of $U(A_k\otimes\mathbf{1})U^{-1}$ where A_k are a dense sequence in $\mathcal{B}(l_2)$. Being constant on each gG_0, the Borel map $G\to\mathsf{A}$ leads to a Borel map $G/G_0\to\mathsf{A}$. The latter map is bijective, and A is a part of a standard measurable space. By a Lusin-Souslin theorem [17, (15.2)], A is a Borel subset, which proves (a), and the inverse map $\mathsf{A}\to G/G_0$ is Borel. The map $\mathsf{A}\to G/G_0\xrightarrow{s}G$ ensures (b). $\square$

END OF DIGRESSION

We return to a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ and assume for simplicity that all $H_{s,t}$ are infinite-dimensional. The family $(H_{-\infty,t}\otimes H_{t,\infty})_{t\in\mathbb{R}}$ of Hilbert spaces evidently carries a measurable structure (according to the given unitary operators $H_{-\infty,t}\otimes H_{t,\infty}\to H_{-\infty,\infty}$). We will see that the measurable structure is factorizing,[1] which is close to Theorem 3b1. Indeed, it means existence of measurable structures on $(H_{-\infty,t})_{t\in\mathbb{R}}$ and $(H_{t,\infty})_{t\in\mathbb{R}}$ that make the given map $H_{-\infty,t}\otimes H_{t,\infty}\to H_{-\infty,\infty}$ Borel measurable in t. Note that such measurable structures on $(H_{-\infty,t})_{t\in\mathbb{R}}$ and $(H_{t,\infty})_{t\in\mathbb{R}}$ are unique up to scalar factors $(c_t)_{t\in\mathbb{R}}$ according to Lemma 3a4.

For convenience we let $H_{-\infty,\infty}=H=l_2\otimes l_2$. For any $t\in\mathbb{R}$ the given unitary operator $W_t:H_{-\infty,t}\otimes H_{t,\infty}\to H$ sends $\mathcal{B}(H_{-\infty,t})\otimes\mathbf{1}$ to an algebra $\mathcal{A}_{-\infty,t}\in\mathsf{A}$. The function $t\mapsto\mathcal{A}_{-\infty,t}$ is increasing ($s<t$ implies $\mathcal{A}_{-\infty,s}\subset\mathcal{A}_{-\infty,t}$), therefore Borel measurable (and in fact, continuous outside a finite or countable set).

Lemma 3b2 gives us unitary operators V_t on $H=l_2\otimes l_2$ such that $\mathcal{A}_{-\infty,t}=V_t(\mathcal{B}(l_2)\otimes\mathbf{1})V_t^{-1}$ for all $t\in\mathbb{R}$, and the map $t\mapsto V_t$ is Borel measurable. On

[1] Similarly one can prove a more general fact: the natural measurable structure on $\biguplus_{\mathcal{A}\in\mathsf{A}}H'_{\mathcal{A}}\otimes H''_{\mathcal{A}}$ is factorizing. Here the disjoint union is taken, roughly speaking, over all possible decompositions of l_2 (or $l_2\otimes l_2$) into the tensor product of two infinite-dimensional Hilbert spaces. The exact formulation is left to the reader.

the other hand, $\mathcal{A}_{-\infty,t} = W_t(\mathcal{B}(H_{-\infty,t}) \otimes \mathbf{1})W_t^{-1}$. Thus, $\mathcal{B}(H_{-\infty,t}) \otimes \mathbf{1} = (V_t^{-1}W_t)^{-1}(\mathcal{B}(l_2) \otimes \mathbf{1})V_t^{-1}W_t$, which means that $(V_t^{-1}W_t)^{-1}$ is a factorizing operator $l_2 \otimes l_2 \to H_{-\infty,t} \otimes H_{t,\infty}$;

$$W_t^{-1}V_t = U_{-\infty,t} \otimes U_{t,\infty}$$

for some unitary operators $U_{-\infty,t} : l_2 \to H_{-\infty,t}$ and $U_{t,\infty} : l_2 \to H_{t,\infty}$. Operators $U_{-\infty,t}$ define a measurable structure on $(H_{-\infty,t})_{t\in\mathbb{R}}$. The same for $U_{t,\infty}$ and $(H_{t,\infty})_{t\in\mathbb{R}}$. The partial binary operation $\big((t,x),(t,y)\big) \mapsto xy$ becomes Borel measurable, since $xy = W_t(x \otimes y) = V_t(U_{-\infty,t}^{-1}x \otimes U_{t,\infty}^{-1}y)$ and V_t is measurable in t.

The proof of Theorem 3b1 is similar. Algebras $\mathcal{A}_{s,t} \in \mathsf{A}$, corresponding to $H_{s,t}$, are used. Joint measurability of $\mathcal{A}_{s,t}$ in s and t follows from the formula $\mathcal{A}_{s,t} = \mathcal{A}_{-\infty,t} \cap \mathcal{A}_{s,\infty}$ and a general fact: on a *compact* metric space, the intersection of two closed subsets is a jointly Borel measurable function of these two subsets [17, (27.7)].

Non-uniqueness of the measurable structure on $(H_{s,t})_{s<t}$ is described by scalar factors $(c_{s,t})_{s<t}$, $c_{s,t} \in \mathbb{C}$, $|c_{s,t}| = 1$ such that

$$c_{r,s}c_{s,t} = c_{r,t} \quad \text{whenever } r < s < t$$

(which means that $c_{s,t} = c_t/c_s$ for some $(c_t)_{t\in\mathbb{R}}$; for example, one may take $c_t = c_{0,t}$ for $t > 0$, $c_t = 1/c_{t,0}$ for $t < 0$, and $c_0 = 1$). The transformation $(s,t,h) \mapsto (s,t,c_{s,t}h)$ of $\biguplus_{s<t} H_{s,t}$ preserves the given maps $H_{r,s} \otimes H_{s,t} \to H_{r,t}$ but changes the measurable structure (unless $c_{s,t}$ is measurable in s,t).

See also [39, Sect. 1].

3c Stationary case; Arveson systems

Let $(\Omega,\mathcal{F},P)$, $(\mathcal{F}_{s,t})_{s<t}$, $(T_h)_{h\in\mathbb{R}}$ be a noise (as defined by 2d1), then $(\mathcal{F}_{s,t})_{s<t}$, being a continuous product of probability spaces, leads to a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, while each T_h, being a measure preserving transformation of $(\Omega,\mathcal{F},P)$, leads to a unitary operator $\theta^h : H \to H$ (where $H = H_{-\infty,\infty} = L_2(\Omega,\mathcal{F},P)$); namely, $\theta^h f = f \circ T_h$ for $f \in H$. The one-parameter group $(\theta^h)_{h\in\mathbb{R}}$, being measurable (in h), is of the form $\theta^h = \exp(ihX)$, where X (the generator) is a self-adjoint operator. Knowing that T_h sends $\mathcal{F}_{s,t}$ to $\mathcal{F}_{s+h,t+h}$ we get unitary operators $\theta^h_{s,t} : H_{s,t} \to H_{s+h,t+h}$ satisfying $\theta^h_{r,s} \otimes \theta^h_{s,t} = \theta^h_{r,t}$ and $\theta^{h_2}_{s+h_1,t+h_1}\theta^{h_1}_{s,t} = \theta^{h_1+h_2}_{s,t}$.

The property 2d1(c) ensures that the global algebra $\mathcal{A}_{-\infty,\infty} = \mathcal{B}(H_{-\infty,\infty})$ is generated by (the union of all) local algebras $\mathcal{A}_{s,t}$, $-\infty < s < t < \infty$. See the proof of 6e1(c $\Longrightarrow$ a); the same argument works here. As before, $\mathcal{A}_{s,t}$ is the image of $\mathbf{1} \otimes \mathcal{B}(H_{s,t}) \otimes \mathbf{1}$ under the given map $H_{-\infty,s} \otimes H_{s,t} \otimes H_{t,\infty} \to$

$H_{-\infty,\infty} = H$; 'generated by' means here 'is the closure of' (in the weak operator topology).

3c1 Definition. A *homogeneous continuous product of Hilbert spaces* consists of a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ and unitary operators $\theta^h_{s,t} : H_{s,t} \to H_{s+h,t+h}$ (given for all $h \in \mathbb{R}$ and $s,t \in [-\infty,\infty]$, $s<t$; of course, $(-\infty)+h = (-\infty)$ and $(+\infty)+h = (+\infty)$) satisfying

(a) $\theta^h_{r,s} \otimes \theta^h_{s,t} = \theta^h_{r,t}$ for $-\infty \le r < s < t \le \infty$ and $h \in \mathbb{R}$;

(b) $\theta^{h_2}_{s+h_1,t+h_1} \theta^{h_1}_{s,t} = \theta^{h_1+h_2}_{s,t}$ for $-\infty \le s < t \le \infty$ and $h_1, h_2 \in \mathbb{R}$;

(c) there exists a self-adjoint operator X such that $\theta^h_{-\infty,\infty} = \exp(ihX)$ for $h \in \mathbb{R}$;

(d) $\mathcal{A}_{-\infty,\infty}$ is the weak closure of the union of all $\mathcal{A}_{s,t}$ for $-\infty < s < t < \infty$.

Every noise leads to a homogeneous continuous product of Hilbert spaces. Here are counterparts of Proposition 2d2 and Corollary 2d4.

3c2 Proposition. (Liebscher [19, Prop. 3.4]; see also [4, 4.2.1]). Every homogeneous continuous product of Hilbert spaces satisfies the 'upward continuity' condition

$$\mathcal{A}_{s,t} \text{ is generated by } \bigcup_{\varepsilon>0} \mathcal{A}_{s+\varepsilon,t-\varepsilon} \quad \text{for } -\infty \le s < t \le \infty\,; \tag{3c3}$$

here $-\infty+\varepsilon$ is interpreted as $-1/\varepsilon$ and $\infty-\varepsilon$ as $1/\varepsilon$.

Proof (sketch). Assume $s,t \in \mathbb{R}$ (other cases, $s=-\infty$ and $t=\infty$, follow via 3c1(d)). It is enough to prove that $\mathcal{A}_{r,s}\mathcal{A}_{s,t}$ is generated by $\bigcup \mathcal{A}_{r+\varepsilon,t-\varepsilon}$. For any $A \in \mathcal{A}_{r,s}$ and $B \in \mathcal{A}_{s,t}$,

$$\underbrace{\mathrm{e}^{\mathrm{i}\varepsilon X} A \mathrm{e}^{-\mathrm{i}\varepsilon X}}_{\in \mathcal{A}_{r+\varepsilon,s+\varepsilon}} \underbrace{\mathrm{e}^{-\mathrm{i}\varepsilon X} B \mathrm{e}^{\mathrm{i}\varepsilon X}}_{\in \mathcal{A}_{s-\varepsilon,t-\varepsilon}} \to AB \quad \text{as } \varepsilon \to 0$$

weakly and even strongly, since $\| \mathrm{e}^{\mathrm{i}\varepsilon X} A \mathrm{e}^{-2\mathrm{i}\varepsilon X} B \mathrm{e}^{\mathrm{i}\varepsilon X} f - ABf \| \le \|A\| \|B\| \| \mathrm{e}^{\mathrm{i}\varepsilon X} f - f \| + \|A\| \| \mathrm{e}^{-2\mathrm{i}\varepsilon X} Bf - Bf \| + \| \mathrm{e}^{\mathrm{i}\varepsilon X} ABf - ABf \| \to 0$. $\square$

3c4 Corollary. (Liebscher [19, Prop. 3.4]; see also [4, 4.2.1]). Every homogeneous continuous product of Hilbert spaces satisfies the 'downward continuity' condition

$$\mathcal{A}_{s,t} = \bigcap_{\varepsilon>0} \mathcal{A}_{s-\varepsilon,t+\varepsilon} \quad \text{for all } s,t \in \mathbb{R},\ s \le t\,; \tag{3c5}$$

here $\mathcal{A}_{t,t}$ is the trivial subalgebra, and $-\infty-\varepsilon = -\infty$, $\infty+\varepsilon = \infty$.

Proof (sketch). Every operator $A \in \bigcap \mathcal{A}_{s-\varepsilon,t+\varepsilon}$ commutes with $\bigcup \mathcal{A}_{-\infty,s-\varepsilon}$ and $\bigcup \mathcal{A}_{t+\varepsilon,\infty}$, therefore (using Prop. 3c2) with $\mathcal{A}_{-\infty,s}$ and $\mathcal{A}_{t,\infty}$, which means $A \in \mathcal{A}_{s,t}$. $\square$

The two continuity conditions ('upward' and 'downward') make sense also for (non-homogeneous) continuous products of Hilbert spaces. Still, the upward continuity implies the downward continuity. (Indeed, the proof of 3c4 does not use the homogeneity.) Unlike Sect. 2d, the converse is true. Systems of Sect. 1 do not lead to a counterexample! Especially, for the system of 1d, triviality of the limiting σ-field $\mathcal{F}_{\infty-,\infty}$ means that the limiting operator algebra $\mathcal{A}_{\infty-,\infty}$ contains no multiplication operators; but still, it contains projections to the 'superselection sectors' $H_0, \dots, H_{m-1}$. See also 6e1–6e2, and [41], Lemma 1.5 (and Example 4.3): (mu) $\Longrightarrow$ (md), but (Hu)$\Longleftrightarrow$ (Hd).

Definition 3c1 may seem to be unsatisfactory, since it does not stipulate measurability of $\theta^h_{s,t}$ in h for finite s,t. Recall however the non-uniqueness of the measurable structure on $(H_{s,t})_{s<t}$.

3c6 Theorem. For every homogeneous continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, $(\theta^h_{s,t})_{s<t;h}$ there exists a measurable structure on the family $(H_{s,t})_{s<t}$ of Hilbert spaces that makes the given map $H_{r,s} \otimes H_{s,t} \to H_{r,t}$ Borel measurable in r,s,t, and also makes $\theta^h_{s,t}$ Borel measurable in h,s,t.

Proof (sketch). By Theorem 3b1 (restricted to $(s,t) = (-\infty,t)$ or (s,∞)), there exist unitary $V_{-\infty,t} : l_2 \to H_{-\infty,t}$ and $V_{t,\infty} : l_2 \to H_{t,\infty}$ such that the unitary operator $W_t(V_{-\infty,t} \otimes V_{t,\infty}) : l_2 \otimes l_2 \to H$ is a Borel function of t; here, as before, $H = H_{-\infty,\infty}$ and W_t is the given unitary operator $H_{-\infty,t} \otimes H_{t,\infty} \to H$. The equality $\theta^h_{-\infty,t} \otimes \theta^h_{t,\infty} = \theta^h_{-\infty,\infty}$ (a special case of (3c1)(a)) means in fact $W_{t+h}(\theta^h_{-\infty,t} \otimes \theta^h_{t,\infty}) = \theta^h_{-\infty,\infty} W_t$. We define unitary operators $\alpha_{s,t} : l_2 \otimes l_2 \to l_2 \otimes l_2$ for $s,t \in \mathbb{R}$ by

$$
\begin{aligned}
\alpha_{s,t} = V^{-1}_{-\infty,t}\theta^{t-s}_{-\infty,s}V_{-\infty,s} \otimes V^{-1}_{t,\infty}\theta^{t-s}_{s,\infty}V_{s,\infty} = \\
= (V^{-1}_{-\infty,t} \otimes V^{-1}_{t,\infty})(\theta^{t-s}_{-\infty,s} \otimes \theta^{t-s}_{s,\infty})(V_{-\infty,s} \otimes V_{s,\infty}) = \\
= (V^{-1}_{-\infty,t} \otimes V^{-1}_{t,\infty})W_t^{-1}\theta^{t-s}_{-\infty,\infty}W_s(V_{-\infty,s} \otimes V_{s,\infty})\,.
\end{aligned}
$$

We see that $\alpha_{s,t}$ is a Borel function of s and t, and for every s,t it is a factorizing operator, $\alpha_{s,t} = \beta_{s,t} \otimes \gamma_{s,t}$ for some unitary $\beta_{s,t}, \gamma_{s,t} : l_2 \to l_2$. These $\beta_{s,t}, \gamma_{s,t}$ are unique up to a coefficient: $\alpha_{s,t} = (c\beta_{s,t}) \otimes ((1/c)\gamma_{s,t})$, $c \in \mathbb{C}$, $|c| = 1$. Similarly to the proof of Lemma 3b2, we use a Borel selector $G/G_0 \to G$, but for $G = \mathrm{U}(l_2) \times \mathrm{U}(l_2)$ and $G_0 = \{(c, 1/c) : c \in \mathbb{C}, |c| = 1\}$. This way we make $\beta_{s,t}, \gamma_{s,t}$ Borel measurable in s and t. Also, $\beta_{s,t} = c_{s,t}V^{-1}_{-\infty,t}\theta^{t-s}_{-\infty,s}V_{-\infty,s}$, $c_{s,t} \in \mathbb{C}$, $|c_{s,t}| = 1$. The product $c_{r,s}c_{s,t}c_{t,r}$ is Borel measurable in r,s,t since $\beta_{t,r}\beta_{s,t}\beta_{r,s} = c_{t,r}c_{s,t}c_{r,s} \cdot \mathbf{1}$. Multiplying each $V_{-\infty,t}$ by $1/c_{0,t}$ we get Borel measurability in s, t of $V_{-\infty,t}^{-1} \theta_{-\infty,s}^{t-s} V_{-\infty,s} = (c_{0,s} c_{s,t} c_{t,0})^{-1} \beta_{s,t}$. That is, we get measurable structures on $(H_{-\infty,t})_t$ and $(H_{t,\infty})_t$ that conform to the shifts. It remains to use the relation $H_{-\infty,s} \otimes H_{s,t} = H_{-\infty,t}$; two terms ($H_{-\infty,s}$ and $H_{-\infty,t}$) are understood, the third ($H_{s,t}$) comes out. $\square$

Waiving the infinite points $\pm\infty$ on the time axis we get a *local* homogeneous continuous product of Hilbert spaces. In this case we may treat $H_{s,t}$ as a copy of $H_{0,t-s}$, forget about shift operators $\theta_{s,t}^h$, and stipulate unitary operators $H_{0,s-r} \otimes H_{0,t-s} \to H_{0,t-r}$ instead of $H_{r,s} \otimes H_{s,t} \to H_{r,t}$. See also [50, Prop. 4.1.8].

3c7 Definition. An *algebraic product system of Hilbert spaces* consists of separable Hilbert spaces H_t (given for all $t \in (0, \infty)$; possibly finite-dimensional, but not zero-dimensional), and unitary operators $H_s \otimes H_t \to H_{s+t}$ (given for all $s, t \in (0, \infty)$), satisfying the associativity condition:

$$(fg)h = f(gh) \quad \text{for all } f \in H_r,\ g \in H_s,\ h \in H_t$$

whenever $r, s, t \in (0, \infty)$. Here fg stands for the image of $f \otimes g$ under the given operator $H_r \otimes H_s \to H_{r+s}$.

All spaces H_t are infinite-dimensional, unless they all are one-dimensional; indeed, $\dim H_{s+t} = \dim H_s \cdot \dim H_t$.

Algebraic product systems are in a natural one-to-one correspondence with local homogeneous continuous products of Hilbert spaces.

Every noise leads to a homogeneous continuous product of Hilbert spaces, therefore to a local homogeneous continuous product of Hilbert spaces, therefore to an algebraic product system of Hilbert spaces. In particular, every Lévy process in $\mathbb{R}$ (or $\mathbb{R}^n$) does.

Absence of measurability conditions opens the door to pathologies. An example follows. Consider an isotropic Lévy process in $\mathbb{R}^2$; 'isotropic' means that its distribution is invariant under rotations $(x, y) \mapsto (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi)$ of $\mathbb{R}^2$. (Especially, the standard Brownian motion in $\mathbb{R}^2$ fits.) Rotating sample paths we get (measure preserving) automorphisms of the 'global' probability space (Ω, P), as well as 'local' probability spaces $(\Omega_{s,t}, P_{s,t})$. These automorphisms lead to unitary operators $U_{s,t}^{\varphi}$ on $H_{s,t} = L_2(\Omega_{s,t}, P_{s,t})$; note that

$$U_{r,s}^{\varphi} \otimes U_{s,t}^{\varphi} = U_{r,t}^{\varphi} \quad \text{and} \quad U_{s,t}^{\varphi} U_{s,t}^{\psi} = U_{s,t}^{\varphi+\psi} .$$

Being a group of automorphisms of the homogeneous continuous product of Hilbert spaces, they lead to a group of automorphisms of the corresponding algebraic product system of Hilbert spaces: $H_t = H_{0,t}$; $U_t^\varphi = U_{0,t}^\varphi$;

$$U_{s+t}^\varphi(fg) = (U_s^\varphi f)(U_t^\varphi g) \quad \text{for } f \in H_s,\ g \in H_t\,;$$
$$U_t^\varphi U_t^\psi = U_t^{\varphi+\psi}\,.$$

No doubt, U_t^φ is a Borel function of φ and t. We spoil the algebraic product system of Hilbert spaces, replacing the given operators $W_{s,t} : H_s \otimes H_t \to H_{s+t}$ with operators $\tilde{W}_{s,t}$ defined by

$$\tilde{W}_{s,t}(f \otimes g) = W_{s,t}(f \otimes U_t^{\varphi(s)} g) \quad \text{for } f \in H_s,\ g \in H_t\,;$$

here $\varphi : \mathbb{R} \to \mathbb{R}$ is some non-measurable additive function (that is, $\varphi(s+t) = \varphi(s) + \varphi(t)$ for all $s, t \in \mathbb{R}$). The associativity condition is still satisfied:

$$\begin{gathered}\tilde{W}_{r+s,t}\big(\tilde{W}_{r,s}(f \otimes g) \otimes h\big) = f \cdot (U_s^{\varphi(r)} g) \cdot (U_t^{\varphi(r+s)} h) = \\ = f \cdot (U_s^{\varphi(r)} g) \cdot (U_t^{\varphi(r)} U_t^{\varphi(s)} h) = f \cdot U_{s+t}^{\varphi(r)} (g \cdot U_t^{\varphi(s)} h) = \tilde{W}_{r,s+t}(f \otimes \tilde{W}_{s,t}(g \otimes h))\end{gathered}$$

for $f \in H_r$, $g \in H_s$, $h \in H_t$; here $f \cdot g$ means $W_{r,s}(f \otimes g)$ rather than $\tilde{W}_{r,s}(f \otimes g)$.

We will see in Sect. 3d that the 'spoiled' binary operation is not Borel measurable, no matter which measurable structure is chosen on the family $(H_t)_{t>0}$ of Hilbert spaces.

3c8 Definition. A *product system of Hilbert spaces,* or *Arveson system,* is a family $(H_t)_{t>0}$ of Hilbert spaces, equipped with two structures: first, an algebraic product system of Hilbert spaces, and second, a standard measurable family of Hilbert spaces, such that the binary operation $(f, g) \mapsto fg$ on $\biguplus_{t>0} H_t$ is Borel measurable.

3c9 Corollary. (From Theorem 3c6.) Every homogeneous continuous product of Hilbert spaces leads to an Arveson system.

Existence of a good measurable structure was derived in Theorem 3c6 from measurability of a unitary group of shifts on the 'global' Hilbert space $H_{-\infty,\infty}$. Arveson systems in general seem to need a different idea, since no 'global' Hilbert space is stipulated. Nevertheless the same idea (group of shifts) works, being combined with another idea: cyclic time.

See also [4, Chap. 3], [19, Sect. 3.1 and 7], [39, Sect. 1].

3d Cyclic time; Liebscher's criterion

Till now, our time set was $\mathbb{R}$, or $[-\infty, \infty]$, or a subset of $\mathbb{R}$; in every case it was a linearly ordered set. Now we want to use the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as the time set. It makes no sense for processes with independent increments (every periodic process with independent increments on $\mathbb{R}$ is deterministic), but it makes sense for convolution systems, flow systems, continuous products of probability spaces or Hilbert spaces, noises and product systems. Definitions 2b1, 2b2, 2c1, 2c6, 3a1 may be transferred to $\mathbb{T}$. To this end we just replace '$r,s,t \in \mathbb{R}$' (or '$r,s,t \in [-\infty,\infty]$') with '$r,s,t \in \mathbb{T}$' and interprete '$r < s < t$' according to the cyclic order on $\mathbb{T}$. More formally, $t_1 < \dots < t_n$ means (for $t_1,\dots,t_n \in \mathbb{T}$) that there exist $\tilde t_1,\dots,\tilde t_n \in \mathbb{R}$ such that $t_k = \tilde t_k \bmod 1$ for $k = 1,\dots,n$ and $\tilde t_1 < \dots < \tilde t_n \le \tilde t_1 + 1$. Special cases $n = 2,3,4$ give us relations $s < t$, $r < s < t$, $r < s < t < u$.

The general (non-homogeneous) case is described by probability spaces $(G_{s,t}, \mu_{s,t})$, $G_{s,t}$-valued random variables $X_{s,t}$, sub-σ-fields $\mathcal{F}_{s,t}$, probability spaces $(\Omega_{s,t}, P_{s,t})$ and finally, Hilbert spaces $H_{s,t}$. The degenerate case $\tilde t_n = \tilde t_1 + 1$ is allowed, and leads to $G_{t,t}, \dots, H_{t,t}$ ($t \in \mathbb{T}$). Note that the interval from t to t is of length 1 (zero length intervals are excluded by the strict inequalities $\tilde t_1 < \dots < \tilde t_n$); one could prefer the notation $G_{t,t+1}, \dots, H_{t,t+1}$ (taking into account that $t+1 = t$ in $\mathbb{T}$). For a flow system $(X_{s,t})_{s<t}$, random variables $X_{0,0}$ and $X_{t,t}$ are generally different; $X_{0,0} = X_{0,t}X_{t,0}$ but $X_{t,t} = X_{t,0}X_{0,t}$. (Also $G_{0,0}$ and $G_{t,t}$ are generally different.) For G-flows in a group G these random variables are conjugate: $X_{t,t} = X_{0,t}^{-1} X_{0,0} X_{0,t}$. If G is commutative then $X_{t,t} = X_{0,0}$, but generally $X_{t,t} \ne X_{0,0}$. Nevertheless $\mathcal{F}_{0,0} = \mathcal{F}_{t,t}$ (it is the σ-field generated by the whole flow), which leads to $(\Omega_{0,0}, P_{0,0}) = (\Omega_{t,t}, P_{t,t})$ and $H_{0,0} = H_{t,t}$ where $H_{s,t} = L_2(\mathcal{F}_{s,t}) = L_2(\Omega_{s,t}, P_{s,t})$. Transferring Definition 2c1 to the time set $\mathbb{T}$ we get $\mathcal{F}_{0,0} = \mathcal{F}_{0,t} \otimes \mathcal{F}_{t,0} = \mathcal{F}_{t,0} \otimes \mathcal{F}_{0,t} = \mathcal{F}_{t,t}$. Using the approach of Definition 2c6 we identify $\Omega_{0,0}$ and $\Omega_{t,t}$ according to $\Omega_{0,0} = \Omega_{0,t} \times \Omega_{t,0} = \Omega_{t,0} \times \Omega_{0,t} = \Omega_{t,t}$. Similarly, when transferring Definition 3a1 to $\mathbb{T}$ we identify $H_{0,0}$ and $H_{t,t}$ according to $H_{0,0} = H_{0,t} \otimes H_{t,0} = H_{t,0} \otimes H_{0,t} = H_{t,t}$. We may denote $H_{0,0}$ by $H_{\mathbb{T}}$ and write $H_{t,t} = H_{\mathbb{T}}$ for all $t \in \mathbb{T}$; similarly, $\Omega_{t,t} = \Omega_{\mathbb{T}}$ etc. (However, $X_{\mathbb{T}}$ makes sense only in commutative semigroups.)

Cyclic-time systems (of various kinds) correspond naturally to *periodic* linear-time systems. Here 'periodic' means, invariant under the discrete group of time shifts $t \mapsto t+n$, $n \in \mathbb{Z}$.

Homogeneous linear-time systems correspond to homogeneous cyclic-time systems. Here homogeneity is defined as before (in Definitions 2d1, 3c1) via shifts of the cyclic time set $\mathbb{T}$.

Given a (linear-time) algebraic product system of Hilbert spaces (or equivalently, a local homogeneous continuous product of Hilbert spaces), we may consider the corresponding cyclic-time system. The latter (in contrast to the former) stipulates the 'global' Hilbert space $H_{\mathbb{T}}$, and a group $(\theta^h_{\mathbb{T}})_{h \in \mathbb{T}}$ of unitary operators on $H_{\mathbb{T}}$. In terms of the local homogeneous continuous product of Hilbert spaces, $H_{\mathbb{T}} = H_{0,1}$ and $\theta^t_{\mathbb{T}}(fg) = (\theta^{-t}_{t,1} g)(\theta^{1-t}_{0,t} f)$ for $f \in H_{0,t}$, $g \in H_{t,1}$, $t \in (0,1)$. In terms of the algebraic product system of Hilbert spaces, $H_{\mathbb{T}} = H_{0,1}$ and $\theta^t_{\mathbb{T}}(fg) = gf$ for $f \in H_t$, $g \in H_{1-t}$, $t \in (0,1)$.

3d1 Theorem. (Liebscher [19, Th. 7]) A (linear-time) algebraic product system of Hilbert spaces can be upgraded to an Arveson system if and only if the corresponding cyclic-time shift operators $\theta^h_{\mathbb{T}}$ are a Borel measurable (therefore continuous) function of $h \in \mathbb{T}$.

Let us apply Liebscher's criterion to the pathologic example of Sect. 3c. We have $\tilde\theta^t_{\mathbb{T}}\big(\tilde W_{t,1-t}(f \otimes g)\big) = \tilde W_{1-t,t}(g \otimes f)$ for $f \in H_t$, $g \in H_{1-t}$, $t \in (0,1)$. That is, $\tilde\theta^t_{\mathbb{T}}(f \cdot U^{\varphi(t)}_{1-t} g) = g \cdot U^{\varphi(1-t)}_t f$; as before, $f \cdot g$ means $W_{t,1-t}(f \otimes g)$ rather than $\tilde W_{t,1-t}(f \otimes g)$. We have $\tilde\theta^t_{\mathbb{T}}(f \cdot g) = (U^{\varphi(-t)}_{1-t} g) \cdot (U^{\varphi(1-t)}_t f)$, which means that $\tilde\theta^t_{\mathbb{T}}$ is not a measurable function of t. Indeed, we may take $f = \exp(iX^{(1)}_{0,t})$ and $g = \exp(iX^{(1)}_{0,1-t})$; here $(X^{(1)}_{s,t}, X^{(2)}_{s,t})$ are the increments of the underlying isotropic two-dimensional Lévy process. Then $f \cdot g = \exp(iX^{(1)}_{0,t}) \exp(iX^{(1)}_{t,1}) = \exp(iX^{(1)}_{0,1})$ does not depend on t, but $\tilde\theta^t_{\mathbb{T}}(f \cdot g) = \exp\big(i(X^{(1)}_{0,1-t} \cos\varphi(-t) - X^{(2)}_{0,1-t} \sin\varphi(-t))\big) \exp\big(i(X^{(1)}_{1-t,1} \cos\varphi(1-t) - X^{(2)}_{1-t,1} \sin\varphi(1-t))\big)$. Even in the special case $\varphi(1) = 0$ we get $\exp\big(i(X^{(1)}_{0,1} \cos\varphi(t) + X^{(2)}_{0,1} \sin\varphi(t))\big)$, which is not measurable in t.

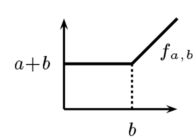
4 Singularity concentrated in space (examples)

4a Coalescence: another way to the white noise

A model described here is itself of little interest, but helps to understand more interesting models introduced afterwards.

Functions $[0,\infty) \to [0,\infty)$ of the form $f_{a,b}$,

$$f_{a,b}(x) = a + \max(x,b)\,,$$



for $a, b \in \mathbb{R}$, $b \ge 0$, $a + b \ge 0$, form a semigroup G. That is, the composition $fg = g \circ f : x \mapsto g(f(x))$ of two such functions is such a function, again:

$$f_{a_1,b_1} f_{a_2,b_2} = f_{a,b}\,, \quad \begin{aligned} a &= a_1 + a_2\,, \\ b &= \max(b_1, b_2 - a_1)\,. \end{aligned}$$

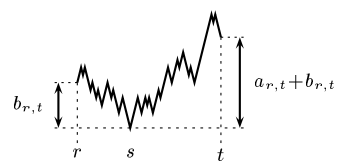
Equipped with the evident topology, G is a two-dimensional topological semigroup. The following probability distributions are a weakly continuous convolution semigroup $(\mu_t)_{t>0}$ in G:

$$\frac{\mu_t(\mathrm{d}a\mathrm{d}b)}{\mathrm{d}a\mathrm{d}b} = \frac{2(a+2b)}{\sqrt{2\pi}\, t^{3/2}} \exp\Big(- \frac{(a+2b)^2}{2t} \Big) .$$

It leads to a stationary G-flow $(X_{s,t})_{s<t}$; $X_{s,t} = f_{a_{s,t},b_{s,t}}$.

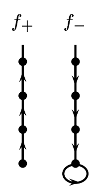
The map $(a,b) \mapsto a$ is a homomorphism $G \to (\mathbb{R},+)$. It sends μ_t to the normal distribution $\mathrm{N}(0,1)$, which means that $a_{s,t}$ is nothing but the increment of the standard Brownian motion $(a_{0,t})_t$ in $\mathbb{R}$. It appears that

$$b_{r,t} = - \min_{s\in[r,t]} a_{r,s} \, ,$$
$$a_{r,t} + b_{r,t} = \max_{s\in[r,t]} a_{s,t} \, .$$



The 'two-dimensional nature' of the flow is a delusion; the second dimension b reduces to the first dimension a. The noise generated by this G-flow is (isomorphic to) the white noise.

The G-flow $(X_{s,t})_{s<t}$ may be treated as the scaling limit of a discrete-time G-flow formed by (compositions of) two functions $f_+, f_- : \mathbb{Z}_+ \to \mathbb{Z}_+$ (chosen equiprobably),



$$f_+(x) = x+1 \, , \quad f_-(x) = \max(0, x-1) \, .$$

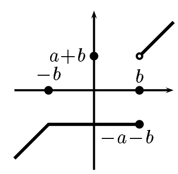
The semigroup spanned by f_-, f_+ may also be treated as the semigroup (with unit, non-commutative) defined by two generators f_-, f_+ and a single relation $f_+f_- = 1$. (The second relation $f_-f_+ = 1$ would turn the semigroup into $\mathbb{Z}$, giving in the scaling limit the homomorphism $G \to \mathbb{R}$ mentioned above.)

The one-point motion $(X_{0,t}(x))_{t>0}$ of our G-flow is (distributed like) the reflecting Brownian motion (starting at x). Two particles starting at x_1, x_2 $(x_1 < x_2)$ keep their distance $(X_{0,t}(x_2) - X_{0,t}(x_1) = x_2 - x_1)$ as long as the boundary is not hit $(X_{0,t}(x_1) > 0)$. In general, the distance decreases in time. At some instant s (when $b_{0,s}$ reaches x_2) the two particles coalesce at the boundary point $(X_{0,s}(x_1) = X_{0,s}(x_2) = 0)$ and never diverge afterwards $(X_{0,t}(x_1) = X_{0,t}(x_2)$ for all $t \in [s,\infty))$.

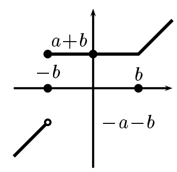
4b Splitting: a nonclassical noise

Functions $\mathbb{R} \to \mathbb{R}$ of two forms, $f^-_{a,b}$ and $f^+_{a,b}$,

$$f^-_{a,b}(x) = \begin{cases} x - a & \text{for } x \in (-\infty, -b), \\ -a - b & \text{for } x \in [-b, b], \\ x + a & \text{for } x \in (b, \infty); \end{cases}$$



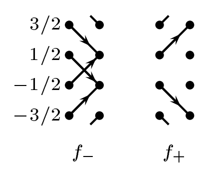
$$f^+_{a,b}(x) = \begin{cases} x - a & \text{for } x \in (-\infty, -b), \\ a + b & \text{for } x \in [-b, b], \\ x + a & \text{for } x \in (b, \infty) \end{cases}$$



for $a, b \in \mathbb{R}$, $b \ge 0$, $a + b \ge 0$, form a semigroup G. It is not a topological semigroup, since the composition is not continuous, but it is a topo-semigroup (as defined by 2d8). The map $f^-_{a,b} \mapsto f^{4a}_{a,b}$, $f^+_{a,b} \mapsto f^{4a}_{a,b}$ is a homomorphism $G \to G^{4a}$; here G^{4a} stands for the semigroup denoted by G in Sect. 4a. We define a measure μ_t on G by two conditions: first, the homomorphism $G \to G^{4a}$ sends μ_t to μ^{4a}_t, and second, μ_t is invariant under the map $f^-_{a,b} \mapsto f^+_{a,b}$, $f^+_{a,b} \mapsto f^-_{a,b}$. In other words, a and b are distributed as in Sect. 4a, while the third parameter is '$-$' or '$+$' with probabilities 1/2, 1/2, independently of a, b. These distributions are a convolution semigroup. Proposition 2d9 gives us a stationary G-flow $(X_{s,t})_{s<t}$ and a noise, — *the noise of splitting.* It is a nonclassical noise! (See Sect. 5d.)

The G-flow $(X_{s,t})_{s<t}$ may be treated as the scaling limit of a discrete-time G-flow formed by (compositions of) two functions $f_+, f_- : \mathbb{Z} + \frac12 \to \mathbb{Z} + \frac12$ (chosen equiprobably),

$$\begin{aligned} &f_-(x) = x - 1\,, \\ &f_+(x) = x + 1 \end{aligned} \quad \text{for } x \in \big(\mathbb{Z} + \tfrac12\big) \cap (0, \infty)\,,$$
$$f_-(-x) = -f_-(x)\,, \quad f_+(-x) = -f_+(x)\,.$$



They satisfy the relation $f_+ f_- = 1$ and generate the same (discrete) semigroup as in Sect. 4a, but the scaling limit is different, since here (in contrast to 4a) the product $(f_-)^b (f_+)^{a+b}$ for large a, b is sensitive to $(-1)^b$.

The G-flow $(X_{s,t})_{s<t}$ is intertwined with the G^{4a}-flow $(X^{4a}_{s,t})_{s<t}$ by the map $\mathbb{R} \to [0, \infty)$, $x \mapsto |x|$. Indeed, $|f^\pm_{a,b}(x)| = f^{4a}_{a,b}(|x|)$. The radial part $|X_{0,t}(x)|$ is (distributed like) the coalescing flow of Sect. 4a. The sign of $X_{0,t}(x)$, being independent of the radial motion, is chosen anew each time when the

radial motion starts an excursion. The one-point motion is just the standard Brownian motion in $\mathbb{R}$.

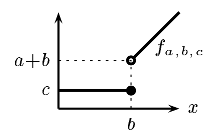
Similarly we may take the space set as the union $\{z \in \mathbb{C} : z^3 \in [0,\infty)\}$ of three (or more) rays on the complex plane and define a splitting flow such that its radial part is the coalescing flow, and the argument (the angular part) is chosen anew (with probabilities $1/3, 1/3, 1/3$) each time when starting an excursion. Then the one-point motion is a complex-valued martingale known as the spider martingale, see [5, Sect. 2].

The noise of splitting was introduced and investigated by J. Warren [43]. See also [46], [40, Example 1d1], and Sections 4d, 5d of this survey.

4c Stickiness: a time-asymmetric noise

Functions $[0,\infty) \to [0,\infty)$ of the form $f_{a,b,c}$,

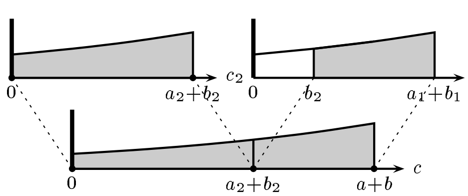
$$f_{a,b,c}(x) = \begin{cases} c & \text{for } x \in [0,b], \\ x+a & \text{for } x \in (b,\infty) \end{cases}$$



for $a,b,c \in \mathbb{R}$, $b \ge 0$, $0 \le c \le a+b$, form a semigroup G (a topo-semigroup, not topological). The map $f_{a,b,c} \mapsto f^{4\mathrm{a}}_{a,b}$ is a homomorphism $G \to G^{4\mathrm{a}}$. In fact, $G^{4\mathrm{a}} = \{f_{a,b,a+b}\}$ is a sub-semigroup of G, therefore the convolution semigroup $(\mu^{4\mathrm{a}}_t)_{t>0}$ in $G^{4\mathrm{a}}$ is also a convolution semigroup in G; it is a degenerate case $(\lambda = 0)$ of a family of convolution semigroups $(\mu^{(\lambda)}_t)_{t>0}$ on G; the parameter λ runs over $(0,\infty)$. Namely,

$$\begin{gathered} \mu^{(\lambda)}_t \text{ is the joint distribution of } a,b \text{ and } c = \max(0, a+b-\lambda\eta), \\ \text{where the pair } (a,b) \text{ is distributed } \mu^{4\mathrm{a}}_t, \\ \text{while } \eta \text{ is independent of } (a,b) \text{ and distributed } \mathrm{Exp}(1)\,; \end{gathered}$$

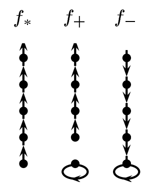
that is, $\mathbb{P}\big(\eta > c\big) = e^{-c}$ for $c \in [0,\infty)$. It is indeed a convolution semigroup, due to a property of the composition in G ($c = a_2 + c_1$ if $c_1 > b_2$, otherwise c_2; about a, b see Sect. 4a): for every a_1, b_1, a_2, b_2, if $c_1 \sim \max(0, a_1 + b_1 - \lambda\eta)$ and $c_2 \sim \max(0, a_2 + b_2 - \lambda\eta)$ are independent then $c \sim \max(0, a+b-\lambda\eta)$.



(The case $a_1 + b_1 > b_2$ is shown; the other case is trivial.)

Note that the measure $\mu_t^{(\lambda)}$ has an absolutely continuous part (its three-dimensional density can be written explicitly, using the two-dimensional density of μ_t^{4a} and the one-dimensional exponential density of η) and a singular part concentrated on the plane $c = 0$; the singular part has a two-dimensional density (it can also be written explicitly). Proposition 2d9 gives us a stationary G-flow $(X_{s,t})_{s<t}$ and a noise, — *the noise of stickiness.* It is a nonclassical noise. Moreover, the noise is time-asymmetric! (See Sect. 4e.)

The G-flow $(X_{s,t})_{s<t}$ may be treated as the scaling limit of a discrete-time G-flow formed by (compositions of) three functions $f_+, f_-, f_* : \mathbb{Z}_+ \to \mathbb{Z}_+$:



$$f_*(x) = x + 1\,, \quad f_-(x) = \max(0, x - 1)\,,$$
$$f_+(x) = \begin{cases} x + 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$

The functions are chosen with probabilities $\mathbb{P}\big(f_-\big) = \frac{1}{2}$, $\mathbb{P}\big(f_*\big) = \frac{1}{2\lambda}\sqrt{\Delta t}$, $\mathbb{P}\big(f_+\big) = \frac{1}{2} - \frac{1}{2\lambda}\sqrt{\Delta t})$, where Δt is the time pitch (tending to 0 in the scaling limit); the space pitch is equal to $\sqrt{\Delta t}$. The semigroup spanned by f_-, f_+, f_* may also be treated as the semigroup defined by three generators f_-, f_+, f_* and three relations $f_+ f_- = 1$, $f_* f_- = 1$, $f_* f_+ = f_* f_*$.

The one-point motion $(X_{0,t}(x))_{t>0}$ of our G-flow is (distributed like) the sticky Brownian motion (starting at x). A particle spends a positive time at the origin, but never sits there during a time interval. Two particles keep a constant distance until one of them reaches the origin. Generally, the distance is non-monotone. But ultimately the two particles coalesce.

The noise of stickiness was introduced and investigated by J. Warren [44]. See also [40, Sect. 4], and Sections 4e, 5d of this survey.

4d Warren's noise of splitting

The noise of splitting consists of σ-fields generated by random variables $a_{s,t}$, $b_{s,t}$ and $\tau_{s,t}$ according to the parameters a, b, τ of an element $f^\tau_{a,b}$ of the semigroup G $(= G^{\text{4b}})$; $b \ge 0$, $a+b \ge 0$, and $\tau = \pm 1$. We may drop $b_{s,t}$ but not $\tau_{s,t}$. The binary operation in G is such that (assuming $r < s < t$) $\tau_{r,t}$ is either $\tau_{r,s}$ or $\tau_{s,t}$ depending on whether the minimum of the Brownian motion $B_u = a_{0,u}$ on $[r, t]$ is reached on $[r, s]$ or $[s, t]$. It means that the random sign $\tau_{r,t}$ may be assigned to the minimizer $s \in [r, t]$ of the Brownian motion on $[r, t]$. "This is a noise richer than white noise: in addition to the increments of a Brownian motion B it carries a countable collection of independent Bernoulli random variables which are attached to the local minima of B" [43, the last phrase].

It may seem that these Bernoulli random variables appear suddenly, having no precursors in the past (like jumps of the Poisson process). However, this is a delusion.

4d1 Definition. A continuous product of probability spaces $(\mathcal{F}_{s,t})_{s<t}$ is *predictable,* if the filtration $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$ admits of no discontinuous martingales.

Equivalently: for every stopping time T (w.r.t. the filtration $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$) there exist stopping times T_n such that $T_n < T$ and $T_n \to T$ a.s.

The white noise is predictable; the Poisson noise is not.

The noise of splitting is predictable.

What is wrong in saying 'each one of these Bernoulli random variables appears suddenly at the corresponding instant'? The very beginning 'each one of these' is misleading. We cannot number them in real time. Rather, we can consider (say) $\tau_{0,1}$, the Bernoulli random variable attached to the minimizer of B on $[0,1]$. Its conditional expectation, given $\mathcal{F}_{-\infty,t}$ $(0 < t < 1)$, does not jump, since we do not know (at t) whether the minimum was already reached or not; the corresponding probability is continuous in t.

Is there anything special in local minima of the Brownian path? Any other random dense countable set could be used equally well, if it satisfies two conditions, locality and stationarity, formalized below. However, what should we mean by a 'random dense countable set'? The set of all dense countable subsets of $\mathbb{R}$ does not carry a natural structure of a standard measurable space. (Could you imagine a function of the set of all Brownian local minimizers that gives a non-degenerate random variable?) They form a *singular space* in the sense of Kechris [18, §2]: a 'bad' quotient space of a 'good' space by a 'good' equivalence relation. Several possible interpretations of 'good' and 'bad' are discussed in [18], but we restrict ourselves to few noise-related examples.

The space $\mathbb{R}^\infty$ of all infinite sequences $(t_1, t_2, \dots)$ of real numbers is naturally a standard measurable space. The group S_∞ of all bijective maps $\{1, 2, \dots\} \to \{1, 2, \dots\}$ acts on $\mathbb{R}^\infty$ by permutations: $(t_1, t_2, \dots) \mapsto (t_{n_1}, t_{n_2}, \dots)$. The Borel subset $\mathbb{R}^\infty_{\ne} \subset \mathbb{R}^\infty$ of all sequences of *pairwise different* numbers $t_1, t_2, \dots$ is S_∞-invariant, and the set of orbits $\mathbb{R}^\infty_{\ne}/S_\infty$ may be identified with the set of all countable subsets of $\mathbb{R}$. The same for $(a, b)^\infty_{\ne}/S_\infty$ and countable subsets of a given interval $(a, b) \subset \mathbb{R}$.

The group $L_0(\Omega \to S_\infty)$ of *random permutations* acts on the space $L_0(\Omega \to \mathbb{R}^\infty)$ of *random sequences.* The subset $L_0(\Omega \to \mathbb{R}^\infty_{\ne})$ is invariant under random permutations. We treat the quotient space $L_0(\Omega \to \mathbb{R}^\infty_{\ne})/L_0(\Omega \to S_\infty)$ as a well-defined substitute of the ill-defined $L_0(\Omega \to \mathbb{R}^\infty_{\ne}/S_\infty)$. A random countable set is treated as a random sequence up to a random permutation.

Local minimizers of a Brownian path are such a random set; that is, they admit a measurable enumeration. Here is a simple construction for $(0,1)$. First, $t_1(\omega)$ is the minimizer on the whole $(0,1)$ (unique almost sure). Second, if $t_1(\omega) \in (0,1/2)$ then $t_2(\omega)$ is the minimizer on $(1/2,1)$, otherwise — on $(0,1/2)$. Third, $t_3(\omega)$ is the minimizer on the first of the four intervals $(0,1/4)$, $(1/4,1/2)$, $(1/2,3/4)$ and $(3/4,1)$ that contains neither $t_1(\omega)$ nor $t_2(\omega)$. And so on.

Random sets $M_{s,t}$ of Brownian minimizers on intervals $(s,t) \subset \mathbb{R}$ satisfy $M_{r,s} \cup M_{s,t} = M_{r,t}$ for $r < s < t$ (almost sure, s is not a local minimizer), and $M_{s,t}$ depends only on the increments of B on (s,t).

4d2 Definition. Let $(\Omega_{s,t}, P_{s,t})_{s<t}$ be a continuous product of probability spaces. A *local random dense countable set* (over the continuous product) is a family $(N_{s,t})_{s<t}$ of random sets

$$N_{s,t} \in L_0(\Omega_{s,t} \to (s,t)^\infty_{\ne}) / L_0(\Omega_{s,t} \to S_\infty)$$

satisfying

$$N_{r,s} \cup N_{s,t} = N_{r,t} \quad \text{a.s.}$$

whenever $r < s < t$.

The same may be said in terms of $(\mathcal{F}_{s,t})_{s<t}$. In this case $N_{s,t}$ is defined via $\mathcal{F}_{s,t}$-measurable random sequences modulo $\mathcal{F}_{s,t}$-measurable random permutations,

$$N_{s,t} \in L_0(\mathcal{F}_{s,t} \to (s,t)^\infty_{\ne}) / L_0(\mathcal{F}_{s,t} \to S_\infty) \, .$$

Now we add stationarity.

4d3 Definition. Let $(\mathcal{F}_{s,t})_{s<t}$, $(T_h)_h$ be a noise. A *stationary local random dense countable set* (over the noise) is a family $(N_{s,t})_{s<t}$ of random sets

$$N_{s,t} \in L_0(\mathcal{F}_{s,t} \to (s,t)^\infty_{\ne}) / L_0(\mathcal{F}_{s,t} \to S_\infty)$$

satisfying

$$\begin{gathered} N_{r,s} \cup N_{s,t} = N_{r,t} \quad \text{a.s.}, \\ N_{s,t} \circ T_h = N_{s+h,t+h} \quad \text{a.s.} \end{gathered}$$

whenever $r < s < t$, $h \in \mathbb{R}$.

Brownian minimizers are an example of a stationarity local random dense countable set over the white noise. Brownian maximizers are another example. Their union is the third example.

4d4 Question. Do these three examples exhaust *all* stationarity local random dense countable sets over the white noise?

New examples could lead to new noises.

4e Warren's noise of stickiness, made by a Poisson snake

The noise of stickiness consists of σ-fields generated by random variables $a_{s,t}$, $b_{s,t}$ and $c_{s,t}$ according to the parameters a, b, c of an element $f_{a,b,c}$ of the semigroup G $(= G^{4c})$; $b \ge 0$, $0 \le c \le a + b$. We may drop $b_{s,t}$ but not $c_{s,t}$.

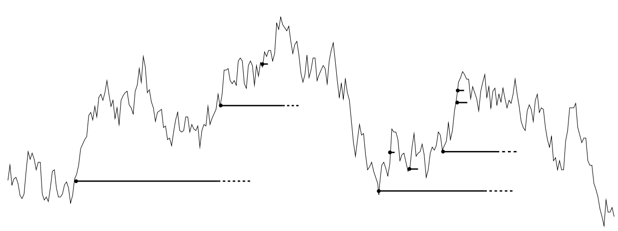
Consider the (random) set $C_t = \{c_{s,t} : s \in (-\infty, t)\} \setminus \{0\}$; its points will be called 'spots'. For a small Δt usually (with probability $1 - O(\sqrt{\Delta t})$) $c_{t,t+\Delta t} = 0$ (since $a + b - \lambda\eta < 0$, recall Sect. 4c), therefore $C_{t+\Delta t} = (C_t + a_{t,t+\Delta t}) \cap (a_{t,t+\Delta t} + b_{t,t+\Delta t}, \infty)$. We see that the spots move up and down, driven by Brownian increments. The boundary annihilates the spots that hit it. However, sometimes the boundary creates new spots. It happens (with probability $\sim \text{const} \cdot \sqrt{\Delta t}$) when $c_{t,t+\Delta t} > 0$.

An observer that moves according to the Brownian increments sees a set $C_t - a_{0,t}$ of fixed spots on the changing ray $(-a_{0,t}, \infty)$. The spotted ray may be called a *Poisson snake.* The movement of its endpoint $(-a_{0,t})$ is Brownian. When the snake shortens, some spots disappear on the moving boundary. When the snake lengthens, new spots appear on the moving boundary. It happens with a rate infinite in time but finite in space. Infinitely many spots appear (and disappear) during any time interval (because of locally infinite variation of a Brownian path); only a finite number of them survive till the end of the interval. In fact, at every instant the spots are (distributed like) a Poisson point process of rate $1/\lambda$ on $(-a_{0,t}, \infty)$.

Being discrete in space, the spots may seem to appear suddenly in time (like jumps of the usual Poisson process). However. this is a delusion (similarly to Sect. 4d).

The noise of stickiness is predictable.

A spot can appear at an instant s only if s is 'visible from the right' in the sense that $a_{s,t} > 0$ for all t close enough to s (that is, $\exists \varepsilon > 0\ \forall t \in (s, s + \varepsilon)\ \ a_{s,t} > 0$).



A random dense countable subset of the continuum of points visible from the right. Few chords are shown, others are too short.

Points visible from the right are (a.s.) a dense Borel set of cardinality continuum but Lebesgue measure zero. Knowing the past (according to $\mathcal{F}_{-\infty,s}$) but not the future ($\mathcal{F}_{s,\infty}$) we cannot guess that s is (or rather, will appear to be) visible from the right. (Compare it with Sect. 4d: knowing the past we cannot guess that s is a local minimizer.)

In contrast, knowing the future ($\mathcal{F}_{s,\infty}$) but not the past ($\mathcal{F}_{-\infty,s}$) we know, whether s is visible from the right or not. (This asymmetry reminds me that we often know the date of death of a great man but not the date of birth...)

The time-reversed noise of stickiness is not predictable.

In other words, the filtration $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$ admits of continuous martingales only, but the filtration $(\mathcal{F}_{-t,\infty})_{t\in\mathbb{R}}$ admits of some discontinuous martingales.

All the birth instants (when new spots appear) are (a.s.) a dense countable subset of the set of points visible from the right. Conditionally, given the Brownian path $(B_t)_{t\in\mathbb{R}} = (a_{0,t})_{t\in\mathbb{R}}$, birth instants are a Poisson random subset of $\mathbb{R}$ whose intensity measure is a singular σ-finite measure $(dB)^+$ concentrated on points visible from the right. Such a measure $(df)^+$ may be defined for *every* continuous function f (not just a Brownian path); note that f need not be of locally finite variation. Namely, $(df)^+$ is the supremum (over $t \in \mathbb{R}$) of images of Lebesgue measure on $(-\infty, f(t))$ under the maps $x \mapsto \max\{s \in (-\infty, t) : f(s) = x\}$, provided that $\inf\{f(s) : s \in (-\infty, t)\} = -\infty$ (which holds a.s. for Brownian paths); otherwise $(-\infty, f(t))$ should be replaced with $\big(\inf\{f(s) : s \in (-\infty, t)\},\, f(t)\big)$. The measure $(df)^+$ is always positive and σ-finite, but need not be locally finite. That is, $\mathbb{R}$ can be decomposed into a sequence of Borel subsets of finite measure $(df)^+$. However, it does not mean that all (or even, some) intervals are of finite measure $(df)^+$.

The σ-finite positive measure $(dB)^+$ is infinite on every interval (because of locally infinite variation of the Brownian path). Such measures are a singular space (recall Sect. 4d); a *random* element of such a space should be treated with great care. Interestingly, the singular space of Sect. 4d is naturally embedded into the singular space considered here. Indeed, every dense countable set may be identified with its counting measure (consisting of atoms of mass 1); the measure is σ-finite, but infinite on every interval. In contrast, the measure $(dB)^+$ is non-atomic.

Similarly to Sect. 4d it should be possible to define a stationary local random σ-finite positive measure, infinite on every interval, over (say) the white noise. One example is $(dB)^+$. Replacing B_t with $-B_t$, or B_{-t}, or $-B_{-t}$ we get three more examples. Similarly to Question 4d4 we may ask: do these four examples and their linear combinations exhaust *all* possible cases? New examples could lead to new noises.

See also [44] and [40, Sect. 4].

5 Stability and sensitivity

5a Morphism, joining, maximal correlation

The idea, presented in Sect. 1c, is formalized below.

5a1 Definition. Let $\big((\Omega_1, P_1), (\mathcal{F}^{(1)}_{s,t})_{s<t}\big)$ and $\big((\Omega_2, P_2), (\mathcal{F}^{(2)}_{s,t})_{s<t}\big)$ be two continuous products of probability spaces.

(a) A *morphism* from the first product to the second is a morphism of probability spaces $\alpha : \Omega_1 \to \Omega_2$, measurable from $(\Omega_1, \mathcal{F}^{(1)}_{s,t})$ to $(\Omega_2, \mathcal{F}^{(2)}_{s,t})$ whenever $s < t$.

(b) An *isomorphism* from the first product to the second is a morphism α such that the inverse map α^{-1} exists and is also a morphism (of the products).

If a morphism of products is an isomorphism of probability spaces then it is an isomorphism of products.

5a2 Example. Let $(B^{(1)}_t, B^{(2)}_t)_{t\in[0,\infty)}$ be the standard Brownian motion in $\mathbb{R}^2$, and $\big((\Omega_1, P_1), (\mathcal{F}^{(1)}_{s,t})_{s<t}\big)$ be the continuous product of probability spaces generated by the (two-dimensional) increments $(B^{(1)}_t - B^{(1)}_s, B^{(2)}_t - B^{(2)}_s)$. Let $\big((\Omega_2, P_2), (\mathcal{F}^{(2)}_{s,t})_{s<t}\big)$ correspond in the same way to the standard Brownian motion $(B_t)_{t\in[0,\infty)}$ in $\mathbb{R}$. Then for every $\varphi \in \mathbb{R}$ the formula

$$B_t = B^{(1)}_t \cos\varphi + B^{(2)}_t \sin\varphi$$

defines a morphism (not an isomorphism, of course) from the first product to the second.

5a3 Definition. A *morphism* from a noise to another noise is a morphism α between the corresponding continuous products of probability spaces that intertwines the corresponding shifts:

$$\alpha \circ T^{(1)}_h = T^{(2)}_h \circ \alpha \quad \text{a.s.}$$

for every $h \in \mathbb{R}$.

Similarly to Example 5a2 we have for each φ a morphism from the two-dimensional white noise to the one-dimensional white noise.

5a4 Example. The homomorphism $f_{a,b,c} \mapsto a$ from the semigroup $G^{4\text{c}}$ ($= G$ of Sect. 4c) to $(\mathbb{R}, +)$ leads to a morphism (not an isomorphism) from the noise of stickiness to the (one-dimensional) white noise. The same holds for the noise of splitting.

5a5 Definition. A *joining* (or *coupling*) of two continuous products of probability spaces $\big((\Omega_1, P_1), (\mathcal{F}^{(1)}_{s,t})_{s<t}\big)$ and $\big((\Omega_2, P_2), (\mathcal{F}^{(2)}_{s,t})_{s<t}\big)$ consists of a third continuous product of probability spaces $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ and two morphisms $\alpha : \Omega \to \Omega_1$, $\beta : \Omega \to \Omega_2$ of these products such that $\mathcal{F}_{-\infty,\infty}$ is generated by α, β (that is, by inverse images of $\mathcal{F}^{(1)}_{-\infty,\infty}$ and $\mathcal{F}^{(2)}_{-\infty,\infty}$).

Each joining leads to a measure on $\Omega_1 \times \Omega_2$ with given projections P_1, P_2; namely, the image of P under the (one-to-one) map $\omega \mapsto (\alpha(\omega), \beta(\omega))$. Two joinings that lead to the same measure (on $\Omega_1 \times \Omega_2$) will be called isomorphic.

A joining of a continuous product of probability spaces with itself will be called a *self-joining.* A *symmetric* self-joining is a self-joining (α, β) isomorphic to (β, α). For example, every pair of angles φ, ψ leads to a symmetric self-joining of the (one-dimensional) white noise,

$$\begin{aligned} B_t \circ \alpha &= B^{(1)}_t \cos\varphi + B^{(2)}_t \sin\varphi \,, \\ B_t \circ \beta &= B^{(1)}_t \cos\psi + B^{(2)}_t \sin\psi \,. \end{aligned} \tag{5a6}$$

Only the difference $|\alpha - \beta|$ matters (up to isomorphism).

Every joining (α, β) of two continuous products of probability spaces has its *maximal correlation*

$$\rho^{\max}(\alpha, \beta) = \sup |\mathbb{E}\,(f \circ \alpha)(g \circ \beta)| \,,$$

where the supremum is taken over all $f \in L_2(\Omega_1, P_1)$, $g \in L_2(\Omega_2, P_2)$ such that $\mathbb{E}\, f = 0$, $\mathbb{E}\, g = 0$, $\operatorname{Var} f \le 1$, $\operatorname{Var} g \le 1$. (All L_2 spaces are real, not complex.) The product structure is irrelevant to the 'global' correlation $\rho^{\max}(\alpha, \beta)$, but relevant to 'local' correlations $\rho^{\max}_{s,t}(\alpha, \beta)$; here the supremum is taken under an additional condition: f is $\mathcal{F}^{(1)}_{s,t}$-measurable, and g is $\mathcal{F}^{(2)}_{s,t}$-measurable. Surprisingly, the global correlation is basically the supremum of local correlations over infinitesimal time intervals.

5a7 Proposition. $\rho^{\max}_{r,t}(\alpha, \beta) = \max\big(\rho^{\max}_{r,s}(\alpha, \beta),\, \rho^{\max}_{s,t}(\alpha, \beta)\big)$ whenever $r < s < t$.

Proof (sketch). Generally, $L_2(\mathcal{F}_{r,t}) = L_2(\mathcal{F}_{r,s}) \otimes L_2(\mathcal{F}_{s,t}) = \big(\mathbb{R} \oplus L_2^0(\mathcal{F}_{r,s})\big) \otimes \big(\mathbb{R} \oplus L_2^0(\mathcal{F}_{s,t})\big) = \mathbb{R}\otimes\mathbb{R} \oplus \mathbb{R}\otimes L_2^0(\mathcal{F}_{s,t}) \oplus L_2^0(\mathcal{F}_{r,s})\otimes\mathbb{R} \oplus L_2^0(\mathcal{F}_{r,s})\otimes L_2^0(\mathcal{F}_{s,t})$, where $\mathbb{R}$ is the one-dimensional space of constants, and $L_2^0(\dots)$ — its orthogonal complement (the zero-mean space). We apply the argument to $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, decompose f and g into three orthogonal summands each ($\mathbb{R} \otimes \mathbb{R}$ does not appear), and get the maximum of $\rho^{\max}_{r,s}$, $\rho^{\max}_{s,t}$ and $\rho^{\max}_{r,s}\rho^{\max}_{s,t}$. $\square$

In fact, $\rho^{\max}(\alpha, \beta) = \cos(\varphi - \psi)$ for the self-joining (5a6).

5b A generalization of the Ornstein-Uhlenbeck semigroup

Here is the 'best' self-joining for a given maximal correlation. (See also [45, Lemma 2.1].)

5b1 Proposition. Let $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ be a continuous product of probability spaces, and $\rho \in [0, 1]$. Then there exists a symmetric self-joining $(\alpha_\rho, \beta_\rho)$ of the given product such that

$$\rho^{\max}(\alpha_\rho, \beta_\rho) \le \rho$$

and

$$|\mathbb{E}\,(f \circ \alpha)(f \circ \beta)| \le \mathbb{E}\,(f \circ \alpha_\rho)(f \circ \beta_\rho)$$

for all $f \in L_2(\Omega)$ and all self-joinings (α, β) satisfying $\rho^{\max}(\alpha, \beta) \le \rho$.

The self-joining $(\alpha_\rho, \beta_\rho)$ is unique up to isomorphism.

DIGRESSION: THE COMPACT SPACE OF JOININGS

Here we forget about continuous products and deal with joinings of two probability spaces. Let them be just $[0, 1]$ with Lebesgue measure; the nonatomic case is thus covered. (Atoms do not invalidate the results and are left to the reader.) Up to isomorphism, joinings are probability measures μ on the square $[0, 1] \times [0, 1]$ with given (Lebesgue) projections to both coordinates. They are a closed subset of the compact metrizable space of all probability measures on the square, equipped with the weak topology (generated by integrals of continuous functions). Surprisingly, the topology of $[0, 1]$ plays no role.

5b2 Lemma. Let $\mu, \mu_1, \mu_2, \dots$ be probability measures on $[0, 1] \times [0, 1]$ with Lebesgue projections to both coordinates. Then the following conditions are equivalent.

(a) $\int f \,\mathrm{d}\mu_n \to \int f \,\mathrm{d}\mu$ for all continuous functions $f : [0, 1] \times [0, 1] \to \mathbb{R}$;

(b) $\mu_n(A \times B) \to \mu(A \times B)$ for all measurable sets $A, B \subset [0, 1]$.

(c) $\int f(x)g(y)\, \mu_n(\mathrm{d}x\mathrm{d}y) \to \int f(x)g(y)\, \mu(\mathrm{d}x\mathrm{d}y)$ for all $f, g \in L_2[0, 1]$.

Proof (sketch). (c) $\Longrightarrow$ (b): trivial; (b) $\Longrightarrow$ (a): we approximate f uniformly by functions constant on each $\big(\frac{k}{n}, \frac{k+1}{n}\big) \times \big(\frac{l}{n}, \frac{l+1}{n}\big)$.

(a) $\Longrightarrow$ (c): we choose continuous $f_\varepsilon, g_\varepsilon$ such that $\|f - f_\varepsilon\| \le \varepsilon$ and $\|g - g_\varepsilon\| \le \varepsilon$ (the norms are in $L_2[0, 1]$). We apply (a) to the continuous function $(x, y) \mapsto f_\varepsilon(x)g_\varepsilon(y)$ and note that $\iint |f(x)g(y) - f_\varepsilon(x)g_\varepsilon(y)|\, \mu_n(\mathrm{d}x\mathrm{d}y) \le \|f\|\|g - g_\varepsilon\| + \|f - f_\varepsilon\|\|g_\varepsilon\|$. $\square$

(It is basically Slutsky's lemma, see [28, Lemma 0.5.7] or [5, Th. 1]; see also [41, Lemma B3].)

We see that all joinings (of two given standard probability spaces) are (naturally) a compact metrizable space.

Here is another (unrelated to the compactness) general fact about joinings, used in the sequel. It is formulated for $[0,1]$ but holds for all probability spaces.

5b3 Lemma. Let $\mu_1, \mu_2, \dots$ be probability measures on $[0,1] \times [0,1]$ with Lebesgue projections to both coordinates. Then there exists a sub-σ-field $\mathcal{F}$ of the Lebesgue σ-field on $[0,1]$ such that

$$\iint |f(x) - f(y)|^2 \, \mu_n(\mathrm{d}x\mathrm{d}y) \to 0 \quad \text{if and only if } f \text{ is } \mathcal{F}\text{-measurable}$$

for all $f \in L_2[0,1]$.

Proof (sketch). For each n the quadratic form $\mathcal{E}_n(f) = \int |f(x) - f(y)|^2 \, \mu_n(\mathrm{d}x\mathrm{d}y)$ satisfies $\mathcal{E}_n(T \circ f) \le \mathcal{E}_n(f)$ for every $T : \mathbb{R} \to \mathbb{R}$ such that $|T(a) - T(b)| \le |a - b|$ for all $a, b \in \mathbb{R}$. (Thus, $\mathcal{E}_n$ is a continuous symmetric Dirichlet form.) The set E of all f such that $\mathcal{E}_n(f) \to 0$ is a (closed) linear subspace of $L_2[0,1]$, and $f \in E$ implies $T \circ f \in E$. Such E is necessarily of the form $L_2(\mathcal{F})$, see for instance [24, Problem IV.3.1]. □

Note also that every joining (α, β) leads to a bilinear form $(f, g) \mapsto \mathbb{E}\,(f \circ \alpha)(g \circ \beta)$ and the corresponding operator $U_{\alpha,\beta} : L_2[0,1] \to L_2[0,1]$, $\langle U_{\alpha,\beta} f, g \rangle = \mathbb{E}\,(f \circ \alpha)(g \circ \beta)$. Generally $U_{\alpha,\beta}$ maps one L_2 space into another, but for a self-joining we deal with a single space. Clearly, $U_{\beta,\alpha} = (U_{\alpha,\beta})^*$; $U_{\alpha,\beta}$ is Hermitian if and only if the joining is symmetric.

END OF DIGRESSION

Proof (*sketch*) *of Proposition* 5b1. Uniqueness: a self-joining (α, β) is uniquely determined by its bilinear form $(f, g) \mapsto \mathbb{E}\,(f \circ \alpha)(g \circ \beta)$, therefore a symmetric self-joining is uniquely determined by its quadratic form $f \mapsto \mathbb{E}\,(f \circ \alpha)(f \circ \beta)$.

Existence. First, on the space $\Omega \times \Omega$ we consider the maps $\alpha_\rho(\omega_1, \omega_2) = \omega_1$, $\beta_\rho(\omega_1, \omega_2) = \omega_2$ and the measure $\tilde{P}_0 = \rho P_{\mathrm{diag}} + (1 - \rho) P \times P$; here P_{diag} is the image of P under the map $\omega \mapsto (\omega, \omega)$. We get a symmetric self-joining of the probability space (Ω, P), but not of the continuous product $(\mathcal{F}_{s,t})_{s<t}$. Note that $|\mathbb{E}\,(f \circ \alpha)(f \circ \beta)| \le (\mathbb{E}\, f)^2 + \rho \operatorname{Var} f = \int (f \circ \alpha_\rho)(f \circ \beta_\rho) \, \mathrm{d}\tilde{P}_0$ for all $f \in L_2(\Omega)$ and all self-joinings (α, β) satisfying $\rho^{\max}(\alpha, \beta) \le \rho$.

Second, we apply the same construction to $\Omega_{-\infty,0}$ and $\Omega_{0,\infty}$ separately, and consider $\tilde{P}_1 = \tilde{P}_{-\infty,0} \times \tilde{P}_{0,\infty}$. Similarly to the proof of Proposition 5a7 we see that $|\mathbb{E}\,(f\circ\alpha)(f\circ\beta)| \le \int (f\circ\alpha_\rho)(f\circ\beta_\rho)\,\mathrm{d}\tilde P_1 \le \int (f\circ\alpha_\rho)(f\circ\beta_\rho)\,\mathrm{d}\tilde P_0$ for all $f\in L_2(\Omega)$ and all self-joinings (α,β) satisfying $\rho^{\max}(\alpha,\beta)\le\rho$.

Third, we do it for every decomposition $\Omega = \Omega_{-\infty,t_1}\times\Omega_{t_1,t_2}\times\dots\times\Omega_{t_{n-1},t_n}\times\Omega_{t_n,\infty}$ and get a net of measures, symmetric self-joinings, and their quadratic forms

$$
\begin{gathered}
\tilde U^\rho_{t_1,\dots,t_n} f = \bigotimes_{k=0}^n \bigl(\rho f_k + (1-\rho)\mathbb{E}\,f_k\bigr)\,, \\
\langle \tilde U^\rho_{t_1,\dots,t_n} f, f\rangle = \prod_{k=0}^n \bigl((\mathbb{E}\,f_k)^2 + \rho\operatorname{Var} f_k \bigr)
\end{gathered}
\tag{5b4}
$$

for $f=f_0\otimes\dots\otimes f_n$, $f_0\in L_2(\Omega_{-\infty,t_1})$, $f_1\in L_2(\Omega_{t_1,t_2})$, ..., $f_n\in L_2(\Omega_{t_n,\infty})$. The net converges (in the compact space of joinings) due to monotonicity of the net of quadratic forms. The limit is a symmetric self-joining (α_ρ,β_ρ) of the continuous product of probability spaces. It majorizes $|\mathbb{E}\,(f\circ\alpha)(f\circ\beta)|$, since every element of the net does. □

See also [40, 5b4]. Basically, each infinitesimal element of the data set is replaced with a fresh copy, independently of others, with probability $1-\rho$. Doing it twice with parameters ρ_1 and ρ_2 is equivalent to doing it once with parameter $\rho=\rho_1\rho_2$. In terms of operators $U^\rho = U_{\alpha_\rho,\beta_\rho} : L_2(\Omega)\to L_2(\Omega)$ it means $U^{\rho_1}U^{\rho_2} = U^{\rho_1\rho_2}$; a one-parameter semigroup! It seems to lead to an Ω-valued stationary Markov process $(X_u)_{u\in\mathbb{R}}$, $X_u:\tilde\Omega\to\Omega$, such that for every $u>0$ the pair (X_0,X_u) is distributed like the pair (α_ρ,β_ρ) where $\rho=e^{-u}$. However, $L_2(\tilde\Omega)$ is separable only in classical cases.

If the given continuous product of probability spaces is the white noise then the Markov process is the well-known Ornstein-Uhlenbeck process (infinite-dimensional, over the Gaussian measure that describes the white noise).

The proof of the relation $U^{\rho_1}U^{\rho_2} = U^{\rho_1\rho_2}$ is an easy supplement to the proof of Proposition 5b1; an elementary check for each element $\tilde U^\rho_{t_1,\dots,t_n}$ of the net, and a passage to the limit. In the same way we prove that the spectrum of the Hermitian operator U^ρ is contained in $\{1,\rho,\rho^2,\dots\}\cup\{0\}$. The spectral theorem gives the following.

5b5 Proposition. Let $\bigl((\Omega,P),(\mathcal{F}_{s,t})_{s<t}\bigr)$ be a continuous product of probability spaces, (α_ρ,β_ρ) the self-joinings given by Prop. 5b1, and U^ρ the corresponding operators, that is, $\mathbb{E}\,(f\circ\alpha_\rho)(g\circ\beta_\rho) = \langle U^\rho f, g\rangle$ for all $f,g\in L_2(\Omega)$. Then there exist (closed linear) subspaces $H_0,H_1,H_2,\dots$ and H_∞ of $L_2(\Omega)$ such that

$$L_2(\Omega) = (H_0\oplus H_1\oplus H_2\oplus\dots)\oplus H_\infty$$

(that is, the subspaces are orthogonal and span the whole $L_2(\Omega)$), and

$$U^\rho f = \rho^n f \quad \text{for } f \in H_n,\ \rho \in [0,1]\,,$$
$$U^\rho f = 0 \quad \text{for } f \in H_\infty,\ \rho \in [0,1)\,.$$

Of course, $U^1 f = f$ for all f. The semigroup $(U^\rho)_\rho$ is strongly continuous if and only if $\dim H_\infty = 0$. Note also that H_0 is the one-dimensional space of constants.

The spaces H_n may be called *chaos spaces,* since for the white noise H_n is the n-th Wiener chaos space (and $\dim H_\infty = 0$).

5c The stable σ-field; classical and nonclassical

5c1 Definition. Let $\big((\Omega,P),(\mathcal{F}_{s,t})_{s<t}\big)$ be a continuous product of probability spaces.

(a) A random variable $f \in L_2(\Omega)$ is *stable* if there exist symmetric self-joinings (α_n,β_n) of the continuous product such that

$$\rho^{\max}(\alpha_n,\beta_n) < 1 \quad \text{for every } n\,,$$
$$\mathbb{E}\,|f\circ\alpha_n - f\circ\beta_n|^2 \to 0 \quad \text{as } n\to\infty\,.$$

(b) A random variable $f \in L_2(\Omega)$ is *sensitive* if $\mathbb{E}\,(f\circ\alpha)(g\circ\beta) = 0$ for all $g \in L_2(\Omega)$ and all symmetric self-joinings (α,β) of the continuous product such that $\rho^{\max}(\alpha,\beta) < 1$.

5c2 Theorem. (Tsirelson [36, 2.5], [40, 5b11]) For every continuous product of probability spaces $\big((\Omega,P),(\mathcal{F}_{s,t})_{s<t}\big)$ there exists a sub-σ-field $\mathcal{F}^{\text{stable}} \subset \mathcal{F}_{-\infty,\infty}$ such that

$$f \text{ is stable if and only if } f \text{ is } \mathcal{F}^{\text{stable}}\text{-measurable}\,,$$
$$f \text{ is sensitive if and only if } \mathbb{E}\big(f\,\big|\,\mathcal{F}^{\text{stable}}\big) = 0$$

for all $f \in L_2(\Omega)$.

Proof (sketch). Let (α,β) be a symmetric self-joining, $\rho^{\max}(\alpha,\beta) \le \rho$. Rewriting the inequality $\mathbb{E}\,(f\circ\alpha)(f\circ\beta) \le \mathbb{E}\,(f\circ\alpha_\rho)(f\circ\beta_\rho)$ as $\mathbb{E}\,|f\circ\alpha - f\circ\beta|^2 \ge \mathbb{E}\,|f\circ\alpha_\rho - f\circ\beta_\rho|^2$ we see that f is stable iff $\mathbb{E}\,|f\circ\alpha_\rho - f\circ\beta_\rho|^2 \to 0$ as $\rho \to 1$. Lemma 5b3 gives us a σ-field $\mathcal{F}^{\text{stable}}$ such that f is stable iff f is $\mathcal{F}^{\text{stable}}$-measurable. Also, f is stable iff $\langle U^\rho f, f\rangle \to \|f\|^2$ as $\rho \to 1-$, that is, f is orthogonal to H_∞.

We have $|\langle U_{\alpha,\beta} f, f\rangle| \le \langle U^\rho f, f\rangle$ for all f, therefore $|\langle U_{\alpha,\beta} f, f\rangle| \le \sqrt{\langle U^\rho f, f\rangle}\sqrt{\langle U^\rho g, g\rangle}$. Rewriting sensitivity of f in the form $\forall \alpha,\beta\,\forall g\ \langle U_{\alpha,\beta} f, g\rangle = 0$ we see that f is sensitive iff $U^\rho f = 0$ for all $\rho < 1$, that is, $f \in H_\infty$. $\square$

5c3 Theorem. For every continuous product of probability spaces $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ there exists a symmetric self-joining $(\alpha_{1-}, \beta_{1-})$ of the given product such that

$$\mathbb{E}\,(f \circ \alpha_{1-})(g \circ \beta_{1-}) = \mathbb{E}\,(fg)$$

if $f, g \in L_2(\Omega)$ are stable, but

$$\mathbb{E}\,(f \circ \alpha_{1-})(g \circ \beta_{1-}) = 0$$

if $f \in L_2(\Omega)$ is sensitive (and $g \in L_2(\Omega)$ is arbitrary). The self-joining $(\alpha_{1-}, \beta_{1-})$ is unique up to isomorphism.

Proof (sketch). Just take the limit of $(\alpha_\rho, \beta_\rho)$ in the (compact!) space of joinings, as $\rho \to 1-$. □

See 'the 1^--joining' in [45, Def. 2.2]; see also [42, Sect. 1 (for $p = 0$)].

5c4 Definition. A continuous product of probability spaces is *classical,* if it satisfies the following equilavent conditions:

(a) all random variables are stable;

(b) no random variable is sensitive;

(c) the stable sub-σ-field $\mathcal{F}^{\text{stable}}$ is the whole σ-field $\mathcal{F}$.

A noise is classical if the underlying continuous product of probability spaces is classical.

Equivalent definitions in terms of $\mathbb{R}$-flows (Lévy processes) exist, see 6b4, 6c2. See also [40, Sect. 5b, especially Def. 5b5], and [36, 2.5].

5c5 Remark. The continuous product of probability spaces, corresponding to a flow system $(X_{s,t})_{s<t}$, is classical if and only if random variables $\varphi(X_{s,t})$ are stable for all $s < t$ and all bounded Borel functions $\varphi : G_{s,t} \to \mathbb{R}$ (a single φ is enough if it is one-to-one).

Proof (sketch). If each $\varphi(X_{s,t})$ is $\mathcal{F}^{\text{stable}}$-measurable then $\mathcal{F}^{\text{stable}} = \mathcal{F}$ since $\mathcal{F}$ is generated by these $\varphi(X_{s,t})$. □

5d Examples

The time set implicit in Sections 5a–5c is not necessarily $\mathbb{R}$; a subset of $\mathbb{R}$ (or any linearly ordered set) is also acceptable. In particular, the theory is applicable to the 'singularity concentrated in time' cases of Sect. 1.

The $\mathbb{Z}_m$-flow $X^{1\text{b}}$ (= X of Sect. 1b) generates a continuous product of probability spaces over the time set $\{0, 1, 2, \dots\} \cup \{\infty\}$. Random variables $X_{s,s+1}$ are stable; indeed, a single (indivisible) element of the data set is replaced with probability $1-\rho$, therefore $\mathbb{P}\big(X_{s,s+1}\circ\alpha_\rho \ne X_{s,s+1}\circ\beta_\rho \big) \le 1-\rho$. It follows that $X_{s,t} = X_{s,s+1} \dots X_{t-1,t}$ is stable whenever $s < t < \infty$. Of course, $X_{s,t}$, being a $\mathbb{Z}_m$-valued random variable, is not an element of $L_2(\Omega)$. By stability of $X_{s,t}$ we mean stability of $f(X_{s,t})$ for every $f : \mathbb{Z}_m \to \mathbb{R}$.

In contrast, the random variable $X_{0,\infty}$ is sensitive by Prop. 1c1. The same holds for $X_{s,\infty}$. More exactly, $g(X_{s,\infty})$ is sensitive for every $g : \mathbb{Z}_m \to \mathbb{R}$ such that $\mathbb{E}\, g(X_{s,\infty}) = 0$. *Sketch of the proof* (see also Sect. 1c for $m = 2$):

$$\begin{aligned} X_{s,\infty}\circ\alpha_\rho - X_{s,\infty}\circ\beta_\rho = (X_{s,s+1}\circ\alpha_\rho - X_{s,s+1}\circ\beta_\rho) + \dots + \\ + (X_{t-1,t}\circ\alpha_\rho - X_{t-1,t}\circ\beta_\rho) + (X_{t,\infty}\circ\alpha_\rho - X_{t,\infty}\circ\beta_\rho) \,, \end{aligned}$$

the summands being independent. For large t the sum from s to t is distributed on $\mathbb{Z}_m$ approximately uniformly, therefore $X_{s,\infty}\circ\alpha_\rho - X_{s,\infty}\circ\beta_\rho$ is uniform. The same holds conditionally, given α_ρ (that is, $X_{r,t}\circ\alpha_\rho$ for all r,t including $t = \infty$).

We see that random variables of the form $f(X_{0,1}, X_{1,2}, \dots)$ are stable, and random variables of the form $f(X_{0,1}, X_{1,2}, \dots) g(X_{0,\infty})$ are sensitive (as before, $\sum_{x\in\mathbb{Z}_m} g(x) = 0$). Their sums exhaust $L_2(\Omega)$. Therefore $\mathcal{F}^{\text{stable}}$ is generated by $X_{0,1}, X_{1,2}, \dots$; random variables $X_{s,\infty}$ are independent of $\mathcal{F}^{\text{stable}}$ (each one separately).

The $\mathbb{T}$-flow Y^{1b} (over the time set $[0,\infty)$) behaves similarly: $\mathcal{F}^{\text{stable}}$ is generated by $Y_{s,t}$ for $0 < s < t < \infty$; random variables $Y_{0,t}$ are independent of $\mathcal{F}^{\text{stable}}$ (each one separately).

We turn to the noises of Sect. 4: splitting and stickiness. These two may be treated uniformly. Below, G is either G^{4b} or G^{4c}. The Brownian motion $(B_t)_t = (a_{0,t})_t$ generates (via increments) sub-σ-fields $\mathcal{F}^{\text{white}}_{s,t} \subset \mathcal{F}_{s,t}$. It will be shown that $\mathcal{F}^{\text{stable}} = \mathcal{F}^{\text{white}}_{-\infty,\infty}$.

The Brownian motion $(B_t)_t$ has the *predictable representation property* w.r.t. the filtration $(\mathcal{F}_{-\infty,t})_t$. That is, every local martingale $(M_t)_t$ in this filtration is of the form $M_t = M_{-\infty} + \int_{-\infty}^t h_s \, \mathrm{d}B_s$ for some predictable process $(h_t)_t$ (in the considered filtration); see [28, Def. V.4.8]. Note that M_t and h_t need not be $\mathcal{F}^{\text{white}}_{-\infty,t}$-measurable.

Sketch of the proof (*of the predictable representation property*). We may restrict ourselves to a dense set of martingales, namely, $M_s = \mathbb{E}\big(\varphi(X_{t_0,t_1}, \dots, X_{t_{n-1},t_n}) \big| \mathcal{F}_{-\infty,s} \big)$ where $\varphi : G^n \to \mathbb{R}$ is a bounded measurable (or even smooth) function, $-\infty < t_0 < \dots < t_n < \infty$, and $(X_{s,t})_{s<t}$ stands for the given G-flow. When $s \in [t_{k-1}, t_k]$, we deal effectively with the case $M_s = \mathbb{E}\big(\psi(X_{r,t}) \big| \mathcal{F}_{r,s} \big)$, $-\infty < r < t < \infty$, to which we may restrict ourselves. By independence of $X_{r,s}$ and $X_{s,t}$,

$$
\begin{aligned}
&M_s = u(X_{r,s}, t-s)\,, \quad \text{where } u : G \times \mathbb{R} \to \mathbb{R} \text{ is defined by} \\
&u(x,t) = \int_G \psi(xy)\, \mu_t(\mathrm{d}y)\,,
\end{aligned}
$$

and $(\mu_t)_t$ is the given convolution semigroup in G.

The semigroup G is in fact a smooth manifold with boundary, and the function u is smooth up to the boundary (which can be checked using explicit formulas for μ_t and the binary operation in G). The random process $(X_{r,s})_{s\in[r,t]}$ is a diffusion process on the smooth manifold G; it is a weak solution of a stochastic differential equation driven by $(B_s)_s$. Itô's formula gives the needed representation. $\square$

By a Brownian motion adapted to a continuous product of probability spaces $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ we mean a family $(B_t)_{t\in\mathbb{R}}$ of random variables B_t such that

$$
B_t - B_s \text{ is } \mathcal{F}_{s,t}\text{-measurable, and distributed normally } \mathrm{N}(0, t-s)
$$

whenever $-\infty < s < t < \infty$; and in addition, $B_0 = 0$.

5d1 Proposition. Let $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ be a continuous product of probability spaces, and $(B_t)_t$ a Brownian motion adapted to the continuous product. If $(B_t)_t$ has the predictable representation property w.r.t. the filtration $(\mathcal{F}_{-\infty,t})_t$, then the sub-$\sigma$-field generated by $(B_t)_t$ is equal to $\mathcal{F}^{\text{stable}}$.

Proof (sketch). The sub-σ-field $\mathcal{F}^{\text{white}}$ generated by $(B_t)_t$ is contained in $\mathcal{F}^{\text{stable}}$, since Wiener chaos spaces (with finite indices) exhaust the corresponding L_2 space. We have to prove that $\mathcal{F}^{\text{white}} \supset \mathcal{F}^{\text{stable}}$.

Every $f \in L_2^0(\Omega)$ is of the form $f = \int_{-\infty}^{\infty} h_t \, \mathrm{d}B_t$. We have

$$
\langle U^\rho f, f \rangle = \rho \int_{-\infty}^{\infty} \langle U^\rho h_t, h_t \rangle \, \mathrm{d}t\,,
$$

since[1]

$$
\begin{aligned}
\langle (\textstyle\int h_t \, \mathrm{d}B_t)\circ\alpha_\rho, (\int h_t \, \mathrm{d}B_t)\circ\beta_\rho \rangle &= \langle \textstyle\int (h_t\circ\alpha_\rho)\, \mathrm{d}(B_t\circ\alpha_\rho), \int (h_t\circ\beta_\rho)\, \mathrm{d}(B_t\circ\beta_\rho) \rangle = \\
&= \textstyle\int (h_t \circ \alpha_\rho)(h_t \circ \beta_\rho)\, \underbrace{\mathrm{d}\langle B_t \circ \alpha_\rho, B_t \circ \beta_\rho \rangle}_{=\rho\, dt}\,.
\end{aligned}
$$

[1]Usually $\langle \cdot, \cdot \rangle$ stands for the scalar product, but now it will denote the predictable quadratic (co)variation.

In particular, if $f \in H_1$ (the first chaos) then

$$\rho \int \langle U^\rho h_t, h_t \rangle \, \mathrm{d}t = \langle U^\rho f, f \rangle = \rho \|f\|^2 = \rho \int \|h_t\|^2 \, \mathrm{d}t \, ,$$

that is, $\|h_t\|^2 - \langle U^\rho h_t, h_t \rangle = 0$, which means that $h_t \in H_0$ is a constant (nonrandom) for almost every t. Therefore $f = \int h_t \, \mathrm{d}B_t$ is $\mathcal{F}^{\text{white}}$-measurable, and we get

$$H_1 \subset L_2(\mathcal{F}^{\text{white}}) \, .$$

Further, let $f \in H_2$, then h_t are orthogonal to H_0 (since f is orthogonal to H_1), therefore $\langle U^\rho h_t, h_t \rangle \le \rho \|h_t\|^2$. On the other hand,

$$\rho \int \langle U^\rho h_t, h_t \rangle \, \mathrm{d}t = \langle U^\rho f, f \rangle = \rho^2 \|f\|^2 = \rho^2 \int \|h_t\|^2 \, \mathrm{d}t \, ,$$

that is, $\rho \|h_t\|^2 - \langle U^\rho h_t, h_t \rangle = 0$, which means that $h_t \in H_1$ for almost all t. It follows that h_t is $\mathcal{F}^{\text{white}}$-measurable; therefore f is $\mathcal{F}^{\text{white}}$-measurable, and we get

$$H_2 \subset L_2(\mathcal{F}^{\text{white}}) \, .$$

And so on. Finally,

$$L_2(\mathcal{F}^{\text{stable}}) = H_0 \oplus H_1 \oplus H_2 \oplus \dots \subset L_2(\mathcal{F}^{\text{white}}) \, . \qquad \square$$

6 Classical part of a continuous product

6a Probability spaces: additive flows

By the *classical part* of a continuous product of probability spaces $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t} \big)$ we mean the quotient space $(\Omega, P)/\mathcal{F}^{\text{stable}}$ equipped with $(\mathcal{F}^{\text{stable}}_{s,t})_{s<t}$ where $\mathcal{F}^{\text{stable}}_{s,t}$ is the stable part $\mathcal{F}_{s,t} \cap \mathcal{F}^{\text{stable}}$ of $\mathcal{F}_{s,t}$ transferred to the quotient space. The classical part is a continuous product of probability spaces; indeed,

$$\mathcal{F}^{\text{stable}}_{r,t} = \mathcal{F}^{\text{stable}}_{r,s} \otimes \mathcal{F}^{\text{stable}}_{s,t} \quad \text{for } r < s < t \, ,$$

since local versions $U^\rho_{s,t}$ of the operators U^ρ satisfy

$$U^\rho_{r,t} = U^\rho_{r,s} \otimes U^\rho_{s,t} \quad \text{for } r < s < t \, .$$

Recall that $L_2(\mathcal{F}^{\text{stable}}) = H_0 \oplus H_1 \oplus H_2 \oplus \dots$, the chaos spaces H_n being defined by $U^\rho f = \rho^n f$ for $f \in H_n$; also, H_0 is the one-dimensional space of constant functions. Similarly, $L_2(\mathcal{F}^{\text{stable}}_{s,t}) = \bigoplus_{n<\infty} H_n(s,t)$, $U^\rho_{s,t} f = \rho^n f$ for $f \in H_n(s,t) = H_n \cap L_2(\mathcal{F}_{s,t})$.

6a1 Proposition. ([36, 2.9]) The following conditions are equivalent for every $f \in L_2(\Omega)$:

(a) $f \in H_1$;

(b) $f = \mathbb{E}\big(f \big| \mathcal{F}_{-\infty,t}\big) + \mathbb{E}\big(f \big| \mathcal{F}_{t,\infty}\big)$ for all $t \in \mathbb{R}$;

(c) $\mathbb{E}\big(f \big| \mathcal{F}_{r,t}\big) = \mathbb{E}\big(f \big| \mathcal{F}_{r,s}\big) + \mathbb{E}\big(f \big| \mathcal{F}_{s,t}\big)$ whenever $-\infty \le r < s < t \le \infty$.

Proof (sketch). (b) $\Longrightarrow$ (c): in terms of the projections $Q_{s,t} : f \mapsto \mathbb{E}\big(f \big| \mathcal{F}_{s,t}\big)$ we have $Q_{r,t} f = Q_{r,t} Q_{-\infty,s} f + Q_{r,t} Q_{s,\infty} f = Q_{r,s} f + Q_{s,t} f$.

(c) $\Longrightarrow$ (a): $U^\rho f = \rho f$, since it holds for each element $\tilde U^\rho_{t_1,\dots,t_n}$ of the net converging to U^ρ (recall (5b4)).

(a) $\Longrightarrow$ (b): eigenvalues of the operator $U^\rho = U^\rho_{-\infty,t} \otimes U^\rho_{t,\infty}$ are products of eigenvalues, $\rho^k \rho^l = \rho^{k+l}$, $k,l \in \{0,1,2,\dots\} \cup \{\infty\}$. We have $H_n = \bigoplus_{k=0}^n H_k(-\infty,t) \otimes H_{n-k}(t,\infty)$. Especially, $H_1 = H_0(-\infty,t) \otimes H_1(t,\infty) \oplus H_1(-\infty,t) \otimes H_0(t,\infty) = H_1(-\infty,t) \oplus H_1(t,\infty)$. Thus, $f = g + h$ for some $g \in H_1(-\infty,t)$ and $h \in H_1(t,\infty)$. However, $\mathbb{E}\big(h \big| \mathcal{F}_{-\infty,t}\big) = \mathbb{E}\, h = 0$, therefore $\mathbb{E}\big(f \big| \mathcal{F}_{-\infty,t}\big) = g$; similarly $\mathbb{E}\big(f \big| \mathcal{F}_{t,\infty}\big) = h$ and we get (b). $\square$

6a2 Corollary. Every square integrable $\mathbb{R}$-flow adapted to a continuous product of probability spaces is adapted to its classical part.

6a3 Corollary. A continuous product of probability spaces generated by square integrable $\mathbb{R}$-flows is classical.

The integrability condition can be removed, see 6b3, 6b4.

6a4 Theorem. (Tsirelson [36, Th. 2.12], [40, Th. 6a3]) The sub-σ-field generated by H_1 is equal to $\mathcal{F}^{\text{stable}}$.

Proof (sketch). Clearly, $\mathcal{F}_1 \subset \mathcal{F}^{\text{stable}}$ ($\mathcal{F}_1$ being generated by H_1); the other inclusion, $\mathcal{F}^{\text{stable}} \subset \mathcal{F}_1$, follows from the next lemma. $\square$

6a5 Lemma. The space $L_2(\mathcal{F}^{\text{stable}})$ is the closure of the union of all subspaces of the form

$$\bigotimes_{k=0}^{n} \big(H_0(t_k, t_{k+1}) \oplus H_1(t_k, t_{k+1}) \big)$$

where $-\infty = t_0 < t_1 < \dots < t_n < t_{n+1} = \infty$, $n = 0,1,2,\dots$

Proof (sketch). U^ρ is the limit of the *decreasing* net of *commuting* operators $\tilde U^\rho_{t_1,\dots,t_n}$ (recall (5b4)). Therefore for each n the spectral subspace $H_0 \oplus \dots \oplus H_n$ of U^ρ corresponding to the upper part $\{\rho^n, \rho^{n-1}, \dots, 1\}$ of its spectrum, is the limit (that is, the intersection) of the decreasing net of the corresponding subspaces for $\tilde U^\rho_{t_1,\dots,t_n}$. Similarly, the subspace $(H_n \oplus H_{n+1} \oplus \dots) \oplus H_\infty$

is the limit (that is, the closure of the union) of the increasing net of the corresponding subspaces for $\tilde U^\rho_{t_1,\dots,t_n}$. The latter subspace, being intersected with H_n, gives a subspace of $\bigotimes_{k=0}^n \bigl(H_0(t_k,t_{k+1}) \oplus H_1(t_k,t_{k+1}) \bigr)$. $\square$

By the way, it follows from the lemma above that

$$\text{(6a6)} \qquad H_n \text{ is the closed linear span of } \bigcup_{t\in\mathbb{R}} H_1(-\infty,t)\otimes H_{n-1}(t,\infty)$$

for each n.

Each $f \in H_1$ leads to an $\mathbb{R}$-flow $(f_{s,t})_{s<t}$, $f_{s,t} = \mathbb{E}\bigl(f \,\big|\, \mathcal{F}_{s,t} \bigr)$, adapted to $(\mathcal{F}_{s,t})_{s<t}$ in the sense that $f_{s,t}$ is $\mathcal{F}_{s,t}$-measurable whenever $s<t$. Choosing a sequence $(f_k)_k$ that spans H_1 we get the following.

6a7 Corollary. For every continuous product of probability spaces, its classical part is generated by (a finite or countable collection of) square integrable adapted $\mathbb{R}$-flows.

These $\mathbb{R}$-flows may be combined into a single vector-valued flow, say, l_2-flow. Assuming the downward continuity (recall 2d5) we may use the infinite-dimensional Lévy-Itô theorem [13, 4.1] for representing the l_2-flow via a Gaussian process and (compensated, nonstationary) Poisson processes, those processes being independent [13, 5.1]. In fact, the whole Poissonian component can be generated by a single $\mathbb{R}$-flow [13, 6.1], in contrast to the Gaussian component. The framework of Feldman [13] is different from ours, but the difference is inessential for the classical part, as explained below.

Spaces $H_1(s,t)$ satisfy the additive relation

$$H_1(r,t) = H_1(r,s) \oplus H_1(s,t) \quad \text{for } r<s<t\,,$$

much simpler than the multiplicative relation $L_2(\mathcal{F}_{r,t}) = L_2(\mathcal{F}_{r,s}) \otimes L_2(\mathcal{F}_{s,t})$. Orthogonal projections $Q_{s,t} : H_1 \to H_1$, $Q_{s,t}H_1 = H_1(s,t)$, lead to a projection-valued measure $(Q_A)_A$; $Q_A : H_1 \to H_1$ for Borel sets $A \subset \mathbb{R}$, $Q_{(s,t]} = Q_{s,t}$ for $s<t$. To this end, however, we must assume right-continuity of $Q_{-\infty,t}$ in t. Otherwise we should split each point t of a finite of countable set in two, t_{left} and t_{right}. (Alternatively, we could replace the time set $[-\infty,\infty]$ with an arbitrary, not just connected, compact subset of $\mathbb{R}$, thus making $Q_{-\infty,t}$ continuous in t.) Assume for simplicity the right-continuity (for a while; the assumption expires before Prop. 6a13). We get (closed linear) subspaces $H_1(A) = Q_A H_1 \subset H_1$ satisfying

$$\text{(6a8)} \qquad \begin{gathered} H_1(A\cup B) = H_1(A) \oplus H_1(B) \quad \text{when } A\cap B = \emptyset\,, \\ H_1(A_1 \cap A_2 \cap \dots) = H_1(A_1) \cap H_1(A_2) \cap \dots\,, \\ H_1\bigl((s,t]\bigr) = H_1(s,t) \quad \text{for } s<t\,. \end{gathered}$$

Defining $\mathcal{F}_A^{\text{stable}}$ as the sub-σ-field generated by $H_1(A)$ we get ([40, 6c4])

$$\mathcal{F}_{A\cup B}^{\text{stable}} = \mathcal{F}_A^{\text{stable}} \otimes \mathcal{F}_B^{\text{stable}} \quad \text{whenever } A \cap B = \emptyset \,, \tag{6a9}$$

$$A_n \uparrow A \quad \text{implies} \quad \mathcal{F}_{A_n}^{\text{stable}} \uparrow \mathcal{F}_A^{\text{stable}} \,, \tag{6a10}$$

$$A_n \downarrow A \quad \text{implies} \quad \mathcal{F}_{A_n}^{\text{stable}} \downarrow \mathcal{F}_A^{\text{stable}} \,, \tag{6a11}$$

$$\mathcal{F}_{(s,t]}^{\text{stable}} = \mathcal{F}_{s,t}^{\text{stable}} \quad \text{for } s < t \,. \tag{6a12}$$

It means that the classical part of any continuous product of probability spaces is a *factored probability space* as defined by Feldman [13, 1.1], which cannot be extended beyond the classical part, see Theorem 11a2.

Proof (sketch). (6a10): if $A_n \uparrow A$ then $H_1(A_n) \uparrow H_1(A)$.

(6a9): for each $f \in H_1$ the equality $\mathbb{E}\, \mathrm{e}^{\mathrm{i}f} = \big(\mathbb{E}\, \mathrm{e}^{\mathrm{i}Q_A f}\big)\big(\mathbb{E}\, \mathrm{e}^{\mathrm{i}Q_{\mathbb{R}\setminus A} f}\big)$ holds (by independence) if the set $A \subset \mathbb{R}$ is an interval or the union of a finite number of intervals. The monotone class theorem extends the equality to all Borel sets A. Thus, $\mathcal{F}_A^{\text{stable}}$ and $\mathcal{F}_{\mathbb{R}\setminus A}^{\text{stable}}$ are independent; (6a9) follows.

(6a11) follows from (6a10) and (6a9) similarly to 2d4.

(6a12): see 6a4. □

6a13 Proposition. The following conditions are equivalent for a *classical* continuous product of probability spaces:

(a) upward continuity (2d3);

(b) downward continuity (2d5);

(c) the subspace $\bigcap_{\varepsilon>0} H_1(t-\varepsilon, t+\varepsilon)$ is trivial for every $t \in \mathbb{R}$.

Proof (sketch). By (6a10), (a) is equivalent to triviality of $H_1(\{s_{\text{right}}\})$ and $H_1(\{t_{\text{left}}\})$ for $s < t$. By (6a11), (b) is equivalent to triviality of $H_1(\{s_{\text{left}}\})$ and $H_1(\{t_{\text{right}}\})$ for $s \le t$. Also, (c) is equivalent to triviality of $H_1(\{t_{\text{left}}\})$ and $H_1(\{t_{\text{right}}\})$ for all t. (At $\pm\infty$ use the non-redundancy stipulated by Def. 2c1.) □

6b Probability spaces: multiplicative flows

We turn to $\mathbb{T}$-flows; the circle $\mathbb{T}$ will now be treated as the complex circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ rather than $\mathbb{R}/\mathbb{Z}$. Accordingly, L_2 spaces over $\mathbb{C}$ are used. Corollaries 6a2, 6a3 fail for $\mathbb{T}$-flows. For counterexamples see Sect. 1b; the singular time point must be finite (not $\pm\infty$), since the upward continuity at $\pm\infty$ is ensured by Def. 2c1. Results presented below (6b1, 6b3, 6b4) are close to [41, 1.7]. The time set is $\mathbb{R}$, but may be enlarged to $[-\infty, \infty]$.

6b1 Proposition. If a continuous product of probability spaces satisfies the upward continuity condition (2d3), then every $\mathbb{T}$-flow adapted to the continuous product is adapted to the classical part.

Proof (sketch). Every $\mathbb{T}$-flow $(X_{s,t})_{s<t}$ satisfies the inequality

$$\langle U^\rho X_{s,t}, X_{s,t}\rangle \ge |\mathbb{E}\, X_{s,t}|^{2(1-\rho)} \quad \text{for } s<t \text{ and } \rho\in[0,1]\,, \tag{6b2}$$

since it holds for each element $\tilde U^\rho_{t_1,\dots,t_n}$ of the net converging to U^ρ (recall (5b4)): $\langle \tilde U^\rho_{t_1,\dots,t_n} X_{s,t}, X_{s,t}\rangle = \prod_{k=0}^n \bigl(\rho + (1-\rho)|\mathbb{E}\, X_{t_k,t_{k+1}}|^2\bigr)$, the logarithm of each factor being concave in ρ. Stability of $X_{s,t}$ is thus ensured, if $\mathbb{E}\, X_{s,t}\ne 0$. The latter follows from the upward continuity: $|\mathbb{E}\, X_{s-\varepsilon,s+\varepsilon}|^2 = \mathbb{E}\,|\mathbb{E}\bigl(X_{r,t} \,\big|\, \mathcal{F}_{r,s-\varepsilon}\vee\mathcal{F}_{s+\varepsilon,t}\bigr)|^2 \to 1$ as $\varepsilon\to 0$; we cover the compact interval $[r,t]$ by a finite number of open intervals $(s-\varepsilon,s+\varepsilon)$ such that $\mathbb{E}\, X_{s-\varepsilon,s+\varepsilon}\ne 0$ and get $\mathbb{E}\, X_{r,t}\ne 0$. $\square$

A stable (that is, adapted to the classical part) $\mathbb{T}$-flow $(X_{s,t})_{s<t}$ can satisfy $\mathbb{E}\, X_{s,t}=0$ for some $s<t$; indeed, $H_1(t-,t+)$ can contain a random variable $X_{t-,t+}=\pm 1$ such that $\mathbb{E}\, X_{t-,t+}=0$. On the other hand, it must be $\mathbb{E}\, X_{s,t}\ne 0$ for some s,t (irrespective of stability), and moreover, the equivalence relation $s\sim t \iff \mathbb{E}\, X_{s,t}\ne 0$ divides $\mathbb{R}$ into at most countable number of intervals (maybe, sometimes degenerate), since $L_2(\Omega)$ is separable.

6b3 Corollary. Every $\mathbb{R}$-flow adapted to a continuous product of probability spaces is adapted to its classical part.

Proof (sketch). $\mathbb{T}$-flows $(\mathrm{e}^{\mathrm{i}\lambda X_{s,t}})_{s<t}$ corresponding to the given $\mathbb{R}$-flow $(X_{s,t})_{s<t}$ satisfy $\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda X_{s,t}}\to 1$ as $\lambda\to 0$. By (6b2), $\mathrm{e}^{\mathrm{i}\lambda X_{s,t}}$ is $\mathcal{F}^{\text{stable}}$-measurable for all λ small enough. Therefore $X_{s,t}$ is $\mathcal{F}^{\text{stable}}$-measurable. $\square$

See also [41, proof of Th. 1.7].

6b4 Corollary. A continuous product of probability spaces is classical if and only if it is generated by (a finite or countable collection of) $\mathbb{R}$-flows.

The same holds for $(\mathbb{C},\cdot)$-flows (valued in the multiplicative semigroup of complex numbers), see [41, 1.7]. Relations between $(\mathbb{C},\cdot)$-flows and $(\mathbb{C},+)$-flows described below appear in different forms in [41, Appendix A] and earlier works cited there.

6b5 Proposition. A stable square integrable $(\mathbb{C},\cdot)$-flow $(X_{s,t})_{s<t}$ is uniquely determined by the projections $Q_0X_{s,t}=\mathbb{E}\, X_{s,t}$ and $Q_1X_{s,t}$ of each $X_{s,t}$ to $H_0(s,t)$ and $H_1(s,t)$.

Proof (sketch). The projection $\prod_{k=0}^n (Q_0+Q_1)X_{t_k,t_{k+1}}$ of $X_{t_0,t_{n+1}}$ to $\bigotimes_{k=0}^n \bigl(H_0(t_k,t_{k+1})\oplus H_1(t_k,t_{k+1})\bigr)$ is uniquely determined. It remains to use Lemma 6a5. $\square$

Clearly, $\mathbb{E}\, X_{r,t} = (\mathbb{E}\, X_{r,s})(\mathbb{E}\, X_{s,t})$ and $Q_1 X_{r,t} = (Q_1 X_{r,s})(\mathbb{E}\, X_{s,t}) + (\mathbb{E}\, X_{r,s})(Q_1 X_{s,t})$. In particular, if $\mathbb{E}\, X_{s,t} = 1$ for all $s < t$, then $Q_1 X_{r,t} = Q_1 X_{r,s} + Q_1 X_{s,t}$, that is, $(Q_1 X_{s,t})_{s<t}$ is a $(\mathbb{C}, +)$-flow.

6b6 Proposition. For every square integrable, zero-mean $(\mathbb{C}, +)$-flow $(Y_{s,t})_{s<t}$ there exists a square integrable $(\mathbb{C}, \cdot)$-flow $(X_{s,t})_{s<t}$ such that

$$\mathbb{E}\, X_{s,t} = 1 \quad \text{and} \quad Q_1 X_{s,t} = Y_{s,t} \quad \text{for } s < t \, .$$

Proof (sketch). Similarly to the proof of 6b5 we calculate the projection of the desired $X_{s,t}$ to subspaces of the form $\bigotimes_{k=0}^{n} \big(H_0(t_k, t_{k+1}) \oplus H_1(t_k, t_{k+1}) \big)$. The subspaces are an increasing net. The projections are consistent, and bounded in L_2:

$$\Big\| \bigotimes_k (1 + Y_{t_k, t_{k+1}}) \Big\|^2 = \prod_k (1 + \| Y_{t_k, t_{k+1}} \|^2) \le \\ \le \exp \Big(\sum_k \| Y_{t_k, t_{k+1}} \|^2 \Big) = \exp\big(\| Y_{s,t} \|^2 \big) \, .$$

Thus, they are a net converging in L_2; its limit is the desired $X_{s,t}$. □

The relation between the flows X and Y as in 6b6 will be denoted by

$$X = \operatorname{Exp} Y \, ; \qquad Y = \operatorname{Log} X \, . \tag{6b7}$$

It is a one-to-one correspondence between (the set of all) square integrable $(\mathbb{C}, \cdot)$-flows $(X_{s,t})_{s<t}$ satisfying $\mathbb{E}\, X_{s,t} = 1$ for $s < t$ (which implies stability by (6b2)), and (the set of all) square integrable $(\mathbb{C}, +)$-flows $(Y_{s,t})_{s<t}$ satisfying $\mathbb{E}\, Y_{s,t} = 0$ for $s < t$ (these are stable by 6a2).

Relations (6b7) do not mean that $X_{s,t} = \exp(Y_{s,t})$. In fact, if Y is sample continuous, therefore Gaussian, then $X_{s,t} = \exp(Y_{s,t} - \frac{1}{2} \| Y_{s,t} \|^2)$ for $s < t$. Of course, 'exp' is the usual exponential function $\mathbb{C} \to \mathbb{C}$, while 'Exp' is introduced by (6b7). If the limit $Y_{-\infty,\infty} = \lim_{s \to -\infty, t \to \infty} Y_{s,t}$ exists (in L_2) then the limit $X_{-\infty,\infty}$ exists, and we may write

$$X_{-\infty,\infty} = \operatorname{Exp}(Y_{-\infty,\infty}) \, ; \qquad \operatorname{Exp} : H_1 \to L_2(\mathcal{F}^{\text{stable}}) \, .$$

6b8 Proposition. The space $L_2(\mathcal{F}^{\text{stable}})$ is the closed linear span of $\{ \operatorname{Exp} f : f \in H_1 \}$.

Proof (sketch). The projection of $\operatorname{Exp} f$ to $H_0 \oplus H_1$ is $1 + f$, thus (using the upper bound from the proof of 6b6)

$$\| (\operatorname{Exp} f) - (1 + f) \|^2 = \| \operatorname{Exp} f \|^2 - \| 1 + f \|^2 \le (\exp \| f \|^2) - (1 + \| f \|^2) \, .$$

It follows that

$$f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big(\operatorname{Exp}(\varepsilon f) - 1 \big) \quad \text{in } L_2 \,,$$

and we see that $H_1 \subset E$, where E is the closed linear span of $\{ \operatorname{Exp} f : f \in H_1 \}$. Similarly, $H_1(s,t) \subset E(s,t)$, where $E(s,t)$ is the closed linear span of $\{ \operatorname{Exp} f : f \in H_1(s,t) \}$. However, $E(r,t) \supset E(r,s) \otimes E(s,t)$ for $r < s < t$. We see that E contains each subspace of the form $\bigotimes_{k=0}^n \big(H_0(t_k, t_{k+1}) \oplus H_1(t_k, t_{k+1}) \big)$; it remains to use 6a5. $\square$

6b9 Proposition. If $Y_{t-,t+} = 0$ for all t, then $\|X_{s,t}\|^2 = \exp(\|Y_{s,t}\|^2)$ for $s < t$.

Proof (sketch). Recall the proof of 6b6 and note that $\exp\big(\|Y_{t_k,t_{k+1}}\|^2 \big) = 1 + \|Y_{t_k,t_{k+1}}\|^2 + o\big(\|Y_{t_k,t_{k+1}}\|^2 \big)$. $\square$

6b10 Proposition. If a classical continuous product of probability spaces satisfies the equivalent continuity conditions 6a13(a–c), then the map $\operatorname{Exp} : H_1 \to L_2(\Omega)$ has the property

$$\langle \operatorname{Exp} f, \operatorname{Exp} g \rangle = \exp \langle f, g \rangle \quad \text{for } f, g \in H_1 \,.$$

Proof (sketch). Similarly to 6b9, $\langle 1 + f_{t_k,t_{k+1}}, 1 + g_{t_k,t_{k+1}} \rangle = 1 + \langle f_{t_k,t_{k+1}}, g_{t_k,t_{k+1}} \rangle \approx \exp \langle f_{t_k,t_{k+1}}, g_{t_k,t_{k+1}} \rangle$. $\square$

It means that $L_2(\Omega)$ is nothing but the Fock space e^{H_1}, see [4, Sect. 2.1.1, especially (2.7)]. More generally,

$$L_2(\mathcal{F}^{\text{stable}}) = \mathrm{e}^{H_1}$$

for all continuous products of probability spaces satisfying the downward continuity condition.

6c Noises

Given a noise $(\mathcal{F}_{s,t})_{s<t}$, $(T_h)_h$, we may consider the classical part of the continuous product of probability spaces $(\mathcal{F}_{s,t})_{s<t}$. It consists of sub-$\sigma$-fields $\mathcal{F}_{s,t}^{\text{stable}} = \mathcal{F}_{s,t} \cap \mathcal{F}^{\text{stable}}$ (transferred to the quotient space $(\Omega, P)/\mathcal{F}^{\text{stable}}$, which does not matter now). Time shifts T_h leave $\mathcal{F}^{\text{stable}}$ invariant (since operators U^ρ evidently commute with time shifts), thus T_h sends $\mathcal{F}_{s,t}^{\text{stable}}$ to $\mathcal{F}_{s+h,t+h}^{\text{stable}}$. It means that the classical part of a noise is a (classical) noise.

A classical noise is generated by $\mathbb{R}$-flows, like any other classical continuous product of probability spaces (see 6b4). However, we want these $\mathbb{R}$-flows $(X_{s,t})_{s<t}$ to be *stationary* in the sense that $X_{s+h,t+h} = X_{s,t} \circ T_h$ (where $T_h : \Omega \to \Omega$ are time shifts). The following result is proven in [35, 2.9] under assumptions excluding the Poisson component, but the argument works in general.

6c1 Theorem. Every classical noise is generated by (a finite or countable collection of) square integrable stationary $\mathbb{R}$-flows.

Proof (sketch). Time shifts T_h on Ω induce unitary operators U_h on $L_2(\Omega)$, commuting with U^ρ and therefore leaving invariant the first chaos space H_1; we will treat U_h as operators $H_1 \to H_1$. They are connected with the projections $Q_{s,t} : H_1 \to H_1$ by the relation $U_h^{-1} Q_{s,t} U_h = Q_{s+h,t+h}$. Integrating the function $t \mapsto \mathrm{e}^{\mathrm{i}\lambda t}$ by the projection-valued measure $(Q_A)_A$ (recall (6a8)) we get unitary operators $V_\lambda : H_1 \to H_1$ satisfying Weyl relations $U_h V_\lambda = \mathrm{e}^{\mathrm{i}\lambda h} V_\lambda U_h$. By the well-known theorem of von Neumann (see [27, Th. VIII.14]), H_1 decomposes into the direct sum of a finite or countable number of irreducible components, — subspaces, each carrying an irreducible representation of Weyl relations. Each irreducible representation is unitarily equivalent to the standard representation in $L_2(\mathbb{R})$, where U_h acts as the shift by h, and V_λ acts as the multiplication by $t \mapsto \mathrm{e}^{\mathrm{i}\lambda t}$. Defining $X_{s,t}^{(k)}$ as the vector that corresponds to the indicator function of the interval (s,t) in the k-th irreducible component of H_1 we get the needed stationary $\mathbb{R}$-flows $(X_{s,t}^{(k)})_{s<t}$. $\square$

6c2 Corollary. A noise is classical if and only if it is generated by (a finite or countable collection of) stationary $\mathbb{R}$-flows.

These $\mathbb{R}$-flows may be combined into a single vector-valued flow ($\mathbb{R}^n$-flow or l_2-flow). The infinite-dimensional Lévy-Itô theorem [13, 4.1] may be used for representing the stationary l_2-flow via Brownian motions and (stationary, compensated) Poisson processes.

We may treat H_1 as the tensor product, $H_1 = L_2(\mathbb{R}) \otimes \mathcal{H} = L_2(\mathbb{R}, \mathcal{H})$, where $L_2(\mathbb{R})$ carries the standard representation of Weyl relations, and $\mathcal{H}$ is the Hilbert space of all square integrable, zero mean, stationary $\mathbb{R}$-flows. Further, the space $\mathcal{H}$ decomposes in two orthogonal subspaces, the Brownian part and the Poissonian part. The (finite or infinite) dimension of the Brownian part is the maximal number of independent Brownian motions adapted to the given classical noise. The Poissonian part may be identified with the L_2 space over the corresponding Lévy-Khinchin measure.

6d Pointed Hilbert spaces

6d1 Definition. Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces. A vector $f \in H_{r,t}$ is *decomposable,* if $f \neq 0$ and for every $s \in (r,t)$ there exist $g \in H_{r,s}$ and $h \in H_{s,t}$ such that $f = gh$.

(As before, gh is the image of $g \otimes h$ under the given unitary operator $H_{r,s} \otimes H_{s,t} \to H_{r,t}$.)

6d2 Lemma. If $g \in H_{r,s}$ and $h \in H_{s,t}$ are such that the vector $gh \in H_{r,t}$ is decomposable then g and h are decomposable.

See also [4, 6.0.2 and 6.2.1].

Proof (sketch). We may assume $\|g\| = 1$, $\|h\| = 1$. Consider the one-dimensional orthogonal projection $Q_g : H_{r,s} \to H_{r,s}$, $Q_g\psi = \langle \psi, g \rangle g$. Note that $\langle (Q_g \otimes \mathbf{1}_{s,t})f, f \rangle = \langle Q_g g, g \rangle \langle h, h \rangle = 1$. Let $s' \in (r, s)$. Then $f = g'h'$ for some vectors $g' \in H_{r,s'}$ and $h \in H_{s',t}$ (of norm 1). As before, $\langle (Q_{g'} \otimes \mathbf{1}_{s',t})f, f \rangle = 1$. Therefore

$$\underbrace{\langle (Q_{g'} \otimes \mathbf{1}_{s',s} \otimes \mathbf{1}_{s,t})f, f \rangle}_{=1} = \langle (Q_{g'} \otimes \mathbf{1}_{s',s})g, g \rangle \underbrace{\langle \mathbf{1}_{s,t}h, h \rangle}_{=1} .$$

The equality $\langle (Q_{g'} \otimes \mathbf{1}_{s',s})g, g \rangle = 1$ means that $g = g'\psi$ for some $\psi \in H_{s',s}$. Thus, g is decomposable. $\square$

Theorem 3b1 gives us a measurable structure on the family $(H_{s,t})_{s<t}$ of Hilbert spaces. The structure is non-unique, but we can adapt factor-vectors to any given structure, as stated below. See also [19, Corollary 5.2].

6d3 Proposition. For every measurable structure as in Theorem 3b1 and every decomposable vector $f \in H_{-\infty,\infty}$ there exists a family $(f_{s,t})_{s<t}$ of vectors $f_{s,t} \in H_{s,t}$ (given for all s, t such that $s < t$) satisfying the conditions

$$\begin{gathered} f_{r,s}f_{s,t} = f_{r,t} \quad \text{whenever } r < s < t \,, \\ f_{s,t} \quad \text{is measurable in } s, t \,. \end{gathered}$$

Proof (sketch). For every $t \in \mathbb{R}$ we choose $g_t \in H_{-\infty,t}$ and $h_t \in H_{t,\infty}$ such that $g_t h_t = f$. Lemma 3a3 gives us complex numbers c_t such that the vectors $f_{-\infty,t} = c_t g_t$ and $f_{t,\infty} = (1/c_t)h_t$ are measurable in t. Now $f_{s,t}$ are uniquely determined by requiring $f_{-\infty,s}f_{s,t}f_{t,\infty} = f$. $\square$

Applying 6d3 to a continuous product of spaces L_2 (recall Sect. 3a) we may get the following.

6d4 Corollary. Every square integrable $(\mathbb{C}, \cdot)$-flow $(X_{s,t})_{s<t}$ adapted to a continuous product of probability spaces can be written as $X_{s,t} = c_{s,t}Y_{s,t}$ where $c_{s,t}$ are complex numbers satisfying $c_{r,s}c_{s,t} = c_{r,t}$ for $r < s < t$, and $(Y_{s,t})_{s<t}$ is a $(\mathbb{C}, \cdot)$-flow such that the map $(s, t) \mapsto Y_{s,t}$ from $\{(s, t) \in \mathbb{R}^2 : s < t\}$ to $L_2(\Omega)$ is Borel measurable.

6d5 Corollary. If a square integrable $(\mathbb{C}, \cdot)$-flow $(X_{s,t})_{s<t}$ satisfies $\mathbb{E}\, X_{s,t} = 1$ whenever $s < t$, then the map $(s, t) \mapsto X_{s,t}$ from $\{(s, t) \in \mathbb{R}^2 : s < t\}$ to $L_2(\Omega)$ is Borel measurable.

One may prove 6d5 via 6d4 or, alternatively, via (6b7) and the proof of 6b5.

Decomposable vectors need not exist in a continuous product of Hilbert spaces in general, but they surely exist in every continuous product of spaces L_2, since constant functions are decomposable vectors.

6d6 Definition. A *continuous product of pointed Hilbert spaces* consists of a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ and vectors $u_{s,t} \in H_{s,t}$ such that

$$\begin{gathered} u_{r,s}u_{s,t} = u_{r,t} \quad \text{whenever } -\infty \le r < s < t \le \infty \,, \\ \|u_{s,t}\| = 1 \quad \text{whenever } -\infty \le s < t \le \infty \,. \end{gathered}$$

Such family $(u_{s,t})_{s<t}$ will be called a *unit* (of the given continuous product of Hilbert spaces). It is basically the same as a decomposable vector of $H = H_{-\infty,\infty}$. Each $H_{s,t}$ may be identified with a subspace of H, namely (the image of) $u_{-\infty,s} \otimes H_{s,t} \otimes u_{t,\infty}$.

Waiving the infinite points $\pm\infty$ on the time axis we get a *local* continuous product of pointed Hilbert spaces. The embeddings $H_{-1,1} \subset H_{-2,2} \subset \dots$ may be used for enlarging the time set $\mathbb{R}$ to $[-\infty,\infty]$; to this end $H_{-\infty,\infty}$ is constructed as the completion of the union of $H_{-n,n}$ (see also (6d16)). In this respect (and many others), continuous products of pointed Hilbert spaces are closer to continuous products of probability spaces than Hilbert spaces.

Every continuous product of spaces L_2 is naturally a continuous product of pointed Hilbert spaces, $u_{s,t}$ being the function that equals to 1 everywhere.

6d7 Question. Does every continuous product of pointed Hilbert spaces emerge from some continuous product of probability spaces (that is, is isomorphic to some continuous product of L_2 spaces with 'probabilistic' units)?

Many results and arguments of Sections 5a, 5b, 6a, 6b may be generalized to continuous products of pointed Hilbert spaces.

6d8 Definition. (a) Let $(H^{(1)}_{s,t})_{s<t}$ and $(H^{(2)}_{s,t})_{s<t}$ be two continuous products of Hilbert spaces. An *embedding*[1] of the first product to the second is a family $(\alpha_{s,t})_{s<t}$ of isometric linear embeddings $\alpha_{s,t} : H^{(1)}_{s,t} \to H^{(2)}_{s,t}$ such that

$$(\alpha_{r,s}f)(\alpha_{s,t}g) = \alpha_{r,t}(fg) \quad \text{for } f \in H^{(1)}_{r,s},\, g \in H^{(1)}_{s,t}$$

(as before, fg is the image of $f \otimes g$ in $H_{r,t}$).

(b) Let $(H^{(1)}_{s,t}, u^{(1)}_{s,t})_{s<t}$ and $(H^{(2)}_{s,t}, u^{(2)}_{s,t})_{s<t}$ be two continuous products of pointed Hilbert spaces. An *embedding* of the first product to the second is an

[1]Not 'morphism' for not contradicting [4, 3.7.1].

embedding $(\alpha_{s,t})_{s<t}$ of $(H^{(1)}_{s,t})_{s<t}$ to $(H^{(2)}_{s,t})_{s<t}$ as in (a) satisfying the additional condition

$$\alpha_{s,t} u^{(1)}_{s,t} = u^{(2)}_{s,t} \quad \text{for } s < t \,.$$

If $\alpha_{s,t}(H^{(1)}_{s,t})$ is the whole $H^{(2)}_{s,t}$ for $s<t$, then $(\alpha_{s,t})_{s<t}$ is an *isomorphism.*

Every morphism between continuous products of probability spaces leads to an embedding of the corresponding continuous product of pointed Hilbert spaces (in the opposite direction). See Examples 5a2, 5a4.

6d9 Definition. A *joining* (or *coupling*) of two continuous products of pointed Hilbert spaces $(H^{(1)}_{s,t}, u^{(1)}_{s,t})_{s<t}$ and $(H^{(2)}_{s,t}, u^{(2)}_{s,t})_{s<t}$ consists of a third continuous product of pointed Hilbert spaces $(H_{s,t}, u_{s,t})_{s<t}$ and two embeddings $(\alpha_{s,t})_{s<t}$, $(\beta_{s,t})_{s<t}$, $\alpha_{s,t} : H^{(1)}_{s,t} \to H_{s,t}$, $\beta_{s,t} : H^{(2)}_{s,t} \to H_{s,t}$ of these products such that $H_{-\infty,\infty}$ is the closed linear span of $\alpha_{-\infty,\infty}(H^{(1)}_{-\infty,\infty}) \cup \beta_{-\infty,\infty}(H^{(2)}_{-\infty,\infty})$.

Each joining leads to bilinear forms $(f,g) \mapsto \langle \alpha_{s,t} f, \beta_{s,t} g \rangle$ for $f \in H^{(1)}_{s,t}$, $g \in H^{(2)}_{s,t}$. Two joinings that lead to the same bilinear forms will be called isomorphic. A joining with itself will be called a *self-joining.* A *symmetric* self-joining is a self-joining (α,β) isomorphic to (β,α).

Every joining of two continuous products of probability spaces leads to a joining of the corresponding continuous products of pointed Hilbert spaces. The same holds for self-joinings and symmetric self-joinings.

Every joining (α,β) of two continuous products of pointed Hilbert spaces has its *maximal correlation*

$$\rho^{\max}(\alpha,\beta) = \rho^{\max}_{-\infty,\infty}(\alpha,\beta) \,,$$
$$\rho^{\max}_{s,t}(\alpha,\beta) = \sup |\langle \alpha_{s,t} f, \beta_{s,t} g \rangle| \,,$$

where the supremum is taken over all $f \in H^{(1)}_{s,t}$, $g \in H^{(2)}_{s,t}$ such that $\|f\| \le 1$, $\|g\| \le 1$, $\langle f, u^{(1)}_{s,t} \rangle = 0$, $\langle g, u^{(2)}_{s,t} \rangle = 0$.

The maximal correlation defined in Sect. 5a for a joining of continuous products of probability spaces is equal to the maximal correlation of the corresponding joining of continuous products of pointed Hilbert spaces.

6d10 Proposition. $\rho^{\max}_{r,t}(\alpha,\beta) = \max\big(\rho^{\max}_{r,s}(\alpha,\beta),\, \rho^{\max}_{s,t}(\alpha,\beta)\big)$ whenever $r < s < t$.

Proof (sketch). Similar to 5a7; each $H_{s,t}$ decomposes into the one-dimensional subspace spanned by $u_{s,t}$ and its orthogonal complement $H^0_{s,t}$. $\square$

6d11 Proposition. For every continuous product of pointed Hilbert spaces and every $\rho \in [0,1]$ there exists a symmetric self-joining $(\alpha_\rho, \beta_\rho)$ of the given product such that

$$\rho^{\max}(\alpha_\rho, \beta_\rho) \le \rho$$

and

$$\langle \alpha_{s,t} f, \beta_{s,t} f \rangle \le \langle (\alpha_\rho)_{s,t} f, (\beta_\rho)_{s,t} f \rangle$$

for all $s < t$, $f \in H_{s,t}$ and all self-joinings (α, β) satisfying $\rho^{\max}(\alpha, \beta) \le \rho$.

The self-joining $(\alpha_\rho, \beta_\rho)$ is unique up to isomorphism.

Proof (sketch). Similar to 5b1 but simpler; we just take the limit of the decreasing net of (commuting) Hermitian operators (or their quadratic forms)

$$\tilde U^\rho_{t_1,\dots,t_n} f = \bigotimes_{k=0}^n \big(\rho f_k + (1-\rho) \langle f_k, u_{t_k,t_{k+1}} \rangle u_{t_k,t_{k+1}} \big) \quad \text{for } f = f_0 \otimes \dots \otimes f_n ,$$
$$f_0 \in H_{-\infty,t_1} ,\; f_1 \in H_{t_1,t_2} ,\; \dots ,\; f_n \in H_{t_n,\infty} . \qquad \square$$

Operators $U^\rho_{s,t}$ satisfy

$$U^\rho_{r,s} \otimes U^\rho_{s,t} = U^\rho_{r,t} \, , \quad U^{\rho_1}_{s,t} U^{\rho_2}_{s,t} = U^{\rho_1 \rho_2}_{s,t} \, ,$$

and the spectrum of $U^\rho_{s,t}$ is contained in $\{1, \rho, \rho^2, \dots\} \cup \{0\}$. Similarly to 5b5 we have the following.

6d12 Proposition. For every continuous product of pointed Hilbert spaces there exist (closed linear) subspaces $H_0, H_1, H_2, \dots$ and H_∞ of $H = H_{-\infty,\infty}$ such that

$$\begin{gathered} H = (H_0 \oplus H_1 \oplus H_2 \oplus \dots) \oplus H_\infty \, , \\ U^\rho f = \rho^n f \quad \text{for } f \in H_n ,\; \rho \in [0,1] \, , \\ U^\rho f = 0 \quad \text{for } f \in H_\infty ,\; \rho \in [0,1) \, . \end{gathered}$$

The space H_0 is one-dimensional, spanned by $u_{-\infty,\infty}$. Similarly we introduce subspaces $H_n(s,t)$. The relation $U^\rho_{r,s} \otimes U^\rho_{s,t} = U^\rho_{r,t}$ implies

$$H_n(r,t) = \bigoplus_{k=0}^n H_k(r,s) \otimes H_{n-k}(s,t) \, . \tag{6d13}$$

We recall the embedding of each $H_{s,t}$ into $H = H_{-\infty,\infty}$ by $f \mapsto u_{-\infty,s} f u_{t,\infty}$ for $f \in H_{s,t}$, and introduce for $s < t$ the orthogonal projection $Q_{s,t}$ of H onto $H_{s,t} \subset H$; clearly,

$$Q_{s,t}(fgh) = \langle f, u_{-\infty,s} \rangle u_{-\infty,s} g \langle h, u_{t,\infty} \rangle u_{t,\infty}$$

for $f \in H_{-\infty,s}$, $g \in H_{s,t}$, $h \in H_{t,\infty}$.

6d14 Proposition. The following conditions are equivalent for every $f \in H$:

(a) $f \in H_1$;

(b) $f = Q_{-\infty,t} f + Q_{t,\infty} f$ for all $t \in \mathbb{R}$;

(c) $Q_{r,t} f = Q_{r,s} f + Q_{s,t} f$ whenever $-\infty \le r < s < t \le \infty$.

Proof (sketch). Similar to 6a1, with $Q_{s,t}$ instead of $\mathbb{E}\big(\cdot \,\big|\, \mathcal{F}_{s,t}\big)$ and projection to the unit instead of expectation. $\square$

6d15 Lemma. The space $H_0 \oplus H_1 \oplus H_2 \oplus \dots$ is the closure of the union of all subspaces of the form

$$\bigotimes_{k=0}^{n} \big(H_0(t_k, t_{k+1}) \oplus H_1(t_k, t_{k+1}) \big)$$

where $-\infty = t_0 < t_1 < \dots < t_n < t_{n+1} = \infty$, $n = 0, 1, 2, \dots$

Proof (sketch). Similar to 6a5. $\square$

The formula (6a6) holds as well. Similarly to (6a8), spaces $H_1(A)$ may be defined for all Borel sets $A \subset \mathbb{R}$.

Given a continuous product of pointed Hilbert spaces $(H_{s,t}, u_{s,t})_{s<t}$, we introduce the 'upward continuity' condition, similar to (2d3),

$$H_{s,t} \text{ is the closure of } \bigcup_{\varepsilon>0} H_{s+\varepsilon,t-\varepsilon} \quad \text{for } -\infty \le s < t \le \infty \tag{6d16}$$

(here $-\infty+\varepsilon$ means $-1/\varepsilon$, $\infty-\varepsilon$ means $1/\varepsilon$), and the 'downward continuity' condition, similar to (2d5),

$$H_{s,t} = \bigcap_{\varepsilon>0} H_{s-\varepsilon,t+\varepsilon} \quad \text{for } -\infty \le s \le t \le \infty \tag{6d17}$$

(here $H_{t,t}$ is the one-dimensional subspace spanned by the unit, $-\infty - \varepsilon$ means $-\infty$, and $\infty + \varepsilon$ means ∞).

Upward continuity (2d3) for continuous products of probability spaces is evidently equivalent to upward continuity (6d16) of the corresponding continuous products of pointed Hilbert spaces. The same holds for downward continuity. As noted in Sect. 2d (after 2d4), downward continuity does not imply upward continuity. The argument of 2d4 may be generalized as follows.

6d18 Proposition. Upward continuity implies downward continuity.

Proof (sketch). It is sufficient to prove that $\bigcap_{\varepsilon>0} H_{s,s+\varepsilon}$ is one-dimensional (spanned by the unit), since $\bigcap_{\varepsilon>0} H_{-\infty,s+\varepsilon} = H_{-\infty,s} \otimes \bigcap_{\varepsilon>0} H_{s,s+\varepsilon}$. Assuming the contrary, we take $f \in \bigcap_{\varepsilon>0} H_{s,s+\varepsilon}$, $\|f\| = 1$, orthogonal to the unit. Using upward continuity we approximate f by $g \in H_{s+\varepsilon,\infty}$, $\|g\| = 1$. We have $f = f_{s,s+\varepsilon} u_{s+\varepsilon,\infty}$, $g = u_{s,s+\varepsilon} g_{s+\varepsilon,\infty}$; thus, $|\langle f_{s,s+\varepsilon}, u_{s,s+\varepsilon}\rangle| \cdot |\langle u_{s+\varepsilon,\infty}, g_{s+\varepsilon,\infty}\rangle| = |\langle f, g\rangle|$ is close to 1, while $|\langle f_{s,s+\varepsilon}, u_{s,s+\varepsilon}\rangle|$ is small; a contradiction. □

6d19 Proposition. (Zacharias [50, Lemma 2.2.1], Arveson [4, Th. 6.2.3].) Let $(H_{s,t}, u_{s,t})_{s<t}$ be a continuous product of pointed Hilbert spaces, satisfying the upward continuity condition (6d16). Let $-\infty \le r < t \le \infty$ be given, and $f \in H_{r,t}$ be a decomposable vector. Then $\langle f, u_{r,t}\rangle \ne 0$.

Proof (sketch). Similar to the last argument of the proof of 6b1. Namely, f is the limit of the projection $g_\varepsilon = f_{r,s-\varepsilon}\langle f_{s-\varepsilon,s+\varepsilon}, u_{s-\varepsilon,s+\varepsilon}\rangle u_{s-\varepsilon,s+\varepsilon} f_{s+\varepsilon,t}$ of f to $H_{r,s-\varepsilon} \otimes u_{s-\varepsilon,s+\varepsilon} \otimes H_{s+\varepsilon,t}$. Therefore (see also [4, Sect. 6.1])

$$\frac{|\langle f_{s-\varepsilon,s+\varepsilon}, u_{s-\varepsilon,s+\varepsilon}\rangle|}{\|f_{s-\varepsilon,s+\varepsilon}\|} = \frac{\|f_{r,s-\varepsilon}\langle f_{s-\varepsilon,s+\varepsilon}, u_{s-\varepsilon,s+\varepsilon}\rangle u_{s-\varepsilon,s+\varepsilon} f_{s+\varepsilon,t}\|}{\|f_{r,s-\varepsilon}\|\|f_{s-\varepsilon,s+\varepsilon}\|\|f_{s+\varepsilon,t}\|} = \frac{\|g_\varepsilon\|}{\|f\|} \to 1\,.$$

We cover the compact interval $[r,t]$ by a finite number of open intervals $(s-\varepsilon, s+\varepsilon)$ such that $\langle f_{s-\varepsilon,s+\varepsilon}, u_{s-\varepsilon,s+\varepsilon}\rangle \ne 0$ and get $\langle f, u_{r,t}\rangle \ne 0$. (The argument may be adapted to the compact interval $[-\infty,\infty]$.) □

Now we generalize 6b1, 6b5 and 6b6.

6d20 Proposition. Let $(H_{s,t}, u_{s,t})_{s<t}$ be a continuous product of pointed Hilbert spaces, satisfying the upward continuity condition (6d16). Then all decomposable vectors of $H_{s,t}$ belong to $H_0(s,t) \oplus H_1(s,t) \oplus H_2(s,t) \oplus \dots$ (that is, are orthogonal to $H_\infty(s,t)$), for $-\infty \le s < t \le \infty$.

Proof (sketch). According to 6d12, it is sufficient to prove that $\langle U^\rho f, f\rangle \to \|f\|^2$ as $\rho \to 1$. We use 6d19 and the inequality

$$\langle U^\rho f, f\rangle \ge \|f\|^{2\rho} |\langle f, u_{s,t}\rangle|^{2(1-\rho)} \tag{6d21}$$

proven similarly to (6b2) (irrespective of the upward continuity). □

Let $f \in H_{-\infty,\infty}$ be a decomposable vector such that $\langle f, u_{-\infty,\infty}\rangle = 1$. Then $f_{s,t}$ as in 6d3 may be chosen such that $\langle f_{s,t}, u_{s,t}\rangle = 1$ for $s < t$. Using (6d21) we see that f is orthogonal to H_∞. Similarly to 6b5 (using 6d15), f is uniquely determined by its projection g to H_1. Similarly to 6b6, every $g \in H_1$ is the projection of some decomposable f. Similarly to (6b7) we denote the relation between f and g by

$$f = \operatorname{Exp} g\,; \qquad g = \operatorname{Log} f\,. \tag{6d22}$$

It is a one-to-one correspondence between decomposable vectors $f \in H_{-\infty,\infty}$ satisfying $\langle f, u_{-\infty,\infty} \rangle = 1$, and vectors $g \in H_1$. Still, $\| \operatorname{Exp} g \|^2 \le \exp(\|g\|^2)$ and $g = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big(\operatorname{Exp}(\varepsilon g) - u_{-\infty,\infty} \big)$. Similarly to 6b8, the space $H_0 \oplus H_1 \oplus H_2 \oplus \dots$ is the closed linear span of $\{ \operatorname{Exp} g : g \in H_1 \}$. Similarly to 6b9, the equality $\| \operatorname{Exp} g \|^2 = \exp(\|g\|^2)$ is ensured if $g_{t-,t+} = 0$ for all t, which in turn is ensured by the downward continuity condition (6d17), since $H_1(t-,t+) \subset H_{t-,t+}$ (and H_1 is orthogonal to the unit). Similarly to 6b10 we get the following.

6d23 Proposition. If $(H_{s,t}, u_{s,t})_{s<t}$ is a continuous product of pointed Hilbert spaces satisfying the downward continuity condition (6d17), then the map $\operatorname{Exp} : H_1 \to H_0 \oplus H_1 \oplus H_2 \oplus \dots$ has the property

$$\langle \operatorname{Exp} f, \operatorname{Exp} g \rangle = \exp \langle f, g \rangle \quad \text{for } f, g \in H_1 \, .$$

See also [4, Sect. 6.4], [50, Th. 2.2.4], [41, Appendix A]. We conclude that

$$H_0 \oplus H_1 \oplus H_2 \oplus \dots = \mathrm{e}^{H_1}$$

is the Fock space.

6e Hilbert spaces

We want to know, to which extent results of Sect. 6d depend on the choice of a unit $(u_{s,t})_{s<t}$ in a given continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ (if a unit exists). As before, the time set is $[-\infty, \infty]$. First, note that the enlargement of $\mathbb{R}$ to $[-\infty, \infty]$ mentioned after 6d6 depends heavily on the choice of a unit. Second, we compare two kinds of continuity, one being unit-independent (3c3), (3c5), the other unit-dependent (6d16), (6d17).

6e1 Proposition. Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces, and $(u_{s,t})_{s<t}$ a unit. Then the following three conditions are equivalent:

(a) the upward continuity (3c3) of $(H_{s,t})_{s<t}$;

(b) the downward continuity (3c5) of $(H_{s,t})_{s<t}$;

(c) the upward continuity (6d16) of $(H_{s,t}, u_{s,t})_{s<t}$.

Proof (sketch). (a) $\Longrightarrow$ (b): see the proof of 3c4.

(b) $\Longrightarrow$ (c): [19, Prop. 3.4] Projections $Q_\varepsilon : H \to H$ defined by $Q_\varepsilon(fgh) = f \langle g, u_{-\varepsilon,\varepsilon} \rangle u_{-\varepsilon,\varepsilon} h$ for $f \in H_{-\infty,-\varepsilon}$, $g \in H_{-\varepsilon,\varepsilon}$, $h \in H_{\varepsilon,\infty}$ belong to algebras $\mathcal{A}_{-\varepsilon,\varepsilon}$. Their limit $Q_{0+} = \lim_{\varepsilon \to 0} Q_\varepsilon$ belongs to the trivial algebra $\mathcal{A}_{0,0}$ by (3c5), and $Q_{0+} u_{-\infty,\infty} = u_{-\infty,\infty}$, therefore $Q_{0+} = \mathbf{1}$. It follows that $H_{-\infty,-\varepsilon} \uparrow H_{-\infty,0}$ and $H_{\varepsilon,\infty} \uparrow H_{0,\infty}$.

(c) $\Longrightarrow$ (a): we take $\varepsilon_n \downarrow 0$ and introduce projections Q_n of $H_{s,t}$ onto $H_{s+\varepsilon_n,t-\varepsilon_n} \subset H_{s,t}$, then $Q_n \uparrow \mathbf{1}$ by (6d16). For any operator $A \in \mathcal{A}_{s,t}$ we define $A_n \in \mathcal{A}_{s+\varepsilon_n,t-\varepsilon_n}$ by $A_n f = Q_n A f$ for $f \in H_{s+\varepsilon_n,t-\varepsilon_n} \subset H_{s,t}$ and observe that $A_n \to A$ strongly, since $\|A_n\| \le \|A\|$ and $A_n f \to Af$ for all $f \in \cup_{\varepsilon>0} H_{s+\varepsilon,t-\varepsilon}$. $\square$

See also [4, proof of 6.1.1].

6e2 Question. What about a counterpart of 6e1 for the downward continuity (6d17) of $(H_{s,t}, u_{s,t})_{s<t}$?

Operators U^ρ and subspaces H_n (recall 6d12) depend on the choice of a unit.

6e3 Theorem. For every continuous product of Hilbert spaces, containing at least one unit and satisfying the (equivalent) continuity conditions 6e1(a–c), the subspaces $H_0 \oplus H_1 \oplus \dots$ and H_∞ do not depend on the choice of the unit.

The theorem follows immediately from the next result (or alternatively, from 6e10).

6e4 Proposition. Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces satisfying 6e1(a–c) and containing at least one unit. Then $H_0(s,t) \oplus H_1(s,t) \oplus \dots$ is the closed linear span of (the set of all) decomposable vectors of $H_{s,t}$, for $-\infty \le s < t \le \infty$.

Proof (sketch). Combine Proposition 6d20 and the generalization of 6b8 (mentioned in Sect. 6d). $\square$

Assuming 6e1(a–c) we define $H^{\mathrm{cls}}_{s,t}$ as the closed linear span of decomposable vectors of $H_{s,t}$ and get for $H^{\mathrm{cls}} = H^{\mathrm{cls}}_{-\infty,\infty}$

$$H^{\mathrm{cls}} = H_0 \oplus H_1 \oplus \dots = H \ominus H_\infty \tag{6e5}$$

if at least one unit exists; otherwise $\dim H^{\mathrm{cls}} = 0$. Clearly,

$$H^{\mathrm{cls}}_{r,t} = H^{\mathrm{cls}}_{r,s} \otimes H^{\mathrm{cls}}_{s,t} \quad \text{for } -\infty \le r < s < t \le \infty \,,$$

and we get the classical part $(H^{\mathrm{cls}}_{s,t})_{s<t}$ of a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ provided that $\dim H^{\mathrm{cls}} \ne 0$. Proposition 6d23 shows that the classical part is the Fock space,

$$H^{\mathrm{cls}} = \mathrm{e}^{H_1} \,.$$

Of course, the space H_1 and the map $\operatorname{Exp} : H_1 \to H^{\text{cls}}$ depend on the choice of a unit. We make the dependence explicit by writing uH_1 instead of H_1 and $u\operatorname{Exp}$ instead of Exp. Recall that an *affine* operator between two Hilbert spaces H', H'' is an operator of the form $x \mapsto y_0 + Lx$ (for $x \in H'$) where $L : H' \to H''$ is a linear operator and $y_0 \in H''$. Here and in the following proposition, Hilbert spaces over $\mathbb{R}$ or $\mathbb{C}$ are acceptable.

6e6 Proposition. Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces satisfying 6e1(a–c), and $(u_{s,t})_{s<t}$, $(v_{s,t})_{s<t}$ two units. Then there exists an isometric affine invertible map $A : uH_1 \to vH_1$ such that the following conditions are equivalent for all $f \in uH_1$, $g \in vH_1$:

(a) $A(f) = g$;

(b) tho vectors $u\operatorname{Exp} f$, $v\operatorname{Exp} g$ span the same one-dimensional subspace.

Proof (sketch). It follows from 6d23 that

$$\frac{\langle u\operatorname{Exp} f_1, u\operatorname{Exp} f_2\rangle}{\|u\operatorname{Exp} f_1\|\|u\operatorname{Exp} f_2\|} = \frac{\langle u\operatorname{Exp} f_1, u\rangle}{|\langle u\operatorname{Exp} f_1, u\rangle|}\frac{\langle u, u\operatorname{Exp} f_2\rangle}{|\langle u, u\operatorname{Exp} f_2\rangle|}\exp\bigl(\langle f_1, f_2\rangle - \tfrac12\|f_1\|^2 - \tfrac12\|f_2\|^2\bigr) \tag{6e7}$$

for all $f_1, f_2 \in uH_1$. Therefore

$$\frac{|\langle u\operatorname{Exp} f_1, u\operatorname{Exp} f_2\rangle|}{\|u\operatorname{Exp} f_1\|\|u\operatorname{Exp} f_2\|} = \exp\bigl(-\tfrac12\|f_1 - f_2\|^2\bigr)\,, \tag{6e8}$$

which expresses the distance between f_1 and f_2 in terms of the one-dimensional subspaces spanned by $u\operatorname{Exp} f_1$ and $u\operatorname{Exp} f_2$. We get an isometric invertible map $A : uH_1 \to vH_1$; it remains to prove that A is affine. In the real case (over $\mathbb{R}$) it is well-known (and easy to see) that every isometry is affine. In the complex case (over $\mathbb{C}$) one more implication of (6e7) is used:

$$\exp\bigl(\mathrm{i}\operatorname{Im}\langle f_2 - f_1, f_3 - f_1\rangle = \frac{\langle u\operatorname{Exp} f_1, u\operatorname{Exp} f_2\rangle\langle u\operatorname{Exp} f_2, u\operatorname{Exp} f_3\rangle\langle u\operatorname{Exp} f_3, u\operatorname{Exp} f_1\rangle}{|\langle u\operatorname{Exp} f_1, u\operatorname{Exp} f_2\rangle\langle u\operatorname{Exp} f_2, u\operatorname{Exp} f_3\rangle\langle u\operatorname{Exp} f_3, u\operatorname{Exp} f_1\rangle|}\,. \tag{6e9}$$

□

The dependence of the operators U^ρ on the choice of a unit is estimated below (which gives us another proof of Theorem 6e3).

6e10 Proposition. Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces satisfying 6e1(a–c), and $(u_{s,t})_{s<t}$, $(v_{s,t})_{s<t}$ two units. Then the operators $U^\rho_{s,t}$, $V^\rho_{s,t}$ corresponding to these units satisfy the inequality

$$U^\rho_{s,t} \ge (V^\rho_{s,t})^2 \exp(-4(\ln\rho)\ln|\langle u_{s,t}, v_{s,t}\rangle|)$$

for $-\infty \le s < t \le \infty$ and $\rho \in (0,1]$.

(Inequalities between operators are treated as inequalities between their quadratic forms, of course.)

Proof (sketch). First, a general inequality

$$(2|\langle x,z\rangle|^2-1)(2|\langle y,z\rangle|^2-1) \le |\langle x,y\rangle|^2 \tag{6e11}$$

for every three unit vectors x,y,z of a Hilbert space satisfying $2|\langle x,z\rangle|^2 \ge 1$. We prove it introducing $\alpha,\beta,\gamma \in [0,\pi/2]$ by $\cos\alpha = |\langle x,y\rangle|$, $\cos\beta = |\langle x,z\rangle|$, $\cos\gamma = |\langle y,z\rangle|$. We have $\alpha \le \beta+\gamma$. However, $\beta \le \pi/4$; also $\gamma \le \pi/4$, otherwise there is nothing to prove. Therefore $\cos 2\alpha \ge \cos 2(\beta+\gamma)$, $2(\cos 2\beta)(\cos 2\gamma) = \cos 2(\beta-\gamma) + \cos 2(\beta+\gamma) \le 1+\cos 2\alpha$ and $(2\cos^2\beta - 1)(2\cos^2\gamma-1) \le \cos^2\alpha$, which is (6e11).

Second, for a given $\rho \in [\frac12, 1]$ and $s<t$ we define operators U,V by

$$Uf = \rho f + (1-\rho)\langle f, u_{s,t}\rangle u_{s,t}\,, \quad Vf = (2\rho-1)f + 2(1-\rho)\langle f, v_{s,t}\rangle v_{s,t}\,.$$

Assuming $\|f\|=1$ and applying (6e11) to $x=u_{s,t}$, $y=f$, $z=v_{s,t}$ we get

$$\bigl(2|\langle u_{s,t}, v_{s,t}\rangle|^2-1\bigr)\bigl(2|\langle f, v_{s,t}\rangle|^2-1\bigr) \le |\langle f, u_{s,t}\rangle|^2$$

provided that $2|\langle u_{s,t}, v_{s,t}\rangle|^2 \ge 1$. It may be written as an operator inequality

$$\begin{gathered}
\bigl(2|\langle u_{s,t}, v_{s,t}\rangle|^2-1\bigr)\biggl(\frac{V-(2\rho-1)\mathbf{1}}{1-\rho}-\mathbf{1}\biggr) \le \frac{U-\rho\mathbf{1}}{1-\rho}\,;\\
\bigl(2|\langle u_{s,t}, v_{s,t}\rangle|^2-1\bigr)(V-\rho\mathbf{1}) \le U-\rho\mathbf{1}\,;\\
\bigl(2|\langle u_{s,t}, v_{s,t}\rangle|^2-1\bigr)V + 2\rho\bigl(1-|\langle u_{s,t}, v_{s,t}\rangle|^2\bigr)\mathbf{1} = \\
= \bigl(2|\langle u_{s,t}, v_{s,t}\rangle|^2-1\bigr)(V-\rho\mathbf{1}) + \rho\mathbf{1} \le U\,.
\end{gathered}$$

Taking into account that $V \le \mathbf{1}$ we have

$$\bigl(2\rho-1+2(1-\rho)|\langle u_{s,t}, v_{s,t}\rangle|^2\bigr)V \le U\,. \tag{6e12}$$

Third, using the upward continuity condition (similarly to 6d19), for any ε we can choose $s=t_0<t_1<\dots<t_n=t$ such that $|\langle u_{t_{k-1},t_k}, v_{t_{k-1},t_k}\rangle| > 1-\varepsilon$ for $k=1,\dots,n$. We apply (6e12) on each interval (t_{k-1},t_k) rather than (s,t) and multiply the inequalities:

$$\biggl(\prod_{k=1}^n \bigl(2\rho-1+2(1-\rho)|\langle u_{t_{k-1},t_k}, v_{t_{k-1},t_k}\rangle|^2\bigr)\biggr)\tilde V^{2\rho-1}_{t_1,\dots,t_n} \le \tilde U^{\rho}_{t_1,\dots,t_n}\,;$$

taking the limit of the net we get for $\rho \in [\frac12,1]$

$$|\langle u_{s,t}, v_{s,t}\rangle|^{4(1-\rho)} V^{2\rho-1}_{s,t} \le U^{\rho}_{s,t}\,, \tag{6e13}$$

since $\ln(2\rho - 1 + 2(1-\rho)a^2) \sim -2(1-\rho)(1-a^2) \sim 4(1-\rho)\ln a$ as $a \to 1$.

Fourth, we apply (6e13) to $\rho = 1 - \frac{c}{n}$ (for an arbitrary $c > 0$) and raise to the n-th power, using the semigroup property of U^ρ, V^ρ:

$$|\langle u_{s,t}, v_{s,t}\rangle|^{\frac{4c}{n}} V_{s,t}^{1-\frac{2c}{n}} \le U_{s,t}^{1-\frac{c}{n}}\,; \qquad |\langle u_{s,t}, v_{s,t}\rangle|^{4c} V_{s,t}^{(1-\frac{2c}{n})^n} \le U_{s,t}^{(1-\frac{c}{n})^n}\,;$$

the limit for $n \to \infty$ gives

$$|\langle u_{s,t}, v_{s,t}\rangle|^{4c} V_{s,t}^{\exp(-2c)} \le U_{s,t}^{\exp(-c)}\,.$$

Substituting $\rho = \exp(-c)$ we get $|\langle u_{s,t}, v_{s,t}\rangle|^{-4\ln\rho} V_{s,t}^{\rho^2} \le U_{s,t}^{\rho}$. □

Definition 3a1 of a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$ stipulates the global space $H_{-\infty,\infty}$. However, all said about local spaces (for $s,t \ne \pm\infty$) holds for *local* continuous products of Hilbert spaces (defined similarly to 3a1 but waiving the infinite points $\pm\infty$ on the time axis).

6f Homogeneous case; Arveson systems of type I

Given a homogeneous continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, $(\theta^h_{s,t})_{s<t;h}$, we may consider the classical part of the continuous product of Hilbert spaces $(H_{s,t})_{s<t}$. It consists of spaces $H^{\mathrm{cls}}_{s,t}$ spanned by decomposable vectors. The time shift $\theta^h_{s,t}$ sends $H^{\mathrm{cls}}_{s,t}$ to $H^{\mathrm{cls}}_{s+h,t+h}$, since decomposable vectors go to decomposable vectors. We see that the classical part of a homogeneous continuous product of Hilbert spaces is a *homogeneous* continuous product of Hilbert spaces, provided that $\dim H^{\mathrm{cls}} > 0$.

If the homogeneous continuous product of Hilbert spaces corresponds to a noise then surely $\dim H^{\mathrm{cls}} > 0$, since constant functions are decomposable vectors. They are also shift-invariant, that is, invariant under the group of shifts $(\theta^h)_h$ (that is, $(\theta^h_{-\infty,\infty})_h$).

As before, the time set is $[-\infty, \infty]$, which is crucial below.

6f1 Proposition. For every homogeneous continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, $(\theta^h_{s,t})_{s<t;h}$, the subspace spanned by all shift-invariant decomposable vectors is either 0-dimensional or 1-dimensional.

Proof (sketch). Let u, v be two such vectors, $\|u\| = 1$, $\|v\| = 1$. By 6e1, the upward continuity (3c3) of $(H_{s,t})_{s<t}$, ensured by 3c2, implies the upward continuity (6d16) of $(H_{s,t}, u_{s,t})_{s<t}$. By 6d19, $\langle u, v\rangle \ne 0$.

Clearly, $|\langle u, v\rangle| \le \prod_{k=1}^n |\langle u_{k-1,k}, v_{k-1,k}\rangle|$. However, $|\langle u_{k-1,k}, v_{k-1,k}\rangle|$ does not depend on k by the shift invariance. Thus, $0 < |\langle u, v\rangle| \le |\langle u_{0,1}, v_{0,1}\rangle|^n$ for all n, which means that $|\langle u_{0,1}, v_{0,1}\rangle| = 1$.

Similarly to the proof of 6d19, $|\langle u_{-\infty,-n}, v_{-\infty,-n}\rangle| \to 1$ and $|\langle u_{n,\infty}, v_{n,\infty}\rangle| \to 1$ for $n \to \infty$. Therefore $|\langle u, v\rangle| = \lim_n |\langle u_{-n,n}, v_{-n,n}\rangle| = \lim_n |\langle u_{0,1}, v_{0,1}\rangle|^{2n} = 1$. $\square$

6f2 Example. It can happen that decomposable vectors exist, but no one of them is shift-invariant. An example will be constructed from a (non-stationary) $\mathbb{R}$-flow $(X_{s,t})_{s<t}$ such that each $X_{s,t}$ is distributed normally, $\operatorname{Var}(X_{s,t}) = t - s$ (just like Brownian increments) and $\mathbb{E}\, X_{s,t} = 0$ for $-\infty < s < t \le 0$, but $\mathbb{E}\, X_{s,t} = t - s$ for $0 \le s < t < \infty$ (a drift after 0). Time shifts $T_h : \Omega \to \Omega$ defined by $X_{s,t} \circ T_h = X_{s+h,t+h}$ do not preserve the measure P (thus, our object is not a noise), however, they transform P into an equivalent (that is, mutually absolutely continuous) measure. These time shifts lead to unitary operators θ_h on $H = L_2(\Omega, P)$; unitarity is achieved by multiplying the given function at $T_h\omega$ by the square root of the corresponding density (that is, Radon-Nikodym derivative), see Sect. 10a for details. The absolute continuity of measures is not uniform in h. Moreover, $\theta_h \to 0$ in the weak operator topology (as $h \to \pm\infty$), which excludes any non-zero shift-invariant vector (decomposable or not). On the other hand, waiving time shifts we get a *classical* system; decomposable vectors span H.

Shift-invariant decomposable vectors are scarce, as far as global vectors are meant. Think about $u = \exp\bigl(\mathrm{i}\lambda(B_\infty - B_{-\infty})\bigr)$ for a Brownian motion B; u is ill-defined (unless $\lambda = 0$), however $u_{s,t} = \exp\bigl(\mathrm{i}\lambda(B_t - B_s)\bigr)$ is well-defined (for each λ).

Waiving the global space $H_{-\infty,\infty}$ we get *local* homogeneous continuous products of Hilbert spaces, or equivalently, algebraic product systems of Hilbert spaces (recall 3c7). In order to exclude pathologies we impose a natural condition of measurability, thus turning to Arveson systems (recall 3c8).

6f3 Definition. A *unit* of an Arveson system $(H_t)_{t>0}$ is a family $(u_t)_{t>0}$ of vectors $u_t \in H_t$ such that

(a) $u_s u_t = u_{s+t}$ for $s, t \in (0, \infty)$,

(b) the map $t \mapsto u_t$ from $(0, \infty)$ to $\biguplus_{t>0} H_t$ is Borel measurable,

(c) $\|u_t\| = 1$ for $t \in (0, \infty)$.

Arveson [4, 3.6.1] admits $\|u_t\| = \mathrm{e}^{ct}$ for any $c \in \mathbb{R}$, but I prefer $\|u_t\| = 1$ for compatibility with 6d6.

Translating 6f3 into the language of local homogeneous continuous products of Hilbert spaces we get units $(u_{s,t})_{-\infty<s<t<\infty}$ satisfying $\theta^h_{s,t} u_{s,t} = u_{s+h,t+h}$ and measurability. The latter can be enforced by appropriate scalar coefficients, see 6d3. Accordingly, 6f3(b) can be enforced by replacing u_t with $c_t u_t$

for some $(c_t)_{t>0}$ such that $c_s c_t = c_{s+t}$. Continuity conditions 6e1(a–c) are satisfied locally (on every finite time interval).

6f4 Lemma. Let $(H_t)_{t>0}$ be an Arveson system such that H_t contains at least one decomposable vector, for at least one t. Then the system contains a unit.

Proof (sketch). Assuming that H_1 contains a decomposable vector, we get a decomposable vector $v \in H_{\mathbb{T}}$ of the cyclic-time system corresponding to the given system (recall Sect. 3d). The space vH_1 need not be shift-invariant, however, the set $\{c \cdot v \operatorname{Exp} f : c \in \mathbb{C} \setminus \{0\}, f \in vH_1\}$ of all decomposable vectors is shift-invariant. Applying 6e6 we see that $\mathbb{T}$ acts on vH_1 by affine transformations $A^h : vH_1 \to vH_1$; namely, $v \operatorname{Exp}(A^h f) = c\theta^h_{\mathbb{T}}(v \operatorname{Exp} f)$ for some $c \in \mathbb{C} \setminus \{0\}$ (that depends on f and h). The action is continuous (recall (6e8)), and has at least one fixed point, since the point $\int_{\mathbb{T}} A^h f \, dh$ evidently is fixed, irrespective of $f \in vH_1$. (Roughly speaking, the geometric mean of all shifts of a decomposable vector is a shift-invariant decomposable vector.) Let f be a fixed point, then the decomposable vector $u = v \operatorname{Exp} f$ satisfies $\theta^h_{\mathbb{T}} u = c(h)u$, therefore $\theta^h_{\mathbb{T}} u = \mathrm{e}^{2\pi \mathrm{i} n h} u$ for some $n \in \mathbb{Z}$.

Now it will be shown that $n = 0$, by checking the equality $\theta^{1/n}_{\mathbb{T}} u = u$ for all $n = 1, 2, \dots$ Instead of u we use here another vector of the same one-dimensional subspace, namely, $\bigotimes_{k=0}^{n-1} \theta^{\frac{k}{n}}_{0,\frac{1}{n}} u_{0,\frac{1}{n}}$. Applying to it the operator $\theta^{\frac{1}{n}}_{\mathbb{T}} = \bigotimes_{k=0}^{n-1} \theta^{\frac{1}{n}}_{\frac{k}{n},\frac{k+1}{n}}$ we get $\bigotimes_{k=0}^{n-1} \theta^{\frac{1}{n}}_{\frac{k}{n},\frac{k+1}{n}} \theta^{\frac{k}{n}}_{0,\frac{1}{n}} u_{0,\frac{1}{n}} = \bigotimes_{k=0}^{n-1} \theta^{\frac{k+1}{n}}_{0,\frac{1}{n}} u_{0,\frac{1}{n}} = \bigotimes_{k=0}^{n-1} \theta^{\frac{k}{n}}_{0,\frac{1}{n}} u_{0,\frac{1}{n}}$.

Having $\theta^h_{\mathbb{T}} u = u$ for $h \in \mathbb{T}$, we return to the linear time and construct a unit $(u_t)_{t>0}$, namely, $u_{n+t} = \underbrace{u_{\mathbb{T}} \otimes \dots \otimes u_{\mathbb{T}}}_{n} \otimes u_{0,t}$ for $n = 0, 1, 2, \dots$ and $0 \le t < 1$. $\square$

The classical part H^{cls}_t of H_t may be defined as the closed linear span of all decomposable vectors of H_t. Clearly, $H^{\mathrm{cls}}_s \otimes H^{\mathrm{cls}}_t = H^{\mathrm{cls}}_{s+t}$, and we get another Arveson system $(H^{\mathrm{cls}}_t)_{t>0}$, the classical part of the given Arveson system, provided that $\dim H^{\mathrm{cls}}_t \ne 0$.

6f5 Definition. [4, 6.0.3] An Arveson system $(H_t)_{t>0}$ is *decomposable,* if for every $t > 0$, the space H_t is the closed linear span of its decomposable vectors.

The classical part of an Arveson system is decomposable. Here is a counterpart of Theorem 6c1.

6f6 Theorem. Every decomposable Arveson system is generated by its units.

That is, H_t is the closed linear span of vectors of the form $(u_1)_{\frac{t}{n}}(u_2)_{\frac{t}{n}}\dots(u_n)_{\frac{t}{n}}$, where $u_1, u_2, \dots, u_n$ are units. See also [4, 6.0.5], [19, Cor. 6.6]. In order to prove the theorem we first translate everything into the language of a local homogeneous continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, $(\theta^h_{s,t})_{s<t;h}$ and its classical part $(H^{\text{cls}}_{s,t})_{s<t}$, $(\theta^h_{s,t}|_{H^{\text{cls}}_{s,t}})_{s<t;h}$. We know from Sect. 6e that $H^{\text{cls}}_{s,t} = uH_0(s,t) \oplus uH_1(s,t) \oplus \dots = H_{s,t} \ominus uH_\infty(s,t)$ for any unit u (not necessarily shift-invariant; and if $H_{s,t}$ contains no units then $\dim H^{\text{cls}}_{s,t} = 0$). Also, we have the map $u_{s,t}\operatorname{Exp} : u_{s,t}H_1(s,t) \to H^{\text{cls}}_{s,t}$ satisfying $\langle u_{s,t}\operatorname{Exp} f, u_{s,t}\operatorname{Exp} g\rangle = \exp\langle f,g\rangle$. All decomposable vectors of $H_{s,t}$ are of the form $u_{s,t}\operatorname{Exp} f$ up to a coefficient. Thus, $H^{\text{cls}}_{s,t}$ is the Fock space,

$$H^{\text{cls}}_{s,t} = \mathrm{e}^{u_{s,t}H_1(s,t)}\,.$$

We have no global space $H^{\text{cls}}_{-\infty,\infty}$, but still, the global first chaos space $uH_1 = uH_1(-\infty,\infty)$ is well-defined, according to *additive* relations $u_{r,s}H_1(r,s) \oplus u_{s,t}H_1(s,t) = u_{r,t}H_1(r,t)$.

Assume now that the unit u is shift-invariant. Then the subspace uH_1 is shift-invariant. We have no global exponential map $u\operatorname{Exp} : uH_1 \to H^{\text{cls}}$, but we have a family of local exponential maps $u_{s,t}\operatorname{Exp} : u_{s,t}H_1(s,t) \to H^{\text{cls}}_{s,t}$ shift-invariant in the sense that $u_{s+h,t+h}\operatorname{Exp}(\theta^h_{s,t}f) = \theta^h_{s,t}u_{s,t}\operatorname{Exp} f$ for $f \in u_{s,t}H_1(s,t)$.

Similarly to the proof of Theorem 6c1, time shifts induce unitary operators $U_h : uH_1 \to uH_1$ such that $U_h^{-1}Q_{s,t}U_h = Q_{s+h,t+h}$, where $Q_{s,t}$ is the projection of uH_1 onto $u_{s,t}H_1(s,t) \subset uH_1$. They lead to Weyl relations $U_hV_\lambda = \mathrm{e}^{\mathrm{i}\lambda h}V_\lambda U_h$, and so, H_1 decomposes into the direct sum of a finite or countable number of irreducible components, unitarily equivalent to the standard representation of Weyl relations in $L_2(\mathbb{R})$. Similarly to Sect. 6c we may treat uH_1 as the tensor product, $uH_1 = L_2(\mathbb{R}) \otimes \mathcal{H} = L_2(\mathbb{R},\mathcal{H})$, where $L_2(\mathbb{R})$ carries the standard representation of Weyl relations, and $\mathcal{H}$ is the Hilbert space of all families $(g_{s,t})_{s<t}$ of vectors $g_{s,t} \in u_{s,t}H_1$ satisfying $g_{r,t} = g_{r,s} + g_{s,t}$ and shift-invariant in the sense that $\theta^h_{s,t}g_{s,t} = g_{s+h,t+h}$.

Given a decomposable vector $u_{0,t}\operatorname{Exp} g \in H_{0,t}$, we approximate $g \in u_{0,t}H_1(0,t) = L_2((0,t),\mathcal{H})$ by step functions $g_n : (0,t) \to \mathcal{H}$ constant on $(0,\frac{t}{n}), (\frac{t}{n},\frac{2t}{n}), \dots, (\frac{(n-1)t}{n}, t)$. Applying $u_{0,t}\operatorname{Exp}$ to this step function we complete the proof of Theorem 6f6.

At the same time we classify all *classical* Arveson systems up to isomorphism. They consist of Fock spaces,

$$H_t = \mathrm{e}^{L_2((0,t),\mathcal{H})}$$

with the natural multiplication (and Borel structure). The classifying parameter is $\dim \mathcal{H} \in \{0, 1, 2, \dots\} \cup \{\infty\}$. See Arveson [4, Th. 6.7.1, Def. 3.1.6 and Prop. 3.5.1] and Zacharias [50, Th. 4.1.10].

6g Examples

The two noises of Sect. 4 (splitting and stickiness) are nonclassical noises; both satisfy $\mathcal{F}^{\text{stable}} = \mathcal{F}^{\text{white}}_{-\infty,\infty} \subsetneqq \mathcal{F}_{-\infty,\infty}$ (recall Sect. 5d). The corresponding continuous products of spaces L_2 are nonclassical continuous products of *pointed* Hilbert spaces (recall Sect. 6d), their classical parts being $L_2(\mathcal{F}^{\text{stable}})$. According to Sect. 6e, they are also nonclassical continuous products of Hilbert spaces; still, $L_2(\mathcal{F}^{\text{stable}})$ are their classical parts. According to Sect. 6f, these nonclassical products lead to Arveson systems. In both cases (splitting and stickiness), the classical part of the Arveson system is the Fock space $H^{\text{cls}} = L_2(\mathcal{F}^{\text{white}}_{-\infty,\infty})$. We see that the classical part is neither trivial nor the whole system; such Arveson systems (or rather, the corresponding E_0-semigroups) are known as type II systems, see [4, 2.7.6]. Type I means a classical system, while type III means a system with a trivial classical part.

7 Distributed singularity, black noise (according to Le Jan and Raimond)

7a Black noise in general

7a1 Definition. A noise is *black* if its classical part is trivial, but the whole noise is not.

In other words: all stable random variables are constant, but some random variables are not constant. There exist nontrivial centered (that is, zero-mean) random variables, and they all are sensitive. The self-joinings $(\alpha_\rho, \beta_\rho)$ and the operators U^ρ introduced in Sect. 5b are trivial, irrespective of $\rho \in [0, 1)$, if the noise is black. (See also [45, Remark 2.1].)

Existence of some black noise is proven by Vershik and the author [41, Sect. 5], but the term 'black noise' appeared in [35]. Why 'black'? Well, the white noise is called 'white' since its spectral density is constant. It excites harmonic oscillators of all frequencies to the same extent. For a black noise, however, the response of any linear sensor is zero!

What could be a physically reasonable nonlinear sensor able to sense a black noise? Maybe a fluid could do it, which is hinted at by the following words of Shnirelman [30, p. 1263] about a paradoxical motion of an ideal incompressible fluid: '... very strong external forces are present, but they are infinitely fast oscillating in space and therefore are indistinguishable from zero in the sense of distributions. The smooth test functions are not "sensitive" enough to "feel" these forces.'

The very idea of black noises, nonclassical continuous products etc. was suggested to me by Anatoly Vershik in 1994.

Two black noises are presented in Sections 7f, 7j. Two more ways of constructing black noises are available, see [47] and [40, 8b].

By Theorem 6a4, a noise is black if and only if $\dim H_1 = 0$, that is, the first chaos space is trivial. More generally, a continuous product of probability spaces has no classical part (that is, its classical part is trivial) if and only if $\dim H_1 = 0$.

7a2 Lemma. (a) For every continuous product of probability spaces, the first chaos space H_1 is equal to the intersection of spaces of the form

$$L_2^0(\mathcal{F}_{-\infty,t_0}) + L_2^0(\mathcal{F}_{t_0,t_1}) + \dots + L_2^0(\mathcal{F}_{t_{n-1},t_n}) + L_2^0(\mathcal{F}_{t_n,\infty})$$

over all finite sets $\{t_0,\dots,t_n\} \subset \mathbb{R}$, $-\infty < t_0 < \dots < t_n < \infty$. Here $L_2^0(s,t) \subset L_2(\mathcal{F}_{-\infty,\infty})$ consists of all $f \in L_2(\mathcal{F}_{s,t})$ such that $\mathbb{E}\, f = 0$.

(b) If the continuous product of probability spaces satisfies the downward continuity condition (2d5), then the intersection over rational $t_0,\dots,t_n$ is also equal to H_1. The same holds for every dense subset of $\mathbb{R}$.

Proof (sketch). (a): Follows easily from 6a1.

(b): Given an irrational $t \in \mathbb{R}$, we choose rational $r_k \uparrow t$. The downward continuity gives $\mathbb{E}\big(f \,\big|\, \mathcal{F}_{r_k,\infty}\big) \to \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t,\infty}\big)$. If f belongs to the intersection over rationales then $f = \mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,r_k}\big) + \mathbb{E}\big(f \,\big|\, \mathcal{F}_{r_k,\infty}\big)$, thus $\mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,r_k}\big) \to f - \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t,\infty}\big)$. Taking $\mathbb{E}\big(\dots \,\big|\, \mathcal{F}_{-\infty,t}\big)$ we get $\mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,r_k}\big) \to \mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,t}\big)$, therefore $f = \mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,t}\big) + \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t,\infty}\big)$. $\square$

7a3 Corollary. (a) For every continuous product of probability spaces, the orthogonal projection of $f \in L_2(\mathcal{F}_{-\infty,\infty})$ such that $\mathbb{E}\, f = 0$ to H_1 is the limit (in L_2) of the net of random variables

$$\mathbb{E}\big(f \,\big|\, \mathcal{F}_{-\infty,t_0}\big) + \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t_0,t_1}\big) + \dots + \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t_{n-1},t_n}\big) + \mathbb{E}\big(f \,\big|\, \mathcal{F}_{t_n,\infty}\big)$$

indexed by all finite sets $\{t_0,\dots,t_n\} \subset \mathbb{R}$, $-\infty < t_0 < \dots < t_n < \infty$.

(b) For every noise, the orthogonal projection of $f \in L_2(\mathcal{F}_{0,1})$ such that $\mathbb{E}\, f = 0$ to $H_1(0,1)$ is equal to

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \mathbb{E}\big(f \,\big|\, \mathcal{F}_{(k-1)2^{-n},k2^{-n}}\big) \,.$$

(c) A noise is black if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \mathbb{E}\bigl(f \bigm| \mathcal{F}_{(k-1)2^{-n},k2^{-n}} \bigr) = 0 \quad \text{for all } f \in L_2(\mathcal{F}_{0,1}),\ \mathbb{E}\, f = 0$$

(or equivalently, for all f of a dense subset of $L_2^0(\mathcal{F}_{0,1})$).

Proof (sketch). (a), (b): Consider the projections to the spaces treated in 7a2; use 2d4.

(c): If $\dim H_1(0,1) = 0$ then $\dim H_1(n, n+1) = 0$ by homogeneity, therefore $\dim H_1 = 0$. □

See also [40, 6a4].

7a4 Corollary. For every continuous product of probability spaces and every function $f \in L_2(\Omega)$, if

$$\operatorname{Var}\bigl(\mathbb{E}\bigl(f \bigm| \mathcal{F}_{s,s+\varepsilon} \bigr)\bigr) = o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

uniformly in $s \in [r,t]$, then f is orthogonal to $H_1(r,t)$.

Proof (sketch). Assuming $\mathbb{E}\, f = 0$ we see that

$$\Bigl\| \sum_{k=1}^{n} \mathbb{E}\bigl(f \bigm| \mathcal{F}_{s_{k-1},s_k} \bigr) \Bigr\|^2 = \sum_{k=1}^{n} \operatorname{Var}\bigl(\mathbb{E}\bigl(f \bigm| \mathcal{F}_{s_{k-1},s_k} \bigr)\bigr)$$

is much smaller than $\sum_{k=1}^{n} (s_k - s_{k-1}) = t - r$ whenever $r = s_0 < s_1 < \dots < s_n = t$ are such that $\max(s_k - s_{k-1})$ is small enough. Thus, the limit of the net vanishes. □

7b Black noise and flow system

7b1 Proposition. Let a flow system $(X_{s,t})_{s<t}$, $X_{s,t} : \Omega \to G_{s,t}$ be such that for every interval $(r,t) \subset \mathbb{R}$ and bounded measurable function $f : G_{r,t} \to \mathbb{R}$,

$$\operatorname{Var}\bigl(\mathbb{E}\bigl(f(X_{r,t}) \bigm| \mathcal{F}_{s,s+\varepsilon} \bigr)\bigr) = o(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{7b2}$$

uniformly in $s \in [r,t]$. Assume also that the classical part of the corresponding continuous product of probability spaces satisfies the equivalent continuity conditions 6a13(a–c). Then the classical part is trivial.

Proof (sketch). By 7a4 the random variable $f(X_{r,t})$ is orthogonal to $H_1(r,t)$. However, $\big(H_{r,s}\ominus H_1(r,s)\big)\otimes\big(H_{s,t}\ominus H_1(s,t)\big)\subset H_{r,t}\ominus H_1(r,t)$, since $H_{r,s}\ominus H_1(r,s)=\big(H_0\oplus H_2(r,s)\oplus H_3(r,s)\oplus\dots\big)\oplus H_\infty(r,s)$, and $H_k(r,s)\otimes H_l(s,t)\subset H_{k+l}(r,t)$ for $k,l\in\{0,1,2,\dots\}\cup\{\infty\}$. Therefore random variables of the form $f_1(X_{r,s})f_2(X_{s,t})$ are orthogonal to $H_1(r,t)$. Similarly, random variables of the form $f_1(X_{t_0,t_1})f_2(X_{t_1,t_2})\dots f_n(X_{t_{n-1},t_n})$ for $t_0<t_1<\dots<t_n$ are orthogonal to H_1. These random variables being dense in $L_2(\Omega)$, we get $\dim H_1=0$. ☐

We have

$$\mathbb{E}\big(f(X_{r,t})\,\big|\,\mathcal{F}_{s,s+\varepsilon}\big)=\mathbb{E}\big(f(X_{r,s}X_{s,s+\varepsilon}X_{s+\varepsilon,t})\,\big|\,\mathcal{F}_{s,s+\varepsilon}\big)=\\ =\mathbb{E}\big(f(X_{r,s}X_{s,s+\varepsilon}X_{s+\varepsilon,t})\,\big|\,X_{s,s+\varepsilon}\big)=f_{s,s+\varepsilon}(X_{s,s+\varepsilon})\,,$$

where $f_{s,s+\varepsilon}:G_{s,s+\varepsilon}\to\mathbb{R}$ is defined by

$$f_{s,s+\varepsilon}(y)=\int f(xyz)\,\mu_{r,s}(\mathrm{d}x)\mu_{s+\varepsilon,t}(\mathrm{d}z)\,; \tag{7b3}$$

as before, the measure $\mu_{s,t}$ on $G_{s,t}$ is the distribution of $X_{s,t}$. Therefore

$$\operatorname{Var}\big(\mathbb{E}\big(f(X_{r,t})\,\big|\,\mathcal{F}_{s,s+\varepsilon}\big)\big)=\int|f_{s,s+\varepsilon}|^2\,\mathrm{d}\mu_{s,s+\varepsilon}-\bigg|\int f_{s,s+\varepsilon}\,\mathrm{d}\mu_{s,s+\varepsilon}\bigg|^2,$$

and (7b2) becomes

$$\int|f_{s,s+\varepsilon}|^2\,\mathrm{d}\mu_{s,s+\varepsilon}-\bigg|\int f_{s,s+\varepsilon}\,\mathrm{d}\mu_{s,s+\varepsilon}\bigg|^2=o(\varepsilon) \tag{7b4}$$

uniformly in s, for all $f\in L_\infty(G_{r,t},\mu_{r,t})$. It is sufficient to check the condition for all f of a subset of $L_\infty(G_{r,t},\mu_{r,t})$ dense in $L_2(G_{r,t},\mu_{r,t})$ (recall the last argument of the proof of 7b1).

7c About random maps in general

It is often inconvenient to treat a random process as a random function, that is, a map from Ω to the space of functions. Say, a Poisson process has a right-continuous modification, a left-continuous modification and a lot of other modifications, but the choice of a modification is often irrelevant. It is already stipulated in Sect. 2a that "a stochastic flow (and any random process) is generally treated as a family of equivalence classes (rather than functions)", but now we have to be more explicit.

7c1 Definition. (a) An *S-map* from a set A to a standard measurable space B consists of a probability space $(\Omega, \mathcal{F}, P)$ and a family $\Xi = (\Xi_a)_{a \in A}$ of equivalence classes Ξ_a of measurable maps $\Omega \to B$ such that $\mathcal{F}$ is generated by all Ξ_a ('non-redundancy').

(b) The *distribution* $\Lambda = \Lambda_\Xi$ of an S-map Ξ is the family of its finite-dimensional distributions, that is, the joint distributions $\lambda_{a_1,\dots,a_n}$ of B-valued random variables $\Xi_{a_1}, \dots, \Xi_{a_n}$ for all $n = 1, 2, \dots$ and $a_1, \dots, a_n \in A$.

Of course, two maps $\Omega \to B$ are called equivalent iff they are equal almost everywhere on Ω. As always, Ω is a standard probability space. The non-redundancy can be enforced by replacing $(\Omega, \mathcal{F}, P)$ with its quotient space.

One may say that an S-map $A \to B$ is a B-valued random process on A, provided that all modifications are treated as the same process.

Two S-maps Ξ, Ξ' are identically distributed ($\Lambda_\Xi = \Lambda_{\Xi'}$) if and only if they are isomorphic in the following sense: there exists an isomorphism α between the corresponding probability spaces such that $\Xi_a = \Xi'_a \circ \alpha$ for $a \in A$.

7c2 Example. A stationary Gaussian random process on $\mathbb{R}$, continuous in probability, may be treated as a special curve ('helix') in a Hilbert space of Gaussian random variables. Depending on the covariance function, sometimes it has continuous sample paths, sometimes not. In the latter case we have no idea of a 'favorite' modification, but anyway, the corresponding S-map from $\mathbb{R}$ to $\mathbb{R}$ is well-defined (and continuous in probability), and its distribution is uniquely determined by the covariance function.

7c3 Example. Skorokhod [31, Sect. 1.1.1] defines a strong random operator on a Hilbert space H as a continuous linear map from H into the space of H-valued random variables. It may be treated as a linear continuous S-map $H \to H$, but generally not a random linear continuous operator $H \to H$.

My term 'S-map' is derived from Skorokhod's 'strong random operator'; 'S' may allude to 'stochastic', 'Skorokhod', or 'strong'.

Every S-map Ξ from A to B leads to a linear operator $\overleftarrow{T}_1^\Xi$ from the space of all bounded measurable functions on B to the space of all bounded functions on A; namely,

$$(\overleftarrow{T}_1^\Xi \varphi)(a) = \mathbb{E}\, \varphi(\Xi_a)\,.$$

However, $\overleftarrow{T}_1^\Xi$ involves only one-dimensional distributions of Ξ. Joint distributions are involved by operators $\overleftarrow{T}_n^\Xi$ from bounded measurable functions on B^n to bounded functions on A^n; here $n \in \{1, 2, \dots\} \cup \{\infty\}$, and A^∞, B^∞ consist of infinite sequences:

$$\begin{gathered}(\overleftarrow{T}_n^\Xi \varphi)(a_1, \dots, a_n) = \mathbb{E}\, \varphi(\Xi_{a_1}, \dots, \Xi_{a_n}) \quad \text{for } n < \infty\,, \\ (\overleftarrow{T}_\infty^\Xi \varphi)(a_1, a_2, \dots) = \mathbb{E}\, \varphi(\Xi_{a_1}, \Xi_{a_2}, \dots)\,. \end{gathered} \tag{7c4}$$

Clearly, the operator $\overleftarrow{T}_n^\Xi$ determines uniquely (and is determined by) the n-dimensional distributions $\lambda_{a_1,\dots,a_n}$ of Ξ; thus, the distribution Λ_Ξ of Ξ (determines and) is uniquely determined by the operators $\overleftarrow{T}_1^\Xi, \overleftarrow{T}_2^\Xi, \overleftarrow{T}_3^\Xi, \dots$ together (or equivalently, the operator $\overleftarrow{T}_\infty^\Xi$ alone). See also Sections 7g, 7h for a description of the class of all operators of the form $\overleftarrow{T}_\infty^\Xi$.

Assume now that A is also a standard measurable space (like B).

7c5 Proposition. The following two conditions are equivalent for every S-map Ξ from A to B:

(a) the map $a \mapsto \Xi_a$ is measurable from A to the space $L_0(\Omega \to B)$;

(b) there exists a measurable function $\xi : A \times \Omega \to B$ such that for every $a \in A$ the function $\omega \mapsto \xi(a, \omega)$ belongs to the equivalence class Ξ_a.

Proof (sketch). We may assume that $\Omega = A = B = (0, 1)$.

(b) $\Longrightarrow$ (a): the set of all ξ satisfying (a) is closed under pointwise (on $A \times \Omega$) convergence.

(a) $\Longrightarrow$ (b): we may take

$$\xi(\omega, a) = \limsup_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\omega-\varepsilon}^{\omega+\varepsilon} \Xi_a(\omega) \, \mathrm{d}\omega$$

(which is just one out of many appropriate ξ). $\square$

See also [15, Introduction (the proof of (II)$\Longrightarrow$ (I))].

S-maps satisfying the equivalent conditions 7c5(a,b) will be called *measurable.*

Every measurable S-map Ξ from A to B leads to an operator $\overrightarrow{T}_1^\Xi$ from probability measures on A to probability measures on B (or rather, a linear operator on finite signed measures), namely,

$$\int_B \varphi \, \mathrm{d}(\overrightarrow{T}_1^\Xi \nu) = \int_A \mathbb{E}\, \varphi(\Xi_a) \, \nu(\mathrm{d}a) = \int_A (\overleftarrow{T}_1^\Xi \varphi) \, \mathrm{d}\nu$$

for all bounded measurable $\varphi : B \to \mathbb{R}$. In other words, $\overrightarrow{T}_1^\Xi \nu$ is the image of $\nu \times P$ under the map $\xi : A \times \Omega \to B$ corresponding to Ξ as in 7c5(b). In fact, $\overrightarrow{T}_1^\Xi \nu = \mathbb{E}\, \Xi_\nu$ where Ξ_ν is a *random* measure, the image of ν under the map $\xi_\omega : A \to B$, $\xi_\omega(a) = \xi(a, \omega)$. Another choice of ξ (for the given Ξ) may change Ξ_ν only on a set of zero probability. Similarly, for any measure ν on A^n,

$$\int_{B^n} \varphi \, \mathrm{d}(\overrightarrow{T}_n^\Xi \nu) = \int_{A^n} \mathbb{E}\, \varphi(\Xi_{a_1}, \dots, \Xi_{a_n}) \, \nu(\mathrm{d}a_1 \dots \mathrm{d}a_n) = \int_{A_n} (\overleftarrow{T}_n^\Xi \varphi) \, \mathrm{d}\nu$$

for $\varphi : B^n \to \mathbb{R}$. The same holds for $n = \infty$.

Let A, B, C be three standard measurable spaces and Ξ', Ξ'' be measurable S-maps, Ξ' from A to B, Ξ'' from B to C, on probability spaces Ω', Ω'' respectively. Their *composition* $\Xi = \Xi'\Xi''$ is a measurable S-map from A to C on $\Omega = \Omega' \times \Omega''$ (or rather its quotient space, for non-redundancy), defined as follows:

$$\xi\big(a, (\omega', \omega'')\big) = \xi''\big(\xi'(a, \omega'), \omega''\big) \quad \text{for } \omega' \in \Omega',\, \omega'' \in \Omega'' ,$$

where $\xi : A \times (\Omega' \times \Omega'') \to C$, $\xi' : A \times \Omega' \to B$, $\xi'' : B \times \Omega'' \to C$ correspond to Ξ, Ξ', Ξ'' as in 7c5(b). The composition is well-defined by the next lemma.

7c6 Lemma. [22, Prop. 1.2.2/1.1] The composition $\Xi = \Xi'\Xi''$ does not depend on the choice of ξ', ξ''.

Proof (sketch). For a given a, a change of ξ' influences $\Xi_a(\cdot,\cdot)$ on a subset of $\Omega_1 \times \Omega_2$, negligible, since its first projection is negligible. A change of ξ'' influences $\Xi_a(\cdot,\cdot)$ on a subset of $\Omega_1 \times \Omega_2$, negligible, since all its sections ($\omega' = \text{const}$) are negligible. $\square$

The distribution Λ of $\Xi = \Xi'\Xi''$ is uniquely determined by the distributions Λ', Λ'' of Ξ' and Ξ'' (since distributions correspond to isomorphic classes), and will be called the *convolution* of these two distributions, $\Lambda = \Lambda' * \Lambda''$. It is easy to see that $\Xi = \Xi'\Xi''$ implies

$$\overleftarrow{T}_n^{\Xi}\varphi = \overleftarrow{T}_n^{\Xi'}(\overleftarrow{T}_n^{\Xi''}\varphi) \quad \text{and} \quad \overrightarrow{T}_n^{\Xi}\nu = \overrightarrow{T}_n^{\Xi''}(\overrightarrow{T}_n^{\Xi'}\nu) .$$

We may treat Ξ' and Ξ'' as independent S-maps on the same probability space Ω; namely, $\Xi'_a(\omega', \omega'') = \Xi'_a(\omega')$ and $\Xi''_b(\omega', \omega'') = \Xi''_b(\omega'')$. Conditional expectations are given by

$$\mathbb{E}\big(\varphi(\Xi_a)\,\big|\,\Xi'\big) = \big(\overleftarrow{T}_1^{\Xi''}\varphi\big)(\Xi'_a) , \tag{7c7}$$

$$\mathbb{E}\big(\varphi(\Xi_a)\,\big|\,\Xi''\big) = \int_B \varphi(\Xi''_b)\,\big(\overrightarrow{T}_1^{\Xi'}\delta_a\big)(\mathrm{d}b) \tag{7c8}$$

for all bounded measurable $\varphi : C \to \mathbb{R}$, where δ_a is the probability measure concentrated at a, and Ξ''_b means $\xi''(b,\cdot)$; the choice of ξ'' does not matter (similarly to 7c6). More generally,

$$\mathbb{E}\big(\varphi(\Xi_{a_1}, \dots, \Xi_{a_n})\,\big|\,\Xi'\big) = \big(\overleftarrow{T}_n^{\Xi''}\varphi\big)(\Xi'_{a_1}, \dots, \Xi'_{a_n}) , \tag{7c9}$$

$$\begin{aligned} &\mathbb{E}\big(\varphi(\Xi_{a_1}, \dots, \Xi_{a_n})\,\big|\,\Xi''\big) = \\ &\quad = \int_{B^n} \varphi(\Xi''_{b_1}, \dots, \Xi''_{b_n})\,\big(\overrightarrow{T}_n^{\Xi'}\delta_{a_1,\dots,a_n}\big)(\mathrm{d}b_1 \dots \mathrm{d}b_n) \end{aligned} \tag{7c10}$$

for all bounded measurable $\varphi : C^n \to \mathbb{R}$ and $n < \infty$; similar formulas hold for $n = \infty$.

See also [22, Sect. 1] and [40, Sect. 8d].

7d Flow systems of S-maps

Let $\mathcal{X}$ be a compact metrizable space (mostly, the circle will be used). Then $L_0(\Omega \to \mathcal{X})$ is equipped with the (metrizable) topology of convergence in probability.

7d1 Definition. An S-map Ξ from $\mathcal{X}$ to $\mathcal{X}$ is *continuous in probability,* if the map $x \mapsto \Xi_x$ is continuous from $\mathcal{X}$ to $L_0(\Omega \to \mathcal{X})$.

Clearly, 7d1 is stronger than 7c5(a). Continuity in probability is preserved by the composition, which is made clear by the next lemma.

7d2 Lemma. The following three conditions are equivalent for every S-map Ξ from $\mathcal{X}$ to $\mathcal{X}$:

(a) Ξ is continuous in probability;

(b) $\overleftarrow{T}_2^{\Xi}\varphi \in C(\mathcal{X}^2)$ for all $\varphi \in C(\mathcal{X}^2)$;

(c) $\overleftarrow{T}_n^{\Xi}\varphi \in C(\mathcal{X}^n)$ for all $\varphi \in C(\mathcal{X}^n)$ and all $n = 1, 2, 3, \dots$

Proof (sketch). (a) $\Longrightarrow$ (c): the map $(x_1, \dots, x_n) \mapsto (\Xi_{x_1}, \dots, \Xi_{x_n})$ from $\mathcal{X}^n$ to $L_0(\Omega \to \mathcal{X}^n)$ is continuous, therefore $\mathbb{E}\,\varphi(\Xi_{x_1}, \dots, \Xi_{x_n})$ is continuous in $x_1, \dots, x_n$.

(c) $\Longrightarrow$ (b): trivial.

(b) $\Longrightarrow$ (a): let φ be the metric, $\varphi(x_1, x_2) = \operatorname{dist}(x_1, x_2)$, then $(\overleftarrow{T}_2^{\Xi}\varphi)(x_1, x_2) = \mathbb{E}\operatorname{dist}(\Xi_{x_1}, \Xi_{x_2}) = 0$ on the diagonal $x_1 = x_2$; by (b), $\mathbb{E}\operatorname{dist}(\Xi_{x_1}, \Xi_{x_2}) \to 0$ as $\operatorname{dist}(x_1, x_2) \to 0$, which is (a). $\square$

A Lipschitzian version, given below, will be used in Sect. 7f.

7d3 Lemma. The following three conditions are equivalent for every S-map Ξ from $\mathcal{X}$ to $\mathcal{X}$:

(a) there exists C such that $\mathbb{E}\operatorname{dist}(\Xi_{x_1}, \Xi_{x_2}) \le C \operatorname{dist}(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{X}$;

(b) there exists C_2 such that $\operatorname{Lip}(\overleftarrow{T}_2^{\Xi}\varphi) \le C_2 \operatorname{Lip}(\varphi)$ for all Lipschitz functions $\varphi : \mathcal{X}^2 \to \mathbb{R}$;

(c) for each $n = 1, 2, 3, \dots$ there exists C_n such that $\operatorname{Lip}(\overleftarrow{T}_n^{\Xi}\varphi) \le C_n \operatorname{Lip}(\varphi)$ for all Lipschitz functions $\varphi : \mathcal{X}^n \to \mathbb{R}$.

(Any 'reasonable' metric on $\mathcal{X}^n$ may be used, say, $\operatorname{dist}\big((x'_1, \dots, x'_n), (x''_1, \dots, x''_n)\big) = \sum_k \operatorname{dist}(x'_k, x''_k)$, or $\max_k \operatorname{dist}(x'_k, x''_k)$.)

Proof (sketch). (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a): trivial;

(a) $\Longrightarrow$ (c): the same as '(b) $\Longrightarrow$ (a) $\Longrightarrow$ (c)' in the proof of 7d2, but quantitative. $\square$

A *convolution system of S-maps* (over $\mathcal{X}$) may be defined as a family $(\Lambda_{s,t})_{s<t}$, where each $\Lambda_{s,t}$ is the distribution of an S-map from $\mathcal{X}$ to $\mathcal{X}$, continuous in probability, and

$$\Lambda_{r,t} = \Lambda_{r,s} * \Lambda_{s,t}$$

whenever $r < s < t$.

Every convolution system of S-maps leads to a convolution system as defined by 2b1. Namely, each $\Lambda_{s,t}$ leads to a probability space $(G_{s,t}, \mu_{s,t})$[1] carrying an S-map $\Xi_{s,t} = (\Xi_x^{s,t})_{x\in\mathcal{X}}$, $\Xi_x^{s,t} \in L_0(G_{s,t} \to \mathcal{X})$ and unique up to isomorphism (between S-maps). Given $r < s < t$ we have $\Xi_{r,t} = \Xi_{r,s}\Xi_{s,t}$ up to isomorphism, which gives us a representation of $G_{r,t}$ as a quotient space of $G_{r,s} \times G_{s,t}$, that is, a morphism $G_{r,s} \times G_{s,t} \to G_{r,t}$. The convolution system $(G_{s,t}, \mu_{s,t})_{s<t}$ is determined by $(\Lambda_{s,t})_{s<t}$ uniquely up to isomorphism. If it is separable (as defined by 2b4) then it leads to a flow system, that is, all S-maps $\Xi_{s,t} = (\Xi_x^{s,t})_x$ may be defined on a single probability space (Ω, P), satisfying (recall 2b2(a,b))

$$\begin{gathered} \Xi_{t_1,t_2}, \Xi_{t_2,t_3}, \dots, \Xi_{t_{n-1},t_n} \text{ are independent for } t_1 < t_2 < \dots < t_n\,; \\ \Xi_{r,t} = \Xi_{r,s}\Xi_{s,t} \quad \text{for } r < s < t\,. \end{gathered} \tag{7d4}$$

According to Sect. 2c, the flow system leads to a continuous product of probability spaces.

A sufficient condition for the separability is, *temporal continuity in probability* (in addition to the spatial continuity in probability assumed before for each $\Xi_{s,t}$):

$$\text{both } \Xi_x^{s-\varepsilon,s} \text{ and } \Xi_x^{s,s+\varepsilon} \text{ converge to } x \text{ in probability as } \varepsilon \to 0+\,, \tag{7d5}$$

for all $s \in \mathbb{R}$ and $x \in \mathcal{X}$. It involves only one-dimensional distributions, $\lambda_x^{s,t}$, and may be reformulated in terms of the operators $T_1^{s,t} = \overleftarrow{T}_1^{\Xi_{s,t}} : C(\mathcal{X}) \to C(\mathcal{X})$, namely,

$$\text{both } T_1^{s-\varepsilon,s}(\varphi) \text{ and } T_1^{s,s+\varepsilon}(\varphi) \text{ converge to } \varphi \text{ pointwise as } \varepsilon \to 0+\,,$$

for all $s \in \mathbb{R}$ and $\varphi \in C(\mathcal{X})$.

7d6 Lemma. Condition (7d5) implies separability.

Proof (sketch). It is sufficient to prove that $\Xi_x^{s-\varepsilon,t} \to \Xi_x^{s,t}$ and $\Xi_x^{s,t+\varepsilon} \to \Xi_x^{s,t}$ in probability as $\varepsilon \to 0$, for $x \in \mathcal{X}$ and $s < t$. We have

$$\mathbb{E}\ \mathrm{dist}(\Xi_x^{s-\varepsilon,t}, \Xi_x^{s,t}) = \int_{\mathcal{X}} \big(\mathbb{E}\ \mathrm{dist}(\Xi_{x'}^{s,t}, \Xi_x^{s,t})\big) \big(\overrightarrow{T}_1^{\Xi_{s-\varepsilon,s}} \delta_x\big)(\mathrm{d}x')\,.$$

[1]Not $(\Omega_{s,t}, P_{s,t})$ for conformity to Sect. 2b.

By (7d5), $\overrightarrow{T}_1^{\Xi_{s-\varepsilon,s}} \delta_x \to \delta_x$. By continuity in probability, $\mathbb{E}\,\mathrm{dist}(\Xi_{x'}^{s,t}, \Xi_x^{s,t}) \to 0$ as $x' \to x$. Therefore $\mathbb{E}\,\mathrm{dist}(\Xi_x^{s-\varepsilon,t}, \Xi_x^{s,t}) \to 0$.

Also,

$$\mathbb{E}\,\mathrm{dist}(\Xi_x^{s,t+\varepsilon}, \Xi_x^{s,t}) = \int_{\mathcal{X}} \big(\mathbb{E}\,\mathrm{dist}(\Xi_{x'}^{t,t+\varepsilon}, x')\big) \big(\overrightarrow{T}_1^{\Xi_{s,t}} \delta_x\big)(\mathrm{d}x') \to 0 \,,$$

since the integrand converges to 0 for every x'. $\square$

7d7 Remark. The continuous product of probability spaces, corresponding to a flow system $(\Xi_{s,t})_{s<t}$ of S-maps from $\mathcal{X}$ to $\mathcal{X}$, is classical if and only if random variables $\varphi(\Xi_x^{s,t})$ are stable for all $s < t$ and all bounded Borel functions $\varphi : \mathcal{X} \to \mathbb{R}$ (a single φ is enough if it is one-to-one). Proof: similar to 5c5.

See also [22, Sect. 1].

7e From S-maps to black noise

Let $\mathcal{X}$, $(G_{s,t}, \mu_{s,t})_{s<t}$ and $(\Xi_{s,t})_{s<t}$ be as in Sect. 7d, satisfying (7d5).

Sect. 7b gives us a sufficient condition (7b2)-(7b3)-(7b4) (in combination with 6a13(a–c)) for triviality of the classical part of the corresponding continuous product of probability spaces.

Instead of all $f \in L_\infty(G_{s,t})$ we may use a subset of $L_\infty(G_{s,t})$ dense in $L_2(G_{s,t})$.

The σ-field on $G_{r,t}$ is generated by $\mathcal{X}$-valued random variables $\Xi_x^{r,t}$ for $x \in \mathcal{X}$. Therefore functions of the form

(7e1) $\qquad \varphi(\Xi_{x_1}^{r,t}, \dots, \Xi_{x_n}^{r,t}) \quad \text{for } \varphi \in C(\mathcal{X}^n),\ (x_1, \dots, x_n) \in \mathcal{X}_n$

and $n = 1, 2, \dots$ are dense in $L_2(G_{r,t})$.

7e2 Lemma. Assume that $n \in \{1, 2, \dots\}$ is given, a linear subset F_n of $C(\mathcal{X}^n)$, dense in the norm topology, and a linear subset N_n of the space of (finite, signed) measures on $\mathcal{X}^n$, dense in the weak topology; and an interval $(r,t) \subset \mathbb{R}$. Then functions of the form

$$\int_{\mathcal{X}^n} \varphi(\Xi_{x_1}^{r,t}, \dots, \Xi_{x_n}^{r,t})\, \nu(\mathrm{d}x_1 \dots \mathrm{d}x_n) \quad \text{for } \varphi \in F_n \text{ and } \nu \in N_n$$

are L_2-dense among functions of the form (7e1) for the given n.

Proof (sketch). For every $\varphi \in F$, the map $\mathcal{X}^n \to L_2(G_{r,t})$ given by $(x_1,\dots,x_n) \mapsto \varphi(\Xi^{r,t}_{x_1},\dots,\Xi^{r,t}_{x_n})$ is continuous (since $\Xi_{r,t}$ is continuous in probability). Therefore, $\nu_k \to \delta_{x_1,\dots,x_n}$ (as $k \to \infty$) implies

$$\int_{\mathcal{X}^n} \varphi(\Xi^{r,t}_{x'_1},\dots,\Xi^{r,t}_{x'_n})\,\nu_k(\mathrm{d}x'_1\dots\mathrm{d}x'_n) \to \varphi(\Xi^{r,t}_{x_1},\dots,\Xi^{r,t}_{x_n}) \quad \text{in } L_2(G_{r,t})\,. \qquad \square$$

Substituting $f = \int \varphi(\Xi^{r,t}_{x_1},\dots,\Xi^{r,t}_{x_n})\,\nu(\mathrm{d}x_1\dots\mathrm{d}x_n)$ to (7b3) we get

$$\text{(7e3)} \qquad f_{s,s+\varepsilon} = \int_{\mathcal{X}^n} \big(\overleftarrow{T}_n^{\Xi_{s+\varepsilon,t}}\varphi\big)(\Xi^{s,s+\varepsilon}_{x_1},\dots,\Xi^{s,s+\varepsilon}_{x_n})\,\big(\overrightarrow{T}_n^{\Xi_{r,s}}\nu\big)(\mathrm{d}x_1\dots\mathrm{d}x_n)\,.$$

7e4 Proposition. (Le Jan and Raimond; implicit in [23].) Let a flow system $(\Xi_{s,t})_{s<t}$ of S-maps from $\mathcal{X}$ to $\mathcal{X}$ and a probability measure ν_0 on $\mathcal{X}$ satisfy the conditions

(a) (stationarity) the distribution of $\Xi_{s,s+h}$ does not depend on s;

(b) (invariant measure) $\overrightarrow{T}_1^{\Xi_{s,t}}\nu_0 = \nu_0$ for $s<t$;

(c) (Lipschitz boundedness) if $\varphi \in C(\mathcal{X}^n)$ is a Lipschitz function, then $\overleftarrow{T}_n^{\Xi_{s,t}}\varphi$ is also a Lipschitz function, with a Lipschitz constant

$$\mathrm{Lip}(\overleftarrow{T}_n^{\Xi_{s,t}}\varphi) \le C_n\,\mathrm{Lip}(\varphi) \quad \text{for } 0 \le s < t \le 1\,,$$

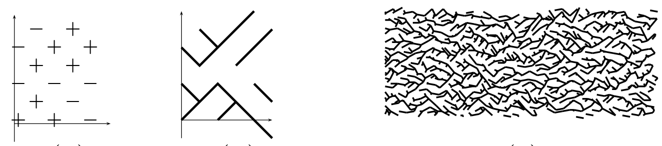
where $C_n < \infty$ depends only on n;

(d) $\displaystyle\sup_{\varphi,\nu} \mathrm{Var}\bigg(\int_{\mathcal{X}^n} \varphi\big(\Xi^{0,\varepsilon}_{x_1},\dots,\Xi^{0,\varepsilon}_{x_n}\big)\,\nu(\mathrm{d}x_1\dots\mathrm{d}x_n)\bigg) = o(\varepsilon)$ as $\varepsilon \to 0$, where the supremum is taken over all $\varphi \in C(\mathcal{X}^n)$ such that $\mathrm{Lip}(\varphi) \le 1$ and all positive measures ν on $\mathcal{X}^n$ such that $\nu_1 \le \nu_0,\dots,\nu_n \le \nu_0$; here $\nu_1,\dots,\nu_n$ are coordinate projections of ν, that is, $\int \varphi(x_k)\,\nu(\mathrm{d}x_1\dots\mathrm{d}x_n) = \int \varphi(x)\,\nu_k(\mathrm{d}x)$ for $\varphi \in C(\mathcal{X})$.

Then the corresponding noise is black.

Proof (sketch). The continuity condition 6a13(a–c) is ensured by (a). By Lemma 7e2 it is sufficient to verify (7b2) in the form (7b4) when $f_{s,s+\varepsilon}$ is given by (7e3), assuming that φ and ν have the properties formulated in (d). In order to apply (d) to the function $\varphi_{s+\varepsilon} = \overleftarrow{T}_n^{\Xi_{s+\varepsilon,t}}\varphi$ (instead of φ) and the measure $\nu_s = \overrightarrow{T}_n^{\Xi_{r,s}}\nu$ (instead of ν) we only need to check that these properties of φ and ν are inherited by $\varphi_{s+\varepsilon}$ and ν_s. The Lipschitz property of $\varphi_{s+\varepsilon}$ follows from (c), up to the (harmless) constant C_n. The property of ν_s (majorization of its coordinate projections) follows from (b). $\square$

7f Example: Arratia's coalescing flow, or the Brownian web



(a) (b) (c)

A two-dimensional array of random signs (a) produces a system of coalescing random walks (b) that converges in the scaling limit (c) to a flow system of S-maps, introduced by Arratia in 1979 [3] and investigated further by Tóth, Werner, Soucaliuc [34], [33], Fontes, Isopi, Newman, Ravishankar [14], Le Jan and Raimond [23, Sect. 3]. It consists of infinitely many coalescing Brownian motions, independent before coalescence. Our approach, based on S-maps, deals with equivalence classes $\Xi_x^{s,t}$ rather than sample functions $(a, s, t) \mapsto \xi_{s,t}(\omega, x)$ (recall 7c5(b)); fine properties of sample functions, examined in some of the works cited above, are irrelevant here.

The space $\mathcal{X}$ is the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The distribution of $\Xi_{s,s+t}$ does not depend on s (stationarity). The (one-dimensional) distribution of $\Xi_x^{0,t}$ is the normal distribution $N(x, t)$ (or rather, the distribution of $x + \zeta \bmod 1$ where $\zeta \sim N(0, t)$). It may be thought of as the distribution of $x + B_t$ where $(B_s)_s$ is the (standard) Brownian motion in $\mathbb{T}$. The (two-dimensional) joint distribution of $\Xi_{x_1}^{0,t}$ and $\Xi_{x_2}^{0,t}$ is the joint distribution of $x_1 + B_t^{(1)}$ and $x_k + B_t^{(k)}$ where $(B_s^{(1)})_s, (B_s^{(2)})_s$ are two independent Brownian motions in $\mathbb{T}$, and $k : \Omega \to \{1, 2\}$ is a random variable, defined as follows:

$$(7f1) \qquad k = \begin{cases} 1 & \text{if } \min\{s : x_1 + B_s^{(1)} = x_2 + B_s^{(2)}\} \le t, \\ 2 & \text{otherwise.} \end{cases}$$

That is, the second Brownian motion $\bigl(x_{k(s)} + B_s^{(k(s))}\bigr)_s$ is independent of the first one, as long as they do not meet. Afterwards they are equal. In spite of the asymmetry (the second motion joins the first), the resulting distribution does not depend on the order of initial points. Joint distributions of higher dimensions are defined similarly. We have

$$(7f2) \qquad \mathbb{E} \operatorname{dist}\bigl(\Xi_{x_1}^{0,t}, \Xi_{x_2}^{0,t}\bigr) \le \operatorname{dist}(x_1, x_2)$$

since the process $t \mapsto \operatorname{dist}\bigl(\Xi_{x_1}^{0,t}, \Xi_{x_2}^{0,t}\bigr)$ is a supermartingale. (On $\mathbb{R}$ it would be a martingale, but we are on the circle.) Thus, for every t the S-map $\Xi_{0,t}$ is continuous in probability. The temporal continuity in probability (7d5) is evident. The uniform distribution ν_0 on the circle evidently is an invariant measure in the sense of 7e4(b).

By (7f2), $\Xi_{0,t}$ satisfies 7d3(a), which implies 7d3(c), the constants C_n not depending on t. Thus, 7e4(c) holds (as well as 7e4(a,b)). In order to get a black noise, it remains to verify 7e4(d).

Let us start with the case $n = 1$. We consider $\operatorname{Var}\bigl(\int \varphi(\Xi_x^{0,\varepsilon})\,\nu(\mathrm{d}x)\bigr)$ assuming $\operatorname{Lip}(\varphi) \le 1$ and $0 \le \nu \le \nu_0$. Note that

$$\operatorname{Var}\biggl(\int \dots\biggr) = \iint \operatorname{Cov}\bigl(\varphi(\Xi_{x_1}^{0,\varepsilon}), \varphi(\Xi_{x_2}^{0,\varepsilon})\bigr)\,\nu(\mathrm{d}x_1)\nu(\mathrm{d}x_2)\,.$$

On one hand,

$$|\operatorname{Cov}\bigl(\varphi(\Xi_{x_1}^{0,\varepsilon}), \varphi(\Xi_{x_2}^{0,\varepsilon})\bigr)| \le \sqrt{\operatorname{Var}\varphi(\Xi_{x_1}^{0,\varepsilon})}\sqrt{\operatorname{Var}\varphi(\Xi_{x_2}^{0,\varepsilon})} \le \varepsilon$$

since $|\varphi(\Xi_x^{0,\varepsilon}) - \varphi(x)| \le |\Xi_x^{0,\varepsilon} - x|$ and $\mathbb{E}\,|\Xi_x^{0,\varepsilon} - x|^2 \le \varepsilon$. (The latter would be '$=\varepsilon$' on $\mathbb{R}$, but we are on the circle.) On the other hand, we may assume that $\|\varphi\|_{C(\mathcal{X})} \le \mathrm{const}\cdot\operatorname{Lip}(\varphi) \le \mathrm{const}$; using (7f1),

$$\begin{aligned}&|\operatorname{Cov}\bigl(\varphi(\Xi_{x_1}^{0,\varepsilon}), \varphi(\Xi_{x_2}^{0,\varepsilon})\bigr)| = |\operatorname{Cov}\bigl(\varphi(x_1 + B_\varepsilon^{(1)}), \varphi(x_k + B_\varepsilon^{(k)})\bigr)| = \\ &= |\operatorname{Cov}\bigl(\varphi(x_1 + B_\varepsilon^{(1)}), \varphi(x_k + B_\varepsilon^{(k)}) - \varphi(x_2 + B_\varepsilon^{(2)})\bigr)| \le \|\varphi\|^2_{C(\mathcal{X})}\mathbb{P}\bigl(k=1\bigr)\,.\end{aligned}$$

However, the probability of meeting, $\mathbb{P}\bigl(k = 1\bigr)$, is (exponentially) small for $\operatorname{dist}(x_1, x_2) \gg \sqrt{\varepsilon}$, therefore $\iint_{\operatorname{dist}(x_1,x_2)\ge\delta} |\operatorname{Cov}(\dots)|\,\nu(\mathrm{d}x_1)\nu(\mathrm{d}x_2)$ is $o(\varepsilon)$ (in fact, exponentially small) as $\varepsilon \to 0$, for every $\delta > 0$. Also, $\iint_{\operatorname{dist}(x_1,x_2)\le\delta} |\operatorname{Cov}(\dots)|\,\nu(\mathrm{d}x_1)\nu(\mathrm{d}x_2) \le \iint_{\operatorname{dist}(x_1,x_2)\le\delta} \varepsilon\,\nu_0(\mathrm{d}x_1)\nu_0(\mathrm{d}x_2) = \varepsilon\cdot 2\delta$. It follows that $\operatorname{Var}\bigl(\int\dots\bigr) = o(\varepsilon)$ as $\varepsilon \to 0$, uniformly in φ and ν.

Generalization for $n = 2, 3, \dots$ is straightforward. One estimates the $(\nu\times\nu)$-measure of the set $\bigl\{\bigl((x'_1, \dots, x'_n), (x''_1, \dots, x''_n)\bigr) \in \mathcal{X}^n \times \mathcal{X}^n : \operatorname{dist}(x'_k, x''_l) \le \delta\bigr\}$ for each pair (k, l) separately (taking into account that $\nu_k \le \nu_0$, $\nu_l \le \nu_0$), and consider the union of these n^2 sets.

By Proposition 7e4, the noise corresponding to Arratia's coalescing flow is black.

The proof presented above, due to Le Jan and Raimond [23, Sect. 3], is simpler than [40, Sect. 7].

7g Random kernels

By a *kernel* from a set A to a standard measurable space B we mean a map $A \to \mathcal{P}(B)$. Here $\mathcal{P}(B)$ is the standard measurable space of all probability measures on B, equipped with the σ-field generated by the functions $\mathcal{P}(B) \to \mathbb{R}$ of the form $\mu \to \int \varphi\,\mathrm{d}\mu$, where φ runs over bounded measurable functions $B \to \mathbb{R}$, see [17, Sect. 17.E].

7g1 Definition. An *S-kernel* from a set A to a standard measurable space B is an S-map from A to $\mathcal{P}(B)$.

This idea was introduced by Le Jan and Raimond in order to describe "turbulent evolutions where [...] two points thrown initially at the same place separate" [22, Introduction].

Note that B is naturally embedded into $\mathcal{P}(B)$ (by $b \mapsto \delta_b$, the measure concentrated at b). Accordingly, a map $A \to B$ may be treated as a special case of a kernel, $A \to B \subset \mathcal{P}(B)$. Similarly, an S-map Ξ (from A to B) may be treated as a special case of an S-kernel K (from A to B); namely, $K_a = \delta_{\Xi_a}$.

By the distribution Λ_K of an S-kernel K from A to B we mean the distribution of K as an S-map from A to $\mathcal{P}(B)$; it consists (recall 7c1(b)) of the joint distributions $\lambda_{a_1,\dots,a_n}$ of $\mathcal{P}(B)$-valued random variables $K_{a_1}, \dots, K_{a_n}$.

Usually it is difficult to construct an S-kernel (or a flow of S-kernels) directly, by specifying joint distributions of *measures* (or corresponding infinitesimal-time data). It is easier to do it indirectly, by specifying joint distributions of *points* and using a moment method described below.

Let K be an S-kernel from A to B. Combining formally 7g1 and (7c4) one could treat $\overleftarrow{T}_1^K$ as defined on the (huge) space of functions on $\mathcal{P}(B)$, but we prefer it to be defined on the same (modest) space as $\overleftarrow{T}_1^\Xi$ in Sect. 7c. Namely, we define a linear operator $\overleftarrow{T}_1^K$ from the space of all bounded measurable functions on B to the space of all bounded functions on A by

$$(\overleftarrow{T}_1^K \varphi)(a) = \mathbb{E} \int_B \varphi \, \mathrm{d}K_a \, .$$

In other words, we restrict ourselves to *linear* functions on $\mathcal{P}(B)$, $\mu \mapsto \int \varphi \, \mathrm{d}\mu$. For an S-map Ξ from A to B, treated as (a special case of) an S-kernel K, we have $\overleftarrow{T}_1^\Xi = \overleftarrow{T}_1^K$, since $\int \varphi \, \mathrm{d}\delta_{\Xi_a} = \varphi(\Xi_a)$. Generally, for $n \in \{1, 2, \dots\} \cup \{\infty\}$ we define a linear operator $\overleftarrow{T}_n^K$ from bounded measurable functions on B^n to bounded functions on A^n by

$$\begin{gathered}(\overleftarrow{T}_n^K \varphi)(a_1, \dots, a_n) = \mathbb{E} \int_{B^n} \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n}) \quad \text{for } n < \infty \, , \\ (\overleftarrow{T}_\infty^K \varphi)(a_1, a_2, \dots) = \mathbb{E} \int_{B^\infty} \varphi \, \mathrm{d}(K_{a_1} \times K_{a_2} \times \dots) \, .\end{gathered} \tag{7g2}$$

(Of course, $\int \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n})$ means $\int \varphi(b_1, \dots, b_n) \, K_{a_1}(\mathrm{d}b_1) \dots K_{a_n}(\mathrm{d}b_n)$.) These expectations of multilinear functions of $K_{a_1}, \dots, K_{a_n}$ are sometimes called the moments of K. A solution of the corresponding moment problem is given below, see 7g3 (uniqueness) and 7g6, 7h3 (existence).

7g3 Lemma. The distribution Λ_K is uniquely determined by the operators $\overleftarrow{T}_n^K$, $n = 1, 2, 3, \dots$ (or equivalently, by a single operator $\overleftarrow{T}_\infty^K$).

Proof (sketch). The n-th moment of the (bounded) random variable $\int \varphi \, \mathrm{d}K_a$ is equal to

$$\mathbb{E} \int_{B^n} \varphi(b_1) \dots \varphi(b_n) \, K_{a_1}(\mathrm{d}b_1) \dots K_{a_n}(\mathrm{d}b_n) = \big(\overleftarrow{T}_n^K (\varphi \otimes \dots \otimes \varphi) \big)(a, \dots, a) \, ;$$

thus, the distribution of $\int \varphi \, \mathrm{d}K_a$ is determined uniquely via its moments. Similarly, the joint distribution of $\int \varphi_1 \, \mathrm{d}K_{a_1}$ and $\int \varphi_2 \, \mathrm{d}K_{a_2}$ is determined via its (mixed) moments; and so on. $\square$

The moments are basically the same as (non-random) kernels T_n^K from A^n to B^n defined by

$$T_n^K(a_1, \dots, a_n) = \mathbb{E} \, (K_{a_1} \times \dots \times K_{a_n}) \, ,$$

that is,

$$\int \varphi \, \mathrm{d}\big(T_n^K(a_1, \dots, a_n) \big) = \mathbb{E} \int \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n}) = (\overleftarrow{T}_n^K \varphi)(a_1, \dots, a_n)$$

for bounded measurable $\varphi : B^n \to \mathbb{R}$. Also $n = \infty$ is admitted,

$$T_\infty^K(a_1, a_2, \dots) = \mathbb{E} \, (K_{a_1} \times K_{a_2} \times \dots) \, .$$

Each T_n^K is a marginal of T_∞^K in the sense that

$$\int \varphi(b_1, \dots, b_n) \, \big(T_\infty^K(a_1, a_2, \dots) \big)(\mathrm{d}b_1 \mathrm{d}b_2 \dots) = \\ = \int \varphi(b_1, \dots, b_n) \, \big(T_n^K(a_1, \dots, a_n) \big)(\mathrm{d}b_1 \dots \mathrm{d}b_n) \, ;$$

similarly, T_n^K is a marginal of T_{n+1}^K (*consistency*). Everyone knows that a probability distribution on B^∞ is basically the same as a consistent family of probability distributions on B^n, $n < \infty$. Accordingly, a consistent family of kernels from A^n to B^n is basically the same as a kernel from A^∞ to B^∞ satisfying the condition

$$\text{(7g4)} \quad \int \varphi(b_1, \dots, b_n) \, T(a_1, a_2, \dots)(\mathrm{d}b_1 \mathrm{d}b_2 \dots) \quad \text{depend on } a_1, \dots, a_n \text{ only.}$$

Measures $T_2^K(a_1, a_2)$ and $T_2^K(a_2, a_1)$ are mutually symmetric: $T_2^K(a_2, a_1)(\mathrm{d}b_2 \mathrm{d}b_1) = T_2^K(a_1, a_2)(\mathrm{d}b_1 \mathrm{d}b_2)$; more formally, $\int \varphi(b_2, b_1) \, T_2^K(a_2, a_1)(\mathrm{d}b_1 \mathrm{d}b_2) = \int \varphi(b_1, b_2) \, T_2^K(a_1, a_2)(\mathrm{d}b_1 \mathrm{d}b_2)$. Similarly,

$$T_n^K(a_{\sigma(1)}, \dots, a_{\sigma(n)})(\mathrm{d}b_{\sigma(1)} \dots \mathrm{d}b_{\sigma(n)}) \quad \text{does not depend on } \sigma \, ,$$

σ being a permutation, $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ bijectively. Also $n = \infty$ is admitted,

$$T_\infty^K(a_{\sigma(1)}, a_{\sigma(2)}, \dots)(\mathrm{d}b_{\sigma(1)}\mathrm{d}b_{\sigma(2)} \dots) \quad \text{does not depend on } \sigma\,, \tag{7g5}$$

$\sigma : \{1, 2, \dots\} \to \{1, 2, \dots\}$ bijectively.

7g6 Lemma. Let A be a finite or countable set, and B a standard measurable space. A kernel T from A^∞ to B^∞ is of the form $T = T_\infty^K$ for some S-kernel K from A to B if and only if T satisfies (7g4) and (7g5).

Proof (sketch). We know that T_∞^K satisfies (7g4), (7g5).

Assume that T satisfies (7g4), (7g5). For every $a \in A$ the measure $T(a, a, \dots)$ on B^∞ is invariant under permutations. The general form of such a measure is well-known (de Finetti type theorem on exchangeability, see [20, Th. 4.2]), it is $\int (\nu \times \nu \times \dots)\, \mu_a(\mathrm{d}\nu)$, the mixture of product measures $\nu \times \nu \times \dots$ over $\nu \in \mathcal{P}(B)$ distributed according to some (uniquely determined) measure $\mu_a \in \mathcal{P}(\mathcal{P}(B))$. The distribution μ_a of a single $\mathcal{P}(B)$-valued random variable K_a is thus constructed.

Given $a_1, a_2 \in A$, the measure $T(a_1, a_2, a_1, a_2, \dots)$ on $(A \times A)^\infty = A^\infty \times A^\infty$ is invariant under (the product of) two permutation groups, each acting on only one of the two A^∞ factors. The general form of such a measure is also well-known (de Finetti type theorem on partial exchangeability [20, Th. 4.1]), it is a mixture of products, namely, $\int (\nu_1 \times \nu_2 \times \nu_1 \times \nu_2 \dots)\, \mu_{a_1,a_2}(\mathrm{d}\nu_1 \mathrm{d}\nu_2)$ for some (uniquely determined) measure $\mu_{a_1,a_2} \in \mathcal{P}(\mathcal{P}(B) \times \mathcal{P}(B))$; this is the joint distribution of K_{a_1} and K_{a_2}. And so on. $\square$

See also [22, Sect. 2.5.1] and [40, 8d3].

The statement of Lemma 7g6 does not hold for uncountable sets A (unless separability is stipulated in the spirit of 2b4); here is a counterexample. Let B contain only two points. We define the measure $T(a_1, a_2, \dots)$ as the uniform distribution on the set of all sequences $(b_1, b_2, \dots)$ such that $\forall k, l \big(a_k = a_l \implies b_k = b_l \big)$. Thus, if the sequence $(a_1, a_2, \dots)$ contains a finite number n of different points, then $T(a_1, a_2, \dots)$ consists of 2^n equiprobable atoms (and if $n = \infty$ then the measure is continuous). Such T is of the form T_∞^K if and only if A is finite or countable.

Separability is naturally treated via continuity in probability, see Sect. 7h.

Assume now that A is also a measurable space. An S-kernel from A to B will be called *measurable,* if it is a measurable S-map (from A to $\mathcal{P}(B)$).

Every measurable S-kernel K from A to B leads to an operator $\vec{T}_1^K$: $\mathcal{P}(A) \to \mathcal{P}(B)$ (or rather, a linear operator on finite signed measures), namely,

$$\int_B \varphi \, \mathrm{d}(\overrightarrow{T}_1^K \nu) = \int_A \biggl(\mathbb{E} \int_B \varphi \, \mathrm{d}K_a \biggr) \, \nu(\mathrm{d}a) = \int_A \overleftarrow{T}_1^K \varphi \, \mathrm{d}\nu$$

for all bounded measurable $\varphi : B \to \mathbb{R}$. In other words, $\overrightarrow{T}_1^K \nu$ is the barycenter of the measure $\overrightarrow{T}_1^\Xi \nu$ on $\mathcal{P}(B)$ (that is, in $\mathcal{P}(\mathcal{P}(B))$), where Ξ is the same as K but treated as an S-map from A to $\mathcal{P}(B)$ (thus, $\overrightarrow{T}_1^\Xi$ is defined according to Sect. 7c). The well-known 'barycenter' map $\mathcal{P}(\mathcal{P}(B)) \to \mathcal{P}(B)$ is used, $\mu \mapsto \int \nu(\cdot) \, \mu(\mathrm{d}\nu)$. Similarly, for any measure ν on A^n,

$$\int\limits_{B^n} \varphi \, \mathrm{d}(\overrightarrow{T}_n^K \nu) = \int\limits_{A^n} \biggl(\mathbb{E} \int\limits_{B^n} \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n}) \biggr) \nu(\mathrm{d}a_1 \dots \mathrm{d}a_n) = \int\limits_{A^n} (\overleftarrow{T}_n^K \varphi) \, \mathrm{d}\nu$$

for $\varphi : B^n \to \mathbb{R}$. The same for $n = \infty$. For an S-map Ξ from A to B, treated as an S-kernel K, we have $\overrightarrow{T}_n^\Xi = \overrightarrow{T}_n^K$.

Integrating out a while keeping ω one may get a *random* measure $K_\nu = \int K_a \, \nu(\mathrm{d}a)$ on B. To this end we consider a measurable function $\xi : A \times \Omega \to \mathcal{P}(B)$ related to Ξ as in 7c5(b), Ξ being related to K as before. For almost every $\omega \in \Omega$ we have a measurable function $\xi_\omega : A \to \mathcal{P}(B)$, $\xi_\omega(a) = \xi(a, \omega)$. The function ξ_ω sends ν into a measure on $\mathcal{P}(B)$; its barycenter is K_ν. The choice of ξ does not matter, since $K_\nu \in L_0(\Omega \to \mathcal{P}(B))$ is treated mod 0. For an S-map Ξ from A to B, treated as an S-kernel K, we have $\Xi_\nu = K_\nu$. In general,

$$\int_B \varphi \, \mathrm{d}K_\nu = \int_A \biggl(\int_B \varphi \, \mathrm{d}K_a \biggr) \, \nu(\mathrm{d}a) \quad \text{a.s.}$$

for every bounded measurable $\varphi : B \to \mathbb{R}$, and

$$\mathbb{E} \, K_\nu = \overrightarrow{T}_1^K \nu \, .$$

The map $\nu \mapsto K_\nu$ is a linear map $\mathcal{P}(A) \to L_0(\Omega \to \mathcal{P}(B))$. The family $(K_\nu)_{\nu \in \mathcal{P}(A)}$ of $\mathcal{P}(B)$-valued random variables K_ν is an S-map from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Unlike an arbitrary S-map from $\mathcal{P}(A)$ to $\mathcal{P}(B)$, the S-map $(K_\nu)_\nu$ is linear (in ν).

Let A, B, C be three standard measurable spaces and K', K'' be measurable S-kernels, K' from A to B, K'' from B to C, on probability spaces Ω', Ω'' respectively. In order to define the composition $K = K'K''$ of S-kernels we may turn to the corresponding S-maps $(K'_\nu)_\nu$, $(K''_\nu)_\nu$. Their composition is a *linear* S-map from $\mathcal{P}(A)$ to $\mathcal{P}(C)$, it is of the form $(K_\nu)_\nu$, which defines $K = K'K''$, an S-kernel from A to C on $\Omega' \times \Omega''$; roughly speaking,

$$(K'K'')_a(\omega', \omega'') = \int_B K''_b(\omega'') \, K'_a(\omega')(\mathrm{d}b) \, ,$$

but rigorously, ξ', ξ'' should be used (as in 7c6). Similarly to Sect. 7c, the composition of S-kernels, $K = K'K''$, is related to convolution of their distributions, $\Lambda_\Xi = \Lambda_{\Xi'} * \Lambda_{\Xi''}$, and composition of operators,

$$\overleftarrow{T}_n^K \varphi = \overleftarrow{T}_n^{K'} (\overleftarrow{T}_n^{K''} \varphi) \quad \text{and} \quad \overrightarrow{T}_n^K \nu = \overrightarrow{T}_n^{K''} (\overrightarrow{T}_n^{K'} \nu) \, .$$

For S-maps from $\mathcal{X}$ to $\mathcal{X}$, treated as S-kernels, the composition defined here conforms to that of Sect. 7c. In general, treating K' and K'' as two independent S-kernels on the same probability space, we generalize (7c7)–(7c10),

$$\mathbb{E} \biggl(\int_C \varphi \, \mathrm{d}K_a \biggm| K' \biggr) = \int_B (\overleftarrow{T}_1^{K''} \varphi) \, \mathrm{d}K'_a \, ; \tag{7g7}$$

$$\mathbb{E} \biggl(\int_C \varphi \, \mathrm{d}K_a \biggm| K'' \biggr) = \tag{7g8}$$

$$= \int_B \biggl(\int_C \varphi \, \mathrm{d}K''_b \biggr) (\overrightarrow{T}_1^{K'} \delta_a)(\mathrm{d}b) = \int_C \varphi \, \mathrm{d}K''_\nu \, , \quad \text{where } \nu = \overrightarrow{T}_1^{K'} \delta_a \, ;$$

$$\mathbb{E} \biggl(\int_{C^n} \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n}) \biggm| K' \biggr) = \tag{7g9}$$

$$= \int_{B^n} (\overleftarrow{T}_n^{K''} \varphi) \, \mathrm{d}(K'_{a_1} \times \dots \times K'_{a_n}) \, ;$$

$$\mathbb{E} \biggl(\int_{C^n} \varphi \, \mathrm{d}(K_{a_1} \times \dots \times K_{a_n}) \biggm| K'' \biggr) = \tag{7g10}$$

$$= \int_{B^n} \biggl(\int_{C^n} \varphi \, \mathrm{d}(K''_{b_1} \times \dots \times K''_{b_n}) \biggr) \bigl(\overrightarrow{T}_n^{K'} (\delta_{a_1} \times \dots \times \delta_{a_n}) \bigr) (\mathrm{d}b_1 \dots \mathrm{d}b_n) \, .$$

7h Flow systems of S-kernels

Let $\mathcal{X}$ be a compact metrizable space (mostly, the circle will be used). Then $\mathcal{P}(\mathcal{X})$, equipped with the weak topology, is also a compact metrizable space, and $L_0(\Omega \to \mathcal{P}(\mathcal{X}))$ is equipped with the (metrizable) topology of convergence in probability.

7h1 Definition. An S-kernel K from $\mathcal{X}$ to $\mathcal{X}$ is *continuous in probability,* if the map $x \mapsto K_x$ is continuous from $\mathcal{X}$ to $L_0(\Omega \to \mathcal{P}(\mathcal{X}))$.

Clearly, 7h1 implies measurability of K. For an S-map Ξ from $\mathcal{X}$ to $\mathcal{X}$, treated as an S-kernel K, 7h1 conforms to 7d1 (since the natural embedding $x \mapsto \delta_x$ of $\mathcal{X}$ into $\mathcal{P}(\mathcal{X})$ is homeomorphic). Here is a generalization of Lemma 7d2. It characterizes continuity in probability of an S-kernel in terms of $C(\mathcal{X}^n)$ (while a straightforward use of 7d2 would involve huge spaces $C(\mathcal{P}^n(\mathcal{X}))$). It also shows that continuity in probability is preserved by the composition.

7h2 Lemma. The following three conditions are equivalent for every S-kernel K from $\mathcal{X}$ to $\mathcal{X}$:

(a) K is continuous in probability;

(b) $\overleftarrow{T}_2^K \varphi \in C(\mathcal{X}^2)$ for all $\varphi \in C(\mathcal{X}^2)$;

(c) $\overleftarrow{T}_n^K \varphi \in C(\mathcal{X}^n)$ for all $\varphi \in C(\mathcal{X}^n)$ and all $n = 1, 2, 3, \dots$

Proof (sketch). (a) $\Longrightarrow$ (c): the composition $\overleftarrow{T}_n^K \varphi$ of a chain of continuous maps $(x_1, \dots, x_n) \mapsto (K_{x_1}, \dots, K_{x_n}) \mapsto K_{x_1} \times \dots \times K_{x_n} \mapsto \int \varphi \, \mathrm{d}(K_{x_1} \times \dots \times K_{x_n}) \mapsto \mathbb{E} \int \varphi \, \mathrm{d}(K_{x_1} \times \dots \times K_{x_n})$ between the spaces $\mathcal{X}^n \to L_0(\Omega \to \mathcal{P}^n(\mathcal{X})) \to L_0(\Omega \to \mathcal{P}(\mathcal{X}^n)) \to L_0(\Omega, [-\|\varphi\|, \|\varphi\|]) \to \mathbb{R}$ is continuous.

(c) $\Longrightarrow$ (b): trivial.

(b) $\Longrightarrow$ (a): let $\varphi \in C(\mathcal{X})$, then

$$
\begin{aligned}
\mathbb{E} \left| \int \varphi \, \mathrm{d}K_{x_1} - \int \varphi \, \mathrm{d}K_{x_2} \right|^2 &= \\
= \mathbb{E} \iint \varphi(x)\varphi(y) \big(K_{x_1}(\mathrm{d}x) K_{x_1}(\mathrm{d}y) &- K_{x_1}(\mathrm{d}x) K_{x_2}(\mathrm{d}y) - \\
- K_{x_2}(\mathrm{d}x) K_{x_1}(\mathrm{d}y) &+ K_{x_2}(\mathrm{d}x) K_{x_2}(\mathrm{d}y) \big) = \\
&= \psi(x_1, x_1) - \psi(x_1, x_2) - \psi(x_2, x_1) + \psi(x_2, x_2) \, ,
\end{aligned}
$$

where $\psi = \overleftarrow{T}_2^K (\varphi \otimes \varphi)$. By (b), ψ is a continuous function on $\mathcal{X}^2$. We see that $(x_1, x_2) \mapsto \mathbb{E} \left| \int \varphi \, \mathrm{d}K_{x_1} - \int \varphi \, \mathrm{d}K_{x_2} \right|^2$ is a continuous function on $\mathcal{X}^2$ vanishing on the diagonal $x_1 = x_2$. Therefore $x \mapsto \int \varphi \, \mathrm{d}K_x$ is a continuous map $\mathcal{X} \to L_2(\Omega)$, which is (a). $\square$

Condition 7h2(b) may be reformulated in terms of the kernel T_2^K (non-random, from $\mathcal{X}^2$ to $\mathcal{X}^2$),

$$
T_2^K \text{ is a continuous map } \mathcal{X}^2 \to \mathcal{P}(\mathcal{X}^2) \, .
$$

Similarly, 7h2(c) means continuity of all $T_n^K : \mathcal{X}^n \to \mathcal{P}(\mathcal{X}^n)$, or equivalently,

$$
T_\infty^K \text{ is a continuous map } \mathcal{X}^\infty \to \mathcal{P}(\mathcal{X}^\infty) \, ,
$$

since finite-dimensional functions $(x_1, x_2, \dots) \mapsto \varphi(x_1, \dots, x_n)$ are dense in $C(\mathcal{X}^\infty)$. Of course, $\mathcal{X}^\infty$ is equipped with the product topology, and is a compact metrizable space.

7h3 Proposition. The following two conditions are equivalent for every kernel T from $\mathcal{X}^\infty$ to $\mathcal{X}^\infty$:

(a) $T = T_\infty^K$ for some S-kernel K from $\mathcal{X}$ to $\mathcal{X}$, continuous in probability;

(b) T satisfies (7g4), (7g5), and is a continuous map $\mathcal{X}^\infty \to \mathcal{P}(\mathcal{X}^\infty)$.

Proof (sketch). (a) $\Longrightarrow$ (b): evident.

(b) $\Longrightarrow$ (a): we choose a countable dense subset $A \subset \mathcal{X}$ and apply Lemma 7g6 to the restriction T_0 of T to A^∞, thus getting an S-kernel K_0 from A to $\mathcal{X}$ such that $T_0 = T_\infty^{K_0}$. For every $\varphi \in C(\mathcal{X}^2)$ the function $\overleftarrow{T}_2^{K_0}\varphi$ is *uniformly* continuous on A^2 (since the map $(x_1, x_2, \dots) \mapsto \int \varphi(x'_1, x'_2) T_{x_1,x_2,\dots}(\mathrm{d}x'_1 \mathrm{d}x'_2 \dots)$ is continuous). Similarly to the proof of 7h2 ((b) $\Longrightarrow$ (a)) we deduce that K_0 is *uniformly* continuous in probability, that is, K_0 is a uniformly continuous map $A \to L_0(\Omega \to \mathcal{P}(\mathcal{X}))$. It remains to extend it to $\mathcal{X}$ by continuity. $\square$

We observe a natural one-to-one correspondence between

- $*$ distributions Λ_K of S-kernels K from $\mathcal{X}$ to $\mathcal{X}$, continuous in probability;
- $*$ kernels T from $\mathcal{X}^\infty$ to $\mathcal{X}^\infty$, satisfying 7h3(b);
- $*$ consistent systems $(T_n)_{n=1}^\infty$ of kernels T_n from $\mathcal{X}^n$ to $\mathcal{X}^n$, satisfying the finite-dimensional counterpart of 7h3(b).

The convolution of distributions corresponds to the composition of kernels.

See also [22, Sect. 2.5.1] and [40, 8d3].

For an S-map Ξ from $\mathcal{X}$ to $\mathcal{X}$, treated as an S-kernel K, the kernel T_2^K satisfies an additional condition: for all $x \in \mathcal{X}$,

(7h4) the measure $T_2^K(x,x)$ is concentrated on the diagonal of $\mathcal{X}^2$.

It is easily reformulated in terms of T_∞^K, and we get a natural one-to-one correspondence between

- $*$ distributions Λ_Ξ of S-maps Ξ from $\mathcal{X}$ to $\mathcal{X}$, continuous in probability;
- $*$ kernels T from $\mathcal{X}^\infty$ to $\mathcal{X}^\infty$, satisfying 7h3(b) and the reformulated additional condition (7h4);
- $*$ consistent systems $(T_n)_{n=1}^\infty$ of kernels T_n from $\mathcal{X}^n$ to $\mathcal{X}^n$, satisfying the finite-dimensional counterpart of 7h3(b), and (7h4).

The convolution of distributions corresponds to the composition of kernels.

A *convolution system of S-kernels* (over $\mathcal{X}$) may be defined as a family $(\Lambda_{s,t})_{s<t}$, where each $\Lambda_{s,t}$ is the distribution of an S-kernel from $\mathcal{X}$ to $\mathcal{X}$, continuous in probability, and

$$\Lambda_{r,t} = \Lambda_{r,s} * \Lambda_{s,t}$$

whenever $r < s < t$. An equivalent description is a family $(T_{s,t})_{s<t}$ of kernels $T_{s,t}$ from $\mathcal{X}^\infty$ to $\mathcal{X}^\infty$, satisfying 7h3(b) and

$$T_{r,t} = T_{r,s} T_{s,t}$$

whenever $r < s < t$.

Similarly to Sect. 7d, every convolution system of S-kernels leads to a convolution system as defined by 2b1, and if separability (as defined by 2b4) holds then we get a flow system, and further, a continuous product of probability spaces. For S-maps from $\mathcal{X}$ to $\mathcal{X}$, treated as S-kernels, the new construction conforms to that of Sect. 7d. In general, S-kernels from $\mathcal{X}$ to $\mathcal{X}$ (or their distributions) may be treated as S-maps from $\mathcal{P}(\mathcal{X})$ to $\mathcal{P}(\mathcal{X})$ (or their distributions, respectively); thus, the 'old' construction (of Sect. 7d) may be applied, as well as the 'new' construction introduced above. The 'old' and 'new' flow systems are isomorphic, and the 'old' and 'new' separability conditions are equivalent (since the relevant 'old' and 'new' σ-fields coincide). The sufficient condition (7d5) for the separability (the *temporal continuity in probability*) may be reformulated in terms of the (non-random) kernels $T_1^{K_{s,t}}$ from $\mathcal{X}$ to $\mathcal{X}$, namely,

$$\text{(7h5)} \qquad \text{both } T_1^{K_{s-\varepsilon,s}}(x) \text{ and } T_1^{K_{s,s+\varepsilon}}(x) \text{ converge to } \delta_x \text{ as } \varepsilon \to 0+$$

for $s \in \mathbb{R}$ and $x \in \mathcal{X}$. It follows that $K_\nu^{s-\varepsilon,s} \to \nu$ and $K_\nu^{s,s+\varepsilon} \to \nu$ in probability, since the expectation of the transportation distance between $K_\nu^{s,t}$ and ν does not exceed

$$\mathbb{E} \iint \operatorname{dist}(x,x')\, K_x^{s,t}(\mathrm{d}x')\nu(\mathrm{d}x) = \iint \operatorname{dist}(x,x')\, T_1^{K_{s,t}}(x)(\mathrm{d}x')\nu(\mathrm{d}x)\,.$$

The conclusion follows.

7h6 Lemma. Condition (7h5) implies separability.

7h7 Remark. The continuous product of probability spaces, corresponding to a flow system $(K_{s,t})_{s<t}$ of S-kernels from $\mathcal{X}$ to $\mathcal{X}$, is classical if and only if random variables $\int \varphi\, \mathrm{d}K_x^{s,t}$ are stable for all $s < t$ and all bounded Borel functions $\varphi : \mathcal{X} \to \mathbb{R}$. Proof: similar to 7d7.

7i From S-kernels to black noise

Sect. 7h gives us $\mathcal{X}$, $(G_{s,t}, \mu_{s,t})_{s<t}$ and $(K_{s,t})_{s<t}$ in the same way as Sect. 7d provided $\mathcal{X}$, $(G_{s,t}, \mu_{s,t})_{s<t}$ and $(\Xi_{s,t})_{s<t}$ to Sect. 7e. The temporal continuity in probability (7h5) is assumed to hold. Similarly to (7e1) we note that functions of the form

$$\text{(7i1)} \qquad \int_{\mathcal{X}^n} \varphi\, \mathrm{d}(K_{x_1}^{r,t} \times \dots \times K_{x_n}^{r,t}) \quad \text{for } \varphi \in C(\mathcal{X}^n),\ (x_1, \dots, x_n) \in \mathcal{X}_n$$

and $n = 1, 2, \dots$ are dense in $L_2(G_{r,t})$. Here are counterparts of 7e2–7e4.

7i2 Lemma. Assume that $n \in \{1, 2, \dots\}$ is given, a linear subset F_n of $C(\mathcal{X}^n)$, dense in the norm topology, and a linear subset N_n of the space of (finite, signed) measures on $\mathcal{X}^n$, dense in the weak topology; and an interval $(r, t) \subset \mathbb{R}$. Then functions of the form

$$\int_{\mathcal{X}^n} \biggl(\int_{\mathcal{X}^n} \varphi \, \mathrm{d}(K_{x_1}^{r,t} \times \dots \times K_{x_n}^{r,t}) \biggr) \nu(\mathrm{d}x_1 \dots \mathrm{d}x_n) \quad \text{for } \varphi \in F_n \text{ and } \nu \in N_n$$

are L_2-dense among functions of the form (7i1) for the given n.

Proof (sketch). Similar to 7e2. □

Substituting $f = \int_{\mathcal{X}^n} \bigl(\int_{\mathcal{X}^n} \varphi \, \mathrm{d}(K_{x_1}^{r,t} \times \dots \times K_{x_n}^{r,t}) \bigr) \nu(\mathrm{d}x_1 \dots \mathrm{d}x_n)$ to (7b3) we get

$$\text{(7i3)} \quad f_{s,s+\varepsilon} = \\ = \int_{\mathcal{X}^n} \biggl(\int_{\mathcal{X}^n} \bigl(\overleftarrow{T}_n^{K_{s+\varepsilon,t}} \varphi \bigr) \, \mathrm{d}(K_{x_1}^{s,s+\varepsilon} \times \dots \times K_{x_n}^{s,s+\varepsilon}) \biggr) \bigl(\overrightarrow{T}_n^{K_{r,s}} \nu \bigr)(\mathrm{d}x_1 \dots \mathrm{d}x_n) \, .$$

7i4 Proposition. (Le Jan and Raimond; implicit in [23].) Let a flow system $(K_{s,t})_{s<t}$ of S-kernels from $\mathcal{X}$ to $\mathcal{X}$ and a probability measure ν_0 on $\mathcal{X}$ satisfy the conditions

(a) (stationarity) the distribution of $K_{s,s+h}$ does not depend on s;

(b) (invariant measure) $\overrightarrow{T}_1^{K_{s,t}} \nu_0 = \nu_0$ for $s < t$;

(c) (Lipschitz boundedness) if $\varphi \in C(\mathcal{X}^n)$ is a Lipschitz function, then $\overleftarrow{T}_n^{K_{s,t}} \varphi$ is also a Lipschitz function, with a Lipschitz constant

$$\operatorname{Lip}(\overleftarrow{T}_n^{K_{s,t}} \varphi) \le C_n \operatorname{Lip}(\varphi) \quad \text{for } 0 \le s < t \le 1 \, ,$$

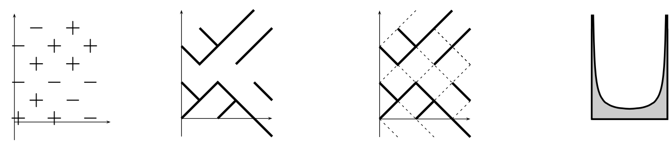
where $C_n < \infty$ depends only on n;

(d) $\sup\limits_{\varphi,\nu} \operatorname{Var} \biggl(\int_{\mathcal{X}^n} \biggl(\int_{\mathcal{X}^n} \varphi \, \mathrm{d}\bigl(K_{x_1}^{0,\varepsilon} \times \dots \times K_{x_n}^{0,\varepsilon}\bigr) \biggr) \nu(\mathrm{d}x_1 \dots \mathrm{d}x_n) \biggr) = o(\varepsilon)$ as $\varepsilon \to 0$, where the supremum is taken over all $\varphi \in C(\mathcal{X}^n)$ such that $\operatorname{Lip}(\varphi) \le 1$ and all positive measures ν on $\mathcal{X}^n$ such that $\nu_1 \le \nu_0, \dots, \nu_n \le \nu_0$; here $\nu_1, \dots, \nu_n$ are coordinate projections of ν, that is, $\int \varphi(x_k) \, \nu(\mathrm{d}x_1 \dots \mathrm{d}x_n) = \int \varphi(x) \, \nu_k(\mathrm{d}x)$ for $\varphi \in C(\mathcal{X})$.

Then the corresponding noise is black.

Proof (sketch). Similar to 7e4. □

7j Example: the sticky flow (Le Jan, Lemaire, Raimond)



(a) (b) (c) (d)

Recall the discrete model of Sect. 7f (see (a), (b)). Every space-time lattice point (t,x) is connected with one of the two points $(t+1,x\pm1)$ according to a two-valued random variable. Now we perturb the model (see (c)): (t,x) is connected strongly with one of the two points $(t+1,x\pm1)$ and weakly with the other, according to a random variable $\theta_{t,x}$ taking on values on the interval $[0,1]$. The case of 7f appears if $\theta_{t,x}$ takes on two values $0,1$ only (equiprobably). Generally, $\theta_{t,x}$ is the strength of the connection to $(t+1,x+1)$, and $1-\theta_{t,x}$ to $(t+1,x-1)$. Random variables $\theta_{t,x}$ are independent, identically distributed; their (common) distribution has a density shown on (d), to be specified later.

Particles move on the lattice. They are conditionally independent, given all $\theta_{t,x}$. Conditional probabilities of transitions from (t,x) to $(t+1,x+1)$ and $(t+1,x-1)$ are equal to $\theta_{t,x}$ and $1-\theta_{t,x}$ respectively. Unconditionally (when $\theta_{t,x}$ are not given), two particles move independently until they meet, after which they prefer moving together, but have a chance to separate, in contrast to the model of 7f; they are sticky rather than coalescing.

Given n particles at (t,x), the probability of k particles to choose $(t+1,x+1)$ (and the other $n-k$ particles to choose $(t+1,x-1)$) is $\binom{n}{k}\mathbb{E}\left(\theta^k(1-\theta)^{n-k}\right)$. A clever choice (specified later) of the distribution of θ makes the expectation a product of the form

(7j1) $$\binom{n}{k}\mathbb{E}\left(\theta^k(1-\theta)^{n-k}\right)=\frac{\alpha(k)\alpha(n-k)}{\beta(n)}$$

whenever $k,n\in\mathbb{Z}$, $0\le k\le n$, for some functions $\alpha,\beta:\{0,1,\dots\}\to(0,\infty)$. This fact leads to a stationary distribution of a simple form described below (Le Jan and Lemaire [21]).

Let x run over the cyclic group $\mathbb{Z}_m=\mathbb{Z}/m\mathbb{Z}$; this is our space ($m$ is a parameter). Each configuration of n unnumbered particles is described by a family of m *occupation numbers*, $s=(s_x)_{x\in\mathbb{Z}_m}$, $\sum_s s_x=n$; namely, s_x is the number of particles situated at x. We ascribe to each configuration a probability

(7j2) $$\mu_n(s)=\text{const}\cdot\prod_{x\in\mathbb{Z}_m}\beta(s_x)$$

(‘const’ being a normalization constant) and claim that such probability measure μ_n is invariant under our dynamics. Moreover, it satisfies the *detailed balance* condition:

$$\mu_n(s')p(s' \to s'') = \mu_n(s'')p(s'' \to s')$$

for all n-particle configurations s', s''; here $p(s' \to s'')$ stands for the transition probability (the probability of s'' at $t+1$ given s' at t).

A transition from s' to s'' may be described by *edge occupation numbers* $s(x \to x-1)$, $s(x \to x+1)$; these must satisfy $s'_x = s(x \to x-1) + s(x \to x+1)$ and $s''_x = s(x-1 \to x) + s(x+1 \to x)$ for all x. Let us call each such family of edge occupation numbers a *transition channel.* Several transition channels from s' to s'' may exist (try increasing each $s(x \to x-1)$ while decreasing each $s(x \to x+1)$); the detailed balance holds for each transition channel separately:

$$\begin{aligned} \mu_n(s')p_{\text{channel}}(s' \to s'') = \prod_x \beta(s_x) \cdot \prod_x \frac{\alpha(s(x \to x+1))\alpha(s(x \to x-1))}{\beta(s_x)} = \\ = \prod_x \big(\alpha(s(x \to x+1))\alpha(s(x \to x-1)) \big) \,, \end{aligned}$$

which evidently is symmetric (that is, time-reversible).

In order to get (7j1), following [21], we choose for θ the beta distribution,

$$\theta \sim \operatorname{Beta}(\varepsilon, \varepsilon) \,,$$

its density being

$$a \mapsto \frac{\Gamma(2\varepsilon)}{\Gamma^2(\varepsilon)} a^{\varepsilon-1}(1-a)^{\varepsilon-1} \quad \text{for } 0 < a < 1 \,;$$

$\varepsilon \in (0,1)$ is a parameter. We have

$$\mathbb{E}\left(\theta^k (1-\theta)^{n-k}\right) = \frac{\Gamma(2\varepsilon)}{\Gamma^2(\varepsilon)} \cdot \frac{\Gamma(k+\varepsilon)\Gamma(n-k+\varepsilon)}{\Gamma(n+2\varepsilon)} \,,$$

thus, (7j1) holds for

$$\begin{aligned} \alpha(k) &= \frac{\Gamma(k+\varepsilon)}{k!\Gamma(\varepsilon)} = \frac{1}{k!}\varepsilon(1+\varepsilon)\dots(k-1+\varepsilon) \,, \\ \beta(n) &= \frac{\Gamma(n+2\varepsilon)}{n!\Gamma(2\varepsilon)} = \frac{1}{n!} 2\varepsilon(1+2\varepsilon)\dots(n-1+2\varepsilon) \,. \end{aligned}$$

In order to take the scaling limit we embed the discrete space $\mathbb{Z}_m$ into a continuous space, the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by $x \mapsto \frac{1}{m}x$, and the discrete time set $\mathbb{Z}$ into the continuous time set $\mathbb{R}$ by $t \mapsto \frac{1}{m^2}t$. We also let ε depend on m, namely,

$$\varepsilon = \frac{a}{m}\,,$$

$a \in (0,\infty)$ being a parameter of the continuous model. Convergence (as $m \to \infty$) is proven by Le Jan and Lemaire [21]; the continuous model is (a special case of) the sticky flow introduced by Le Jan and Raimond [22]. The motion of a single particle is the standard Brownian motion in $\mathbb{T}$. Two particles spend together a non-zero time, but never a time interval (rather, a nowhere dense closed set of non-zero Lebesgue measure on the time axis).

The convergence (as $m \to \infty$) is proven at the level of the kernels $T_n^{K_{s,t}}$ ('moments'); the (continuous) sticky flow is a flow system $(K_{s,t})_{s<t}$ of S-kernels $K_{s,t}$ from $\mathcal{X}$ to $\mathcal{X}$ (here $\mathcal{X} = \mathbb{T}$) that corresponds to the constructed consistent system of kernels $T_n^{K_{s,t}}$ from $\mathcal{X}^n$ to $\mathcal{X}^n$ (recall Proposition 7h3 and the paragraph after it). A consistent system $(\mu_n)_n$ of invariant measures μ_n on $\mathcal{X}^n$ (or equivalently, an invariant measure on $\mathcal{X}^\infty$) is written out explicitly by a continuous counterpart of (7j2). The kernels $T_n^{K_{s,t}}$ are described for infinitesimal $t-s$ via their Dirichlet forms,

$$\varphi \mapsto \lim_{\varepsilon\to 0+} \int_{\mathcal{X}^n} \biggl(\int_{\mathcal{X}^n} |\varphi(x) - \varphi(\cdot)|^2 \, \mathrm{d}T_n^{K_{0,\varepsilon}}(x) \biggr) \mu_n(\mathrm{d}x)\,,$$

written out explicitly for smooth functions $\varphi : \mathcal{X}^n \to \mathbb{R}$.

The flow system of S-kernels satisfies the conditions 7i4(a–d), therefore the corresponding noise is black.

For details see [21], [22].

8 Unitary flows are classical

8a A question and a theorem

We know (see 6b3) that every $\mathbb{R}$-flow is classical (that is, generates a classical continuous product of probability spaces). The same holds for every stationary $\mathbb{T}$-flow, and moreover, for every $\mathbb{T}$-flow satisfying the upward continuity condition (see 6b1 and 2d2). However, nonclassical stationary G-flows exist in some Borel semigroups (in fact, finite-dimensional topo-semigroups) G, see Sect. 5d.

8a1 Question. (a) Whether every stationary G-flow in the group $G = \mathrm{Homeo}(\mathbb{T})$ of all homeomorphisms of the circle $\mathbb{T}$ is classical, or not?

(b) [35, 1.11] Whether every stationary G-flow in every Polish group G is classical, or not?

It is enough to examine a single Polish group $G = \mathrm{Homeo}([0,1]^\infty)$ of all homeomorphisms of the Hilbert cube $[0,1]^\infty$ rather than all Polish groups, since every Polish group is isomorphic to a subgroup of $\mathrm{Homeo}([0,1]^\infty)$ by a theorem of Uspenskii, see [17, 9.18] or [7, 1.4.1].

We turn to the unitary group $\mathrm{U}(H)$ of a separable infinite-dimensional Hilbert space H (over $\mathbb{R}$ or $\mathbb{C}$), equipped with the strong (or equivalently, weak) operator topology. We know (recall 2d9) that every *weakly continuous* convolution semigroup in $\mathrm{U}(H)$ leads to an $\mathrm{U}(H)$-flow $(X_{s,t})_{s<t}$ and a measurable action $(T_h)_h$ of $\mathbb{R}$ on Ω such that $X_{s,t} \circ T_h = X_{s+h,t+h}$. Such $(X_{s,t})_{s<t}$ will be called a *stationary* $\mathrm{U}(H)$*-flow, continuous in probability.*

8a2 Theorem. Every stationary $\mathrm{U}(H)$-flow, continuous in probability, is classical.

8a3 Question. (a) Does Theorem 8a2 hold without assuming continuity in probability?

(b) Is the following claim true? Every stationary G-flow $(X_{s,t})_{s<t}$ is of the form $X_{s,t} = c_s^{-1} Y_{s,t} c_t$ where $(Y_{s,t})_{s<t}$ is a stationary G-flow continuous in probability, and $c_t \in G$ for $t \in \mathbb{R}$. (The map $t \mapsto c_t$ need not be measurable.)

8a4 Corollary. Let G be a topological semigroup admitting a one-to-one continuous homomorphism to $\mathrm{U}(H)$. Then every stationary G-flow, continuous in probability, is classical.

8a5 Example. (a) The group $\mathrm{Diff}(M)$ of all diffeomorphisms of a smooth manifold M acts unitarily on $L_2(M, m)$, where m is any smooth strictly positive measure on M. Unitarity of the action is achieved multiplying by the root of the Radon-Nikodym derivative, see [4, 14.4.5]. By 8a4, every stationary $\mathrm{Diff}(M)$-flow, continuous in probability, is classical. Brownian motions in $\mathrm{Diff}(M)$ are described by Baxendale [6].

(b) Every Lie group G acts by diffeomorphisms on itself. Therefore every stationary G-flow, continuous in probability, is classical. Brownian motions in Lie groups are described by Yosida [49].

We may also consider the semigroup G of all contractions (that is, linear operators of norm ≤ 1) in a separable infinite-dimensional Hilbert space H (over $\mathbb{R}$ or $\mathbb{C}$), equipped with the strong operator topology. It is a topological semigroup, $G \supset \mathrm{U}(H)$. Here is the corresponding generalization of Theorem 8a2.

8a6 Theorem. Every stationary G-flow, continuous in probability, is classical. (Here G is the semigroup of contractions.)

8a7 Example. The semigroup G of all conformal endomorphisms of the disc acts by contractions on the space L_2 on the disc. Therefore every stationary G-flow, continuous in probability, is classical. Some important Brownian motions in G are now well-known as SLE.

8b From unitary flows to quantum instruments

In order to prove Theorem 8a2 it is sufficient to check stability, namely,

$$U^\rho f_{\varphi,\psi} \to f_{\varphi,\psi} \quad \text{as } \rho \to 1\,. \tag{8b1}$$

Here U^ρ are the operators on $L_2(\Omega)$ introduced in Sect. 5b, and random variables $f_{\varphi,\psi} \in L_2(\Omega)$ are matrix elements of the given random unitary operators,

$$f_{\varphi,\psi}(\omega) = \langle X_{0,1}(\omega)\varphi, \psi \rangle_H \quad \text{for } \omega \in \Omega,\ \varphi, \psi \in H\,.$$

The $\mathrm{U}(H)$-valued random variable $X_{0,1}$ belongs to a given $U(H)$-flow $(X_{s,t})_{s<t}$; $X_{s,t} \in L_0\big(\Omega \to \mathrm{U}(H)\big)$. The time interval $[0,1]$ is used just for convenience.

The operator U^ρ is the limit of (a net of) operators $\tilde U^\rho_{t_1,\dots,t_n}$, see (5b4). Calculating $\tilde U^\rho_{t_1,\dots,t_n} f_{\varphi,\psi}$ we come to an important construction (quantum operations and instruments). In order to simplify notation we do it first for a finite time set $T = \{0,1,\dots,n\}$ rather than a finite subset of a continuum. Accordingly, for now our $\mathrm{U}(H)$-flow consists of n independent random operators $X_{0,1},\dots,X_{n-1,n}$ on H (and their products).

For $n=1$ we have a single random operator $X_{0,1}$, and

$$\begin{aligned}\langle U^\rho f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} &= |\mathbb{E}\, f_{\varphi,\psi}|^2 + \rho\big(\mathbb{E}\,|f_{\varphi,\psi}|^2 - |\mathbb{E}\, f_{\varphi,\psi}|^2\big) = \\ &= |\langle \mathbb{E}\, X_{0,1}\varphi, \psi\rangle_H|^2 + \rho\big(\mathbb{E}\,|\langle X_{0,1}\varphi,\psi\rangle_H|^2 - |\langle \mathbb{E}\, X_{0,1}\varphi,\psi\rangle_H|^2\big)\,.\end{aligned}$$

For a fixed φ we have two positive[1] quadratic forms of ψ; they correspond to some positive Hermitian operators $S^{(0)}, S^{(1)} : H \to H$,

$$\langle U^\rho f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} = \langle S^{(0)}\psi,\psi\rangle_H + \rho\langle S^{(1)}\psi,\psi\rangle_H\,; \quad S^{(0)}, S^{(1)} \ge 0\,. \tag{8b2}$$

In terms of one-dimensional operators S_φ,

$$S_\varphi \psi = \langle \psi, \varphi\rangle_H \varphi\,,$$

[1] Not necessarily strictly positive.

we get $\langle S_\varphi \psi, \psi \rangle_H = |\langle \varphi, \psi \rangle|^2 = \operatorname{trace}(S_\varphi S_\psi)$ (the trace is always taken in H, never in $L_2(\Omega)$), $S^{(0)} = S_{\mathbb{E}\, X_{0,1}\varphi}$ and $S^{(0)} + S^{(1)} = \mathbb{E}\, S_{X_{0,1}\varphi}$. Quadratic dependence of $S^{(0)}, S^{(1)}$ on φ means their linear dependence on S_φ,

$$S^{(0)} = \mathcal{E}^{(0)} S_\varphi \,, \quad S^{(1)} = \mathcal{E}^{(1)} S_\varphi \,;$$

here $\mathcal{E}^{(0)}, \mathcal{E}^{(1)} : \mathrm{V}(H) \to \mathrm{V}(H)$ are linear operators on the space $\mathrm{V}(H)$ of all Hermitian trace class operators $H \to H$; $\mathrm{V}(H)$ is a Banach space (over $\mathbb{R}$), partially ordered by the cone $\mathrm{V}^+(H)$ of all positive operators. The operators $\mathcal{E}^{(0)}, \mathcal{E}^{(1)}$ are positive in the sense that $S \in \mathrm{V}^+(H)$ implies $\mathcal{E}^{(0)} S, \mathcal{E}^{(1)} S \in \mathrm{V}^+(H)$. Also, $\operatorname{trace}\big(\mathcal{E}^{(0)} S\big) \le \operatorname{trace}(S)$ for $S \in V^+(H)$, and the same for $\mathcal{E}^{(1)}$. Such operators on $\mathrm{V}(H)$ are called *quantum operations* in [11, Chap. 2]. A stronger requirement, called complete positivity [11, Chap. 9.2] is satisfied, but will not be used. See [35, Sect. 3] and [11] for details.

The sum $\mathcal{E}^{(0)} + \mathcal{E}^{(1)}$ is also a quantum operation, and in fact a *nonselective*[1] one, in the sense that

(8b3) $$\operatorname{trace}(\mathcal{E}^{(0)} S + \mathcal{E}^{(1)} S) = \operatorname{trace}(S) \quad \text{for } S \in \mathrm{V}(H) \,,$$

since $\operatorname{trace}\big(\mathcal{E}^{(0)} S_\varphi + \mathcal{E}^{(1)} S_\varphi\big) = \operatorname{trace}(\mathbb{E}\, S_{X_{0,1}\varphi}) = \mathbb{E}\, \operatorname{trace}(S_{X_{0,1}\varphi}) = \mathbb{E}\, \|X_{0,1}\varphi\|_H^2 = \|\varphi\|_H^2 = \operatorname{trace}(S_\varphi)$ for $\varphi \in H$. A finite family of quantum operations whose sum is nonselective is a (quantum) *instrument* [11, Chap. 4, Def. 1.1].[2] Thus, a random unitary operator $X_{0,1}$ leads to an instrument $\mathcal{E}_{0,1}$ consisting of $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(1)}$,

$$\mathcal{E}^{(0)} S_\varphi = S_{\mathbb{E}\, X_{0,1}\varphi} \,, \quad (\mathcal{E}^{(0)} + \mathcal{E}^{(1)}) S_\varphi = \mathbb{E}\, S_{X_{0,1}\varphi} \,,$$

which completes the case $n = 1$.

For $n = 2$ we have two independent random operators $X_{0,1}, X_{1,2}$ and their product $X_{0,2} = X_{0,1} X_{1,2}$; the latter means $X_{0,2}\varphi = X_{1,2}(X_{0,1}\varphi)$.[3] Two instruments $\mathcal{E}_{0,1}$ and $\mathcal{E}_{1,2}$ arise as above. Their composition [11, Sect. 4.2] is an instrument $\mathcal{E}_{0,2}$ consisting of four quantum operations $\mathcal{E}_{0,2}^{(0,0)}, \mathcal{E}_{0,2}^{(0,1)}, \mathcal{E}_{0,2}^{(1,0)}, \mathcal{E}_{0,2}^{(1,1)}$ defined by

$$\mathcal{E}_{0,2}^{(k,l)} = \mathcal{E}_{0,1}^{(k)} \mathcal{E}_{1,2}^{(l)} \,, \quad \text{that is,} \quad \mathcal{E}_{0,2}^{(k,l)} S = \mathcal{E}_{1,2}^{(l)} (\mathcal{E}_{0,1}^{(k)} S) \quad \text{for } S \in \mathrm{V}(H) \,.$$

Similarly to (8b2),

(8b4) $$\langle U^\rho f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} = \sum_{k,l \in \{0,1\}} \rho^{k+l} \operatorname{trace}\big((\mathcal{E}_{0,2}^{(k,l)} S_\varphi) S_\psi \big) \,.$$

[1] A nonselective quantum operation is also called a quantum channel.

[2] In general, an instrument is a vector measure, valued in quantum operations. We need only the elementary case when the underlying measurable space is finite.

[3] It could be more convenient to write φX rather than $X\varphi$.

Proof (*sketch*) *of* (8b4). First we introduce one-dimensional operators S_{φ_1,φ_2} for $\varphi_1, \varphi_2 \in H$ by $S_{\varphi_1,\varphi_2}\psi = \langle \psi, \varphi_1 \rangle \varphi_2$ and observe that $\operatorname{trace}(AS_{\varphi_1,\varphi_2}) = \langle A\varphi_2, \varphi_1 \rangle$ for $A \in \mathrm{V}(H)$. Two formulas for $n = 1$ follow from (8b2) by bilinearity:

$$\langle U^\rho f_{\varphi,\psi_1}, f_{\varphi,\psi_2} \rangle_{L_2(\Omega)} = \sum_{k=0,1} \rho^k \operatorname{trace}\bigl((\mathcal{E}^{(k)} S_\varphi) S_{\psi_1,\psi_2} \bigr) \, ,$$

$$\langle U^\rho f_{\varphi_1,\psi}, f_{\varphi_2,\psi} \rangle_{L_2(\Omega)} = \sum_{k=0,1} \rho^k \operatorname{trace}\bigl((\mathcal{E}^{(k)} S_{\varphi_2,\varphi_1}) S_\psi \bigr) \, .$$

Second, we choose an orthonormal basis $(e_n)_n$ of H and note that

$$A = \sum_{m,n} \operatorname{trace}(AS_{n,m}) S_{m,n}$$

for $A \in \mathrm{V}(H)$.

Third,

$$\begin{aligned} f_{\varphi,\psi} = \langle X_{0,2}\varphi, \psi \rangle_H = \langle X_{1,2}(X_{0,1}\varphi), \psi \rangle_H = \\ = \sum_n \langle X_{0,1}\varphi, e_n \rangle_H \langle X_{1,2} e_n, \psi \rangle_H = \sum_n f'_{\varphi,e_n} \otimes f''_{e_n,\psi} \, ; \end{aligned}$$

here $f'_{\dots} \in L_2(\Omega, \mathcal{F}_{0,1})$ and $f''_{\dots} \in L_2(\Omega, \mathcal{F}_{1,2})$. Finally,

$$\begin{aligned} \langle U^\rho f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} &= \Bigl\langle \sum_m (U^\rho_{0,1} f'_{\varphi,e_m}) \otimes (U^\rho_{1,2} f''_{e_m,\psi}), \sum_n f'_{\varphi,e_n} \otimes f''_{e_n,\psi} \Bigr\rangle_{L_2(\Omega)} = \\ &= \sum_{m,n} \langle U^\rho_{0,1} f'_{\varphi,e_m}, f'_{\varphi,e_n} \rangle_{L_2(\Omega)} \langle U^\rho_{1,2} f''_{e_m,\psi}, f''_{e_n,\psi} \rangle_{L_2(\Omega)} = \\ &= \sum_{m,n} \biggl(\sum_{k=0,1} \rho^k \operatorname{trace}\bigl((\mathcal{E}^{(k)}_{0,1} S_\varphi) S_{m,n} \bigr) \biggr) \biggl(\sum_{l=0,1} \rho^l \operatorname{trace}\bigl((\mathcal{E}^{(l)}_{1,2} S_{n,m}) S_\psi \bigr) \biggr) = \\ &= \sum_{k,l \in \{0,1\}} \rho^{k+l} \operatorname{trace}\bigl(S^{(l)}_{1,2} (\mathcal{E}^{(k)}_{0,1} S_\varphi) S_\psi \bigr) = \sum_{k,l \in \{0,1\}} \rho^{k+l} \operatorname{trace}\bigl((\mathcal{E}^{(k,l)}_{0,2} S_\varphi) S_\psi \bigr) \, . \end{aligned}$$

□

Similarly, for $T = \{0, 1, \dots, n\}$, $n = 1, 2, 3, \dots$

(8b5) $$\langle U^\rho f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} = \sum_{k_1,\dots,k_n \in \{0,1\}} \rho^{k_1+\dots+k_n} \operatorname{trace}\bigl((\mathcal{E}^{(k_1,\dots,k_n)}_{0,n} S_\varphi) S_\psi \bigr)$$

where $\mathcal{E}_{0,n}$ is the composition of n instruments,

$$\mathcal{E}^{(k_1,\dots,k_n)}_{0,n} = \mathcal{E}^{(k_1)}_{0,1} \dots \mathcal{E}^{(k_n)}_{n-1,n} \, ,$$

$\mathcal{E}_{m-1,m}$ being defined by

$$\mathcal{E}^{(0)}_{m-1,m} S_\varphi = S_{\mathbb{E}\, X_{m-1,m}\varphi} \, , \quad (\mathcal{E}^{(0)}_{m-1,m} + \mathcal{E}^{(1)}_{m-1,m}) S_\varphi = \mathbb{E}\, S_{X_{m-1,m}\varphi}$$

for $\varphi \in H$, $m = 1, \dots, n$.

8c From quantum instruments to Markov chains and stopping times

We still deal with the finite time set $T = \{0, 1, \dots, n\}$ and an $\mathrm{U}(H)$-flow $(X_{s,t})_{s<t;s,t\in T}$. Let $S_0 \in \mathrm{V}^+_1(H)$ be given; by $\mathrm{V}^+_1(H)$ we denote $\{S \in \mathrm{V}^+(H) : \operatorname{trace}(S) = 1\}$. Together with the instrument $\mathcal{E}_{0,n}$ constructed in Sect. 8b, S_0 leads to a probability distribution $\mu_{0,n}$ on the set $\{0,1\}^n$ of 2^n points $(k_1, \dots, k_n)$,

$$\mu_{0,n}(k_1, \dots, k_n) = \operatorname{trace}(\mathcal{E}^{(k_1,\dots,k_n)}_{0,n} S_0) \, .$$

For any $\varphi, \psi \in H$ such that $\|\varphi\| = 1$, $\|\psi\| = 1$ we have $\operatorname{trace}\big((\mathcal{E}^{(k_1,\dots,k_n)}_{0,n} S_\varphi) S_\psi \big) \le \operatorname{trace}(\mathcal{E}^{(k_1,\dots,k_n)}_{0,n} S_\varphi)$, thus, (8b5) gives

$$\text{(8c1)} \quad \langle (\mathbf{1} - U^\rho) f_{\varphi,\psi}, f_{\varphi,\psi} \rangle_{L_2(\Omega)} \le \sum_{k_1,\dots,k_n \in \{0,1\}} (1 - \rho^{k_1+\dots+k_n}) \mu_{0,n}(k_1, \dots, k_n) \, ;$$

striving to prove (8b1) we will estimate this sum from above, showing that $k_1 + \dots + k_n$ is not too large for the most of $(k_1, \dots, k_n)$ according to $\mu_{0,n}$.

The probability measure $\mu_{0,n}$ turns the set $\{0,1\}^n$ into another probability space (in addition to the original probability space Ω that carries $(X_{s,t})_{s<t}$); the 'coordinate' random process $k_1, \dots, k_n$ generates the natural filtration on $(\{0,1\}^n, \mu_{0,n})$. We introduce a Markov chain $S_0, \dots, S_n$ on the (filtered) probability space $(\{0,1\}^n, \mu_{0,n})$; each S_t is a random element of $\mathrm{V}^+_1(H)$. The initial state S_0 is chosen from the beginning (and non-random). For $t \in \{1, \dots, n\}$ the state S_t is a function of $k_1, \dots, k_t$, namely,

$$S_t = \frac{1}{\operatorname{trace}(\mathcal{E}^{(k_1,\dots,k_t)}_{0,t} S_0)} \mathcal{E}^{(k_1,\dots,k_t)}_{0,t} S_0 \, ;$$

of course, $\mathcal{E}^{(k_1,\dots,k_t)}_{0,t} = \mathcal{E}^{(k_1)}_{0,1} \dots \mathcal{E}^{(k_t)}_{t-1,t}$. It may happen that the denominator vanishes for some $(k_1, \dots, k_t)$, but such cases are of probability 0 and may be ignored. We have

$$\begin{gathered} \mathbb{P}\big(k_{t+1} = 1 \,\big|\, k_1, \dots, k_t \big) = \operatorname{trace}(\mathcal{E}^{(1)}_{t,t+1} S_t) \, , \\ S_{t+1} = \frac{1}{\mathbb{P}\big(k_{t+1} \,\big|\, k_1, \dots, k_t \big)} \mathcal{E}^{(k_{t+1})}_{t,t+1} S_t \, , \end{gathered}$$

since the joint distribution of $k_1, \dots, k_t$ is given by $\operatorname{trace}(\mathcal{E}_{0,t}^{(k_1,\dots,k_t)} S_0)$.

Striving to estimate $k_1 + \dots + k_n$ from above we introduce stopping times $\tau_1, \tau_2, \dots$ on the (filtered) probability space $(\{0,1\}^n, \mu_{0,n})$ by

$$\begin{aligned} \tau_1 &= \min \big\{ t \in \{1, \dots, n\} : k_t = 1 \big\} \,, \\ \tau_{m+1} &= \min \big\{ t \in \{\tau_m + 1, \dots, n\} : k_t = 1 \big\} \quad \text{for } m = 1, 2, \dots \end{aligned}$$

The minimum of the empty set is, by definition, infinite; thus, $\tau_1 = \infty$ if $k_1 = \dots = k_n = 0$, and more generally, $\tau_m = \infty$ if (and only if) $k_1 + \dots + k_n < m$.

8c2 Lemma.

(a) $$1 - \mathbb{E} \exp(-\tau_1/n) \ge \frac{1}{3n} \sum_{t=1}^n \operatorname{trace}(\mathcal{E}_{0,t}^{(0,\dots,0)} S_0) \,;$$

(b) $$1 - \mathbb{E} \Big(\exp\Big(-\frac{\tau_{m+1} - \tau_m}{n} \Big) \Big| k_1, \dots, k_{\tau_m} \Big) \ge \frac{1}{3n} \sum_{t=\tau_m+1}^n \operatorname{trace}(\mathcal{E}_{\tau_m,t}^{(0,\dots,0)} S_{\tau_m})$$

for $m = 1, 2, \dots$

Proof (sketch). (a):

$$\begin{aligned} 1 - \mathbb{E} \exp(-\tau_1/n) &= \sum_{t=0}^{n-1} \big(\exp(-t/n) - \exp(-(t+1)/n) \big) \mathbb{P} \big(k_1 = \dots = k_t = 0 \big) \\ &\quad + \exp(-1) \mathbb{P} \big(k_1 = \dots = k_n = 0 \big) \ge \\ &\ge \frac{1}{3n} \sum_{t=1}^n \mathbb{P} \big(k_1 = \dots = k_t = 0 \big) = \frac{1}{3n} \sum_{t=1}^n \operatorname{trace}(\mathcal{E}_{0,t}^{(0,\dots,0)} S_0) \,; \end{aligned}$$

(b): similar. □

The next lemma holds for any increasing sequence of stopping times $(\tau_m)_m$, irrespective of instruments etc.

8c3 Lemma. For any $\theta \in (0,1)$ and $m = 1, 2, \dots$ the probability of the event

$$\tau_m \le n \quad \text{and} \quad \mathbb{E} \big(\exp(-(\tau_{l+1} - \tau_l)/n) \big| k_1, \dots, k_{\tau_l} \big) \le \theta \text{ for } l = 0, 1, \dots, m-1$$

does not exceed $\mathrm{e}\theta^m$. (Here $\tau_0 = 0$.)

Proof (sketch). Denote by M the (random) least $l \in \{0, \dots, m-1\}$ such that $\mathbb{E} \big(\exp(-(\tau_{l+1} - \tau_l)/n) \big| k_1, \dots, k_{\tau_l} \big) > \theta$ and let $M = m$ if there is no such l. The expectation $\mathbb{E} \, (\theta^{-(M \wedge l)} \exp(-\tau_{M \wedge l}/n))$ decreases in l, therefore $\mathbb{E} \, (\theta^{-M} \exp(-\tau_M/n)) \le 1$, which implies $\mathbb{P} \big(\tau_m \le n, \, M = m \big) \le \mathrm{e}\theta^m$. □

8d A compactness argument

We return from the finite time set to the continuum $\mathbb{R}$. All said in 8b and 8c is applicable to any finite subset of $\mathbb{R}$, but for now we only need the operation $\mathcal{E}^{(0)}_{s,t} : \mathrm{V}(H) \to \mathrm{V}(H)$ for $s < t$,

$$\mathcal{E}^{(0)}_{s,t} S_\varphi = S_{\mathbb{E}\, X_{s,t}\varphi} \quad \text{for } \varphi \in H\,.$$

By stationarity, $\mathcal{E}^{(0)}_{s,t} = \mathcal{E}^{(0)}_{0,t-s}$.

8d1 Lemma. (a) $\operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S) \to \operatorname{trace}(S)$ as $t \to 0+$, for every $S \in \mathrm{V}(H)$;

(b) convergence in (a) is uniform in $S \in K$ whenever K is a compact subset of $\mathrm{V}(H)$.

Proof (sketch). (a):

$$\operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S_\varphi) = \operatorname{trace}(S_{\mathbb{E}\, X_{s,t}\varphi}) = \|\mathbb{E}\, X_{s,t}\varphi\|^2 \to \|\varphi\|^2 = \operatorname{trace}(S_\varphi)\,,$$

since $X_{0,t}\varphi \to \varphi$ in probability, and $\|X_{0,t}\varphi\|^2 \le \|\varphi\|^2$.

(b): Use monotonicity of $\operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S)$ in t. Or alternatively, use uniform continuity: the linear functionals $S \mapsto \operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S)$ are of norm ≤ 1. $\square$

Given a finite set $T \subset \mathbb{R}$, $T = \{t_0, \dots, t_n\}$, $t_0 < t_1 < \dots < t_n$, we may consider the restricted $\mathrm{U}(H)$-flow $(X_{s,t})_{s<t;s,t\in T}$. Applying to it the construction of Sect. 8c we get a Markov chain $(S_t)_{t\in T}$ provided that an initial state $S_{t_0} \in \mathrm{V}^+_1(H)$ is chosen; each S_t is a random element of $\mathrm{V}^+_1(H)$.

8d2 Lemma. For every $S_0 \in \mathrm{V}^+_1(H)$ and $\varepsilon > 0$ there exists a compact set $K \subset \mathrm{V}^+_1(H)$ such that for every finite set $T = \{t_0, \dots, t_n\}$, $0 = t_0 < t_1 < \dots < t_n = 1$,

$$\mathbb{P}\big(S_{t_0} \in K, \, \dots, \, S_{t_n} \in K \big) \ge 1 - \varepsilon\,,$$

where $(S_t)_{t\in T}$ is the Markov chain starting with $S_{t_0} = S_0$.

Proof (sketch). We take finite-dimensional projection operators $Q_m : H \to H$ such that

$$\mathbb{E}\, \operatorname{trace}(Q_m X_{0,1} S_0 X^*_{0,1}) \ge 1 - \frac{1}{m^3}$$

and (given T and m) define a martingale M_m by

$$\begin{aligned} M_m(t_k) &= \mathbb{E}\big(\operatorname{trace}(Q_m S_{t_n}) \,\big|\, S_{t_0}, \dots, S_{t_k} \big) = \\ &= \operatorname{trace}\big(Q_m \mathbb{E}\big(S_{t_n} \,\big|\, S_{t_k} \big) \big) = \operatorname{trace}\big(Q_m \mathbb{E}\,(X_{t_k,1} S_{t_k} X^*_{t_k,1}) \big) = \\ &= \operatorname{trace}\big(S_{t_k} \mathbb{E}\,(X^*_{t_k,1} Q_m X_{t_k,1}) \big) = \operatorname{trace}\big(S_{t_k} Q_m(t_k) \big)\,, \end{aligned}$$

where

$$Q_m(t) = \mathbb{E}\,(X^*_{t,1} Q_m X_{t,1}) \,.$$

That is,

$$\mathbb{P}\big(S_{t_0} \in A_m, \dots, S_{t_n} \in A_m \big) \ge 1 - \frac{1}{m^2} \,,$$

where

$$A_m = \{ S \in \mathrm{V}^+_1(H) : \sup_{t\in[0,1]} \operatorname{trace}\big(S Q_m(t) \big) \ge 1 - \tfrac1m \} \,.$$

(Note that A_m does not depend on T.) It remains to prove that for every m the set $A_m \cap A_{m+1} \cap \dots$ is compact.

We have $Q_m(t) = \mathcal{T}^*_{1-t}(Q_m)$ where linear operators $\mathcal{T}^*_t : \mathrm{V}(H) \to \mathrm{V}(H)$ defined by $\mathcal{T}^*_t(S) = \mathbb{E}\,(X^*_{0,t} S X_{0,t})$ are a one-parameter semigroup. The semigroup is strongly continuous, since $X^*_{0,t} S_\varphi X_{0,t} = S_{X^*_{0,t}\varphi} \to S_\varphi$ a.s. as $t \to 0$. Therefore sets $\{ Q_m(t) : t \in [0,1] \}$ are compact in $\mathrm{V}(H)$, and we may choose finite-dimensional projections $\tilde Q_m$ such that

$$Q_m(t) \le \tilde Q_m + \frac1m \mathbf{1} \quad \text{for } t \in [0,1] \,,$$

which implies $\operatorname{trace}\big(S Q_m(t) \big) \le \operatorname{trace}(S \tilde Q_m) + \frac1m$, thus

$$A_m \subset \{ S \in \mathrm{V}^+_1(H) : \operatorname{trace}(S \tilde Q_m) \ge 1 - \tfrac2m \} \,.$$

Compactness of $A_m \cap A_{m+1} \cap \dots$ follows easily. $\square$

Proof (*sketch*) *of Theorem* 8a2. According to (8b1) and Sect. 5b it is sufficient to prove that $\tilde U^\rho_{t_1,\dots,t_n} f_{\varphi,\psi} \xrightarrow[\rho\to1]{} f_{\varphi,\psi}$ uniformly in T, where T runs over all finite sets $T = \{t_1,\dots,t_n\} \subset (0,1)$. According to (8c1) it is sufficient to prove that $\mathbb{E}\,(1 - \rho^{k_1+\dots+k_n}) \xrightarrow[\rho\to1]{} 0$ uniformly in T. Here T runs over all finite sets $T = \{t_0,\dots,t_n\}$, $0 = t_0 < t_1 < \dots < t_n = 1$, and $(k_1,\dots,k_n)$ is the random process introduced in Sect. 8c (distributed $\mu_{0,n}$). The initial state $S_0 \in \mathrm{V}^+_1(H)$ is arbitrary but fixed (that is, the convergence need not be uniform in S_0).

Let $\varepsilon > 0$ be given; we have to find $\rho < 1$ such that $\mathbb{E}\,(1 - \rho^{k_1+\dots+k_n}) \le \varepsilon$ (or rather, 3ε) for all T (and n).

Lemma 8d2 gives us a compact set $K \subset \mathrm{V}^+_1(H)$ such that $\mathbb{P}\big(S_{t_0} \in K, \dots, S_{t_n} \in K \big) \ge 1 - \varepsilon$. By Lemma 8d1, $\operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S) \xrightarrow[t\to0]{} 1$ uniformly in $S \in K$. It follows that $\inf_T \inf_{S\in K} \int_0^1 \operatorname{trace}(\mathcal{E}^{(0)}_{0,t} S) \, \mathrm{d}t > 0$. Similarly, $\inf_T \inf_{S\in K} \frac1n \sum_{k=1}^n \operatorname{trace}(\mathcal{E}^{(0)}_{0,t_k} S) > 0$ provided that T is distributed on $[0,1]$ uniformly enough, say, $t_{k+1} - t_k \le \frac2n$ for all k, which can be ensured by enlarging T appropriately. Using Lemma 8c2(b) and taking into account that $\tau_m = \infty$ for $m > k_1 + \dots + k_n$ we get $\theta < 1$ such that the inequality

$$\min_{m=1,\dots,k_1+\dots+k_n} \mathbb{E}\Big(\exp\Big(-\frac{\tau_{m+1}-\tau_m}{n}\Big)\,\Big|\, k_1,\dots,k_{\tau_m}\Big) \le \theta$$

holds with probability $\ge \mathbb{P}\big(S_{t_0} \in K, \dots, S_{t_n} \in K\big) \ge 1-\varepsilon$; note that θ depends on K but not T, n. Combining it with Lemma 8c3 we have for $m = 1, 2, \dots$

$$\mathbb{P}\big(\tau_m \le n\big) \le \mathrm{e}\theta^m + \varepsilon\,,$$

that is, $\mathbb{P}\big(k_1 + \dots + k_n \ge m\big) \le \mathrm{e}\theta^m + \varepsilon$. Choosing m such that $\mathrm{e}\theta^m \le \varepsilon$ we get $\mathbb{P}\big(k_1 + \dots + k_n \ge m\big) \le 2\varepsilon$; note that m depends on K but not T, n.

Finally, $\mathbb{E}\,(1-\rho^{k_1+\dots+k_n}) \le \mathbb{P}\big(k_1+\dots+k_n \ge m\big)+(1-\rho^m) \le 2\varepsilon+(1-\rho^m)$. Choosing $\rho < 1$ such that $1-\rho^m \le \varepsilon$ we get $\mathbb{E}\,(1-\rho^{k_1+\dots+k_n}) \le 3\varepsilon$. □

The proof of Theorem 8a6 is similar. However, the quantum operation $\mathcal{E}^{(0)} + \mathcal{E}^{(1)}$ becomes selective (recall (8b2)), which leads to killing for the random process $k_1, \dots, k_n$ (introduced in 8c). Accordingly, we enlarge the state space of the Markov chain $S_0, \dots, S_n$ (also introduced in 8c) by an absorbing state 0; now, each S_t is a random element of $\mathrm{V}_1^+(H) \cup \{0\} \subset \mathrm{V}^+(H)$. If S_t jumps to 0 then the next stopping time τ_m is, by definition, infinite. In the proof of Lemma 8d2 each inequality of the form $\mathbb{E}\,\mathrm{trace}(Q_m S) \ge 1-\delta$ should be first rewritten as $\mathbb{E}\,\mathrm{trace}\big((1-Q_m)S\big) \le \delta$; the latter form is applicable in the more general situation.

9 Random sets as degrees of nonclassicality

9a Discrete time (toy models)

For a finite time set $T = \{1, \dots, n\}$ a continuous product $(\Omega_{s,t}, P_{s,t})_{s<t;s,t\in T}$ of probability spaces is just the usual product,

$$(\Omega_{s,t}, P_{s,t}) = (\Omega_{s,s+1}, P_{s,s+1}) \times \dots \times (\Omega_{t-1,t}, P_{t-1,t})\,.$$

Accordingly,

$$H_{s,t} = H_{s,s+1} \otimes \dots \otimes H_{t-1,t}$$

for Hilbert spaces $H_{s,t} = L_2(\Omega_{s,t}, P_{s,t})$. Each $H_{t,t+1}$ is a direct sum,

$$H_{t,t+1} = H_{t,t+1}^{(1)} \oplus H_{t,t+1}^{(0)}\,,$$

where $H^{(1)}_{t,t+1}$ is the one-dimensional subspace of constants, and $H^{(0)}_{t,t+1}$ is its orthogonal complement, the subspace of centered (zero mean) random variables. We open the brackets:

$$\begin{aligned}H = H_{1,n} &= H_{1,2} \otimes \dots \otimes H_{n-1,n} = \\ &= \big(H^{(1)}_{1,2} \oplus H^{(0)}_{1,2} \big) \otimes \dots \otimes \big(H^{(1)}_{n-1,n} \oplus H^{(0)}_{n-1,n} \big) = \bigoplus_{C \in \operatorname{Comp}(T)} H_C \,,\end{aligned}$$

where $\operatorname{Comp}(T) = 2^{T'}$ is the set of all subsets C of the set $T' = \{1.5, 2.5, \dots, n-0.5\}$, and

$$H_C = \bigotimes_{t=1}^{n-1} H^{(k(t))}_{t,t+1} \,, \qquad k(t) = \begin{cases} 1, & \text{if } t+0.5 \in C, \\ 0, & \text{otherwise.} \end{cases}$$

(Later, $\operatorname{Comp}(T)$ will consist of *compact* sets.) In other words, H_C is spanned by products $\prod_{t:t+0.5 \in C} f_{t,t+1}$ for $f_{t,t+1} \in L_2(\Omega_{t,t+1}, P_{t,t+1})$, $\mathbb{E}\, f_{t,t+1} = 0$.

The orthogonal decomposition of H leads to a *projection-valued measure* Q on $\operatorname{Comp}(T)$. That is, a Hermitian projection operator $Q(A) : H \to H$ corresponds to every $A \subset \operatorname{Comp}(T)$, satisfying

$$\begin{gathered} Q(A \cap B) = Q(A) Q(B) \,, \\ Q(\operatorname{Comp}(T) \setminus A) = \mathbf{1}_H - Q(A) \,, \\ Q(A \cup B) = Q(A) + Q(B) \quad \text{if } A \cap B = \emptyset \end{gathered} \tag{9a1}$$

for $A, B \subset \operatorname{Comp}(T)$. Such Q is uniquely determined by

$$Q(\{C\}) H = H_C \quad \text{for } C \in \operatorname{Comp}(T)$$

or alternatively, by

$$Q(\{C : t+0.5 \notin C\}) = \mathbb{E}\big(\cdot \,\big|\, \mathcal{F}_{1,t} \vee \mathcal{F}_{t+1,n} \big) \quad \text{for } t = 1, \dots, n-1$$

(the operator of conditional expectation, given $\omega_{1,2}, \dots, \omega_{t-1,t}, \omega_{t+1,t+2}, \dots, \omega_{n-1,n}$).

The operators U^ρ introduced in Sect. 5b are easily expressed in terms of H_C,

$$U^\rho = \int_{\operatorname{Comp}(T)} \rho^{|C|} \, Q(\mathrm{d}C) = \sum_{C \in \operatorname{Comp}(T)} \rho^{|C|} Q(\{C\}) = \bigoplus_{C \in \operatorname{Comp}(T)} \rho^{|C|} \,;$$

that is, each H_C is an eigenspace, its eigenvalue being $\rho^{|C|}$ (here $|C|$ stands for the number of elements in C). Accordingly, the eigenspaces H_k introduced in Sect. 5b are

$$H_k = Q(\{C : |C| = k\}) H = \bigoplus_{C:|C|=k} H_C \,.$$

Every $f \in H$ leads to a measure μ_f on the set $\mathrm{Comp}(T)$, called the *spectral measure* of f; namely,

$$\mu_f(A) = \langle Q(A)f, f \rangle = \|Q_A f\|^2 \quad \text{for } A \subset \mathrm{Comp}(T)\,.$$

Clearly, $\mu_f(\mathrm{Comp}(T)) = \|f\|^2$. Assuming $\|f\| = 1$ we get a probability measure μ_f, thus, C may be thought of as a random set. However, this random set is defined on the probability space $(\mathrm{Comp}(T), \mu_f)$ rather than (Ω, P).

Let $g \in H_1 = \bigoplus_{|C|=1} H_C$ and $f = \operatorname{Exp} g$ in the sense of Sect. 6b; that is, $g = g_{1,2} + \dots + g_{n-1,n}$ and $f = (1 + g_{1,2}) \dots (1 + g_{n-1,n})$. Then μ_f is a product measure, $\mu_f(\{C\}) = \prod_{t:t+0.5 \in C} \|g_{t,t+1}\|^2$. That is, the probability measure $\mu_{f/\|f\|}$ describes a random set C that contains each point $t+0.5$ with probability $\|g_{t,t+1}\|^2/(1+\|g_{t,t+1}\|^2)$, independently of others. In contrast, the probability measure $\mu_{g/\|g\|}$ describes a single-point random set C, equal to $\{t + 0.5\}$ with probability $\|g_{t,t+1}\|^2/\|g\|^2$.

Spaces L_2 may be replaced with arbitrary *pointed Hilbert spaces* (recall Sect. 6d).

Now we turn to the discrete example of Sect. 1b, the $\mathbb{Z}_m$-flow $(X^{1\mathrm{b}}_{s,t})_{s<t;s,t\in T}$ over the *infinite* time set $T = \{0, 1, 2, \dots\} \cup \{\infty\}$ and the corresponding continuous product of probability spaces. The set $\mathrm{Comp}(T)$ of all compact subsets of the space $T' = \{0.5, 1.5, \dots\} \cup \{\infty\}$ is a standard measurable space, it may be identified with the set $2^{T' \setminus \{\infty\}}$ of *all* subsets of $\{0.5, 1.5, \dots\}$. The σ-field on $2^{T' \setminus \{\infty\}}$ is generated by the algebra of cylinder subsets. On this algebra we define a projection-valued measure (or rather, additive set function) Q by opening (a finite number of) brackets in $H = \big(H^{(1)}_{0,1} \oplus H^{(0)}_{0,1} \big) \otimes \dots \otimes \big(H^{(1)}_{n-1,n} \oplus H^{(0)}_{n-1,n} \big) \otimes H_{n,\infty}$. Its σ-additive extension to the σ-field, evidently unique, exists by Kolmogorov's theorem combined with a simple argument [40, 3d11].

The spectral measure μ_f of the random variable $f = \exp\big(\frac{2\pi i}{m} X^{1\mathrm{b}}_{0,\infty} \big)$ is easy to calculate. For any n we have a product, $f = \exp\big(\frac{2\pi i}{m} X^{1\mathrm{b}}_{0,1} \big) \dots \exp\big(\frac{2\pi i}{m} X^{1\mathrm{b}}_{n-1,n} \big) \exp\big(\frac{2\pi i}{m} X^{1\mathrm{b}}_{n,\infty} \big)$, therefore the random set contains each of the points $\frac12, \frac32, \dots, n - \frac12$ with probability $1 - |\mathbb{E} \exp\big(\frac{2\pi i}{m} X^{1\mathrm{b}}_{0,1} \big)|^2 = \sin^2 \frac{\pi}{m}$, *independently of others.* It is just a Bernoulli process, an infinite sequence of independent equiprobable events. The random set is infinite a.s., which means that $f \in H_\infty$, the sensitive space orthogonal to the stable space $H_0 \oplus H_1 \oplus \dots$ See also 1c1 and Sect. 5d.

Continuous-time counterparts of these ideas for the commutative setup (see 9b) are introduced by Tsirelson [35, Sect. 2]; for the noncommutative setup — by Liebscher [19] and Tsirelson [39, Sect. 2] independently.

9b Probability spaces

Dealing with the time set $\mathbb{R}$ we introduce the set $\operatorname{Comp}(\mathbb{R})$ of all compact subsets $C \subset \mathbb{R}$. Endowed with the σ-field generated by the sets of the form $\{C \in \operatorname{Comp}(\mathbb{R}) : C \cap U = \emptyset\}$, where U varies over all open subsets of $\mathbb{R}$, $\operatorname{Comp}(\mathbb{R})$ becomes a standard measurable space [17, 12.6]. Nowhere dense compact sets $C \subset \mathbb{R}$ (that is, with no interior points) are a measurable subset of $\operatorname{Comp}(\mathbb{R})$.

By a projection-valued measure on $\operatorname{Comp}(\mathbb{R})$ (over a Hilbert space H) we mean a family of Hermitian projection operators $Q(A) : H \to H$ given for all measurable $A \subset \operatorname{Comp}(\mathbb{R})$, satisfying (9a1) and countable additivity:

$$\text{if } A_n \uparrow A \text{ then } Q(A_n) \to Q(A) \text{ strongly}$$

(that is, $\|Q(A_n)f - Q(A)f\| \to 0$ for every $f \in H$).

9b1 Theorem. [40, Th. 3d12 and (3d3)] Let $(\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s<t}$ be a continuous product of probability spaces, satisfying the upward continuity condition (2d3). Then

(a) there exists one and only one projection-valued measure Q on $\operatorname{Comp}(\mathbb{R})$ (over $H = L_2(\Omega, \mathcal{F}, P)$) such that

$$Q(\{C : C \cap (s,t) = \emptyset\}) = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s} \vee \mathcal{F}_{t,\infty}\right) \tag{9b2}$$

whenever $-\infty < s < t < \infty$;

(b) Q is concentrated on (the set of all) nowhere dense compact sets $C \subset \mathbb{R}$;

(c) $Q(\{C : t \in C\}) = 0$ for every $t \in \mathbb{R}$.

Throughout Sect. 9b, the upward continuity condition is assumed for all continuous products of probability spaces. The time set $\mathbb{R}$ may be enlarged to $[-\infty, \infty]$, but it is the same, since a compact subset of $[-\infty, \infty]$ not containing $\pm\infty$ (like 9b1(c)) is in fact a compact subset of $\mathbb{R}$.

Using the relation $Q(A \cap B) = Q(A)Q(B)$ we get

$$Q(\{C : C \cap ((r,s) \cup (t,u)) = \emptyset\}) = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,r} \vee \mathcal{F}_{s,t} \vee \mathcal{F}_{u,\infty}\right) \tag{9b3}$$

for $-\infty < r < s < t < u < \infty$, and the same for any finite number of intervals.

It is convenient to express a relation of the form $Q(A) = \mathbf{1}_H$ by saying that 'almost all spectral sets belong to A'. Thus, (b) says that almost all spectral sets are nowhere dense, while (c) says that for every t, almost all spectral sets do not contain t. By the way, the latter shows that $Q(\{C :$

$C \cap (s,t) = \emptyset\}) = Q(\{C : C \cap [s,t] = \emptyset\})$. Also, applying Fubini theorem we see that almost every spectral set is of zero Lebesgue (or other) measure.

As before, every $f \in H$, $\|f\| = 1$, has its *spectral measure*

$$\mu_f(A) = \langle Q(A)f, f \rangle = \|Q_A f\|^2 \,,$$

a probability measure on $\mathrm{Comp}(\mathbb{R})$. The relation $Q(A) = \mathbf{1}_H$ holds if and only if $\mu_f(A) = 1$ for all f.

As before,

$$U^\rho = \int_{\mathrm{Comp}(\mathbb{R})} \rho^{|C|} \, Q(\mathrm{d}C) \,, \tag{9b4}$$

but now C may be infinite, in which case $\rho^{|C|} = 0$ for all $\rho \in [0,1)$. We have

$$L_2(\mathcal{F}_{\mathrm{stable}}) = Q(\{C : |C| < \infty\}) \,;$$

accordingly, the sensitive subspace is $Q(\{C : |C| = \infty\})$. A continuous product of probability spaces is classical if and only if almost all spectral sets are finite. The classical part of a continuous product of probability spaces is trivial if and only if almost all nonempty spectral sets are infinite. Also,

$$\langle U^\rho f, f \rangle = \int_{\mathrm{Comp}(\mathbb{R})} \rho^{|C|} \, \mu_f(\mathrm{d}C) \,; \tag{9b5}$$

f is stable if and only if μ_f-almost all spectral sets are finite; f is sensitive if and only if μ_f-almost all nonempty spectral sets are infinite.

Let $g \in H_1 = Q(\{C : |C| = 1\})$ and $f = \operatorname{Exp} g$ (in the sense of Sect. 6b), then $\mu_{f/\|f\|}$ is the distribution of a Poisson point process; the mean number of points on (s,t) is equal to $\|g_{s,t}\|^2$ where $g_{s,t} = \mathbb{E}\big(g \,\big|\, \mathcal{F}_{s,t}\big)$.

If $Q(A) = 0$ then $\mu_f(A) = 0$ for all f. For some f (for instance, $f = 1$) the relation $\mu_f(A) = 0$ does not imply $Q(A) = 0$; however, such f are exceptional (in fact, they are a meager subset of H). For other, typical $f \in H$ the class of A such that $\mu_f(A) = 0$ does not depend on f. It means that all typical spectral measures are equivalent (that is, mutually absolutely continuous). Thus, each continuous product of probability spaces leads to a *measure type* (or 'class') $\mathcal{M}$, — an equivalence class of probability measures on $\mathrm{Comp}(\mathbb{R})$.

A measure belonging to the class $\mathcal{M}$ has one and only one atom. Namely, a spectral set has a chance to be empty.

The same construction may be applied to the restriction of a given continuous product of probability spaces to a given time interval $(s,t) \subset \mathbb{R}$. We get a projection-valued measure $Q_{s,t}$ on the space $\mathrm{Comp}(s,t)$ of all compact subsets of (s,t), over $H_{s,t} = L_2(\mathcal{F}_{s,t})$, and a measure type $\mathcal{M}_{s,t}$ on $\operatorname{Comp}(s,t)$. It appears that

$$Q_{r,t} = Q_{r,s} \otimes Q_{s,t} \quad \text{for } -\infty \le r < s < t \le \infty$$

in the sense that $Q_{r,t}(A \times B) = Q_{r,s}(A) \otimes Q_{s,t}(B)$ for all measurable $A \subset \operatorname{Comp}(r,s)$ and $B \subset \operatorname{Comp}(s,t)$. Here $A \times B = \{C_1 \cup C_2 : C_1 \in A, C_2 \in B\}$. Note that $\operatorname{Comp}(r,s) \times \operatorname{Comp}(s,t)$ is a subset of $\operatorname{Comp}(r,t)$ consisting of all $C \in \operatorname{Comp}(r,t)$ such that $s \notin C$; it does not harm, since $Q_{r,t}(\{C : s \in C\}) = 0$. We get

$$\mathcal{M}_{r,t} = \mathcal{M}_{r,s} \times \mathcal{M}_{s,t} \quad \text{for } -\infty \le r < s < t \le \infty \tag{9b6}$$

in the sense that the product measure $\mu = \mu_1 \times \mu_2$ belongs to $\mathcal{M}_{r,t}$ for some (therefore, all) $\mu_1 \in \mathcal{M}_{r,s}$, $\mu_2 \in \mathcal{M}_{s,t}$. Also, $\mathcal{M}_{s,t}$ is the marginal distribution of $\mathcal{M}$ in the sense that the marginal of μ belongs to $\mathcal{M}_{s,t}$ for some (therefore, all) $\mu \in \mathcal{M}$.

9b7 Definition. A measure type $\mathcal{M}$ on $\operatorname{Comp}(\mathbb{R})$ is *factorizing,* if $\{C : s \in C\}$ is $\mathcal{M}$-negligible for each s, and the marginals of $\mathcal{M}$ satisfy (9b6).

See also [19, Def. 4.1]. Every continuous product of probability spaces (satisfying the upward continuity condition) leads to a factorizing measure type on $\operatorname{Comp}(\mathbb{R})$.

9b8 Question. Does every factorizing measure type on $\operatorname{Comp}(\mathbb{R})$ correspond to some continuous product of probability spaces? (See also 9c2 and 10a1.)

A fragment $C \cap [s,t]$ of a spectral set C is also a spectral set. More formally, for every interval $[s,t] \subset \mathbb{R}$ the map $C \mapsto C \cap [s,t]$ of $\operatorname{Comp}(\mathbb{R})$ to itself is $\mathcal{M}$-nonsingular. That is, the inverse image of each negligible set is negligible.

The following three conditions are thus equivalent:

* almost all nonempty spectral sets are infinite;
* almost all spectral sets are perfect (no isolated points; the empty set is perfect);
* the classical part of the continuous product of probability spaces is trivial.

9b9 Example. All classical noises (except for the trivial case, that is, assuming $\dim L_2(\Omega) > 1$) lead to the same measure type $\mathcal{M} = \mathcal{M}_{\text{Poisson}}$ on $\operatorname{Comp}(\mathbb{R})$. Namely, $\mathcal{M}_{\text{Poisson}}$ contains the distribution of the Poisson random subset of $\mathbb{R}$ whose intensity measure is finite and equivalent to the Lebesgue measure on $\mathbb{R}$. (Equivalent finite intensity measures lead to equivalent Poisson measures.) Clearly, $\mathcal{M}_{\text{Poisson}}$ is factorizing and shift-invariant.

9b10 Example. For the noise of splitting, considered in Sect. 4d, spectral sets are described by Warren [43], see also Watanabe [46]. They are at most countable. Moreover, almost all spectral sets C satisfy $|C'| < \infty$ (or equivalently, $C'' = \emptyset$); here C' stands for the set of all limit (that is, accumulation) points of C. The relation $|C'| < \infty$ holds also for the noise of stickiness, considered in Sect. 4e [40, 6b4].

9b11 Example. For the black noise of coalescence, considered in Sect. 7f, almost all nonempty spectral sets are perfect (therefore, uncountable). For some especially simple random variable f the spectral measure μ_f is described by Tsirelson (see [40, Sect. 7d] and references there); spectral sets are of Hausdorff dimension 1/2.

For every noise, finite spectral sets select a subnoise (namely, the classical part of the noise). Similarly, spectral sets C satisfying $C'' = \emptyset$ select a subnoise [40, Th. 6b2(b) for $\alpha = 1$, and 6b10]. The same holds for $C''' = \emptyset$ ($\alpha = 2$) and higher levels of the Cantor-Bendixson hierarchy, but for now we have no examples.

For every $\alpha \in (0, 1)$ spectral sets of Hausdorff dimension at most α select a subnoise [40, Th. 6b9, 6b10]. (See also Sect. 9c.)

Consider (necessarily countable or finite) spectral sets C such that $\forall t \in C$ $\exists \varepsilon > 0$ $\big(C \cap (t, t+\varepsilon) = \emptyset\big)$; that is, accumulation is allowed from the left but not from the right. Do these select a subnoise? I do not know. (See also [40, Question 6b12].)

All said above holds for (not just homogeneous) continuous products of probability spaces.

Here is a generalization of Theorem 5c3. In Item (b), $\dim C$ stands for the Hausdorff dimension of C. In Item (a), $C^{(\kappa)}$ is defined recursively: $C^{(0)} = C$, $C^{(\kappa+1)} = (C^{(\kappa)})'$ and $C^{(\kappa)} = \cap_{\kappa_1 < \kappa} C^{(\kappa_1)}$ for limit ordinals κ. The case $\kappa = 1$ of (a) returns us to 5c3.

9b12 Theorem. Let $\big((\Omega, P), (\mathcal{F}_{s,t})_{s<t}\big)$ be a continuous product of probability spaces, satisfying the upward continuity condition (2d3). Then:

(a) For every finite or countable ordinal κ there exists a symmetric self-joining $(\alpha_\kappa, \beta_\kappa)$ of the given product such that

* $\mathbb{E}\,(f \circ \alpha_\kappa)(g \circ \beta_\kappa) = \mathbb{E}\,(fg)$ for all $f, g \in L_2(\Omega)$ such that the relation $C^{(\kappa)} = \emptyset$ holds both for μ_f-almost all C and for μ_g-almost all C;
* $\mathbb{E}\,(f \circ \alpha_\kappa)(g \circ \beta_\kappa) = 0$ for all $f \in L_2(\Omega)$ such that $C^{(\kappa)} \neq \emptyset$ for μ_f-almost all C, and all $g \in L_2(\Omega)$.

The self-joining $(\alpha_\kappa, \beta_\kappa)$ is unique up to isomorphism.

(b) For every $\theta \in (0, 1)$ there exists a symmetric self-joining $(\alpha_\theta, \beta_\theta)$ of the given product such that

* $\mathbb{E}\,(f \circ \alpha_\theta)(g \circ \beta_\theta) = \mathbb{E}\,(fg)$ for all $f, g \in L_2(\Omega)$ such that the relation $\dim C \le \theta$ holds both for μ_f-almost all C and for μ_g-almost all C;
* $\mathbb{E}\,(f \circ \alpha_\theta)(g \circ \beta_\theta) = 0$ for all $f \in L_2(\Omega)$ such that $\dim C > \theta$ for μ_f-almost all C, and all $g \in L_2(\Omega)$.

The self-joining $(\alpha_\theta, \beta_\theta)$ is unique up to isomorphism.

Proof (sketch). We combine the arguments of [40, Sect. 6b] with the compactness of the space of joinings. A symmetric self-joining corresponds to every element of the set S (of some Borel functions $\operatorname{Comp}(\mathbb{R}) \to [0,1]$) introduced in [40, Sect. 5b] and used in [40, Sect. 6b, especially 6b1, 6b8]. $\square$

9b13 Corollary. The following two conditions are equivalent for any $\theta \in (0,1)$ and any continuous product of probability spaces, corresponding to a flow system $(X_{s,t})_{s<t}$:

(a) $\dim C \le \theta$ for almost all spectral sets C;

(b) $\dim C \le \theta$ for μ_f-almost all C, where $f = \varphi(X_{s,t})$, for all $s < t$ and all bounded Borel functions $\varphi : G_{s,t} \to \mathbb{R}$ (a single φ is enough if it is one-to-one).

Proof: similar to 5c5, using 9b12(b). See also [45, Proof of Th. 1.3, especially (4.2)].

9b14 Remark. Five more corollaries similar to 9b13 are left to the reader. Namely, each one of 5c5, 7d7, 7h7 may be generalized using each one of 9b12(a), 9b12(b), giving $3 \times 2 = 6$ corollaries, 9b13 being one of them.

Similarly to (9b2) we have for $s < t$

$$(9b15)\quad Q(\{C : C' \cap (s,t) = \emptyset\}) = Q(\{C : |C \cap (s,t)| < \infty\}) = \\ = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s} \vee \mathcal{F}_{s,t}^{\text{stable}} \vee \mathcal{F}_{t,\infty}\right).$$

The counterpart of (9b3) for C' is left to the reader. The operator (9b2) corresponds to a self-joining (of the continuous product of probability spaces), a combination of $(\alpha_\rho, \beta_\rho)$ with different ρ; namely, $\rho = 0$ on (s,t) and $\rho = 1$ on $(-\infty, s) \cup (t, \infty)$. The same holds for the operator (9b15); still, $\rho = 1$ on $(-\infty, s) \cup (t, \infty)$, but on (s,t) we use $\rho = 1-$ (recall 5c3)!

Similarly to (9b4), (9b5) we may construct operators $V^\rho : L_2(\Omega) \to L_2(\Omega)$ by

$$V^\rho = \int_{\operatorname{Comp}(\mathbb{R})} \rho^{|C'|}\, Q(\mathrm{d}C)\,,$$

$$\langle V^\rho f, f \rangle = \int_{\operatorname{Comp}(\mathbb{R})} \rho^{|C'|}\, \mu_f(\mathrm{d}C)$$

(see also [40, Th. 6b3 and before]). This is another generalization of the Ornstein-Uhlenbeck semigroup; it perturb sensitive random variables, while stable random variables are untouched.

9c Example: nonclassical Harris flows (Warren and Watanabe)

Similarly to the Arratia flow (considered in Sect. 7f), a Harris flow is a flow system of S-maps (from $\mathcal{X}$ to $\mathcal{X}$, where $\mathcal{X}$ may be the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or the line $\mathbb{R}$), homogeneous both in time and in space, such that the one-point motion is the (standard) Brownian motion in $\mathcal{X}$. However, Arratia's particles are independent before coalescence, while Harris' particles are correlated all the time. Namely,

$$\mathrm{d}X_t \mathrm{d}Y_t = b(X_t - Y_t)\,\mathrm{d}t\,,$$

where X_t, Y_t are coordinates of two particles at time t, and the *correlation function* $b(\cdot)$ is a given positive definite function $\mathcal{X} \to \mathbb{R}$; $b(0) = 1$, $b(-x) = b(x)$. Thus, $(X_t - Y_t)_t$ is a diffusion process, it becomes the (standard) Brownian motion in $\mathcal{X}$ under a random time change, the new time being $2\int_0^t \bigl(1 - b(X_s - Y_s)\bigr)\,\mathrm{d}s$, as long as $X_s \ne Y_s$. Three cases emerge (see also [22, Sect. 7.4]):

∗ non-coalescing case:

$$\text{the function } x \mapsto \frac{x}{1 - b(x)} \text{ is non-integrable near } 0\,;$$

∗ classical coalescing case:

$$x \mapsto \frac{x}{1 - b(x)} \text{ is integrable near 0, but } x \mapsto \frac{1}{1 - b(x)} \text{ is not}\,;$$

∗ nonclassical case:

$$x \mapsto \frac{1}{1 - b(x)} \text{ is integrable near } 0\,.$$

The non-coalescing case: $X_t - Y_t$ never vanishes; rather, $X_t - Y_t \to 0$ as $t \to \infty$, and $\int_0^\infty \bigl(1 - b(X_s - Y_s)\bigr)\,\mathrm{d}s < \infty$. Two particles cannot meet, since their rapprochement is infinitely long. The origin is a natural boundary for the diffusion process $(X_t - Y_t)_t$. In particular it happens if $b(\cdot)$ is twice continuously differentiable (which leads to a flow of homeomorphisms, see [16, Sect. 8]).

The classical coalescing case: $X_t - Y_t$ vanishes at some t and remains 0 forever. Two particles cannot diverge after meeting; even a small divergence would take infinite time (and no wonder: it must involve infinitely many more meetings). The origin is an exit boundary. In particular it happens if $1 - b(x) \sim |x|^\alpha$ (as $x \to 0$), $1 \le \alpha < 2$.

In the nonclassical case (in particular, $1-b(x) \sim |x|^\alpha$, $0 < \alpha < 1$) we have an additional freedom. The origin is a regular boundary; we may postulate it to be absorbing, sticky or reflecting. Only absorption leads to a flow system of S-maps (*'the coalescing nonclassical case'*); stickiness and reflection lead rather to flow systems of S-kernels.

The corresponding noise is classical in the non-coalescing case, as well as in the classical coalescing case [22, Sect. 7.4], [45, Th. 1.1].

We turn to the coalescing nonclassical case. Here, the noise is nonclassical [22, Sect. 7.4], [45, Th. 1.1]. If the correlation function $b(\cdot)$ is continuous (and smooth outside the origin, and strictly positive definite) then the classical part of the noise is generated by infinitely many independent Brownian motions [22, Sects. 6.4, 7.4], [45, Th. 1.1]. By the way, the Arratia flow does not fit into this framework (it needs a discontinuous correlation function, $b(0) = 1$, $b(x) = 0$ for $x \ne 0$), and leads to a black noise (recall Sect. 7f).

Assuming $1 - b(x) \asymp |x|^\alpha$ as $x \to 0$ for some $\alpha \in (0,1)$ (and some additional technical conditions on $b(\cdot)$), Warren and Watanabe [45, Th. 1.3] manage to find the Hausdorff dimension; the inequality

$$\dim C \le \frac{1-\alpha}{2-\alpha} \tag{9c1}$$

holds for almost all spectral sets C, but the strict inequality $\dim C < \frac{1-\alpha}{2-\alpha}$ does not.

By 9b14 it is sufficient to prove (9c1) for μ_f-almost all C, where f is a random variable of the form $\Xi_x^{s,t}$ (the coordinate at t of a particle starting at s, x). The set C' of limit points of C is related to the self-joining $(\alpha_{1-}, \beta_{1-})$ (recall the paragraph after (9b15)). An explicit description of the joining in terms of diffusions, found by Warren and Watanabe, leads them to an explicit description of the random set C'. It is (distributed like) the set of zeros of a diffusion process. If C' happens to be nonempty then $\dim C' = \frac{1-\alpha}{2-\alpha}$.

9c2 Question. Can spectral sets of a noise be of Hausdorff dimension greater than $\frac{1}{2}$? (See also 9b8 and 10a1.)

9d Hilbert spaces

Throughout Sect. 9b the spaces L_2 may be replaced with arbitrary *pointed Hilbert spaces* (recall Sect. 6d). I do it explicitly for Theorem 9b1; the rest of the work is left to the reader.

9d1 Theorem. Let $(H_{s,t}, u_{s,t})_{s<t}$ be a continuous product of pointed Hilbert spaces, satisfying the upward continuity condition (6d16). Then

(a) there exists one and only one projection-valued measure Q on Comp($\mathbb{R}$) (over $H = H_{-\infty,\infty}$) such that

$$Q(\{C : C \cap (s,t) = \emptyset\})H = H_{-\infty,s} u_{s,t} H_{t,\infty} \tag{9d2}$$

whenever $-\infty < s < t < \infty$;

(b,c): the same as in 9b1.

(See also Theorem 9d5.) As before, $H_{-\infty,s} u_{s,t} H_{t,\infty}$ stands for the image of $H_{-\infty,s} \otimes u_{s,t} \otimes H_{t,\infty}$ under the given unitary operator $H_{-\infty,s} \otimes H_{s,t} \otimes H_{t,\infty} \to H_{-\infty,\infty}$.

The upward continuity condition will be assumed for all continuous products of pointed Hilbert spaces.

Every continuous product of pointed Hilbert spaces leads to a factorizing measure type on Comp($\mathbb{R}$). The continuous product is classical if and only if almost all spectral sets are finite. In this case the factorizing measure type is Poissonian (but the underlying measure type on $\mathbb{R}$ need not be shift-invariant). By Theorem 6e3, classicality does not depend on the choice of a unit.

9d3 Question. ([19, Notes 3.11, 10.2]) Does the factorizing measure type depend on the choice of a unit?

9d4 Question. (See [9, Def. 8.2] and [19, Notes 3.6, 5.8 and Sect. 11 (question 1)].) Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces, satisfying 6e1(a-c), and $(u_{s,t})_{s<t}$, $(v_{s,t})_{s<t}$ two units. Does there exist an automorphism of $(H_{s,t})_{s<t}$ sending $(u_{s,t})_{s<t}$ to $(v_{s,t})_{s<t}$? In other words, are the two continuous products of pointed Hilbert spaces $(H_{s,t}, u_{s,t})_{s<t}$, $(H_{s,t}, v_{s,t})_{s<t}$ isomorphic?

(An automorphism may be defined as an invertible embedding to itself, recall 6d8.) If the answer to 9d4 is 'exists', then the answer to 9d3 evidently is 'does not depend'.

Every continuous product of probability spaces $(\Omega_{s,t}, P_{s,t})_{s<t}$ leads to a continuous product of pointed Hilbert spaces $(H_{s,t}, u_{s,t})_{s<t}$ (namely, $H_{s,t} = L_2(\Omega_{s,t}, P_{s,t})$ and $u_{s,t} = 1$ on $\Omega_{s,t}$) and further, to a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$. The classical part of $(\Omega_{s,t}, P_{s,t})_{s<t}$ corresponds to the classical part of $(H_{s,t}, u_{s,t})_{s<t}$, in fact, the classical part of $(H_{s,t})_{s<t}$.

We say that the classical part of $(\Omega_{s,t}, P_{s,t})_{s<t}$ is trivial, if all stable random variables are constant, that is, $\dim L_2(\mathcal{F}^{\text{stable}}) = 1$. On the other hand,

we say that the classical part of $(H_{s,t})_{s<t}$ is trivial, if $\dim H^{\mathrm{cls}} = 0$ (recall (6e5)), which means, no decomposable vectors at all. The latter never happens for $(H_{s,t})_{s<t}$ obtained from some $(\Omega_{s,t}, P_{s,t})_{s<t}$; indeed, $\dim H^{\mathrm{cls}} = \dim L_2(\mathcal{F}^{\mathrm{stable}}) \ge 1$. However, it may happen that $\dim H^{\mathrm{cls}} = 1$; every black noise is an example.

If $\dim H^{\mathrm{cls}} = 1$ then all units are basically the same, and we may treat the corresponding factorizing measure type on $\mathrm{Comp}(\mathbb{R})$ as an (isomorphic) invariant of $(H_{s,t})_{s<t}$. In this case the projection-valued measure Q may be attributed to the embedding $H^{\mathrm{cls}} \subset H$ (rather than the unit). More generally, some Q may be attributed to the embedding $H^{\mathrm{cls}} \subset H$ assuming only $\dim H^{\mathrm{cls}} \ge 1$ (as explained below), which leads to invariants of (not pointed) continuous products of Hilbert spaces.

Let $(H_{s,t})_{s<t}$ be a continuous product of Hilbert spaces, and $H'_{s,t} \subset H_{s,t}$ be (closed linear) subspaces satisfying $\dim H'_{s,t} \ge 1$ and $H'_{r,s} H'_{s,t} = H'_{r,t}$ for $r < s < t$. Then $(H'_{s,t})_{s<t}$ is also a continuous product of Hilbert spaces, and identical maps $H'_{s,t} \to H_{s,t}$ are an embedding of the latter product to the former (recall 6d8). Such a pair of continuous products may be called an *embedded pair.*

Instead of the subspaces $H'_{s,t}$ we may consider (following [19, Sect. 3.2]) Hermitian projections $P_{s,t} : H \to H$ satisfying $P_{s,t} \in \mathcal{A}_{s,t}$, $P_{s,t} \ne 0$ and $P_{r,s} P_{s,t} = P_{r,t}$ for $r < s < t$ (the algebras $\mathcal{A}_{s,t}$ appear in Sect. 3b). Namely, $P_{s,t} H = H_{-\infty,s} H'_{s,t} H_{t,\infty}$. Monotonicity of $P_{s,t}$ in s and t ensures existence of the limit $P_{s-,t+} = \lim_{\varepsilon \to 0+} P_{s-\varepsilon,t+\varepsilon}$ in the strong operator topology. (As before, $-\infty - \varepsilon = -\infty$, $\infty + \varepsilon = \infty$.) In fact, $P_{s-,t+} = P_{s,t}$ unless s or t belongs to a finite or countable set of discontinuity points. Theorem 9d1 is a special case of the following fact.

9d5 Theorem. (Liebscher [19, Th. 1], Tsirelson [39, 2.9]) Let $(H'_{s,t} \subset H_{s,t})_{-\infty \le s < t \le \infty}$ be an embedded pair of continuous products of Hilbert spaces, and $(P_{s,t})_{s<t}$ the corresponding family of projections. Then there exists one and only one projection-valued measure Q on $\mathrm{Comp}([-\infty, \infty])$ (over $H = H_{-\infty,\infty}$) such that

$$Q(\{C : C \cap [s,t] = \emptyset\}) = P_{s-,t+} \tag{9d6}$$

whenever $-\infty \le s \le t \le \infty$.

Proof (sketch). We define Q on the algebra generated by sets of the form $\{C : C \cap [s,t] = \emptyset\}$ by $Q(\{C : C \cap [s,t] = \emptyset\}) = P_{s-,t+}$ and $Q(A \cap B) = Q(A) Q(B)$ (and additivity). The algebra generates the Borel σ-field on $\mathrm{Comp}([-\infty, \infty])$, and we extend Q to the σ-field using [39, 2.4, 2.5, 2.6]. $\square$

The time set $[-\infty, \infty]$ is essential. In the local case (the time set $\mathbb{R}$) we have no global space $H_{-\infty,\infty}$ (and no embeddings $H_{-1,1} \subset H_{-2,2} \subset \dots$), thus, no global Q. Still, we have $Q_{s,t}$ for $s < t$ (since the time set $[s, t]$ is similar to $[-\infty, \infty]$).

A measure type on $\operatorname{Comp}([-\infty, \infty])$ corresponds to each embedded pair. Still, for a given t we have $t \notin C$ for almost all spectral sets C, unless t belongs to a finite or countable set of discontinuity points. Therefore almost all spectral sets are nowhere dense, of Lebesgue measure zero. They are compact subsets of $\mathbb{R}$, unless $-\infty$ or $+\infty$ is a discontinuity point. If there is no discontinuity points at all then we get a factorizing measure type (as defined by 9b7). Especially, the factorizing measure type corresponding to the embedding of the classical part is an invariant of a continuous product of Hilbert spaces [19, Th. 2]. In the local case (the time set $\mathbb{R}$) we get instead a consistent family of factorizing measure types on $\operatorname{Comp}([s, t])$, $-\infty < s < t < \infty$.

Let $(H_{s,t}, u_{s,t})_{s<t}$ be a continuous product of pointed Hilbert spaces (satisfying the upward continuity condition), Q^u the corresponding projection-valued measure, and Q^{U} the projection-valued measure corresponding to the embedding of the classical part. Similarly to (9b15),

$$Q^u(\{C : C' \cap (s, t) = \emptyset\}) = Q^u(\{C : |C \cap (s, t)| < \infty\}) = P_{s+,t-} = P_{s,t}\,,$$

thus $Q^u(\{C : C' \cap (s, t) = \emptyset\}) = Q^{\mathrm{U}}(\{C : C \cap (s, t) = \emptyset\})$, which implies

$$Q^u(\{C : C' \in A\}) = Q^{\mathrm{U}}(A) \tag{9d7}$$

for all Borel sets $A \subset \operatorname{Comp}(\mathbb{R})$. This is a useful relation, due to Liebscher [19, Prop. 3.9], between unit-dependent spectral sets C and unit-independent spectral sets; the latter spectral set consists of the limit points of the former!

The factorizing measure types corresponding to Q^u and Q^{U} will be denoted by $\mathcal{M}^u$ and $\mathcal{M}^{\mathrm{U}}$ respectively, and the corresponding spectral sets by C_u and C_{U} (these make sense when saying that C satisfies something a.s.). In some sense, (9d7) allows us to say that $C'_u = C_{\mathrm{U}}$ in distribution (for every u).

If $(H_{s,t})_{s<t}$ is classical then $Q^{\mathrm{U}}(\{\emptyset\}) = \mathbf{1}$, C_u is finite a.s., and C_{U} is empty a.s. In general, $Q^{\mathrm{U}}(\{\emptyset\})H = H^{\mathrm{cls}}$; C_{U} has a chance to be empty.

Dealing with the probabilistic case ($H_{s,t} = L_2(\Omega_{s,t}, P_{s,t})$) we assume that u is 'the probabilistic unit' ($u_{s,t}(\cdot) = 1$ on $\Omega_{s,t}$), unless otherwise stated.

9d8 Example. For the noise of splitting (recall 9b10) C'_u is finite a.s., thus C_{U} is finite a.s. In fact, $\mathcal{M}^{\mathrm{U}} = \mathcal{M}_{\mathrm{Poisson}}$ (for $\mathcal{M}_{\mathrm{Poisson}}$ see 9b9). The same holds for the noise of stickiness.

For a black noise (for instance, 9b11), spectral sets are perfect a.s.; $C_{\mathrm{U}} = C'_u = C_u$.

9d9 Question. The two noises (of splitting and stickiness) mentioned in 9d8 lead to two continuous products of Hilbert spaces (not pointed!) $(H^{\mathrm{split}}_{s,t})_{s<t}$ and $(H^{\mathrm{stick}}_{s,t})_{s<t}$. Are these products isomorphic?

We know that $\mathcal{M}^{\mathrm{U}}_{\mathrm{split}} = \mathcal{M}^{\mathrm{U}}_{\mathrm{stick}} = \mathcal{M}_{\mathrm{Poisson}}$. However, $\mathcal{M}^u_{\mathrm{split}}$ and $\mathcal{M}^u_{\mathrm{stick}}$ are different (u being the probabilistic unit in both cases). Namely, accumulation of C^u_{split} is two-sided, while accumulation of C^u_{stick} is one-sided (from the right only).

Note that the continuous products of Hilbert spaces $(H^{\mathrm{split}}_{s,t})_{s<t}$ and $(H^{\mathrm{stick}}_{s,t})_{s<t}$ in Question 9d9 are not treated as homogeneous, despite the fact that they originate from noises. The desired isomorphism need not intertwine time shifts, it need not be an isomorphism of the two homogeneous continuous products of Hilbert spaces $\big((H^{\mathrm{split}}_{s,t})_{s<t},(\theta^{\mathrm{split},h}_{s,t})_{s<t;h}\big)$ and $\big((H^{\mathrm{stick}}_{s,t})_{s<t},(\theta^{\mathrm{stick},h}_{s,t})_{s<t;h}\big)$ that arise naturally from the two noises (recall Sect. 3c).

In fact, the two *homogeneous* (and not local!) continuous products of Hilbert spaces are non-isomorphic. Indeed, Prop. 6f1 shows that the 'probabilistic' units $u^{\mathrm{split}}, u^{\mathrm{stick}}$ are basically the only shift-invariant units. Therefore the relation $\mathcal{M}^u_{\mathrm{split}} \ne \mathcal{M}^u_{\mathrm{stick}}$ denies isomorphism.

However, Prop. 6f1 does not apply to Arveson systems (recall Def. 6f3 and Th. 6f6). The Arveson systems $(H^{\mathrm{split}}_t)_{t>0}$, $(H^{\mathrm{stick}}_t)_{t>0}$ have units $u_t = \exp\big((\alpha+\beta\mathrm{i})B_t - \frac12\alpha^2 + \mathrm{i}\gamma t\big)$ parametrized by $\alpha,\beta,\gamma\in\mathbb{R}$; here $(B_t)_t = (a_{0,t})_t$ is the corresponding Brownian motion. An isomorphism must send a unit into a unit, but may change the parameters α,β,γ.

9d10 Question. The two noises (of splitting and stickiness) mentioned in 9d8 lead to two Arveson systems. Are these systems isomorphic?

In 9d9 (unlike 9d10) on one hand, the isomorphism need not be shift-invariant; on the other hand, it must act on the global space $H_{-\infty,\infty}$.

Let $(H_t)_{t>0}$ be an Arveson system and $(u_t)_{t>0}$ a unit. They lead to a homogeneous local continuous product of pointed Hilbert spaces, and we may enlarge the time set $\mathbb{R}$ to $[-\infty,\infty]$, thus getting a shift-invariant factorizing measure type $\mathcal{M}^u$ on $\mathrm{Comp}(\mathbb{R})$; but the enlargement depends on the unit u. Waiving the unit we have a homogeneous *local* continuous product of Hilbert spaces, thus, a consistent shift-invariant factorizing family of measure types $(\mathcal{M}_{s,t})_{-\infty<s<t<\infty}$; they correspond to embeddings $H^{\mathrm{cls}}_{s,t}\subset H_{s,t}$. However, the absence of the 'global' embedding $H^{\mathrm{cls}}_{-\infty,\infty}\subset H_{-\infty,\infty}$ does not prevent us from introducing a 'global' measure type $\mathcal{M}$ on $\mathrm{Comp}(\mathbb{R})$. To this ens we note that spaces $\mathrm{Comp}([-n,n])$ are related not only by projections $\mathrm{Comp}([-n-1,n+1]) \to \mathrm{Comp}([-n,n])$, $C \mapsto C \cap [-n,n]$, but also by embeddings $\mathrm{Comp}([-n,n]) \subset \mathrm{Comp}([-n-1,n+1])$. The measure classes (see 10a2) $(\mathrm{Comp}([-n,n]), \mathcal{M}_{-n,n})$ form not only a projective (inverse) system, but also an inductive (direct) system. In contrast to probability spaces (suitable for projective but not inductive limits), for measure classes we may take inductive limits (but not projective limits). Thus, we may define $\mathcal{M}$ as the measure type on $\mathrm{Comp}(\mathbb{R})$ compatible with all $\mathcal{M}_{-n,n}$ in the sense that the conditional distribution of C given that $C \subset [-n,n]$ belongs to $\mathcal{M}_{-n,n}$. (Existence and uniqueness of such $\mathcal{M}$ is easy to check.)

9d11 Corollary. Every Arveson system of type II leads to a measure type $\mathcal{M}^{\mathrm{U}}$ on $\mathrm{Comp}(\mathbb{R})$; every unit u of the Arveson system leads to a measure type $\mathcal{M}^u$ on $\mathrm{Comp}(\mathbb{R})$; and $\mathcal{M}^{\mathrm{U}}$ is the image of $\mathcal{M}^u$ under the map $C \mapsto C'$.

Do not think, however, that type II systems are simpler than type III systems. In fact, invariants like $\mathcal{M}^{\mathrm{U}}$ catch only a small part of the structure of type II systems. The rest of the structure cannot be simpler than the whole structure of type III systems! See [19, Sect. 6.4].

10 Continuous products of measure classes

10a From random sets to Hilbert spaces

We start with a result that involves an idea of Anatoly Vershik (private communication, 1994) of a continuous product of measure classes as a source of a continuous product of Hilbert spaces, and an idea of Jonathan Warren (private communication, 1999) of constructing a continuous product of measure classes out of a given random set. See Liebscher [19, Prop. 4.1 and Sect. 8.2] and Tsirelson [39, Lemma 5.3]. For now this is the richest source of (non-isomorphic) continuous products of Hilbert spaces with nontrivial classical part. (See also 9b8.) First, recall that every continuous product of pointed Hilbert spaces, satisfying the upward continuity condition, leads to a factorizing measure type on $\mathrm{Comp}(\mathbb{R})$, as explained after Theorem 9d1.

10a1 Theorem. For every factorizing measure type $\mathcal{M}$ on $\mathrm{Comp}(\mathbb{R})$ (as defined by 9b7) there exists a continuous product of pointed Hilbert spaces, satisfying the upward continuity condition (6d16), such that the corresponding factorizing measure type is equal to $\mathcal{M}$.

A proof will be sketched later. 'Square roots of measures', introduced by Accardi [1], are instrumental. For more definitions and basic facts see

[4, Sect. 14.4] and [39, Sect. 3]. My definition (below) is somewhat more restrictive than Arveson's, since I restrict myself to measure classes generated by a single measure (and standard probability spaces).

10a2 Definition. A *measure class* is a triple $(\Omega, \mathcal{F}, \mathcal{M})$ consisting of a set Ω, a σ-field $\mathcal{F}$ on Ω and a set $\mathcal{M}$ of probability measures on $(\Omega, \mathcal{F})$ such that for some $\mu \in \mathcal{M}$ the probability space $(\Omega, \mathcal{F}, \mu)$ is standard and for every probability measure ν on $(\Omega, \mathcal{F})$,

$$\nu \sim \mu \quad \text{if and only if} \quad \nu \in \mathcal{M}\,,$$

$\nu \sim \mu$ denoting mutual absolute continuity.

Hilbert spaces $L_2(\mu)$, $L_2(\nu)$ for $\mu, \nu \in \mathcal{M}$ may be glued together via the unitary operator $L_2(\mu) \to L_2(\nu)$,

$$f \mapsto \sqrt{\frac{\mu}{\nu}} f\,;$$

here $\frac{\mu}{\nu}$ is the Radon-Nikodym derivative (denoted also by $\frac{d\mu}{d\nu}$). These spaces may be treated as 'incarnations' of a single Hilbert space $L_2(\Omega, \mathcal{F}, \mathcal{M})$. The general form of an element of $L_2(\Omega, \mathcal{F}, \mathcal{M})$ is $f\sqrt{\mu}$, where $\mu \in \mathcal{M}$ and $f \in L_2(\Omega, \mathcal{F}, \mu)$, taking into account the relation

$$f\sqrt{\mu} = \biggl(\sqrt{\frac{\mu}{\nu}} f \biggr) \sqrt{\nu}\,.$$

Any isomorphism of measure classes induces naturally a unitary operator between the corresponding Hilbert spaces.

The product of two measure classes is defined naturally, and

$$L_2\bigl((\Omega, \mathcal{F}, \mathcal{M}) \times (\Omega', \mathcal{F}', \mathcal{M}') \bigr) = L_2(\Omega, \mathcal{F}, \mathcal{M}) \otimes L_2(\Omega', \mathcal{F}', \mathcal{M}')\,;$$

that is, we have a canonical unitary operator between these spaces, namely, $f\sqrt{\mu} \otimes f'\sqrt{\mu'} \mapsto (f \otimes f')\sqrt{\mu \otimes \mu'}$, where $(f \otimes f')(\omega, \omega') = f(\omega) f'(\omega')$. I'll write in short $L_2(\mathcal{M})$ instead of $L_2(\Omega, \mathcal{F}, \mathcal{M})$; thus, $L_2(\mathcal{M} \times \mathcal{M}') = L_2(\mathcal{M}) \otimes L_2(\mathcal{M}')$. Everyone knows the similar fact for measure spaces, $L_2(\mu \times \mu') = L_2(\mu) \otimes L_2(\mu')$. Here is a counterpart of Def. 2c6.

10a3 Definition. A *continuous product of measure classes* consists of measure classes $(\Omega_{s,t}, \mathcal{M}_{s,t})$ (given for all $s, t \in [-\infty, \infty]$, $s < t$), and isomorphisms $(\Omega_{r,s}, \mathcal{M}_{r,s}) \times (\Omega_{s,t}, \mathcal{M}_{s,t}) \to (\Omega_{r,t}, \mathcal{M}_{r,t})$ (given for all $r, s, t \in [-\infty, \infty]$, $r < s < t$) satisfying the associativity condition:

$$(\omega_1 \omega_2)\omega_3 = \omega_1(\omega_2 \omega_3) \quad \text{for almost all } \omega_1 \in \Omega_{r,s},\ \omega_2 \in \Omega_{s,t},\ \omega_3 \in \Omega_{t,u}$$

whenever $-\infty \le r < s < t \le \infty$.

Note the time set $[-\infty,\infty]$ rather than $\mathbb{R}$. Enlarging $\mathbb{R}$ to $[-\infty,\infty]$ is easy when dealing with probability spaces (as noted after Def. 2c1) but not measure classes (nor Hilbert spaces, as noted after 3a1). Having a *local* continuous product of measure classes (over the time set $\mathbb{R}$) we may choose $\mu_{n,n+1} \in \mathcal{M}_{n,n+1}$ for each $n \in \mathbb{Z}$ and define $\mathcal{M}_{-\infty,\infty}$ as the equivalence class that contains the product of these $\mu_{n,n+1}$. However, another choice of $\mu_{n,n+1}$ may lead to another $\mathcal{M}_{-\infty,\infty}$.

Given a continuous product of measure classes $(\mathcal{M}_{s,t})_{s<t}$, we may construct the corresponding continuous product of Hilbert spaces $(H_{s,t})_{s<t}$; just $H_{s,t} = L_2(\mathcal{M}_{s,t})$.

10a4 Question. Does every continuous product of Hilbert spaces (up to isomorphism) emerge from some continuous product of measure classes?

See also [19, Note 8.4 and Sect. 11 (question 9)]. A counterpart of Def. 2c1 (see 10a7) needs some preparation.

10a5 Definition. Sub-σ-fields $\mathcal{F}_1,\dots,\mathcal{F}_n$ on a measure class $(\Omega,\mathcal{F},\mathcal{M})$ are *independent,* if there exists a probability measure $\mu \in \mathcal{M}$ such that

$$\mu(A_1 \cap \dots \cap A_n) = \mu(A_1)\dots\mu(A_n) \quad \text{for all } A_1 \in \mathcal{F}_1,\dots,A_n \in \mathcal{F}_n\,. \tag{10a6}$$

For independent $\mathcal{F}_1,\dots,\mathcal{F}_n$ the sub-σ-field $\mathcal{F}_1 \vee \dots \vee \mathcal{F}_n$ generated by them will be denoted also by $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$.

Given a product of two measure classes,

$$(\Omega,\mathcal{F},\mathcal{M}) = (\Omega_1,\mathcal{F}_1,\mathcal{M}_1) \times (\Omega_2,\mathcal{F}_2,\mathcal{M}_2)\,,$$

we have two independent sub-σ-fields $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2$ such that $\mathcal{F} = \tilde{\mathcal{F}}_1 \otimes \tilde{\mathcal{F}}_2$; roughly,

$$\tilde{\mathcal{F}}_1 = \{A \times \Omega_2 : A \in \mathcal{F}_1\}\,, \quad \tilde{\mathcal{F}}_2 = \{\Omega_1 \times B : B \in \mathcal{F}_2\}$$

(however, all negligible sets must be added).

And conversely, every two independent sub-σ-fields $\mathcal{F}_1,\mathcal{F}_2$ such that $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ emerge from a representation of $(\Omega,\mathcal{F},\mathcal{M})$ (up to isomorphism) as a product; in fact, $(\Omega_k,\mathcal{F}_k,\mathcal{M}_k) = (\Omega,\mathcal{F},\mathcal{M})/\mathcal{F}_k$ is the quotient space.

The following definition (in the style of 2c1) is equivalent to 10a3.

10a7 Definition. A *continuous product of measure classes* consists of a measure class $(\Omega,\mathcal{F},\mathcal{M})$ and sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ (given for all $s,t \in [-\infty,\infty]$, $s<t$) such that $\mathcal{F}_{-\infty,\infty} = \mathcal{F}$ ('non-redundancy'), and

$$\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t} \quad \text{whenever } -\infty \le r < s < t \le \infty\,. \tag{10a8}$$

Every factorizing measure type $\mathcal{M}$ on $\mathrm{Comp}(\mathbb{R})$ leads to a continuous product of measure classes $(\mathrm{Comp}(s,t), \mathcal{M}_{s,t})_{s<t}$; recall (9b6).

Sketch of the proof of Theorem 10a1. The factorizing measure type $\mathcal{M}$ on $\mathrm{Comp}(\mathbb{R})$ leads to a continuous product of measure classes $(\mathrm{Comp}(s,t), \mathcal{M}_{s,t})_{s<t}$ and further, to a continuous product of Hilbert spaces $(H_{s,t})_{s<t}$, $H_{s,t} = L_2(\mathcal{M}_{s,t})$. Measures of $\mathcal{M}$ have an atom at $\emptyset$, that is, $\mu(\{\emptyset\}) > 0$. (Indeed, on small intervals $\mu_{s,t}(\{\emptyset\}) > 0$ since it is close to 1; cover $[-\infty, \infty]$ by a finite number of such intervals, and multiply.) We define $u_{s,t}$ as the root of the probability measure concentrated at the atom, thus getting a continuous product of pointed Hilbert spaces $(H_{s,t}, u_{s,t})_{s<t}$. The upward continuity follows from the fact that $\mathrm{Comp}(s,t) = \bigcup_{\varepsilon>0} \mathrm{Comp}(s+\varepsilon, t-\varepsilon) \bmod 0$. The projection-valued measure Q given by Theorem 9d1 satisfies $Q(\{C : C \cap (s,t) = \emptyset\})H = H_{-\infty,s} u_{s,t} H_{t,\infty}$, the latter being the space of all vectors $\psi = f\sqrt{\mu}$ such that the measure $|\psi|^2 = |f|^2 \cdot \mu$ on $\mathrm{Comp}(\mathbb{R})$ is concentrated on the set $\{C : C \cap (s,t) = \emptyset\}$. Thus, $\langle Q(A)\psi, \psi\rangle = |\psi|^2(A)$ for sets $A \subset \mathrm{Comp}(\mathbb{R})$ of the form $A = \{C : C \cap (s,t) = \emptyset\}$. The same holds for finite intersections of such sets, therefore, for all measurable $A \subset \mathrm{Comp}(\mathbb{R})$. It means that the spectral measure of $\psi = f\sqrt{\mu}$ is $|\psi|^2 = |f|^2 \cdot \mu$. $\square$

If $\mathcal{M}$ is shift-invariant then the continuous product constructed above is homogeneous, and leads to an Arveson system (recall 3c9). If, in addition, $\mathcal{M}$-almost all C are perfect then the constructed Arveson system has no other units, that is, is of type II_0.

10a9 Corollary. (a) Every shift-invariant factorizing measure type $\mathcal{M}$ on $\mathrm{Comp}(\mathbb{R})$ is equal to $\mathcal{M}^u$ for some unit u of some Arveson system.

(b) If, in addition, $\mathcal{M}$ is concentrated on (the set of all) perfect subsets of $\mathbb{R}$ then $\mathcal{M} = \mathcal{M}^U$ for some Arveson system of type II_0.

(See also [19, Th. 3] for a stronger result.) Thus, Arveson systems of type II_0 are at least as diverse as shift-invariant factorizing measure types on the space of all perfect subsets of $\mathbb{R}$.

The set of zeros of a Brownian motion is an example of such measure type. More exactly, we may consider the random set $\{s \in (0,t) : a + B_s = 0\}$ for given $t, a \in (0, \infty)$; the distribution of the random set depends on a, but its measure type does not. The corresponding shift-invariant factorizing measure type on $\mathrm{Comp}(\mathbb{R})$ exists and is unique. The random set is perfect, of Hausdorff dimension $\frac{1}{2}$ (unless empty).

Similarly, an example of a random set of any desired Hausdorff dimension between 0 and 1 is given by zeros of a Bessel process [37, Sect. 3]. See also [19, Sect. 4.4].

However, random sets are much more diverse than Bessel processes. For every perfect set of Hausdorff dimension less than $\frac{1}{2}$ there exists a random set (I mean, a shift-invariant factorizing measure type) obtained from the given (nonrandom) set by a random perturbation preserving almost all the microstructure of the given set. It may be called the *barcode construction,* see [39, Sect. 6]. In contrast to Bessel zeros, random sets obtained from the barcode construction are in general not invariant under time reversal ($t \mapsto -t$) and time rescaling ($t \mapsto ct$ for $c \in (0, \infty)$, $c \neq 1$); they gives us a continuum of mutually non-isomorphic asymmetric Arveson systems of type II_0 [39, Th. 7.3].

10b From off-white noises to Hilbert spaces

Random compact subsets of $\mathbb{R}$ are one out of many sources of continuous products of measure classes. One may use random compact (or closed) subsets of $\mathbb{R} \times L$ for some locally compact space L [19, Sect. 4.1 and Sect. 11 (question 3)], random measures [19, Note 6.8 and Sect. 8.3], etc. But first of all we should try Gaussian processes, for several reasons: they occupy a prominent place among random processes; relations between sub-σ-fields reduce to relations between subspaces of a Hilbert space; the white noise is a Gaussian process. Indeed, random sets used before generalize the Poisson process, while off-white noises used below generalize the white noise, and appear to lead to type III Arveson systems (see 10b6).

Of course, the white noise cannot be treated as a random function on $\mathbb{R}$. Gaussian random variables correspond to test functions rather than points, which is harmless; we need only sub-σ-fields $\mathcal{F}_{s,t}$ that correspond to intervals (s, t), not points. The same holds for other Gaussian processes considered here. Being stationary, such process is described by its *spectral measure,* a positive σ-finite measure ν on $[0, \infty)$ such that the Gaussian random variable corresponding to a test function f has mean 0 and variance $\int |\hat{f}(\lambda)|^2 \, \nu(\mathrm{d}\lambda)$; here $\hat{f}(\lambda) = (2\pi)^{-1/2} \int f(t) \mathrm{e}^{-\mathrm{i}\lambda t} \, \mathrm{d}t$ is the Fourier transform of f. We restrict ourselves to measures ν majorized by Lebesgue measure (that is, $\nu(\mathrm{d}\lambda) \le C \mathrm{d}\lambda$ for some C; see [38] and [39, Sect. 9] for the general case). Thus, every $f \in L_2(\mathbb{R})$ is an admissible test function. The space G of Gaussian random variables may be identified with the Hilbert space $L_2(\nu)$. Each interval $(s, t) \subset \mathbb{R}$ leads to a subspace $G_{s,t} \subset G$ defined as the closure of $\{\hat{f} : f \in L_2(s, t)\}$, and the corresponding sub-σ-field $\mathcal{F}_{s,t} \subset \mathcal{F}$.

For the white noise the 'past' and 'future' spaces $G_{-\infty,0}$, $G_{0,\infty}$ are orthogonal; the 'past' and 'future' sub-σ-field $\mathcal{F}_{-\infty,0}$, $\mathcal{F}_{0,\infty}$ are independent, and we have a continuous product of *probability spaces.* More generally, in order to give us a continuous product of *measure classes,* these sub-σ-fields should be independent for some equivalent measure. A necessary and sufficient condition is well-known (see [38, Th. 3.2], [39, Th. 9.7]): $\nu(\mathrm{d}\lambda) = \mathrm{e}^{\varphi(\lambda)}\mathrm{d}\lambda$ for some $\varphi : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty \int_0^\infty \frac{|\varphi(\lambda_1) - \varphi(\lambda_2)|^2}{|\lambda_1 - \lambda_2|^2} \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 < \infty \, . \tag{10b1}$$

A sufficient condition for (10b1) is available (see [38, Prop. 3.6(b)], [39, (9.11)]):

$$\varphi \text{ is continuously differentiable, and} \quad \int_0^\infty \left| \frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi(\lambda) \right|^2 \lambda \, \mathrm{d}\lambda < \infty \, . \tag{10b2}$$

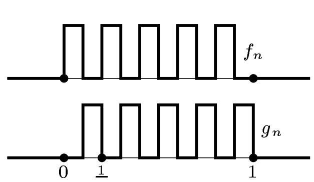
In particular, the sufficient condition is satisfied by any strictly positive smooth function $\lambda \mapsto \mathrm{e}^{\varphi(\lambda)} = \nu(\mathrm{d}\lambda)/\mathrm{d}\lambda$ such that for λ large enough, one of the following equalities holds:

$$\frac{\nu(\mathrm{d}\lambda)}{\mathrm{d}\lambda} = \ln^{-\alpha} \lambda \, , \quad 0 \le \alpha < \infty \, ; \tag{10b3}$$

$$\frac{\nu(\mathrm{d}\lambda)}{\mathrm{d}\lambda} = \exp(- \ln^{\beta} \lambda) \, , \quad 0 < \beta < \frac{1}{2} \, ; \tag{10b4}$$

see [38, Examples 3.11, 3.12], [39, (9.12)–(9.13)]. Every such ν leads to a continuous product of measure classes $(\Omega, \mathcal{F}, \mathcal{M}), (\mathcal{F}_{s,t})_{s<t}$. Namely, $\mathcal{M}$ is the equivalence class containing the Gaussian measure γ whose spectral measure is ν. The group $(T_h)_{h\in\mathbb{R}}$ of time shifts leaves invariant the equivalence class $\mathcal{M}$ and moreover, the measure γ. The corresponding Hilbert spaces $H_{s,t} = L_2(\Omega, \mathcal{F}_{s,t}, \mathcal{M})$, being a homogeneous continuous product of Hilbert spaces (over the time set $[-\infty, \infty]$), lead to an Arveson system.

The white noise, contained in (10b3) as the case $\alpha = 0$, leads to a classical continuous product of probability spaces $(\Omega, \mathcal{F}, \gamma), (\mathcal{F}_{s,t})_{s<t}$ and classical (type I) Arveson system. Generally, $(\Omega, \mathcal{F}, \gamma), (\mathcal{F}_{s,t})_{s<t}$ is not a continuous product of probability spaces, since the past and the future are not independent on $(\Omega, \mathcal{F}, \gamma)$. However, a decomposable vector $\psi = f\sqrt{\mu}$, $\|\psi\| = 1$ (if any) gives us a probability measure $|\psi|^2 = |f|^2 \cdot \mu$, decomposable in the sense that $(\Omega, \mathcal{F}, |\psi|^2), (\mathcal{F}_{s,t})_{s<t}$ is a continuous product of probability spaces. Especially, the measure $|\psi|^2$ makes independent the pair of random variables corresponding to such 'comb' test functions f_n, g_n (for any given n):



$$f_n(t) + g_n(t) = 1 \, , \quad f_n(t) - g_n(t) = \operatorname{sgn} \sin \pi n t \quad \text{for } t \in (0,1) \, .$$

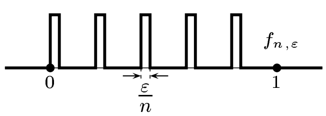
If ν satisfies the condition

$$\frac{\nu(\mathrm{d}\lambda)}{\mathrm{d}\lambda} \to 0 \quad \text{for } \lambda \to \infty \tag{10b5}$$

(which excludes the white noise), then $\|\hat{f}_n - \hat{g}_n\|_{L_2(\nu)} \to 0$ (see [39, 10.2]), therefore the *independent* random variables on $(\Omega, \mathcal{F}, |\psi|^2)$ corresponding to f_n and g_n converge (as $n \to \infty$, in probability) to *the same* random variable Z corresponding to the test function $\frac{1}{2} \cdot \mathbf{1}_{(0,1)}$. We see that Z is constant on $(\Omega, \mathcal{F}, |\psi|^2)$, therefore, has an atom on $(\Omega, \mathcal{F}, \gamma)$ (since the measure $|\psi|^2$ is absolutely continuous w.r.t. γ). However, the *normal* distribution of Z on $(\Omega, \mathcal{F}, \gamma)$ is evidently nonatomic! The conclusion follows.

10b6 Proposition. [39, 10.3] If ν satisfies (10b5) then the corresponding continuous product of Hilbert spaces has no decomposable vectors, and the corresponding Arveson system is of type III.

Every $\alpha > 0$ in (10b3) (as well as every β in (10b4)) gives us a nonclassical Arveson system. One may guess that, the larger the parameter α, the more nonclassical the system. Striving to an invariant able to confirm the guess, we introduce 'spaced comb' test functions $f_{n,\varepsilon}$,



$$f_{n,\varepsilon}(t) = 1 \text{ if } (nt \bmod 1) \in (0, \varepsilon),$$

and the corresponding Gaussian random variables $Z_{n,\varepsilon}$ on $(\Omega, \mathcal{F}, \gamma)$. It is instructive to consider the correlation coefficient

$$\rho_{n,\varepsilon} = \frac{\int Z_{n,\varepsilon} Z \,\mathrm{d}\gamma}{(\int Z_{n,\varepsilon}^2 \,\mathrm{d}\gamma)^{1/2} (\int Z^2 \,\mathrm{d}\gamma)^{1/2}}$$

between $Z_{n,\varepsilon}$ and the random variable Z corresponding to the test function $\mathbf{1}_{(0,1)}$. For the white noise, $\rho_{n,\varepsilon}$ does not depend on n (in fact, $\rho_{n,\varepsilon} = \sqrt{\varepsilon}$). However, (10b5) implies $\rho_{n,\varepsilon} \to 1$ as $n \to \infty$ (for every $\varepsilon \in (0,1)$). On the other hand, $\rho_{n,\varepsilon} \to 0$ as $\varepsilon \to 0$ (for every n). It is more interesting to take $\lim_n \rho_{n,\varepsilon_n}$ when $\varepsilon_n \to 0$. Especially, for ν of the form (10b3) with $\alpha > 0$,

$$\begin{gathered} \text{if } \varepsilon_n \ln^\alpha n \to \infty \text{ then } \rho_{n,\varepsilon_n} \to 1\,, \\ \text{if } \varepsilon_n \ln^\alpha n \to 0 \text{ then } \rho_{n,\varepsilon_n} \to 0\,, \end{gathered}$$

which gives us a clue to a useful invariant. We should consider 'spaced comb' sets $E_n \subset (0,1)$ (namely, $E_n = \{t \in (0,1) : f_{n,\varepsilon_n}(t) = 1\}$), the decompositions $(0,1) = E_n \cup E_n^{\mathrm{c}}$ (where $E_n^{\mathrm{c}} = (0,1) \setminus E_n$) of the interval $(0,1)$, and the corresponding decompositions (see also (11a1))

$$\begin{gathered}
\mathcal{M}_{0,1} = \mathcal{M}_{E_n} \times \mathcal{M}_{E_n^{\mathrm{c}}} \,, \\
G_{0,1} = G_{E_n} \oplus G_{E_n^{\mathrm{c}}} \,, \\
H_{0,1} = H_{E_n} \otimes H_{E_n^{\mathrm{c}}} \,, \\
\mathcal{A}_{0,1} = \mathcal{A}_{E_n} \otimes \mathcal{A}_{E_n^{\mathrm{c}}}
\end{gathered}$$

of the measure class $\mathcal{M}_{0,1}$, the Gaussian space $G_{0,1}$ (a linear subspace of G), the Hilbert space $H_{0,1} = L_2(\mathcal{M}_{0,1})$ and the algebra $\mathcal{A}_{0,1}$ of operators on $H_{0,1}$. Their asymptotic behavior (as $n \to \infty$) should be sensitive to the asymptotic behavior (as $\lambda \to \infty$) of $\nu(\mathrm{d}\lambda)/\mathrm{d}\lambda$.

The rest of the story, sketched below, belongs to functional analysis rather than probability. (In fact, the whole story is translated into the language of analysis by Bhat and Srinivasan [10].) The norm on the Hilbert space $H_{0,1}$ is singled out by the Gaussian measure γ. However, the equivalence class $\mathcal{M}_{0,1}$ contains many Gaussian measures; γ is just one of them. Accordingly, G should not be treated as a Hilbert space. Its natural structure is given by an equivalence class of norms (rather than a single norm), but the equivalence is much stronger than topological, it may be called FHS-equivalence, and G may be called an FHS-space [39, 8.5]. The decomposition $G_{0,1} = G_{E_n} \oplus G_{E_n^{\mathrm{c}}}$ is orthogonal in the FHS sense, that is, orthogonal in *some* (depending on n) norm of the given class. However, the decomposition $\mathcal{A}_{0,1} = \mathcal{A}_{E_n} \otimes \mathcal{A}_{E_n^{\mathrm{c}}}$ is treated as usual; $\mathcal{A}_{E_n^{\mathrm{c}}}$ is the commutant of $\mathcal{A}_{E_n}$. In fact, *every* FHS space G leads to a Hilbert space $H = \mathrm{Exp}(G)$, and *every* orthogonal decomposition of the FHS space G leads to a decomposition of the operator algebra of H into tensor product.

The desired invariant (of an Arveson system) is the set of all sequences $\varepsilon_n \to 0$ such that

$$\limsup_{n\to\infty} \mathcal{A}_{E_n} \quad \text{is trivial}\,; \tag{10b7}$$

the latter means that for all $A_1 \in \mathcal{A}_{E_1}, A_2 \in \mathcal{A}_{E_2}, \dots$ all limit points (in the weak operator topology) of the sequence $A_1, A_2, \dots$ are scalar operators. Condition (10b7), taken from [10, Def. 26 and Th. 30], is equivalent to the condition [39, 2.2]: for every trace-class operator $R : H_{0,1} \to H_{0,1}$ satisfying $\mathrm{trace}(R) = 0$,

$$\sup_{A \in \mathcal{A}_{E_n}, \|A\| \le 1} |\,\mathrm{trace}(AR)| \to 0 \quad \text{as } n \to \infty\,. \tag{10b8}$$

(Note that the sequence $(E_n)_n$ is not decreasing, in contrast to [19, Prop. 10.1 and Cor. 10.2].) Fortunately, the condition can be reformulated in terms of G_{E_n} and $G_{E_n^{\mathrm{c}}}$.

10b9 Proposition. Condition (10b7) holds if and only if

$$\limsup_{n\to\infty} G_{E_n} = \{0\} \quad \text{and} \quad \liminf_{n\to\infty} G_{E_n^c} = G \,. \tag{10b10}$$

The relation $\limsup G_{E_n} = \{0\}$ means that for all $g_1 \in G_{E_1}, g_2 \in G_{E_2}, \dots$ the only possible limit point of the sequence $g_1, g_2, \dots$ is 0. The relation $\liminf G_{E_n^c} = G$ means that every $g \in G$ is the limit of some sequence $g_1, g_2, \dots$ such that $g_1 \in G_{E_1^c}, g_2 \in G_{E_2^c}, \dots$

Proposition 10b9 appeared first in [39, 11.3] with a long, complicated proof (occupying Sections 11 and 12 of [39]). For a substantially simpler proof see [10, Th. 30].

10b11 Proposition. (see [39, 13.10]) Let spectral measures ν_1, ν_2 satisfy (10b3) with parameters α_1, α_2 respectively, $0 < \alpha_1 < \alpha_2$. Then there exists a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$ and the corresponding 'spaced comb' sets E_n satisfy (10b7) for the Arveson system corresponding to ν_1 but not ν_2.

In fact, the sequence $\varepsilon_n = \ln^{-c} n$ fits for $\alpha_1 < c \le \alpha_2$. Of course, (10b10) is checked instead of (10b7). Still, the proof uses tedious calculations [39, Sect. 13]. Here is the conclusion.

10b12 Theorem. (Tsirelson [39, 13.11]) There is a continuum of mutually non-isomorphic Arveson systems of type III.

11 Beyond the one-dimensional time

11a Boolean base

The definitions of continuous products (2c6, 3a1, 6d6, 10a3) center round the relations

$$\Omega_{r,t} = \Omega_{r,s} \times \Omega_{s,t} \,, \quad H_{r,t} = H_{r,s} \otimes H_{s,t} \quad \text{etc.}$$

for $r < s < t$. These are special cases (for $A = (r, s)$ and $B = (s, t)$) of more general relations

$$\Omega_{A \uplus B} = \Omega_A \times \Omega_B \,, \quad H_{A \uplus B} = H_A \otimes H_B \quad \text{etc.} \tag{11a1}$$

for *elementary sets* A, B satisfying $A \cap B = \emptyset$. By an elementary set I mean a union of finitely many intervals, treated modulo finite sets. (For example, $(-5, 1) \cup \{2\} \cup [9, \infty)$ is an elementary set, and $[-5, 1) \cup (9, \infty)$ is the same elementary set.) The disjoint union $A \uplus B$ is just $A \cup B$ provided that $A \cap B = \emptyset$. Of course,

$$\Omega_{(r,s)\cup(t,u)} = \Omega_{r,s} \times \Omega_{t,u}\,, \quad H_{(r,s)\cup(t,u)} = H_{r,s} \otimes H_{t,u} \quad \text{etc.}$$

for $r < s < t < u$; the same for any finite number of intervals.

Elementary sets are a Boolean algebra. More generally, we may consider an arbitrary Boolean algebra $\mathcal{A}$ (instead of the time set T) and define continuous products (of probability spaces, Hilbert spaces etc.) over $\mathcal{A}$ by requiring (11a1) for all disjoint $A, B \in \mathcal{A}$. This 'boolean base' approach is used in [2], [13], [41, Sect. 1], [36]. Early works [2], [13] concentrate on *complete* Boolean algebras (which means that every subset of the Boolean algebra has a supremum in the algebra), striving to prove that all continuous products (satisfying appropriate continuity conditions) are classical. More recent works [41], [36, Sect. 2] prefer incomplete Boolean algebras and nonclassical continuous products.

The following result answers a question of Feldman [13, 1.9].

11a2 Theorem. (Tsirelson [40, 6c7], see also [36, 3.2]) A continuous product of probability spaces, satisfying the upward continuity condition, is classical if and only if the map $E \mapsto \mathcal{F}_E$ can be extended from the algebra of elementary sets to the Borel σ-field, satisfying $\mathcal{F}_{A \uplus B} = \mathcal{F}_A \otimes \mathcal{F}_B$ and the upward continuity ($A_n \uparrow A$ implies $\mathcal{F}_{A_n} \uparrow \mathcal{F}_A$) for Borel sets A, B, A_n.

11b Two-dimensional base

The two black noises considered in Sections 7f, 7j are scaling limits of discrete models driven by two-dimensional arrays of independent random variables. Their one-dimensional time is just one of the two dimensions. The sub-σ-field $\mathcal{F}_{s,t}$ corresponds to the strip $(s,t) \times \mathbb{R} \subset \mathbb{R}^2$. It should be possible to define sub-σ-fields $\mathcal{F}_A$ for more general sets $A \subset \mathbb{R}^2$. The whole Borel σ-field of $\mathbb{R}^2$ is too big (recall Theorem 11a2); the Boolean algebra generated by rectangles $(s,t) \times (a,b)$ is a modest choice. No such theory is available for now.

The class of appropriate sets $A \subset \mathbb{R}^2$ should depend on the model. The same may be said about the one-dimensional time, if we do not restrict ourselves to intervals (as in Sections 2–10) or elementary sets (as in Sect. 11a). The Hausdorff dimension $\dim(\partial A)$ of the boundary of A could be relevant. See also [41, end of Sect. 2d]. It could be related to the Hausdorff dimension of spectral sets (recall 9b, 9c).

The model of Sect. 7f, being a kind of oriented percolation, is much simpler than the true percolation.

11b1 Question. (See also [40, 8a1].) For the (conformally invariant) scaling limit of the critical site percolation on the triangular lattice, invent an appropriate conformally invariant Boolean algebra of sets on the plane and define the corresponding sub-σ-fields $\mathcal{F}_A$ satisfying $\mathcal{F}_{A \uplus B} = \mathcal{F}_A \otimes \mathcal{F}_B$. Is it possible?

Hopefully, the answer is affirmative, that is, the two-dimensional noise of percolation will be defined. Then it should appear to be a (two-dimensional) black noise, see [40, 8a2].

It would be the most important example of a black noise!

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Index

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